10.4 Coupled motion in 1D

The primary emphasis of this section is setting up correct differential equations (without sign errors) and solving these equations on the computer.

Preparatory Problems

10.4.1 Write the following set of coupled second order ODE's as a system of first order ODE's.

$$\begin{aligned}\ddot{x}_1 &= k_2(x_2 - x_1) - k_1 x_1\\ \ddot{x}_2 &= k_3 x_2 - k_2(x_2 - x_1)\end{aligned}$$

10.4.2 The solution of a set of second order differential equations is:

$$\begin{aligned}\xi(t) &= A\sin\omega t + B\cos\omega t + \xi^*\\ \dot{\xi}(t) &= A\omega\cos\omega t - B\omega\sin\omega t,\end{aligned}$$

where A and B are constants to be determined from initial conditions and ξ^* is a known constant. Assume A and B are the only unknowns.

a) Write the equations in matrix form which you would need to solve in order to find A and B in terms of $\xi(0)$ and $\dot{\xi}(0)$.

b) Solve the equations in symbols.

c) Solve for the numerical constants A and B using the matrix form, if $\xi(0) = 0$, $\dot{\xi}(0) = 0.5$, $\omega = 0.5\,\mathrm{rad/s}$ and $\xi^* = 0.2$.

10.4.3 A set of first order linear differential equations is given:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 + kx_1 + cx_2 &= 0.\end{aligned}$$

Write these equations in the form $\dot{\vec{x}} = [\boldsymbol{A}]\vec{x}$, where $\vec{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$.

10.4.4 Write the following pair of coupled ODE's as a set of first order ODE's.

$$\begin{aligned}\ddot{x}_1 + x_1 &= \dot{x}_2\sin t\\ \ddot{x}_2 + x_2 &= \dot{x}_1\cos t\end{aligned}$$

10.4.5 The following set of differential equations can be written in first order form, and in particular, in matrix form $\dot{\vec{x}} = [\boldsymbol{A}]\vec{x} + \vec{c}$. In general equations of motion are not so simple, but *linear* cases like this are prevalent in the analytic study of dynamical systems.

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_4\\ \dot{x}_3 + 5\Omega^2 x_1 - 4\Omega^2 x_2 &= 2\Omega^2 v_1^*\\ \dot{x}_4 - 4\Omega^2 x_1 + 5\Omega^2 x_2 &= -\Omega^2 v_1^*\end{aligned}$$

10.4.6 Write each of the following equations as a system of first order ODE's.

a) $\ddot{\theta} + \lambda^2\theta = \cos t$,

b) $\ddot{x} + 2p\dot{x} + kx = 0$,

c) $\ddot{x} + 2c\dot{x} + k\sin x = 0$.

10.4.7 A train moves at a constant absolute velocity $v\hat{\imath}$. A passenger, idealized as a point mass, walks at an absolute velocity $u\hat{\imath}$, where $u > v$. What is the velocity of the passenger relative to the train?

10.4.8 Two equal masses, each denoted by the letter m, are on an air track. One mass is connected by a spring to the end of the track. The other mass is connected by a spring to the first mass. The two spring constants are equal and represented by the letter k. In the rest configuration (springs are relaxed) the masses are a distance ℓ apart. Motion of the two masses x_1 and x_2 is measured relative to this configuration.

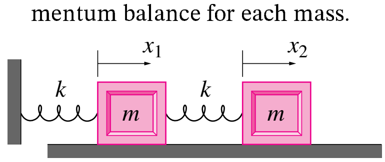
a) Write the potential energy of the system for arbitrary displacements x_1 and x_2 at some time t.

b) Write the kinetic energy of the system at the same time t in terms of $\dot{x}_1$, $\dot{x}_2$, m, and k.

c) Write the total energy of the system.

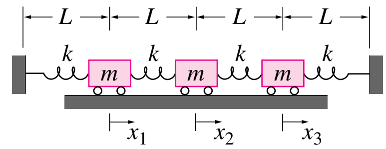
d) Draw a free-body diagram for each mass.

e) Write the equation of linear momentum balance for each mass.



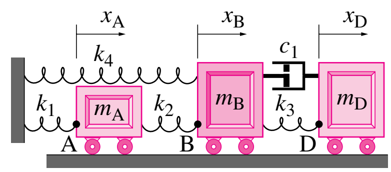
Problem 10.4.8

10.4.9 For the three-mass system shown, draw a free-body diagram of each mass. Write the spring forces in terms of the displacements x_1, x_2, and x_3.



