The primary emphasis of this section is setting up correct differential equations (without sign errors) and solving these equations on the computer.

10.4.5 The following set of differential equations can be written in first order form, and in particular, in matrix form $\dot{\vec{x}} = [A]\vec{x} + \vec{c}$. In general equations of motion are not so simple, but *linear* cases like this are prevalent in the analytic study of dynamical systems.

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 + 5\Omega^2 x_1 - 4\Omega^2 x_2 &= 2\Omega^2 v_1^* \\ \dot{x}_4 - 4\Omega^2 x_1 + 5\Omega^2 x_2 &= -\Omega^2 v_1^* \end{aligned}$$

9.4.5

$$\dot{x}_1 = x_3 \quad , \quad \dot{x}_3 = 4\Omega^2 x_2 - 5\Omega^2 x_1 + 2\Omega^2 V_1^*$$

$$\dot{x}_2 = x_4 \quad , \quad \dot{x}_4 = 4\Omega^2 x_1 - 5\Omega^2 x_2 - \Omega^2 V_1^*$$

$$\begin{Bmatrix} x_1 = x \\ x_2 = y \\ x_3 = \dot{x} \\ x_4 = \dot{y} \end{Bmatrix} = \underline{x}$$

$$[A] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5\Omega^2 & 4\Omega^2 & 0 & 0 \\ 4\Omega^2 & -5\Omega^2 & 0 & 0 \end{bmatrix} \quad \& \quad \underline{c} = \begin{bmatrix} 0 \\ 0 \\ 2\Omega^2 \\ -\Omega^2 \end{bmatrix} V_1^*$$

$$\therefore \quad \dot{\underline{x}} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{Bmatrix}$$

$$\dot{\underline{x}} = [A]\,\underline{x} + \underline{c}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.4.6 Write each of the following equations as a system of first order ODE's.

a) $\ddot{\theta} + \lambda^2\theta = \cos t$,

b) $\ddot{x} + 2p\dot{x} + kx = 0$,

c) $\ddot{x} + 2c\dot{x} + k\sin x = 0$.

9.4.6

a) $\ddot{\theta} + \lambda^2\theta = \cos t \Rightarrow \ddot{\theta} = -\lambda^2\theta + \cos t$

Let

$$\underline{z} = \begin{Bmatrix} z_1 = \theta \\ z_2 = \dot{\theta} \end{Bmatrix}$$

$$\dot{\underline{z}} = \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t$$

b) $\ddot{x} = -2p\dot{x} - Kx$,

$$\underline{z} = \begin{bmatrix} z_1 = x \\ z_2 = \dot{x} \end{bmatrix}$$

$$\dot{\underline{z}} = \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -2p \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

c) $\ddot{x} = -2c\dot{x} - K\sin x$

$\Rightarrow \ddot{x} = -2c\dot{x} - Kx$

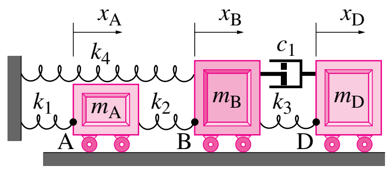
Assuming x is $< 10^\circ$ or small , $\sin x = x$

$\Rightarrow \ddot{x} = -2c\dot{x} - Kx$

$$\Rightarrow \dot{\underline{z}} = \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -K & -2c \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

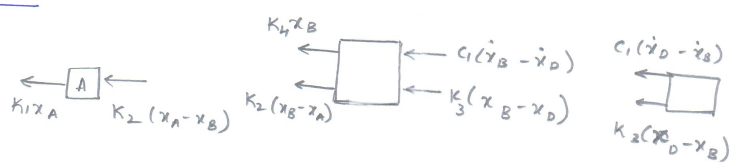
10.4.10 The springs shown are relaxed when $x_A = x_B = x_D = 0$. In terms of some or all of m_A, m_B, m_D, x_A, x_B, x_D, $\dot{x}_A, \dot{x}_B, \dot{x}_C$, and k_1, k_2, k_3, k_4 and c_1, find the acceleration of block B.



Problem 10.10

Answer:
$\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - k_2(x_B - x_A) + c_1(\dot{x}_D - \dot{x}_B) + k_3(x_D - x_B)]\hat{\imath}$.

9.4.10

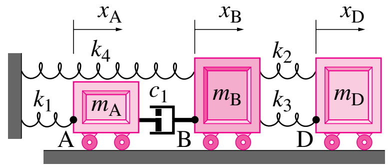


Acceleration of 'B'.

$$a_B = -\frac{1}{m_B}\left[\left[-K_2 x_A + (K_2 + K_3 + K_4)x_B - K_3 x_D\right] + C_1 \dot{x}_B - C_1 \dot{x}_D\right]$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.4.11 A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given $k_1, k_2, k_3, k_4, c_1, m_A, m_B, m_D, x_A, x_B, x_D, \dot{x}_A, \dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\vec{a}_B = \ddot{x}_B \hat{\imath}$.

