10.4 Coupled motions in 1D

Thinking of a car, a plane, a person on a bicycle or a satellite as a single particle is often edifying, and sufficient for many engineering purposes. However, the one-particle model is also often inadequate. That the parts of a machine or structure move relative to each other is obviously sometimes important; many important engineering systems have parts that move independently.

Here we begin the study of independent, but coupled, motions of parts. The independent motions are coupled in that the motion of each part may affect the motion of the others.

Example: Car suspension.
A model of a car suspension treats the wheel as one particle and the car as another. The wheel is coupled to the ground by a tire and to the car by the suspension. In a first analysis the only motion to consider would be vertical for both the wheel and the car. Think of the ground as moving up and down and 'forcing' the motion of the car and wheel system.

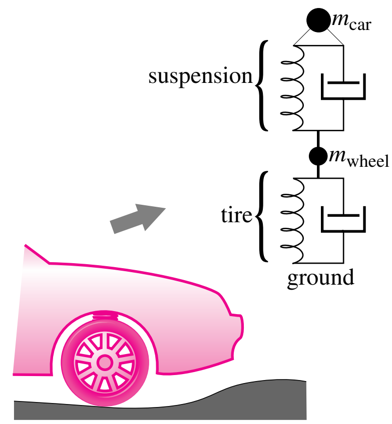

Figure 10.34: The vertical motions of a car and wheel might be considered as coupled motions in 1D.

Still using one-dimensional mechanics, we consider systems that can be modeled as two or more particles. Such one-dimensional coupled motion analysis is common in engineering practice in situations where there are connected parts that all move in about the same direction, but the parts do not move the same amount or necessarily at the same time. Many of the ideas generalize to systems where parts, each with one degree of freedom, are coupled together. Many generalizations apply even if each degree of freedom is quite different from the others. These generalizations to more general coupled motions come later in the book.

The primary goal in this section is to develop two skills:

- To write correct equations of motion for a line of particles connected to each other with springs and dashpots, and
- To simulate the motions of such systems on a computer.
- (the third of the two things, really implicit in the first two) To use the simulation results to find errors in the equations.

The concept of 'normal modes' is postponed to Section 11.3 The simplest way of dealing with the coupled motion of two or more particles is

- to write $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each particle and then
- to use the forces on the free-body diagrams to evaluate the forces.

Because the most common models for the interaction forces are springs and dashpots (see chapter 3), one needs to account for the relative positions and velocities of the particles.

Relative motion in one dimension

If the position of A is $\vec{\boldsymbol{r}}_{\text{A}}$, and B's position is $\vec{\boldsymbol{r}}_{\text{B}}$, then B's position *relative* to A is

$$\vec{\boldsymbol{r}}_{B/A} = \vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A.$$

Relative velocity and acceleration are similarly defined by subtraction, or by differentiating the above expression, as

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A \text{ and } \vec{\boldsymbol{a}}_{B/A} = \vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A.$$

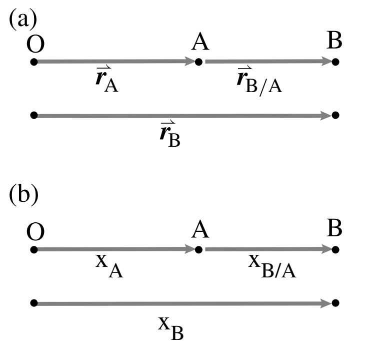

Figure 10.35: The relative position of points A and B in one dimension.

In one dimension, the relative position diagram of fig. 1.5 on page 51 becomes fig. 10.35. $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. So, we can write,

$$\begin{aligned} x_{B/A} &\equiv x_B - x_A, \\ v_{B/A} &\equiv v_B - v_A = \frac{d}{dt}x_{B/A}, \text{ and} \\ a_{B/A} &\equiv a_B - a_A = \frac{d}{dt}v_{B/A} = \frac{d^2}{dt^2}x_{B/A}. \end{aligned}$$

An alternative notation, discussed in Chapter 2, is x_{AB} where the directed line AB is equivalent to the position of B relative to A:

$$x_{AB} = x_{B/A}$$

Example: Two masses connected by a spring.
Consider the two masses on a frictionless support (fig. 10.36). Assume the spring is un-stretched when $x_1 = x_2 = 0$. After drawing free-body diagrams of the two masses we can write $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each mass:

