SAMPLE 10.16 For the given quantities and initial conditions, find $x_1(t)$ and $x_2(t)$. Assume the spring is unstretched at its rest length of L_0 when $x_1 = x_2$.

$$\begin{array}{llll} m_1 = 1\,\text{kg}, & m_2 = 2\,\text{kg}, & k = 3\,\text{N/m}, & c = 5\,\text{N/(m/s)} \\ x_1(0) = 1\,\text{m}, & \dot{x}_1(0) = 0, & x_2(0) = 2\,\text{m}, & \dot{x}_2(0) = 0. \end{array}$$

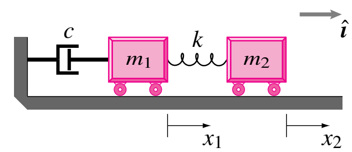

Figure 10.39

Solution The free-body diagrams of all components of the given system are shown below.

FBDs

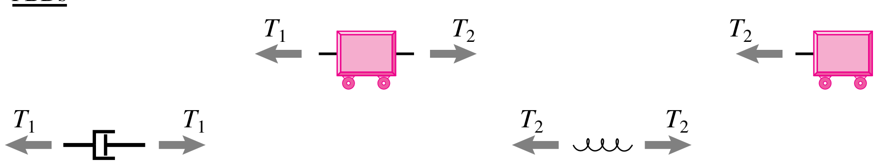

Figure 10.40:

The spring and dashpot laws give

$$T_1 = c\dot{x}_1 \qquad T_2 = k(x_2 - x_1). \tag{10.48}$$

The linear momentum balance for the two masses gives

$$\begin{array}{rl} & \sum \vec{F} = m\vec{a} \\ \text{mass 1:} & -T_1\hat{\imath} + T_2\hat{\imath} = m_1\ddot{x}_1\hat{\imath} \\ \text{mass 2:} & -T_2\hat{\imath} = m_2\ddot{x}_2\hat{\imath}. \end{array} \tag{10.49}$$

Applying the constitutive laws (10.48) to the momentum balance equations (10.49) gives

$$\begin{aligned} \ddot{x}_1 &= [k(x_2 - x_1) - c\dot{x}_1]/m_1 \\ \ddot{x}_2 &= [-k(x_2 - x_1)]/m_2. \end{aligned}$$

Defining $z_1 = x_1, z_2 = \dot{x}_1, z_3 = x_2, z_4 = \dot{x}_2$ gives

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= [k(z_3 - z_1) - cz_2]/m_1 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= [-k(z_3 - z_1)]/m_2. \end{aligned}$$

The initial conditions are

$$z_1(0) = 1\,\text{m}, \quad z_2(0) = 0, \quad z_3(0) = 2\,\text{m}, \quad z_4(0) = 0.$$

We are now set for numerical solution. Solving these equations numerically, we plot $x_1(t)$ and $x_2(t)$ as shown in fig. 10.41. From the solution, it is clear that both the masses settle down to the equilibrium position $x_1 = x_2 = 1$ m after the oscillations die down. In this position, the spring exerts no force as it is unstretched. Also note that the two masses move in the opposite direction immediately after being set into motion as they must because of the opposite accelerations.

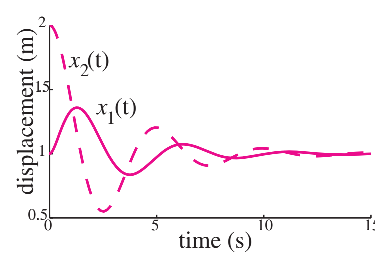

Figure 10.41: Plot of $x_1(t)$ and $x_2(t)$ obtained from the numerical solution of the equations of motion. Note that with $x_1(0) = 1$ m and $x_2(0) = 2$ m, both masses settle down to $x_1 = x_2 = 1$ m after the oscillations die.

SAMPLE 10.17 Flight of a toy hopper. A hopper model ① is made of two masses $m_1 = 0.4\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a spring with stiffness $k = 100\,\text{N/m}$ as shown in fig. 10.42. The unstretched length of the spring is $\ell_0 = 1\,\text{m}$. The model is released from rest from the configuration shown in the figure with $y_1 = 25.5\,\text{m}$ and $y_2 = 24\,\text{m}$.

1. Find and plot $y_1(t)$ and $y_2(t)$ for $t = 0$ to $2\,\text{s}$.
2. Plot the motion of m_1 and m_2 with respect to the center-of-mass of the hopper during the same time interval.
3. Plot the motion of the center-of-mass of the hopper from the solution obtained for $y_1(t)$ and $y_2(t)$ and compare it with analytical values obtained by integrating the center-of-mass motion directly.

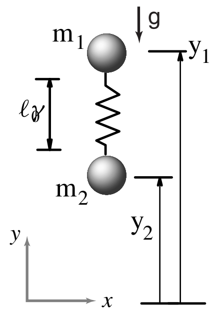

Figure 10.42

① You can think of it as a model of a hopping robot or a hopping animal where the foot mass is modeled with m_2 and the body mass with m_1. The spring models the elasticity of the system. If you restrict this model to one dimensional vertical motion under gravity, it will hop, once released from some initial height. The number of hops depends on the initial conditions. Theoretically, it can hop forever with suitable initial conditions.

