10.3 A mass and spring: the harmonic oscillator

A mass and spring moving in one dimension have the following governing differential equation and solution

$$m\ddot{x} + kx = 0 \quad \Rightarrow \quad x = C\cos\left(\sqrt{(k/m)}t - \phi\right).$$

The rest of this section is a discussion of, and elaboration of, this fact.

Any system with mass and elasticity can oscillate. Because lots of materials are elastic in normal conditions and all real things have mass, most things will oscillate if provoked.

The simplest, and also the most important, example is the spring-mass system of fig. 10.20. When the mass is on the right ($x > 0$) it accelerates to the left; when it is on the left ($x < 0$) it accelerates to the right. So it goes back and forth. As a general rule,

> Oscillations (vibrations) happen when a 'restoring' force pulls something back towards a rest position from both sides.

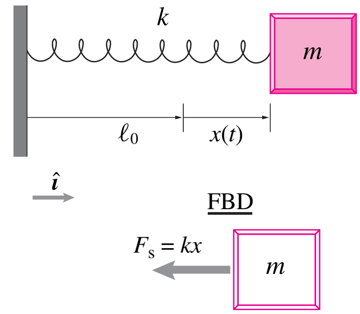


Figure 10.20: A spring mass system. For simplicity in 1D we only show forces along the motion. Although it would be more complete to show gravity and support forces, it would add clutter.

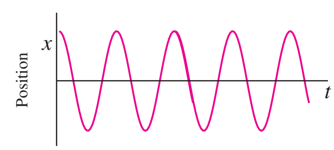


Figure 10.21: Oscillations.

Vibrations occur in the strings of cellos and sitars; the air columns in clarinets, trumpets and organ pipes and the wood blocks in a marimba. A system vibrating like this, whether literally a spring and mass or something more subtle, is called a 'harmonic oscillator'. More specifically,

> a harmonic oscillator has persistent (non-decaying) oscillations which are sinusoidal in time (*e.g.*, $x = \sin t$).

With varying degrees of approximation, car suspensions, buildings responding to earthquakes, earthquake faults themselves, quartz timing crystals and vibrating machines of all kinds are also modeled as mass-spring harmonic oscillators. The subject 'vibration theory' is based on the harmonic oscillator.

> The unforced oscillation of a spring and mass is the basic model for all vibrating systems.

Even structures which you think of as rigid (for example, when doing statics) will vibrate if encouraged to do so by the shaking of an unbalanced motor, the rumbling of a truck, a party upstairs, or the ground motion of an earthquake. And the vibrations of one thing can excite oscillations of another. This mutual excitement of fluids and solids can cause music, as in the vibrations in a clarinet reed (fig. 10.22), or trouble. The Tacoma Narrows bridge infamously collapsed because of the mutual excitement of the air and bridge.

All music, even if electrically powered, is mechanical vibrations (at least of the air in your ear), and so are all annoying sounds. Vibrations are the main function of a vibrating massager, and the main defect of a squeaking hinge. Mechanical vibrations in pendula or quartz crystals are used to measure time, but vibrations can cause a machine to go out of control (e.g., bicycle shimmy), or a bridge to collapse. So the generation of good vibrations and the inhibition of bad vibrations is an important application of dynamics.

The harmonic oscillator

The mother of all vibrating machines is the simple harmonic oscillator of fig. 10.20. The mass slides on a frictionless surface. The spring is relaxed at $x = 0$. The spring is thus stretched from ℓ_0 to $\ell_0 + \Delta\ell$, a stretch of $\Delta\ell = x$. A free-body diagram of the mass, cut 'free' from the spring in its extended state, is shown in the lower part of fig. 10.20. Linear momentum balance in the x direction (the balance of forces) gives:

$$\begin{aligned} \sum F_x &= \dot{L}_x \\ -kx &= m\ddot{x}. \end{aligned}$$

Rearranging, we get one of the most famous and useful differential equations of all time①:

①**Caution:** If you make a sign error you might get $\ddot{x} - (k/m)x = 0$ or $\ddot{x} = (k/m)x$. That not-so-innocent error changes the equation solutions from oscillations to exponential blow up. See appendix 3.1 on page 203.