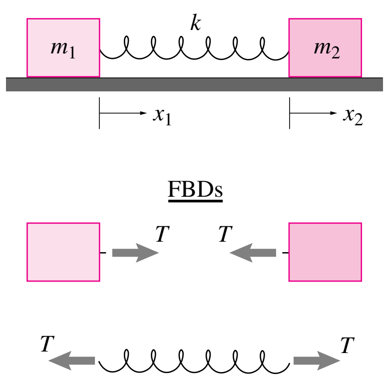

Figure 10.36

$$\begin{aligned} &\textbf{mass 1:} \quad \vec{\boldsymbol{F}}_1 = m\vec{\boldsymbol{a}}_1 \quad \Rightarrow \quad T\hat{\boldsymbol{\imath}} = m_1\ddot{x}_1\hat{\boldsymbol{\imath}} \\ &\textbf{mass 2:} \quad \vec{\boldsymbol{F}}_2 = m\vec{\boldsymbol{a}}_2 \quad \Rightarrow \quad -T\hat{\boldsymbol{\imath}} = m_2\ddot{x}_2\hat{\boldsymbol{\imath}} \end{aligned} \tag{10.41}$$

The stretch of the spring is

$$\Delta\ell = x_2 - x_1$$

$$\text{so} \quad T = k\Delta\ell = k(x_2 - x_1). \tag{10.42}$$

Combining (10.41) and (10.42) we get

$$\begin{aligned} \ddot{x}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\ \ddot{x}_2 &= \left(\frac{1}{m_2}\right)(-k(x_2 - x_1)) \end{aligned} \tag{10.43}$$

Note: Take care with signs when setting up this type of problem. You should check, for example, that if $x_2 > x_1$, mass 1 accelerates to the right ($\ddot{x}_1 > 0$) and mass 2 accelerates to the left($\ddot{x}_2 < 0$). It is easy to make sign errors. You've been warned!

The differential equations that result from writing $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for the separate particles are coupled second-order equations. The equations are 'coupled' in that the equation for m_1, say, includes the position x_2 or velocity v_2 of mass 2. Such systems of second order coupled equations are often solved on a computer by writing them as a system of first-order equations. You have two first-order equations for each of the second order equations because of the addition of equations like, for example, $\dot{x}_{17} = v_{17}$.

Example: Writing second-order ODEs as first-order ODEs.
Refer again to fig. 10.36 If we define $v_1 = \dot{x}_1$ and $v_2 = \dot{x}_2$ we can rewrite equation 10.43 as

$$\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \left(\frac{1}{m_2}\right)(-k)(x_2 - x_1)
\end{aligned}$$

or, defining $z_1 = x_1, z_2 = v_1, z_3 = x_2, z_4 = v_2$, we get

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\left(\frac{k}{m_1}\right) z_1 + \left(\frac{k}{m_1}\right) z_3 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= \frac{k}{m_2} z_1 - \frac{k}{m_2} z_3 \,.
\end{aligned}$$

Most numerical solutions depend on specifying numerical values for the various constants and initial conditions.

Example: computer solution
If we take, in consistent units, $m_1 = 1, k = 1, m_2 = 1, x_1(0) = 0, x_2(0) = 0, v_1(0) = 1$, and $v_2(0) = 0$, we can set up a well defined computer problem (please see the preface for a discussion of the computer notation). This problem corresponds to finding the motion just after the left mass was hit on the left side with a hammer:

```
ODEs = {z1dot =  z2
        z2dot = -z1 + z3
        z3dot =  z4
        z4dot =  z1 - z3}
ICs  = {z1(0) =0, z2(0)=1, z3(0)=0, z4(0)=0}
solve ODEs with ICs from t=0 to t=10
plot z1 vs t.
```

This yields the plot shown in fig. 10.37.

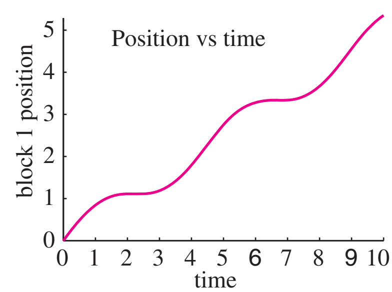

Figure 10.37: Plot of the position of the left mass *vs.* time.

The same methods work for problems involving connections with dashpots.

Example: Multi-DOF system with a dashpot.
Consider m_B in fig. 10.38. Using the free-body diagram shown linear momentum balance gives

$$\begin{aligned}
\sum \vec{F}_i &= m\vec{a}_B \\
\left\{(-T_{k_4} - T_{k_2} + T_{c_1} + T_{k_3})\hat{\imath}\right. &= \left. m a_B \hat{\imath}\right\} \\
\{\}\cdot\hat{\imath} \quad \Rightarrow \quad -\underbrace{T_{k_4}}_{k_4 x_B} - \underbrace{T_{k_2}}_{k_2(x_B - x_A)} + \underbrace{T_{c_1}}_{c_1(\dot{x}_D - \dot{x}_B)} + \underbrace{T_{k_3}}_{k_3(x_D - x_B)} &= m \underbrace{a_B}_{\ddot{x}_B} \\
k_2 x_A - (k_2 + k_4 + k_3) x_B + k_3 x_D - c_1 \dot{x}_B + c_1 \dot{x}_D &= m\ddot{x}_B
\end{aligned}$$

