10.4 Coupled motions in 1D

Thinking of a car, a plane, a person on a bicycle or a satellite as a single particle is often edifying, and sufficient for many engineering purposes. However, the one-particle model is also often inadequate. That the parts of a machine or structure move relative to each other is obviously sometimes important; many important engineering systems have parts that move independently.

Here we begin the study of independent, but coupled, motions of parts. The independent motions are coupled in that the motion of each part may affect the motion of the others.

Example: Car suspension.
A model of a car suspension treats the wheel as one particle and the car as another. The wheel is coupled to the ground by a tire and to the car by the suspension. In a first analysis the only motion to consider would be vertical for both the wheel and the car. Think of the ground as moving up and down and 'forcing' the motion of the car and wheel system.

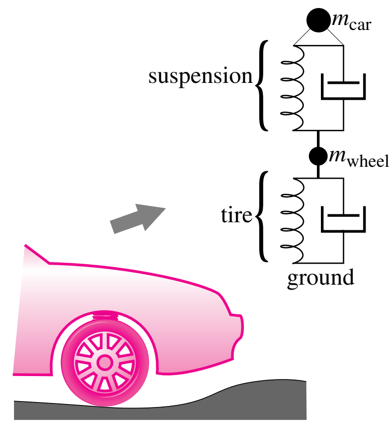


Figure 10.34: The vertical motions of a car and wheel might be considered as coupled motions in 1D.

Still using one-dimensional mechanics, we consider systems that can be modeled as two or more particles. Such one-dimensional coupled motion analysis is common in engineering practice in situations where there are connected parts that all move in about the same direction, but the parts do not move the same amount or necessarily at the same time. Many of the ideas generalize to systems where parts, each with one degree of freedom, are coupled together. Many generalizations apply even if each degree of freedom is quite different from the others. These generalizations to more general coupled motions come later in the book.

The primary goal in this section is to develop two skills:

- To write correct equations of motion for a line of particles connected to each other with springs and dashpots, and
- To simulate the motions of such systems on a computer.
- (the third of the two things, really implicit in the first two) To use the simulation results to find errors in the equations.

The concept of 'normal modes' is postponed to Section 11.3 The simplest way of dealing with the coupled motion of two or more particles is

- to write $\vec{F} = m\vec{a}$ for each particle and then
- to use the forces on the free-body diagrams to evaluate the forces.

Because the most common models for the interaction forces are springs and dashpots (see chapter 3), one needs to account for the relative positions and velocities of the particles.

Relative motion in one dimension

If the position of A is $\vec{r}_A$, and B's position is $\vec{r}_B$, then B's position *relative* to A is

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A.$$

Relative velocity and acceleration are similarly defined by subtraction, or by differentiating the above expression, as

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A \text{ and } \vec{\boldsymbol{a}}_{B/A} = \vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A.$$

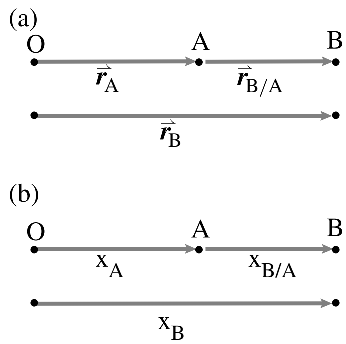


Figure 10.35: The relative position of points A and B in one dimension.

In one dimension, the relative position diagram of fig. 1.5 on page 51 becomes fig. 10.35. $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. So, we can write,

$$\begin{aligned} x_{B/A} &\equiv x_B - x_A, \\ v_{B/A} &\equiv v_B - v_A = \frac{d}{dt}x_{B/A}, \text{ and} \\ a_{B/A} &\equiv a_B - a_A = \frac{d}{dt}v_{B/A} = \frac{d^2}{dt^2}x_{B/A}. \end{aligned}$$

An alternative notation, discussed in Chapter 2, is x_{AB} where the directed line AB is equivalent to the position of B relative to A:

$$x_{AB} = x_{B/A}$$

Example: Two masses connected by a spring.
Consider the two masses on a frictionless support (fig. 10.36). Assume the spring is un-stretched when $x_1 = x_2 = 0$. After drawing free-body diagrams of the two masses we can write $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each mass:

$$\begin{aligned} &\textbf{mass 1:} \quad \vec{\boldsymbol{F}}_1 = m\vec{\boldsymbol{a}}_1 \quad \Rightarrow \quad T\hat{\boldsymbol{\imath}} = m_1\ddot{x}_1\hat{\boldsymbol{\imath}} \\ &\textbf{mass 2:} \quad \vec{\boldsymbol{F}}_2 = m\vec{\boldsymbol{a}}_2 \quad \Rightarrow \quad -T\hat{\boldsymbol{\imath}} = m_2\ddot{x}_2\hat{\boldsymbol{\imath}} \end{aligned} \tag{10.41}$$