Solution The free-body diagrams of the two masses are shown in fig. 10.43. From the linear momentum balance in the y direction, we can write the equations of motion at once.

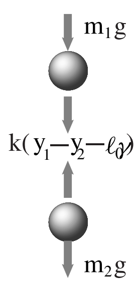

Figure 10.43: Free-body diagram of the two masses m_1 and m_2

$$\begin{aligned}
m_1\ddot{y}_1 &= -k(y_1 - y_2 - \ell_0) - m_1 g \\
\Rightarrow \quad \ddot{y}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g
\end{aligned} \tag{10.50}$$

$$\begin{aligned}
m_2\ddot{y}_2 &= k(y_1 - y_2 - \ell_0) - m_2 g \\
\Rightarrow \quad \ddot{y}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g.
\end{aligned} \tag{10.51}$$

1. The equations of motion obtained above are coupled linear differential equations of second order. We can solve for $y_1(t)$ and $y_2(t)$ by numerical integration of these equations. As we have shown in previous examples, we first need to set up these equations as a set of first order equations.
   Letting $\dot{y}_1 = v_1$ and $\dot{y}_2 = v_2$, we get

$$\begin{aligned}
\dot{y}_1 &= v_1 \\
\dot{v}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \\
\dot{y}_2 &= v_2 \\
\dot{v}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g.
\end{aligned}$$

   Now we solve this set of equations numerically using some ODE solver and the following pseudocode.

```
ODEs = {y1dot = v1,
        v1dot = -k/m1*(y1-y2-l0) - g,
        y2dot = v2,
        v1dot = k/m1*(y1-y2-l0) - g}
IC   = {y1(0)=25.5, v1(0)=0, y2(0)=24, v2(0)=0}
Set  k=100, m1=0.4, m2=1, l0=1
Solve ODEs with IC for t=0 to t=2
Plot y1(t) and y2(t)
```

   The solution obtained thus is shown in fig. 10.44.

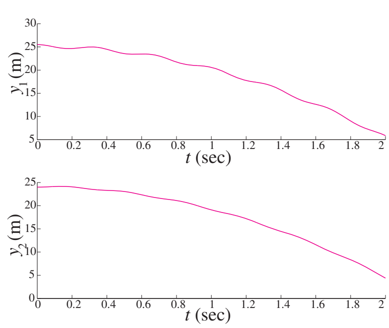

Figure 10.44: Numerically obtained solutions $y_1(t)$ and $y_2(t)$

2. We can find the motion of m_1 and m_2 with respect to the center-of-mass by subtracting the motion of the center-of-mass, y_{cm} from y_1 and y_2. Since,

$$y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \tag{10.52}$$

we get,

$$\begin{aligned} y_{1/\text{cm}} &= y_1 - y_{\text{cm}} = \frac{m_2}{m_1+m_2}(y_1 - y_2) \\ y_{2/\text{cm}} &= y_2 - y_{\text{cm}} = -\frac{m_1}{m_1+m_2}(y_1 - y_2). \end{aligned}$$

The relative motions thus obtained are shown in fig. 10.45. We note that the motions of m_1 and m_2, as seen by an observer sitting at the center-of-mass, are simple harmonic oscillations.

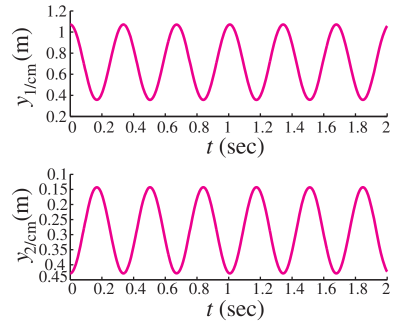

Figure 10.45: Numerically obtained solutions $y_{1/\text{cm}}(t)$ and $y_{1/\text{cm}}(t)$.

3. We can find the center-of-mass motion $y_{\text{cm}}(t)$ from y_1 and y_2 by using eqn. (10.52). The solution obtained thus is shown as a solid line in fig. 10.47. We can also solve for the center-of-mass motion analytically by first writing the equation of motion of the center-of-mass and then integrating it analytically.

   The free-body diagram of the hopper as a single system is shown in fig. 10.46. The linear momentum balance for the system in the vertical direction gives

$$\begin{aligned} (m_1 + m_2)\ddot{y}_{\text{cm}} &= -m_1 g - m_2 g \\ \Rightarrow \quad \ddot{y}_{\text{cm}} &= -g. \end{aligned}$$

   We recognize this equation as the equation of motion of a freely falling body under gravity. We can integrate this equation twice to get

$$y_{\text{cm}}(t) = y_{\text{cm}}(0) + \dot{y}_{\text{cm}}(0)t - \frac{1}{2}gt^2.$$

   Noting that $y_{\text{cm}}(0) = 24.43\,\text{m}$ (from eqn. (10.52)), and $\dot{y}_{\text{cm}}(0) = 0$ (the system is released from rest), we get

$$y_{\text{cm}}(t) = 24.43\,\text{m} - \frac{1}{2}\cdot 9.81\,\text{m/s}^2 \cdot t^2.$$

   The values obtained for the center-of-mass position from the above expression are shown in fig. 10.47 by small circles.