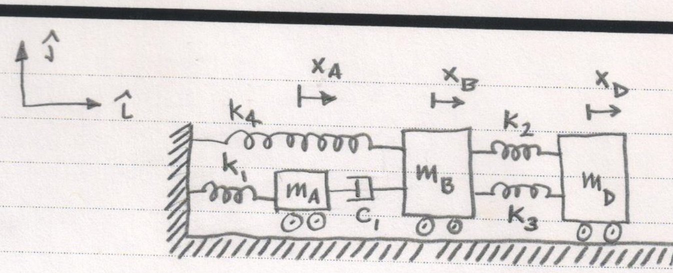


Problem 10.11

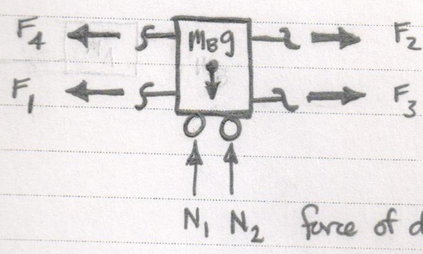
Answer:
$\vec{a}_B = \ddot{x}_B \hat{\imath} = \frac{1}{m_B}[-k_4 x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2 + k_3)(x_D - x_B)]$.

\# 3.91

Find $\underline{a}_B = \ddot{x}_B \hat{\imath}$



FBD



force of spring: $F_s = k \Delta \ell$

$F_2 = k_2(x_D - x_B)$

$F_3 = k_3(x_D - x_B)$

$F_4 = k_4 x_B$

force of damper: $F_d = k \frac{d}{dt}(\Delta \ell)$

$F_1 = c_1(\dot{x}_B - \dot{x}_A)$

LMB $\quad \sum \underline{F} = m_B \underline{a}_B$

$$N_1 \hat{\jmath} + N_2 \hat{\jmath} - mg \hat{\jmath} - F_4 \hat{\imath} - F_1 \hat{\imath} + F_2 \hat{\imath} + F_3 \hat{\imath} = m_B \ddot{x}_B \hat{\imath}$$

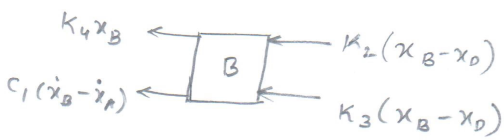
$$\ddot{x}_B = \frac{1}{m_B}\left[-k_4 x_B - c_1(\dot{x}_B - \dot{x}_A) + k_2(x_D - x_B) + k_3(x_D - x_B)\right]$$

cleaning up a bit

$$\boxed{\ddot{x}_B = \frac{1}{m_B}\left[-(k_2 + k_3 + k_4)x_B + (k_2 + k_3)x_D - c_1 \dot{x}_B + c_1 \dot{x}_A\right]}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.4.11

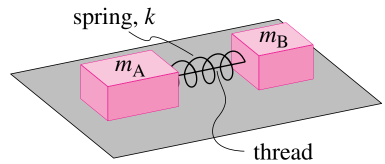


$$m_B \ddot{x}_B \hat{i} = -K_4 x_B - K_2(x_B - x_D) - K_3(x_B - x_D) - C_1(\dot{x}_B - \dot{x}_A)$$

$$\ddot{x}_B = \frac{-1}{m_B}\left[(K_2 + K_3 + K_4)x_B - (K_2 + K_3)x_D + (C_1 \dot{x}_B + C_1 \dot{x}_A)\right]$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

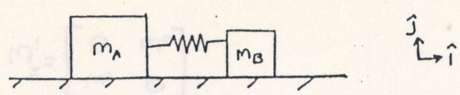
10.4.12 A massless spring with constant k is held compressed a distance δ from its relaxed length by a thread connecting blocks A and B which are still on a frictionless table. The blocks have mass m_A and m_B, respectively. The thread is suddenly but gently cut, the blocks fly apart and the spring falls to the ground. Find the speed of block A as it slides away.



Problem 10.12

Answer:
$v_A = \sqrt{\frac{m_B k\delta^2}{m_A^2 + m_B m_A}}$.

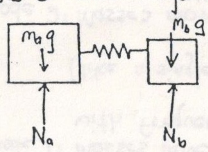
3.38 Page 2



Blocks of mass m_A and m_B are held together by a spring of constant k, compressed a distance δ from it's relaxed length. Find the speed of A after the spring pushes the blocks apart.

Apply conservation of energy and conservation of momentum.

FBD of entire system:



$\Sigma \underline{F} = \frac{d\underline{L}}{dt}$

$\left[-(m_a + m_b)g\,\hat{j} + (N_a + N_b)\hat{j} = \frac{d\underline{L}}{dt}\right]\cdot\hat{i}$

$0 = \frac{d}{dt}(L_x)$

$L_x = \text{constant}$

So the momentum of the entire system in the x-direction is constant throughout time.

$L_{1,x} = L_{2,x}$ where 1: before blocks pushed apart
2: after blocks pushed apart.

$0 = -m_a v_a + m_B v_B$

$v_B = \frac{m_A v_A}{m_B}$ ①

Also, energy is conserved since no external forces do work on the system.