The stretch of the spring is

$$\Delta\ell = x_2 - x_1$$

$$\text{so} \quad T = k\Delta\ell = k(x_2 - x_1). \tag{10.42}$$

Combining (10.41) and (10.42) we get

$$\begin{aligned} \ddot{x}_1 &= \left(\frac{1}{m_1}\right)k(x_2 - x_1) \\ \ddot{x}_2 &= \left(\frac{1}{m_2}\right)(-k(x_2 - x_1)) \end{aligned} \tag{10.43}$$

Note: Take care with signs when setting up this type of problem. You should check, for example, that if $x_2 > x_1$, mass 1 accelerates to the right ($\ddot{x}_1 > 0$) and mass 2 accelerates to the left($\ddot{x}_2 < 0$). It is easy to make sign errors. You've been warned!

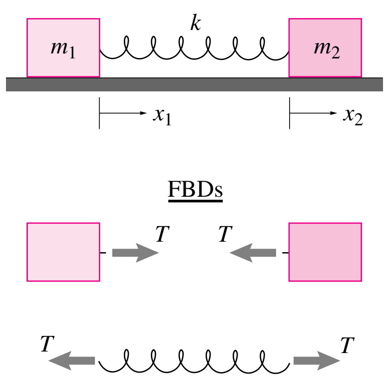


Figure 10.36

The differential equations that result from writing $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for the separate particles are coupled second-order equations. The equations are 'coupled' in that the equation for m_1, say, includes the position x_2 or velocity v_2 of mass 2. Such systems of second order coupled equations are often solved on a computer by writing them as a system of first-order equations. You have two first-order equations for each of the second order equations because of the addition of equations like, for example, $\dot{x}_{17} = v_{17}$.

Example: Writing second-order ODEs as first-order ODEs.
Refer again to fig. 10.36 If we define $v_1 = \dot{x}_1$ and $v_2 = \dot{x}_2$ we can rewrite equation 10.43 as

$$\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \left(\frac{1}{m_2}\right)(-k)(x_2 - x_1)
\end{aligned}$$

or, defining $z_1 = x_1, z_2 = v_1, z_3 = x_2, z_4 = v_2$, we get

$$\begin{aligned}
\dot{z}_1 &= && z_2 && && \\
\dot{z}_2 &= -\left(\frac{k}{m_1}\right) z_1 && && +\left(\frac{k}{m_1}\right) z_3 && \\
\dot{z}_3 &= && && && z_4 \\
\dot{z}_4 &= \frac{k}{m_2} z_1 && && -\frac{k}{m_2} z_3\,. &&
\end{aligned}$$

Most numerical solutions depend on specifying numerical values for the various constants and initial conditions.

Example: computer solution
If we take, in consistent units, $m_1 = 1, k = 1, m_2 = 1, x_1(0) = 0, x_2(0) = 0, v_1(0) = 1$, and $v_2(0) = 0$, we can set up a well defined computer problem (please see the preface for a discussion of the computer notation). This problem corresponds to finding the motion just after the left mass was hit on the left side with a hammer:

```
ODEs = {z1dot =  z2
        z2dot = -z1 + z3
        z3dot =  z4
        z4dot =  z1 - z3}
ICs  = {z1(0) =0, z2(0)=1, z3(0)=0, z4(0)=0}
solve ODEs with ICs from t=0 to t=10
plot z1 vs t.
```

This yields the plot shown in fig. 10.37.

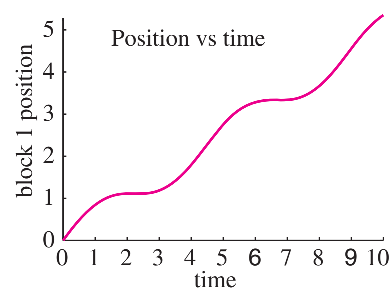


Figure 10.37: Plot of the position of the left mass *vs.* time.

The same methods work for problems involving connections with dashpots.

Example: Multi-DOF system with a dashpot.
Consider m_B in fig. 10.38. Using the free-body diagram shown linear momentum balance gives

$$\begin{aligned}
\sum \vec{F}_i &= m\vec{a}_B \\
\left\{(-T_{k_4} - T_{k_2} + T_{c_1} + T_{k_3})\hat{\imath}\right. &= \left. m a_B \hat{\imath}\right\} \\
\{\}\cdot\hat{\imath} \quad \Rightarrow \quad -\underbrace{T_{k_4}}_{k_4 x_B} - \underbrace{T_{k_2}}_{k_2(x_B - x_A)} + \underbrace{T_{c_1}}_{c_1(\dot{x}_D - \dot{x}_B)} + \underbrace{T_{k_3}}_{k_3(x_D - x_B)} &= m \underbrace{a_B}_{\ddot{x}_B} \\
k_2 x_A - (k_2 + k_4 + k_3) x_B + k_3 x_D - c_1 \dot{x}_B + c_1 \dot{x}_D &= m\ddot{x}_B
\end{aligned}$$