$$\ddot{x} + \frac{k}{m}x = 0. \tag{10.24}$$

The harmonic oscillator ODE. This ubiquitous is a *2nd order linear constant-coefficient homogeneous ordinary differential equation*. Basically you can read this as that it is a 'simple simple simple simple simple' differential equation. It's 2nd order rather than, say 9th order. It's linear as opposed to non-linear. It's constant coefficient as opposed to having crazy functions of time where the constant k/m now sits. It's homogeneous (zero on the right hand side) instead of having crazy functions of time on the right hand side. And it involves ordinary rather than partial derivatives.

Figure 10.22: **Blowing grass.** If you squeeze a blade of grass between your parallel thumbs and then blow in the slot between the left and right thumb joints, at the grass, you can make a squeal sound that is similar (kind of) to that of a clarinet. This can be done with a strip of paper between the lips too. These are self-excited oscillations. The elastic restoring force is related to the tension in the grass which you can control by straightening and bending your thumbs. With practice you can adjust the pitch. Music, kind of. Photo by Katja Ojala.

Ordinary 'd', rather than partial ∂ derivatives

Linear: x always appears alone, never as, say x^2 or $\sin x$

$$\frac{d^2}{dt^2}x + \lambda^2 x = 0. \tag{10.25}$$

Constant coefficient, not, say, $\sin t$ or t^2 here

Homogeneous means zero on the right hand side

We have used λ^2 instead of k/m because Eqn. (10.25) shows up all over the place, both in and out of dynamics. Thus x doesn't necessarily represent displacement. In electronics, for example, x might represent a voltage and the term corresponding to k/m might be $1/LC$, where C is a capacitance and L an inductance. But even in dynamics eqn. (10.25) applies to other things besides a single spring and mass. For example, x might represent rotation of a pendulum.

Why λ^2 instead of just λ? Two reasons:

1. It shows that λ^2 is positive,
2. In the solution we need the square root of this coefficient, so it is convenient to start with $\sqrt{\lambda^2} = \lambda$.

For the spring-block system, λ^2 is k/m and in other problems λ^2 is a combination of other physical quantities.

Why λ instead of another letter? In a large book we can't avoid all notational conflicts, there are more quantities of interest than there are letters. The other common choices, p and ω, are more problematic, however②,

②**Notation: p, ω or λ.** Many books use p^2 or ω^2 in the place we have put λ^2. But using ω ('omega') can lead to confusion because we will later use ω for angular velocity. If one is studying vibrations of a rotating shaft then there would be two very different ω's in the problem. One, the coefficient of a differential equation and, the other, the angular velocity.

To add to the confusion, simple harmonic oscillations and circular motion have a deep connection, so the coincidence of notation is not accidental. Deep connection or not, the ω in the harmonic oscillator equation is not the same thing as the ω describing angular motion of a physical object. We avoid this confusion by using λ instead of ω here. (Also note that this λ is unrelated to the magnitude of the unit vector $\hat{\lambda}$).

Solution of the harmonic oscillator differential equation

We need to find a solution to the differential equation 10.25. A solution is a function $x(t)$ whose second derivative is the negative of the original function multiplied by the constant λ^2. Although math classes may spend some time on finding the solution to such equations (see 10.3 on page 12, for starters it is ok to remember the solution like this: when the mass is on the left it accelerates right when it's on the right it accelerates left. So it goes back and forth, so it seems like a sine wave. And it is. Until you know this (which should be soon) you can look it up in appendix 3.1 on page 203.

The general solution is

$$\begin{array}{rrcl}
 & x(t) & = & A\cos(\lambda\, t) + B\sin(\lambda\, t), \\
\text{or} & x(t) & = & C_1\cos(\lambda\, t) + C_2\sin(\lambda\, t).
\end{array} \qquad (10.26)$$

This sum of two sine waves③ is a solution of differential equation 10.25 for any values of the constants A (or C_1) and B (or C_2).

③A graph of the cosine function is also a sine wave.