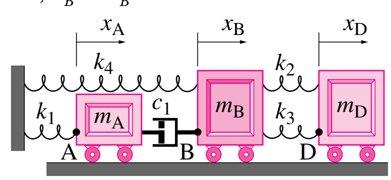
Problem 10.4.9

10.4.10 The springs shown are relaxed when $x_A = x_B = x_D = 0$. In terms of some or all of m_A, m_B, m_D, x_A, x_B, x_D, $\dot{x}_A$, $\dot{x}_B$, $\dot{x}_C$, and k_1, k_2, k_3, k_4 and c_1, find the acceleration of block B. *



Problem 10.4.10

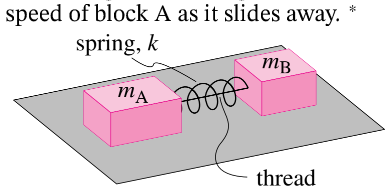
10.4.11 A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given $k_1, k_2, k_3, k_4, c_1, m_A, m_B, m_D, x_A, x_B, x_D, \dot{x}_A, \dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\vec{\boldsymbol{a}}_B = \ddot{x}_B\hat{\boldsymbol{\imath}}$. *



Problem 10.4.11

More-Involved Problems

10.4.12 A massless spring with constant k is held compressed a distance δ from its relaxed length by a thread connecting blocks A and B which are still on a frictionless table. The blocks have mass m_A and m_B, respectively. The thread is suddenly but gently cut, the blocks fly apart and the spring falls to the ground. Find the speed of block A as it slides away. *



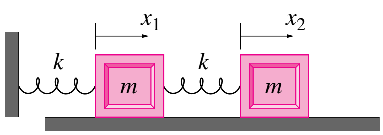
Problem 10.4.12

10.4.13 In the system below the masses are in equilibrium with the springs when $x_1 = x_2 = 0$.

a) First do problem 10.4.8.

b) Pick parameter values and initial conditions of your choice and simulate a motion of this system. Make a plot of the motion of, say, one of the masses vs time,

c) Explain how your plot does or does not make sense in terms of your understanding of this system. Is the initial motion in the right direction? Are the solutions periodic? Bounded? etc.



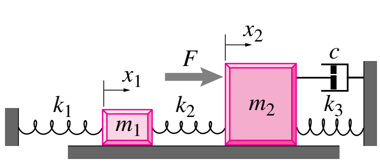
Problem 10.4.13

10.4.14 Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force F acts on mass 2. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be written in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.

a) Write the governing equations in a neat first order form. Your equations should be in terms of any or all of the constants m_1, m_2, k_1, k_2,k_3, C, the constant force F, and t. Getting the signs right is important.

b) Write computer commands to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions.

c) For constants and initial conditions of your choosing, plot x_1 vs t for enough time so that decaying erratic oscillations can be observed.



Problem 10.4.14

10.4.15 The three beads of masses m, $2m$, and m connected by massless linear springs of constant k slide freely on a straight rod. Let x_i denote the displacement of the i^{th} bead from its equilibrium position at rest.

a) Write expressions for the total kinetic and potential energies.

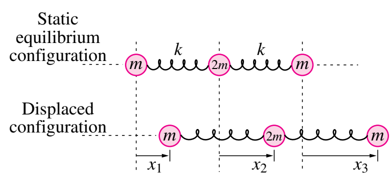
b) Write an expression for the total linear momentum.

c) Draw free-body diagrams for the beads and use Newton's second law to derive the equations for motion for the system.

d) Verify that total energy and linear momentum are both conserved.

e) Show that the center of mass must either remain at rest or move at constant velocity.

f) What can you say about vibratory (sinusoidal) motions of the system?



Problem 10.4.15

10.4.16 Two blocks with masses M and m are connected by a spring with constant k and free length ℓ_0 that can sustain compression. Mass$M$ is resting on the ground at the start. There is gravity. The upwards vertical displacement of mass m is x, which is zero when the spring is at its rest length and M is on the ground.

a) For what value of x is the system in static equilibrium?

b) Find a differential equation governing the motion of m assuming M remains on the ground.

c) Draw a free-body diagram of M.

d) For what value of x is M on the verge of lifting off the ground.

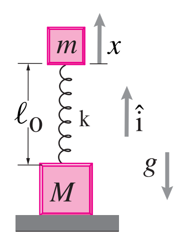
e) Defining y as the height of the lower mass, write two coupled differential equations for the motion of m and M if both masses are in the air.

f) Find the value of $x < 0$ so that if the system is started from rest with that x and $y = 0$ that the ground reaction force on M just goes to zero.

g) Starting here, this problem is more of a project than a typical homework problem. Assume $x(t = 0)$ is less than the value computed above. Write a computer program that integrates the equations of motion until M lifts off and then switches to integrating the equations for the two masses in the air.

h) modify your program so that if M hits the ground again, it sticks until the ground reaction force goes to zero again.

i) By playing around, this way or that, see if you can find a special value for $x(t = 0)$ so that the bouncing continues indefinitely. (This is a perhaps surprising result, that a system with plastic collisions can continue to bounce indefinitely.)



Problem 10.4.16