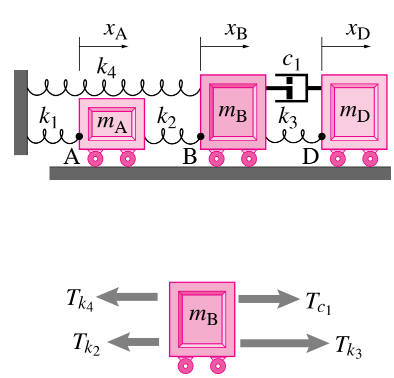


Figure 10.38: A system with 3 masses, 4 springs and a dashpot. Also shown is free-body diagram of mass B.

Similar equations could be written for masses A and C. Some things to note

- We assumed zeros for the displacements so that the system is in static equilibrium if $x_A = x_B = x_D = 0$.

- We have taken the sign convention that tension is positive for all springs and dashpots.
- All of the spring coefficients of x_B have a minus sign in front. That is because all springs, whether to the right or the left of mass B, provide a restoring force if mass B is displaced.
- All of the spring coefficients of x_A and x_D make a positive contribution because motion to the right of mass A or mass D causes a force to the right on mass B.

As for the example above, for any system of masses, linear springs and linear dashpots the set of momentum balance equations can be written in the form

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0} \tag{10.44}$$

where $\boldsymbol{x}$ is a list of positions of the masses. The mass matrix $[M]$ is diagonal because each equation corresponds to $F = ma$ for one mass. The damping and stiffness matrices $[C]$ and $[K]$ are symmetric because, as Jim Marley said, 'every action has a reaction'; if motion of mass 7 causes a stretch on the spring between it and mass 19 then motion of mass 19 causes a stretch on the same spring, similarly affecting mass 7. So row 7 column 19 has the same entry as row 19 column 7. As noted in the example below, the diagonal elements of $[M]$, $[C]$ and $[K]$ are positive (or zero).

Example: Matrix form

When the three momentum balance equations for fig. 10.38 are written, one for each mass, they can be assembled in matrix form as

$$\underbrace{\begin{bmatrix} m_A & 0 & 0 \\ 0 & m_B & 0 \\ 0 & 0 & m_C \end{bmatrix}}_{[M]} \underbrace{\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \\ \ddot{x}_D \end{bmatrix}}_{\ddot{\boldsymbol{x}}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & c_1 & -c_1 \\ 0 & -c_1 & c_1 \end{bmatrix}}_{[C]} \underbrace{\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \\ \dot{x}_D \end{bmatrix}}_{\dot{\boldsymbol{x}}} + \underbrace{\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_4+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}}_{[K]} \underbrace{\begin{bmatrix} x_A \\ x_B \\ x_D \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{0}}.$$

The equation for m_B worked out at the start of this example corresponds to the second row of these matrices.

This form is convenient for numerical solution if it is written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -[M]^{-1}\left[[C]\boldsymbol{v} + [K]\boldsymbol{x}\right] \end{aligned}$$

For the three mass example this would represent 6 first-order differential equations.

Center of mass

For both theoretical and practical reasons it is often useful to pay attention to the motion of the average position of mass in the system. This average position is called the center-of-mass. For a collection of particles in one dimension the center-of-mass is

$$x_{\text{CM}} = \frac{\sum x_i m_i}{m_{\text{tot}}}, \tag{10.45}$$

where $m_{\text{tot}} = \sum m_i$ is the total mass of the system. The velocity and acceleration of the center-of-mass are found by differentiation to be

$$v_{\text{CM}} = \frac{\sum v_i m_i}{m_{\text{tot}}} \quad \text{and} \quad a_{\text{CM}} = \frac{\sum a_i m_i}{m_{\text{tot}}}. \tag{10.46}$$

If we imagine a system of interconnected masses and add the $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ equations from all the separate masses we can get on the left hand side only the forces from the outside; the interaction forces cancel because they come in equal and opposite (action and reaction) pairs. So we get:

$$\sum F_{\text{external}} = \sum a_i m_i = m_{\text{tot}} a_{\text{CM}}. \tag{10.47}$$

So the center-of-mass of a system (a system that may be deforming wildly) obeys the same simple governing equation as a single particle. Although our demonstration here was for particles in one dimension. The result holds for any bodies of any type in 1,2, or 3 dimensions.