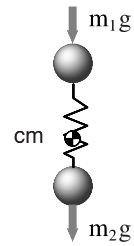

Figure 10.46: Free-body diagram of the hopper as a single system. The spring force does not show up here since it becomes an internal force to the system

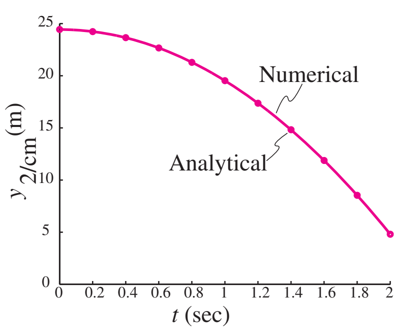

Figure 10.47: Numerically obtained solution for the position of the center-of-mass, $y_{\text{cm}}(t)$.

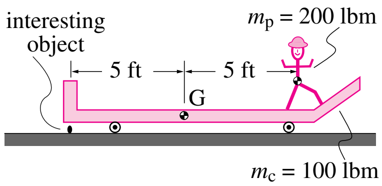

Figure 10.48: Mr. P spots an interesting object.

SAMPLE 10.18 Conservation of linear momentum. Mr. P with mass $m_p = 200$ lbm is standing on a cart with frictionless and massless wheels. The cart weighs half as much as Mr. P. Standing at one end of the cart, Mr. P spots an interesting object at the other end of the cart. Mr. P decides to walk to the other end of the cart to pick up the object. How far does he find himself from the object after he reaches the end of the cart?

Solution From your own experience in small boats perhaps, you know that when Mr. P walks to the left the cart moves to the right. Here, we want to find how far the cart moves.

Consider the cart and Mr. P together to be the system of interest. The free-body diagram of the system is shown in Fig. 10.49(a).

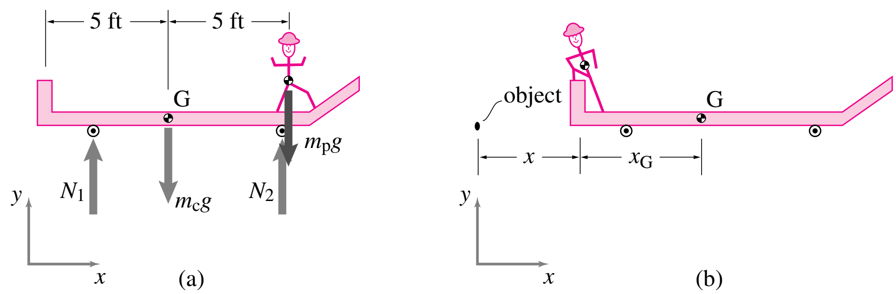

Figure 10.49: (a) Free-body diagram of Mr. P-and-the-cart system. (b) The cart has moved to the right by distance x when Mr. P reaches the other end.

From the diagram it is clear that there are no external forces in the x-direction. Therefore,

$$\dot{L}_x = \sum F_x = 0 \quad \Rightarrow \quad L_x = \text{constant}$$

that is, the linear momentum of the system in the x-direction is 'conserved'. But the initial linear momentum of the system is zero. Therefore,

$$L_x = m_{tot}(v_{cm})_x = 0 \text{ all the time} \quad \Rightarrow \quad (v_{cm})_x = 0 \text{ all the time.}$$

Because the horizontal velocity of the center-of-mass is always zero, the center-of-mass does not change its horizontal position. Now let x_{cm} and x'_{cm} be the x-coordinates of the center-of-mass of the system at the beginning and at the end, respectively. Then,

$$x'_{cm} = x_{cm}.$$

Now, from the given dimensions and the stipulated position at the end in Fig. 10.49(b),

$$x_{cm} = \frac{m_c x_G + m_p x_p}{m_c + m_p} \quad \text{and} \quad x'_{cm} = \frac{m_c(x_G + x) + m_p x}{m_c + m_p}.$$

Equating the two distances we get,

$$\begin{aligned} m_c x_G + m_p x_p &= m_c(x_G + x) + m_p x \\ &= m_c x_G + x(m_c + m_p) \\ \Rightarrow \quad x &= \frac{m_p x_p}{m_c + m_p} \\ &= \frac{200\,\text{lbm} \cdot 10\,\text{ft}}{300\,\text{lbm}} = 6\frac{2}{3}\,\text{ft.} \end{aligned}$$

6.67 ft

[Note: if Mr. P and the cart have the same mass, the cart moves to the right the same distance Mr. P moves to the left.]