$E_{tot,1} = E_{tot,2}$

$\cancel{\tfrac{1}{2}} k\delta^2 = \cancel{\tfrac{1}{2}} m_A v_A^2 + \cancel{\tfrac{1}{2}} m_B v_B^2$

$m_B v_B^2 = k\delta^2 - m_A v_A^2$

$v_B = \sqrt{\frac{1}{m_B}(k\delta^2 - m_A v_A^2)}$ ②

Substitute ① into ②

$v_A = \frac{m_B}{m_A}\sqrt{\frac{1}{m_B}(k\delta^2 - m_A v_A^2)}$

$v_A^2 = \frac{m_B}{m_A^2}(k\delta^2 - m_A v_A^2)$

$v_A^2\left(1 + \frac{m_B}{m_A}\right) = \frac{m_B}{m_A^2} k\delta^2$

$v_A^2(m_A^2 + m_B m_A) = m_B k\delta^2$

$\boxed{v_A = \sqrt{\frac{m_B k\delta^2}{m_A^2 + m_B m_A}}}$

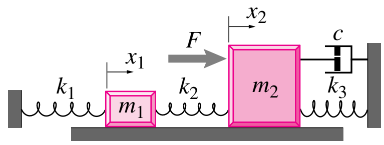
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.4.14 Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force F acts on mass 2. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be written in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.

a) Write the governing equations in a neat first order form. Your equations should be in terms of any or all of the constants m_1, m_2, k_1, k_2, k_3, C, the constant force F, and t. Getting the signs right is important.

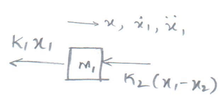
b) Write computer commands to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions.

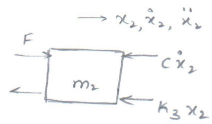
c) For constants and initial conditions of your choosing, plot x_1 vs t for enough time so that decaying erratic oscillations can be observed.



Problem 10.14

9.4.14





$$m_1\ddot{x}_1 = -K_1x_1 - K_2x_1 + K_2x_2$$

$$m_2\ddot{x}_2 = -K_3x_2 - K_2x_2 + K_2x_1 - c\dot{x}_2 + F$$

Let

$z_1 = x_1$ $\qquad \dot{z}_1 = z_2$

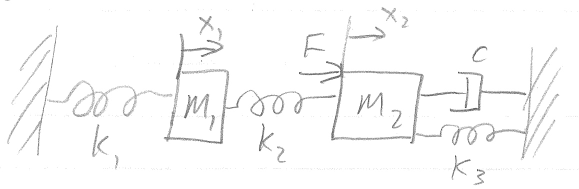
$z_2 = \dot{x}_1$ $\qquad \dot{z}_2 = \left[-(K_1+K_2)z_1 + K_2z_3\right] \times \left(\frac{1}{m_1}\right)$

$z_3 = x_2$ $\qquad \dot{z}_3 = z_4$

$z_4 = \dot{x}_2$ $\qquad \dot{z}_4 = \left(\frac{1}{m_2}\right)\left[K_2x_1 - (K_2+K_3)x_2 - c\dot{x}_2\right] + \frac{F}{m_2}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

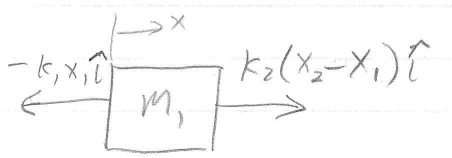
9.76



$X_1 = X_2 = 0$ when springs unstretched.
ground is frictionless

a) Write governing equation in first order form.

FBD



$\{m_1\ddot{x}_1\hat{\imath} = k_2(x_2-x_1)i - k_1x_1\hat{\imath}\}$

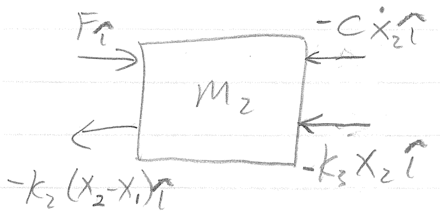
$\{\}\cdot\hat{\imath} \Rightarrow$

$\ddot{x}_1 = \frac{k_2}{m}(x_2-x_1) - \frac{k_1}{m}x_1$

since $\boxed{\dot{x}_1 = v_1}$ ①

$\ddot{x}_1 = \dot{v}_1$

$\boxed{\dot{v}_1 = \frac{k_2}{m}(x_2-x_1) - \frac{k_1}{m}x_1}$ ③



$\{m_2\ddot{x}_2\hat{\imath} = F\hat{\imath} - c\dot{x}_2\hat{\imath} - k_3x_2\hat{\imath} - k_2(x_2-x_1)\hat{\imath}\}$

$\{\}\cdot\hat{\imath} \Rightarrow$

$m_2\ddot{x}_2 = F - cx_2 - k_3x_2 - k_2(x_2-x_1)$

since $\boxed{\dot{x}_2 = v_2}$ ②

$\ddot{x}_2 = \dot{v}_2$

$\boxed{\dot{v}_2 = \frac{F}{m} - \frac{c}{m}\dot{x}_2 - \frac{k_3}{m}x_2 - \frac{k_2}{m}(x_2-x_1)}$ ④