SAMPLE 10.16 For the given quantities and initial conditions, find $x_1(t)$ and $x_2(t)$. Assume the spring is unstretched at its rest length of L_0 when $x_1 = x_2$.

$$m_1 = 1\,\text{kg}, \quad m_2 = 2\,\text{kg}, \quad k = 3\,\text{N/m}, \quad c = 5\,\text{N/(m/s)}$$
$$x_1(0) = 1\,\text{m}, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 2\,\text{m}, \quad \dot{x}_2(0) = 0.$$

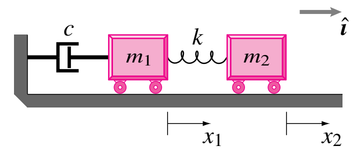


Figure 10.39

Solution The free-body diagrams of all components of the given system are shown below.

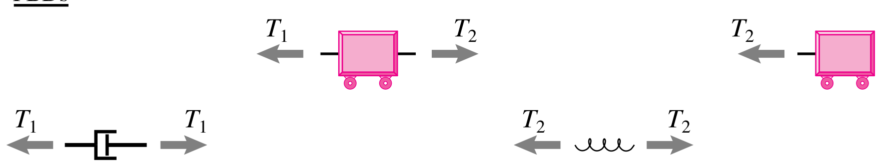


Figure 10.40:

The spring and dashpot laws give

$$T_1 = c\dot{x}_1 \qquad T_2 = k(x_2 - x_1). \tag{10.48}$$

The linear momentum balance for the two masses gives

$$\begin{aligned} &\textstyle\sum \vec{F} = m\vec{a} \\ \text{mass 1:}\quad &-T_1\hat{\imath} + T_2\hat{\imath} = m_1\ddot{x}_1\hat{\imath} \\ \text{mass 2:}\quad &-T_2\hat{\imath} = m_2\ddot{x}_2\hat{\imath}. \end{aligned} \tag{10.49}$$

Applying the constitutive laws (10.48) to the momentum balance equations (10.49) gives

$$\begin{aligned} \ddot{x}_1 &= [k(x_2 - x_1) - c\dot{x}_1]/m_1 \\ \ddot{x}_2 &= [-k(x_2 - x_1)]/m_2. \end{aligned}$$

Defining $z_1 = x_1, z_2 = \dot{x}_1, z_3 = x_2, z_4 = \dot{x}_2$ gives

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= [k(z_3 - z_1) - cz_2]/m_1 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= [-k(z_3 - z_1)]/m_2. \end{aligned}$$

The initial conditions are

$$z_1(0) = 1\,\text{m}, \quad z_2(0) = 0, \quad z_3(0) = 2\,\text{m}, \quad z_4(0) = 0.$$

We are now set for numerical solution. Solving these equations numerically, we plot $x_1(t)$ and $x_2(t)$ as shown in fig. 10.41. From the solution, it is clear that both the masses settle down to the equilibrium position $x_1 = x_2 = 1$ m after the oscillations die down. In this position, the spring exerts no force as it is unstretched. Also note that the two masses move in the opposite direction immediately after being set into motion as they must because of the opposite accelerations.

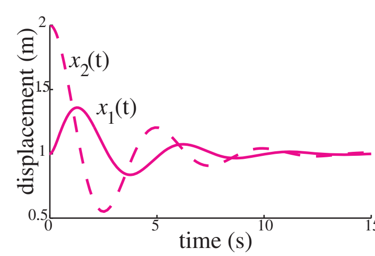


Figure 10.41: Plot of $x_1(t)$ and $x_2(t)$ obtained from the numerical solution of the equations of motion. Note that with $x_1(0) = 1$ m and $x_2(0) = 2$ m, both masses settle down to $x_1 = x_2 = 1$ m after the oscillations die.

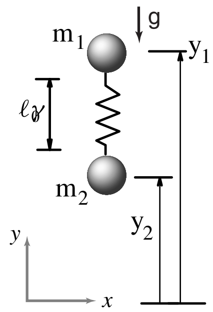


Figure 10.42

SAMPLE 10.17 Flight of a toy hopper.

A hopper model ① is made of two masses $m_1 = 0.4\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a spring with stiffness $k = 100\,\text{N/m}$ as shown in fig. 10.42. The unstretched length of the spring is $\ell_0 = 1\,\text{m}$. The model is released from rest from the configuration shown in the figure with $y_1 = 25.5\,\text{m}$ and $y_2 = 24\,\text{m}$.

1. Find and plot $y_1(t)$ and $y_2(t)$ for $t = 0$ to $2\,\text{s}$.
2. Plot the motion of m_1 and m_2 with respect to the center-of-mass of the hopper during the same time interval.
3. Plot the motion of the center-of-mass of the hopper from the solution obtained for $y_1(t)$ and $y_2(t)$ and compare it with analytical values obtained by integrating the center-of-mass motion directly.