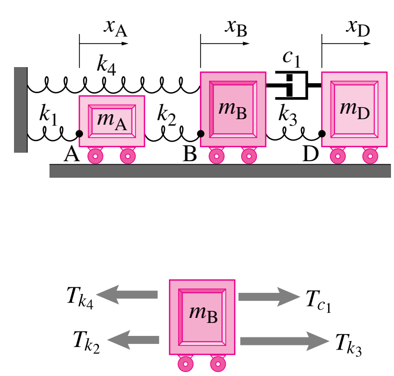

Figure 10.38: A system with 3 masses, 4 springs and a dashpot. Also shown is free-body diagram of mass B.

Similar equations could be written for masses A and C. Some things to note

- We assumed zeros for the displacements so that the system is in static equilibrium if $x_A = x_B = x_D = 0$.

- We have taken the sign convention that tension is positive for all springs and dashpots.
- All of the spring coefficients of x_B have a minus sign in front. That is because all springs, whether to the right or the left of mass B, provide a restoring force if mass B is displaced.
- All of the spring coefficients of x_A and x_D make a positive contribution because motion to the right of mass A or mass D causes a force to the right on mass B.

As for the example above, for any system of masses, linear springs and linear dashpots the set of momentum balance equations can be written in the form

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0} \tag{10.44}$$

where $\boldsymbol{x}$ is a list of positions of the masses. The mass matrix $[M]$ is diagonal because each equation corresponds to $F = ma$ for one mass. The damping and stiffness matrices $[C]$ and $[K]$ are symmetric because, as Jim Marley said, 'every action has a reaction'; if motion of mass 7 causes a stretch on the spring between it and mass 19 then motion of mass 19 causes a stretch on the same spring, similarly affecting mass 7. So row 7 column 19 has the same entry as row 19 column 7. As noted in the example below, the diagonal elements of $[M]$, $[C]$ and $[K]$ are positive (or zero).

Example: Matrix form

When the three momentum balance equations for fig. 10.38 are written, one for each mass, they can be assembled in matrix form as

$$\underbrace{\begin{bmatrix} m_A & 0 & 0 \\ 0 & m_B & 0 \\ 0 & 0 & m_C \end{bmatrix}}_{[M]} \underbrace{\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \\ \ddot{x}_D \end{bmatrix}}_{\ddot{\boldsymbol{x}}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & c_1 & -c_1 \\ 0 & -c_1 & c_1 \end{bmatrix}}_{[C]} \underbrace{\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \\ \dot{x}_D \end{bmatrix}}_{\dot{\boldsymbol{x}}} + \underbrace{\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_4+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}}_{[K]} \underbrace{\begin{bmatrix} x_A \\ x_B \\ x_D \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{0}}.$$

The equation for m_B worked out at the start of this example corresponds to the second row of these matrices.

This form is convenient for numerical solution if it is written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -[M]^{-1}\left[[C]\boldsymbol{v} + [K]\boldsymbol{x}\right] \end{aligned}$$

For the three mass example this would represent 6 first-order differential equations.

Center of mass

For both theoretical and practical reasons it is often useful to pay attention to the motion of the average position of mass in the system. This average position is called the center-of-mass. For a collection of particles in one dimension the center-of-mass is

$$x_{\mathrm{CM}} = \frac{\sum x_i m_i}{m_{\mathrm{tot}}}, \tag{10.45}$$

where $m_{\mathrm{tot}} = \sum m_i$ is the total mass of the system. The velocity and acceleration of the center-of-mass are found by differentiation to be

$$v_{\mathrm{CM}} = \frac{\sum v_i m_i}{m_{\mathrm{tot}}} \quad \text{and} \quad a_{\mathrm{CM}} = \frac{\sum a_i m_i}{m_{\mathrm{tot}}}. \tag{10.46}$$

If we imagine a system of interconnected masses and add the $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ equations from all the separate masses we can get on the left hand side only the forces from the outside; the interaction forces cancel because they come in equal and opposite (action and reaction) pairs. So we get:

$$\sum F_{\mathrm{external}} = \sum a_i m_i = m_{\mathrm{tot}} a_{\mathrm{CM}}. \tag{10.47}$$

So the center-of-mass of a system (a system that may be deforming wildly) obeys the same simple governing equation as a single particle. Although our demonstration here was for particles in one dimension. The result holds for any bodies of any type in 1,2, or 3 dimensions.