①-④ are gov. eqs in 1st order form

b). See attached files

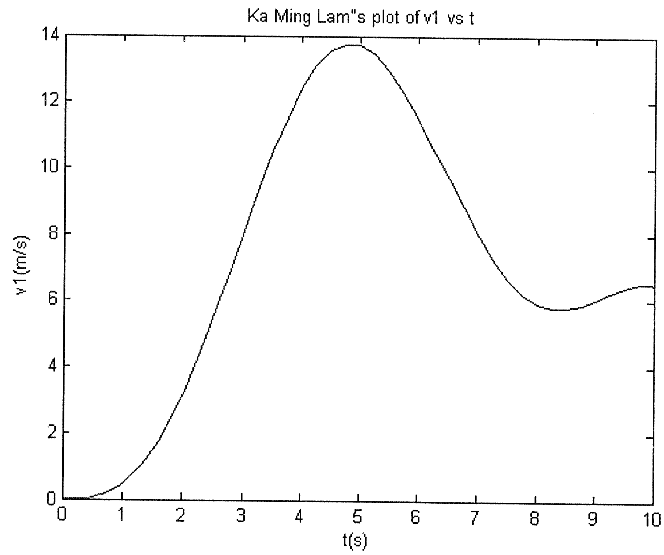
c). Modify the code in b) a little bit to plot $x_1 \sim t$., increase time to see the decaying oscillations.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

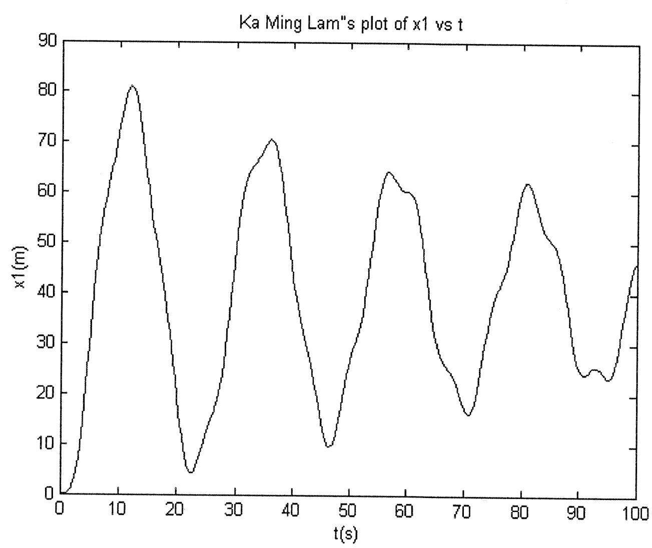
```
% problem 9.76

function question976
%time span
tspan = [0,10];          %integrate for 10 sec
z0 = [0, 0, 0, 0]';      %initial position and velocity
                         %[x0, vx0, y0, vy0]
%solves the ODEs
[t z] = ode45(@rhs,tspan,z0);

%Unpack the variables
x1 = z(:,1);
v1 = z(:,2);
x2 = z(:,3);
v2 = z(:,4);


%plot the results
plot(t,v1)
title('Ka Ming Lam''s plot of v1 vs t')
xlabel('t(s)')
ylabel('v1(m/s)')
%set grid, xmin, xmax, ymin, ymax


end

%-------------------------------------------------------\
function zdot = rhs(t,z)
x1 = z(1);  v1 = z(2);  x2 = z(3);  v2 = z(4);

%put in vlaues for mass, C and g below
m1 = 2;
m2 = 20;  % masses in kg
C = 0.4;  % in kg/s
F = 120;
k1 = 1;
k2 = 1;
k3 = 1;   % k in N/m

% the linear momentum balance eqns:
x1dot = v1;
v1dot = (-k2-k1)/m1*x1+k2/m1*x2;
x2dot = v2;
v2dot = F/m2-C/m2*v2-(k3+k2)/m2*x2+k2/m2*x1;

zdot = [x1dot;v1dot ; x2dot;v2dot];   %this is what the function returns (column vector)


end
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

b). Here is the plot



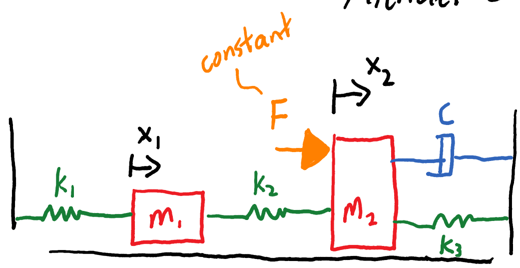
c). In order to have a decaying erratic oscillation we need to increase tspan to [0 100] for this case



HW3-P13-10.4.14

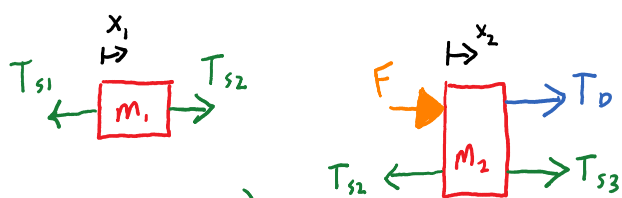
Monday, February 22, 2021 11:50 PM

Solution by Michael Zakaworotny



Displacements defined such that springs are unstretched when $x_1 = x_2 = 0$. Therefore, we need not consider rest lengths.

a)



$T_{s1} = k_1(x_1 - 0)$

$T_{s2} = k_2(x_2 - x_1)$

$T_{s3} = k_3(0 - x_2)$

$T_D = c(0 - \dot{x}_2)$

Pay attention to right end of this spring & dashpot

$$\Sigma F_1 = m_1 \ddot{x}_1$$

$$-T_{s1} + T_{s2} = m_1 \ddot{x}_1$$

$$-k_1 x_1 + k_2(x_2 - x_1) = m_1 \ddot{x}_1$$

$$\boxed{m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0} \qquad (1)$$

$$\Sigma F_2 = m_2 \ddot{x}_2$$

$$-T_{s2} + T_{s3} + T_D + F = m_2 \ddot{x}_2$$

$$-k_2(x_2 - x_1) - k_3 x_2 - c\dot{x}_2 + F = m_2 \ddot{x}_2$$

$$\boxed{m_2 \ddot{x}_2 + c\dot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 = F} \quad (2)$$

Write (1) and (2) as a system of second order equations

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}$$

$$[M][\ddot{x}] + [C][\dot{x}] + [K][x] = [F]$$

State: $[x], [v]$

Second order form:

$$\boxed{\begin{aligned} &[\dot{x}] = [v] \\ &[\dot{v}] = [\ddot{x}] = [M]^{-1}\big([F] - [C][v] - [K][x]\big) \\ &\qquad [M]\backslash\big([F] - [C][v] - [K][x]\big) \end{aligned}}$$

Matlab backslash

MATLAB trick:

To solve a matrix equation of the form:

$$[A][x] = [b]$$

the following syntax can be used:

x = A \ b

This is equivalent to the following:

x = A ^ -1 * b

b)

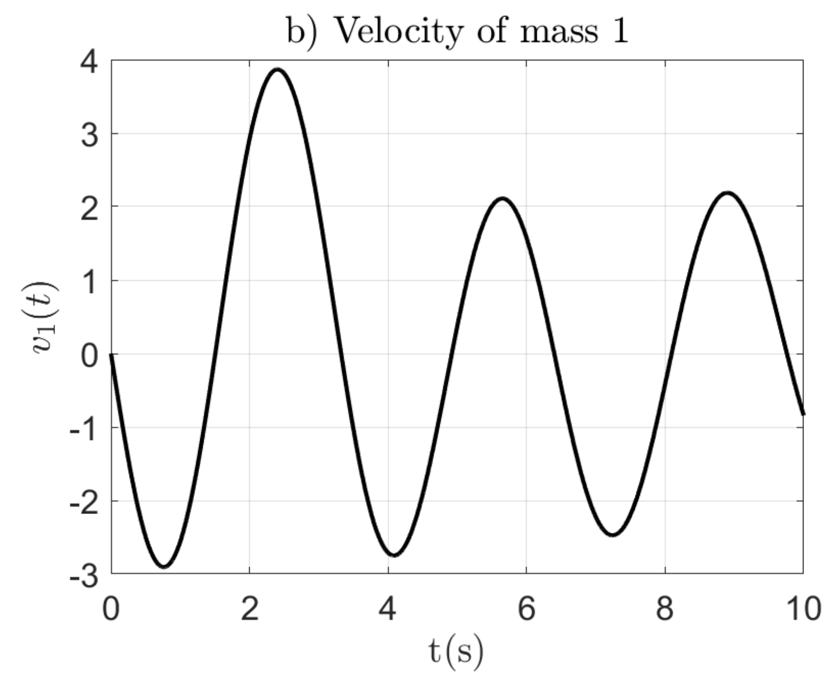
Parameters:

```
m1 = 1; m2 = 2; k1 = 1; k2 = 2; k3 = 1; c = 1; F = 4;
```

Initial conditions:

```
x10 = 2; x20 = 0; v10 = 0; v20 = 0;
```

We expect the resulting motion of mass 1 to be oscillatory, with a velocity that initially becomes negative since it has been displaced in the positive direction.



Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

c)

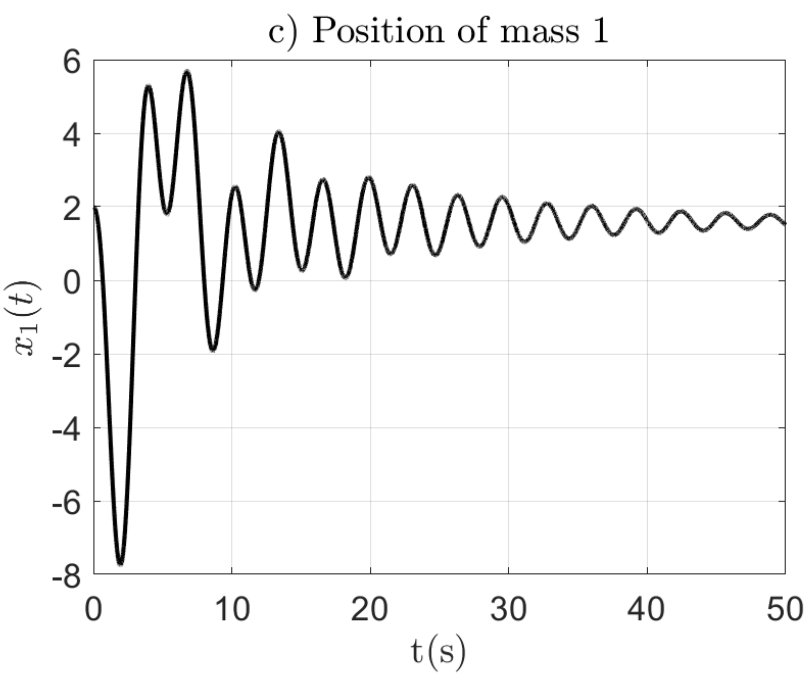
Parameters:

```
m1 = 1; m2 = 2; k1 = 1; k2 = 2; k3 = 1; c = 1; F = 4;
```

Initial conditions:

```
x10 = 2; x20 = 0; v10 = 0; v20 = -10;
```

Initial conditions are selected such both masses move towards each other initially quite aggressively. After a complicated oscillatory motion, the oscillations decay, converging on a steady state value.



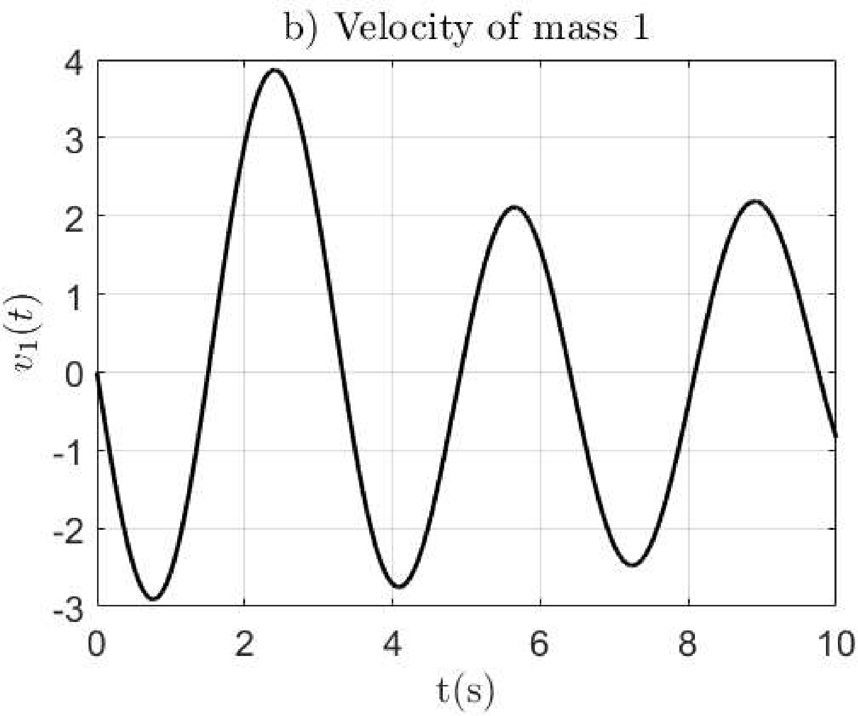
Contents


```
% MAE 2030
% HW 3, Q13 Solution
% 10.4.14


% Clear workspace, clear terminal and close plots
clear all; clc; close all;

% Define parameters - m's, k's, c, constant F
m1 = 1; m2 = 2; k1 = 1; k2 = 2; k3 = 1; c = 1;
F = 4;
% Pack parameters into a struct
p.m1 = m1; p.m2 = m2; p.k1 = k1; p.k2 = k2; p.k3 = k3; p.c = c;
p.F = F;
```

Part b) - Plot velocity 1 for 10 time steps

```
% Initial and final times of simulation
t0 = 0; tend = 10;
tspan = [t0 tend];

% Initial conditions
x10 = 2; x20 = 0; % initial positions
v10 = 0; v20 = 0; % initial velocities
z0 = [x10 x20 v10 v20]';

eqn = @(t,z) myrhs(t,z,p); % anonymous function handle for myrhs function
opts.RelTol = 1e-3; % a "tolerance" on the ode solver
opts.maxtime = 10; % maximum time of solution (will quit if not converged
                   % by this time)
% Call to midpoint solver - returns a 1xN vector for the time divisions
% that satisfy the tolerance, and a 4xN matrix containing the state at each
% time
[tarray, zarray] = myMidpointSolver(eqn, tspan, z0, opts);

x1 = zarray(1,:); x2 = zarray(2,:);
v1 = zarray(3,:); v2 = zarray(4,:);

% Plotting v1(t)
fig = figure; hold on; box on; grid on;
set(gcf,'DefaultTextInterpreter','latex'); % use a latex interpreter
set(gca,'FontSize',16);
plot(tarray,v1,'-k','LineWidth',2);
xlabel('t(s)'); ylabel('$v_1(t)$');
title('b) Velocity of mass 1');
% Save image files for convenience. Will overwrite each run
saveas(fig,'q13_b.fig');
```

```
saveas(fig,'q13_b.png');
```



Part c) - Plot position 1 to observe decaying oscillations

```
% Initial and final times of simulation
t0 = 0; tend = 50;
tspan = [t0 tend];

% Initial conditions
x10 = 2; x20 = 0; % initial positions
v10 = 0; v20 = -10; % initial velocities
z0 = [x10 x20 v10 v20]';

eqn = @(t,z) myrhs(t,z,p); % anonymous function handle for myrhs function
opts.RelTol = 1e-3; % a "tolerance" on the ode solver
opts.maxtime = 10; % maximum time of solution (will quit if not converged
                   % by this time)
% Call to midpoint solver - returns a 1xN vector for the time divisions
% that satisfy the tolerance, and a 4xN matrix containing the state at each
% time
[tarray, zarray] = myMidpointSolver(eqn, tspan, z0, opts);

x1 = zarray(1,:); x2 = zarray(2,:);
v1 = zarray(3,:); v2 = zarray(4,:);