① You can think of it as a model of a hopping robot or a hopping animal where the foot mass is modeled with m_2 and the body mass with m_1. The spring models the elasticity of the system. If you restrict this model to one dimensional vertical motion under gravity, it will hop, once released from some initial height. The number of hops depends on the initial conditions. Theoretically, it can hop forever with suitable initial conditions.

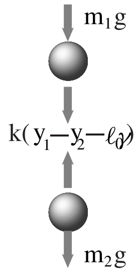


Figure 10.43: Free-body diagram of the two masses m_1 and m_2

Solution The free-body diagrams of the two masses are shown in fig. 10.43. From the linear momentum balance in the y direction, we can write the equations of motion at once.

$$
\begin{aligned}
m_1\ddot{y}_1 &= -k(y_1 - y_2 - \ell_0) - m_1 g \\
\Rightarrow \quad \ddot{y}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \qquad (10.50)\\
m_2\ddot{y}_2 &= k(y_1 - y_2 - \ell_0) - m_2 g \\
\Rightarrow \quad \ddot{y}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g. \qquad (10.51)
\end{aligned}
$$

1. The equations of motion obtained above are coupled linear differential equations of second order. We can solve for $y_1(t)$ and $y_2(t)$ by numerical integration of these equations. As we have shown in previous examples, we first need to set up these equations as a set of first order equations.
   Letting $\dot{y}_1 = v_1$ and $\dot{y}_2 = v_2$, we get

$$
\begin{aligned}
\dot{y}_1 &= v_1 \\
\dot{v}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \\
\dot{y}_2 &= v_2 \\
\dot{v}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g.
\end{aligned}
$$

Now we solve this set of equations numerically using some ODE solver and the following pseudocode.

```
ODEs = {y1dot = v1,
        v1dot = -k/m1*(y1-y2-l0) - g,
        y2dot = v2,
        v1dot = k/m1*(y1-y2-l0) - g}
IC   = {y1(0)=25.5, v1(0)=0, y2(0)=24, v2(0)=0}
Set  k=100, m1=0.4, m2=1, l0=1
Solve ODEs with IC for t=0 to t=2
Plot y1(t) and y2(t)
```

The solution obtained thus is shown in fig. 10.44.

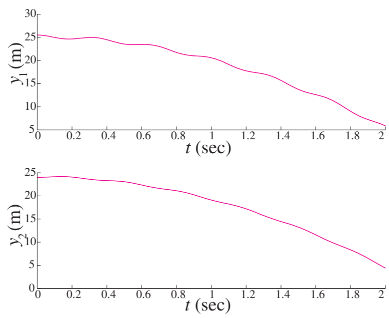


Figure 10.44: Numerically obtained solutions $y_1(t)$ and $y_2(t)$

2. We can find the motion of m_1 and m_2 with respect to the center-of-mass by subtracting the motion of the center-of-mass, y_{cm} from y_1 and y_2. Since,

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \tag{10.52}$$

we get,

$$\begin{aligned} y_{1/cm} &= y_1 - y_{cm} = \frac{m_2}{m_1+m_2}(y_1 - y_2) \\ y_{2/cm} &= y_2 - y_{cm} = -\frac{m_1}{m_1+m_2}(y_1 - y_2). \end{aligned}$$

The relative motions thus obtained are shown in fig. 10.45. We note that the motions of m_1 and m_2, as seen by an observer sitting at the center-of-mass, are simple harmonic oscillations.

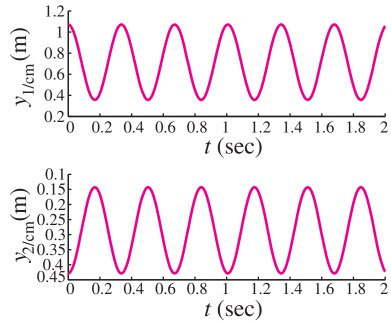


Figure 10.45: Numerically obtained solutions $y_{1/cm}(t)$ and $y_{1/cm}(t)$.

3. We can find the center-of-mass motion $y_{cm}(t)$ from y_1 and y_2 by using eqn. (10.52). The solution obtained thus is shown as a solid line in fig. 10.47. We can also solve for the center-of-mass motion analytically by first writing the equation of motion of the center-of-mass and then integrating it analytically.

   The free-body diagram of the hopper as a single system is shown in fig. 10.46. The linear momentum balance for the system in the vertical direction gives

$$\begin{aligned} (m_1 + m_2)\ddot{y}_{cm} &= -m_1 g - m_2 g \\ \Rightarrow \quad \ddot{y}_{cm} &= -g. \end{aligned}$$

   We recognize this equation as the equation of motion of a freely falling body under gravity. We can integrate this equation twice to get

$$y_{cm}(t) = y_{cm}(0) + \dot{y}_{cm}(0)t - \frac{1}{2}gt^2.$$

   Noting that $y_{cm}(0) = 24.43\,\text{m}$ (from eqn. (10.52)), and $\dot{y}_{cm}(0) = 0$ (the system is released from rest), we get

$$y_{cm}(t) = 24.43\,\text{m} - \frac{1}{2}\cdot 9.81\,\text{m/s}^2 \cdot t^2.$$

   The values obtained for the center-of-mass position from the above expression are shown in fig. 10.47 by small circles.

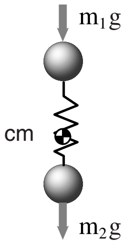


Figure 10.46: Free-body diagram of the hopper as a single system. The spring force does not show up here since it becomes an internal force to the system

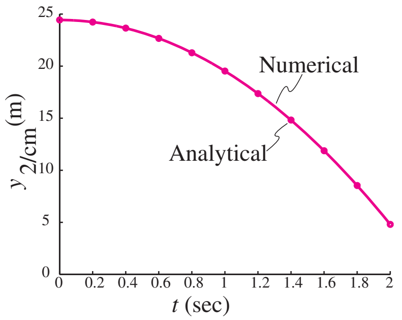


Figure 10.47: Numerically obtained solution for the position of the center-of-mass, $y_{cm}(t)$.

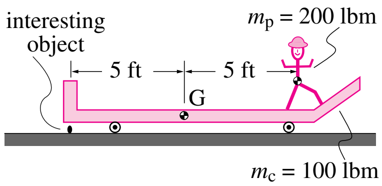


Figure 10.48: Mr. P spots an interesting object.

SAMPLE 10.18 Conservation of linear momentum. Mr. P with mass $m_p = 200$ lbm is standing on a cart with frictionless and massless wheels. The cart weighs half as much as Mr. P. Standing at one end of the cart, Mr. P spots an interesting object at the other end of the cart. Mr. P decides to walk to the other end of the cart to pick up the object. How far does he find himself from the object after he reaches the end of the cart?

Solution From your own experience in small boats perhaps, you know that when Mr. P walks to the left the cart moves to the right. Here, we want to find how far the cart moves.

Consider the cart and Mr. P together to be the system of interest. The free-body diagram of the system is shown in Fig. 10.49(a).

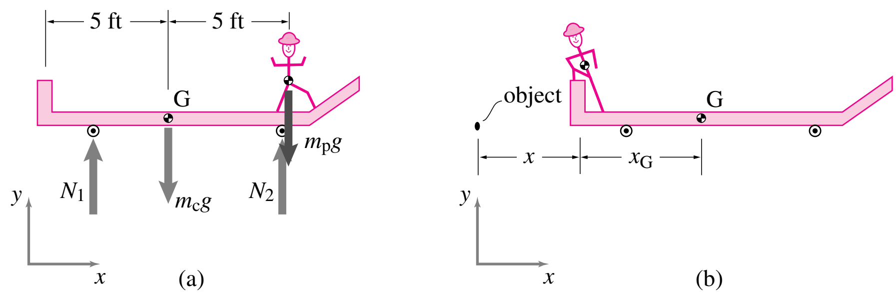


Figure 10.49: (a) Free-body diagram of Mr. P-and-the-cart system. (b) The cart has moved to the right by distance x when Mr. P reaches the other end.

From the diagram it is clear that there are no external forces in the x-direction. Therefore,

$$\dot{L}_x = \sum F_x = 0 \quad \Rightarrow \quad L_x = \text{constant}$$

that is, the linear momentum of the system in the x-direction is 'conserved'. But the initial linear momentum of the system is zero. Therefore,

$$L_x = m_{tot}(v_{cm})_x = 0 \text{ all the time} \quad \Rightarrow \quad (v_{cm})_x = 0 \text{ all the time.}$$

Because the horizontal velocity of the center-of-mass is always zero, the center-of-mass does not change its horizontal position. Now let x_{cm} and x'_{cm} be the x-coordinates of the center-of-mass of the system at the beginning and at the end, respectively. Then,

$$x'_{cm} = x_{cm}.$$

Now, from the given dimensions and the stipulated position at the end in Fig. 10.49(b),

$$x_{cm} = \frac{m_c x_G + m_p x_p}{m_c + m_p} \quad \text{and} \quad x'_{cm} = \frac{m_c(x_G + x) + m_p x}{m_c + m_p}.$$

Equating the two distances we get,

$$\begin{aligned} m_c x_G + m_p x_p &= m_c(x_G + x) + m_p x \\ &= m_c x_G + x(m_c + m_p) \\ \Rightarrow \quad x &= \frac{m_p x_p}{m_c + m_p} \\ &= \frac{200\,\text{lbm} \cdot 10\,\text{ft}}{300\,\text{lbm}} = 6\frac{2}{3}\,\text{ft}. \end{aligned}$$

6.67 ft

[Note: if Mr. P and the cart have the same mass, the cart moves to the right the same distance Mr. P moves to the left.]