% Plotting v1(t)
fig = figure; hold on; box on; grid on;
set(gcf,'DefaultTextInterpreter','latex'); % use latex interpreter for text
```

```
set(gca,'FontSize',16);
plot(tarray,x1,'-k','LineWidth',2);
xlabel('t(s)'); ylabel('$x_1(t)$');
title('c) Position of mass 1');
% Save image files for convenience. Will overwrite each run
saveas(fig,'q13_c.fig');
saveas(fig,'q13_c.png');
```

FUNCTIONS

```
function [tarray zarray] = myMidpointSolver(eqn, tspan, z0, options)
    % Implementation of a midpoint sovler with a tolerance.
    % INPUTS:
    % eqn - a function handle, with inputs (t,z), which returns the state
    %       derivative zdot
    % tspan - a vector containins the initial time (t(1)) and final time
    %         (t(end) of the simulation
    % z0 - a 2*d x 1 vector containing the initial positions of each item
    %      in the system, stacked on top the initial velocities of each
    %      item. d is the number of components in the system
    % options - a struct containing the tolerance, RelTol, and maximum
    %           simulation time
    % OUTPUTS:
    % tarray - a 1xN vector containing the time array at which RelTol is
    %          satisfied
    % zarray - a 2*d x N vector containing the state at each computed time
    %          step

    RelTol = options.RelTol; % desired tolerance in solution
    maxtime = options.maxtime; % maximum time before simulation ends

    d = length(z0); % number of elements in state

    N_array = 5*2.^[1:30]; % a range of potential time divisions to run

    zend = z0; % a vector which will carry the end state from last
               % iteration for now, initialized to the initial state

    tic; % start tracking time
    % loop over
    for N = N_array % number of divisions in time span

        tarray = linspace(tspan(1),tspan(end),N); % make time array with N
        zarray = zeros(d,N);
        zarray(:,1) = z0; % initialize solution matrix with IC

        for i = 1:N-1
            t_i = tarray(i); % current time
            t_i_1 = tarray(i+1); % next time (i+1)
            h = t_i_1 - t_i; % time increment
            t_mid = t_i + h/2; % time at middle of next increment
            %%%%%%%%%%%%%%%%%%%%%%%%% MIDPOINT IMPLEMENTATION %%%%%%%%%%%%%
            z_i = zarray(:,i); % state at current time
            zdot_i = eqn(t_i,z_i); % zdot at current time

            z_mid = z_i + h/2*zdot_i; % state at midpoint of time increment
```

```
            zdot_mid = eqn(t_mid,z_mid); % zdot at midpt of time increment

            z_i_1 = z_i + h*zdot_mid; % state at next time increment -
                                      % "extrapolate" current state using
                                      % derivative at midpoint

            zarray(:,i+1) = z_i_1; % add next state to matrix


        end

        zdiff = zend - zarray(:,end); % get "residual" between last state
        zerr = sum(abs(zdiff)); % get magnitude of residual
        % continue if residual has not yet converged, and time still less
        % than max allowable
        if zerr > RelTol && toc < maxtime
            zend = zarray(:,end); % if so, update end state
        else
            return; % if residual has converged, return
        end

    end

end


function zdot = myrhs(t, z, p)
    % Second order form as a system of equations in matrix form
    % INPUTS:
    % t - the current time, a scalar
    % z - the current state, a 2*d x 1 vector
    % p - a struct containing system parameters
    % OUTPUTS:
    % zdot - a 2*d x 1 vector containing the state derivative

    % Construct mass, damping and stiffness matrices
    M = [p.m1 0;
         0 p.m2];
    C = [0 0;
         0 p.c];
    K = [p.k1+p.k2 -p.k2;
         -p.k2 p.k2+p.k3];
    F = [0;p.F];

    X = z(1:2); V = z(3:4); % Pull X and V from state

    Xdot = V;
    Vdot = M\(F - C*V - K*X); % First order form

    zdot = [Xdot; Vdot]; % pack into zdot

end
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

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10.4.15 The three beads of masses m, $2m$, and m connected by massless linear springs of constant k slide freely on a straight rod. Let x_i denote the displacement of the i^{th} bead from its equilibrium position at rest.

a) Write expressions for the total kinetic and potential energies.

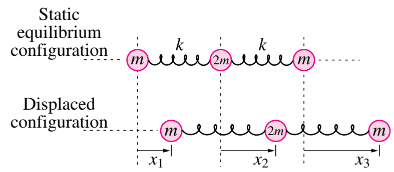
b) Write an expression for the total linear momentum.

c) Draw free-body diagrams for the beads and use Newton's second law to derive the equations for motion for the system.

d) Verify that total energy and linear momentum are both conserved.

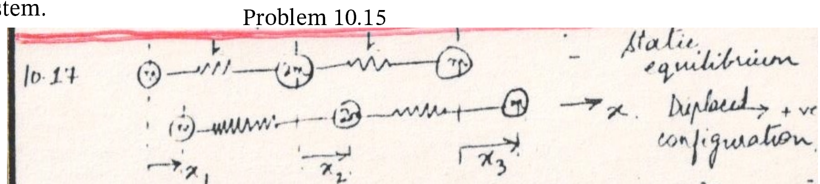
e) Show that the center of mass must either remain at rest or move at constant velocity.

f) What can you say about vibratory (sinusoidal) motions of the system?