10.4 Coupled motion in 1D

The primary emphasis of this section is setting up correct differential equations (without sign errors) and solving these equations on the computer.

Preparatory Problems

10.4.1 Write the following set of coupled second order ODE's as a system of first order ODE's.

$$\begin{aligned} \ddot{x}_1 &= k_2(x_2 - x_1) - k_1 x_1 \\ \ddot{x}_2 &= k_3 x_2 - k_2(x_2 - x_1) \end{aligned}$$

10.4.2 The solution of a set of second order differential equations is:

$$\begin{aligned} \xi(t) &= A \sin \omega t + B \cos \omega t + \xi^* \\ \dot{\xi}(t) &= A\omega \cos \omega t - B\omega \sin \omega t, \end{aligned}$$

where A and B are constants to be determined from initial conditions and ξ^* is a known constant. Assume A and B are the only unknowns.

a) Write the equations in matrix form which you would need to solve in order to find A and B in terms of $\xi(0)$ and $\dot{\xi}(0)$.

b) Solve the equations in symbols.

c) Solve for the numerical constants A and B using the matrix form, if $\xi(0) = 0$, $\dot{\xi}(0) = 0.5$, $\omega = 0.5\,\mathrm{rad/s}$ and $\xi^* = 0.2$.

10.4.3 A set of first order linear differential equations is given:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 + kx_1 + cx_2 &= 0. \end{aligned}$$

Write these equations in the form $\dot{\vec{x}} = [A]\vec{x}$, where $\vec{x} = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\}$.

10.4.4 Write the following pair of coupled ODE's as a set of first order ODE's.

$$\begin{aligned} \ddot{x}_1 + x_1 &= \dot{x}_2 \sin t \\ \ddot{x}_2 + x_2 &= \dot{x}_1 \cos t \end{aligned}$$

10.4.5 The following set of differential equations can be written in first order form, and in particular, in matrix form $\dot{\vec{x}} = [A]\vec{x} + \vec{c}$. In general equations of motion are not so simple, but *linear* cases like this are prevalent in the analytic study of dynamical systems.

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 + 5\Omega^2 x_1 - 4\Omega^2 x_2 &= 2\Omega^2 v_1^* \\ \dot{x}_4 - 4\Omega^2 x_1 + 5\Omega^2 x_2 &= -\Omega^2 v_1^* \end{aligned}$$

10.4.6 Write each of the following equations as a system of first order ODE's.

a) $\ddot{\theta} + \lambda^2\theta = \cos t$,

b) $\ddot{x} + 2p\dot{x} + kx = 0$,

c) $\ddot{x} + 2c\dot{x} + k\sin x = 0$.

10.4.7 A train moves at a constant absolute velocity $v\hat{\imath}$. A passenger, idealized as a point mass, walks at an absolute velocity $u\hat{\imath}$, where $u > v$. What is the velocity of the passenger relative to the train?

10.4.8 Two equal masses, each denoted by the letter m, are on an air track. One mass is connected by a spring to the end of the track. The other mass is connected by a spring to the first mass. The two spring constants are equal and represented by the letter k. In the rest configuration (springs are relaxed) the masses are a distance ℓ apart. Motion of the two masses x_1 and x_2 is measured relative to this configuration.

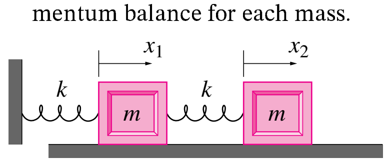
a) Write the potential energy of the system for arbitrary displacements x_1 and x_2 at some time t.

b) Write the kinetic energy of the system at the same time t in terms of $\dot{x}_1$, $\dot{x}_2$, m, and k.

c) Write the total energy of the system.

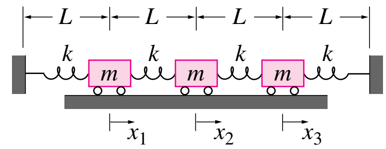
d) Draw a free-body diagram for each mass.

e) Write the equation of linear momentum balance for each mass.



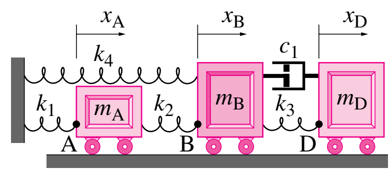
Problem 10.4.8

10.4.9 For the three-mass system shown, draw a free-body diagram of each mass. Write the spring forces in terms of the displacements x_1, x_2, and x_3.



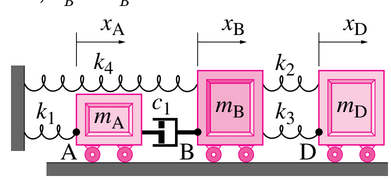
Problem 10.4.9

10.4.10 The springs shown are relaxed when $x_A = x_B = x_D = 0$. In terms of some or all of m_A, m_B, m_D, x_A, x_B, x_D, $\dot{x}_A, \dot{x}_B, \dot{x}_C$, and k_1, k_2, k_3, k_4 and c_1, find the acceleration of block B. *



Problem 10.4.10

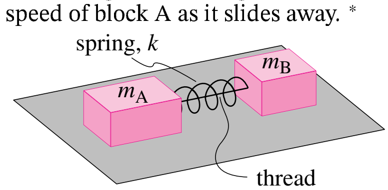
10.4.11 A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given $k_1, k_2, k_3, k_4, c_1, m_A, m_B, m_D, x_A, x_B, x_D$, $\dot{x}_A$, $\dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\vec{a}_B = \ddot{x}_B\hat{\imath}$. *



Problem 10.4.11

More-Involved Problems

10.4.12 A massless spring with constant k is held compressed a distance δ from its relaxed length by a thread connecting blocks A and B which are still on a frictionless table. The blocks have mass m_A and m_B, respectively. The thread is suddenly but gently cut, the blocks fly apart and the spring falls to the ground. Find the speed of block A as it slides away. *



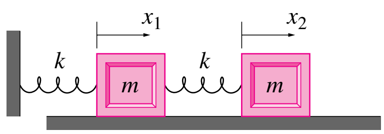
Problem 10.4.12

10.4.13 In the system below the masses are in equilibrium with the springs when $x_1 = x_2 = 0$.

a) First do problem 10.4.8.

b) Pick parameter values and initial conditions of your choice and simulate a motion of this system. Make a plot of the motion of, say, one of the masses vs time,

c) Explain how your plot does or does not make sense in terms of your understanding of this system. Is the initial motion in the right direction? Are the solutions periodic? Bounded? etc.



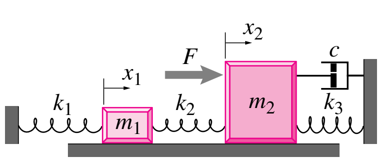
Problem 10.4.13

10.4.14 Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force F acts on mass 2. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be written in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.

a) Write the governing equations in a neat first order form. Your equations should be in terms of any or all of the constants m_1, m_2, k_1, k_2,k_3, C, the constant force F, and t. Getting the signs right is important.

b) Write computer commands to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions.

c) For constants and initial conditions of your choosing, plot x_1 vs t for enough time so that decaying erratic oscillations can be observed.



Problem 10.4.14

10.4.15 The three beads of masses m, $2m$, and m connected by massless linear springs of constant k slide freely on a straight rod. Let x_i denote the displacement of the i^{th} bead from its equilibrium position at rest.

a) Write expressions for the total kinetic and potential energies.

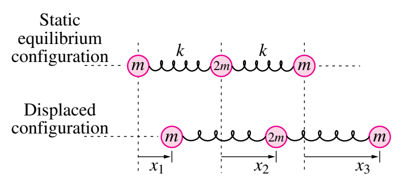
b) Write an expression for the total linear momentum.

c) Draw free-body diagrams for the beads and use Newton's second law to derive the equations for motion for the system.

d) Verify that total energy and linear momentum are both conserved.

e) Show that the center of mass must either remain at rest or move at constant velocity.

f) What can you say about vibratory (sinusoidal) motions of the system?



Problem 10.4.15

10.4.16 Two blocks with masses M and m are connected by a spring with constant k and free length ℓ_0 that can sustain compression. Mass$M$ is resting on the ground at the start. There is gravity. The upwards vertical displacement of mass m is x, which is zero when the spring is at its rest length and M is on the ground.

a) For what value of x is the system in static equilibrium?

b) Find a differential equation governing the motion of m assuming M remains on the ground.

c) Draw a free-body diagram of M.

d) For what value of x is M on the verge of lifting off the ground.

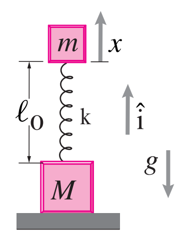
e) Defining y as the height of the lower mass, write two coupled differential equations for the motion of m and M if both masses are in the air.

f) Find the value of $x < 0$ so that if the system is started from rest with that x and $y = 0$ that the ground reaction force on M just goes to zero.

g) Starting here, this problem is more of a project than a typical homework problem. Assume $x(t = 0)$ is less than the value computed above. Write a computer program that integrates the equations of motion until M lifts off and then switches to integrating the equations for the two masses in the air.

h) modify your program so that if M hits the ground again, it sticks until the ground reaction force goes to zero again.

i) By playing around, this way or that, see if you can find a special value for $x(t = 0)$ so that the bouncing continues indefinitely. (This is a perhaps surprising result, that a system with plastic collisions can continue to bounce indefinitely.)



Problem 10.4.16