Problem 10.15

10.17



(a) E_k : total KE $= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2}(2m)\dot{x}_2^2 + \frac{1}{2} m \dot{x}_3^2$

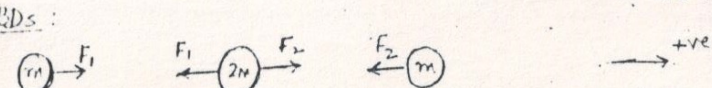
$$\boxed{E_k = \frac{m}{2}\left(\dot{x}_1^2 + 2\dot{x}_2^2 + \dot{x}_3^2\right)}$$

E_p : total PE $= \frac{1}{2}k(x_2-x_1)^2 + \frac{1}{2}k(x_3-x_2)^2$

$$\boxed{E_p = \frac{k}{2}\left[(x_2-x_1)^2 + (x_3-x_2)^2\right]}$$

(b) total linear momentum: $\boxed{L = \sum m v = m\dot{x}_1 + 2m\dot{x}_2 + m\dot{x}_3}$

(c) FBDs :



Equations of motion :

$$\left.\begin{aligned} F_1 &= k(x_2-x_1) = m\ddot{x}_1 \\ F_2 - F_1 &= k[(x_3-x_2)-(x_2-x_1)] = 2m\ddot{x}_2 \\ -F_2 &= -k(x_3-x_2) = m\ddot{x}_3 \end{aligned}\right\} \quad (i)$$

Adding eqs (i) : $m\ddot{x}_1 + 2m\ddot{x}_2 + m\ddot{x}_3 = 0$

(d) $E_{tot} = E_k + E_p$. If E_{total} is conserved then $\frac{dE_{tot}}{dt} = 0$

$\frac{dE_{tot}}{dt} = m\dot{x}_1\ddot{x}_1 + 2m\dot{x}_2\ddot{x}_2 + m\dot{x}_3\ddot{x}_3 + k[(\dot{x}_2-\dot{x}_1)(x_2-x_1) + (\dot{x}_3-\dot{x}_2)(x_3-x_2)]$

Plugging in eqs (i), we obtain $\frac{dE_{tot}}{dt} = 0 \Rightarrow E_{tot}$ is conserved

If L is conserved, then $\dot{L} = 0 = m\ddot{x}_1 + 2m\ddot{x}_2 + m\ddot{x}_3 = 0$ (using eqs (i) from above) Hence L is conserved.

(e) $\dot{L} = 0 = m_{tot}\dot{v}_{cm} \Rightarrow m_{tot}\dot{v}_{cm} = 0$

Since m_{tot} is constant, $\dot{v}_{cm} = 0 \Rightarrow$ the velocity for center of mass is a constant or it is at rest.

(f) Regarding vibratory motion of system, masses (1) and (3) are exactly out of phase with each other.

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10.4.16 Two blocks with masses M and m are connected by a spring with constant k and free length ℓ_0 that can sustain compression. Mass M is resting on the ground at the start. There is gravity. The upwards vertical displacement of mass m is x, which is zero when the spring is at its rest length and M is on the ground.

a) For what value of x is the system in static equilibrium?

b) Find a differential equation governing the motion of m assuming M remains on the ground.

c) Draw a free-body diagram of M.

d) For what value of x is M on the verge of lifting off the ground.

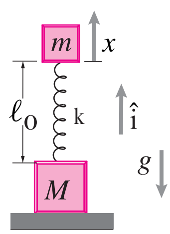
e) Defining y as the height of the lower mass, write two coupled differential equations for the motion of m and M if both masses are in the air.

f) Find the value of $x < 0$ so that if the system is started from rest with that x and $y = 0$ that the ground reaction force on M just goes to zero.

g) Starting here, this problem is more of a project than a typical homework problem. Assume $x(t = 0)$ is less than the value computed above. Write a computer program that integrates the equations of motion until M lifts off and then switches to integrating the equations for the two masses in the air.

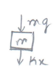
h) modify your program so that if M hits the ground again, it sticks until the ground reaction force goes to zero again.

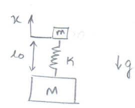
i) By playing around, this way or that, see if you can find a special value for $x(t = 0)$ so that the bouncing continues indefinitely. (This is a perhaps surprising result, that a system with plastic collisions can continue to bounce indefinitely.)



Problem 10.16

9.4.16





$m\ddot{x} = -Kx - mg$

a) Static equilibrium

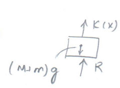
$\ddot{x} = 0$

$\Rightarrow -Kx - mg = 0$

$x = -\frac{mg}{K}$

b) $m\ddot{x} + Kx = mg$

c)



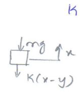
d) M lifts off if $R = 0$

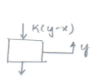
$K(x) + R = (M+m)g$

$K(x) = (M+m)g$

$x = \frac{(M+m)g}{K}$

e)

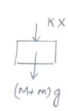




$m\ddot{x} = -K(x-y) - mg$

$M\ddot{y} = -K(y-x) - (m+M)g$

f)



$R = 0$

$X = -\frac{(M+m)g}{K}$

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