Checking the solution in detail

What does it mean to say "$u = C_1\sin(\lambda\ t) + C_2\cos(\lambda\ t)$ satisfies the equation: $\ddot{u} = -\lambda^2 u$?" You satisfy a differential equation by feeding it a function that fully eliminates it. If you plug a candidate solution into a differential equation and get $0 = 0$ you have satisfied (solved) the equation.

Whether or not you learn to derive this solution, you should remember it and be able to check it.

To check if a function is a solution, plug it into the differential equation and see if the right side is equal to the left, like this.

Example: Detailed check of solution

Is this true for the given $u(t)$?

$$\frac{d^2}{dt^2}u = -\lambda^2 u$$

$$\frac{d^2}{dt^2}\overbrace{[C_1\sin(\lambda\ t) + C_2\cos(\lambda\ t)]}^{u(t)} \stackrel{?}{=} -\lambda^2\overbrace{[C_1\sin(\lambda\ t) + C_2\cos(\lambda\ t)]}^{u(t)}$$

$$\frac{d}{dt}\left(\frac{d}{dt}[C_1\sin(\lambda\ t) + C_2\cos(\lambda\ t)]\right) \stackrel{?}{=} -\lambda^2[C_1\sin(\lambda\ t) + C_2\cos(\lambda\ t)]$$

$$\frac{d}{dt}[C_1\lambda\cos(\lambda\ t) - C_2\lambda\sin(\lambda\ t)] \stackrel{?}{=} -\lambda^2[C_1\sin(\lambda\ t) + C_2\cos(\lambda\ t)]$$

$$\overbrace{-C_1\lambda^2\sin(\lambda t) - C_2\lambda^2\cos(\lambda\ t)}^{\ddot{u}} \stackrel{?}{=} -\lambda^2\overbrace{[C_1\sin(\lambda\ t) + C_2\cos(\lambda\ t)]}^{u(t)}$$

The equation $\ddot{u} = -\lambda^2 u$ *does* hold with the given $u(t)$. Right and left sides match. This shows at a glance in the next line.

$$0 \stackrel{:)}{=} 0 \qquad \text{(Satisfied.)}$$

Whatever the constants C_1 and C_2, the proposed solution eqn. (10.26) satisfies the differential equation eqn. (10.25).

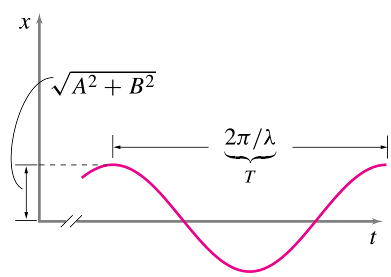


Figure 10.23: **Position versus time for an undamped, unforced harmonic oscillator.** x is the position of the mass, t is time.

Uniqueness. Maybe there are other solutions to this differential equation than eqn. (10.26)? There are not, as the mathematics-for-its-own-sake inclined student can learn to prove elsewhere. ④ The only possible motion of a spring and mass is a sinusoidal oscillation.

④A plausibility argument for uniqueness goes like this. If you release a mass from a given position x_0 at a given speed v_0 it will move in a definite way and no other way. This is a special case of what is called "determinism". But all solutions have some position and speed at $t = 0$ and we can find a C_1 and C_2 in eqn. (10.26) to match each such. Thus we have found *the* motion for every possible situation, and there can be no others.

Angular frequency, period, and frequency

There are three common ways to measure the 'speed' of oscillation: angular frequency, period, and frequency. The simplest of these is *angular frequency* $\lambda = \sqrt{(k/m)}$, sometimes called *circular frequency*.

The period T is the amount of time that it takes to complete one oscillation. One oscillation of both the *sine* function and the *cosine* function occurs when the argument of the function advances by 2π, that is when

$$\lambda T = 2\pi, \qquad \text{so} \qquad T = \frac{2\pi}{\lambda} = \frac{2\pi}{\sqrt{(k/m)}},$$

Some people memorize these formulas in high school.

The *frequency* f, without the modifiers 'angular' or 'circular', is usually taken to mean the reciprocal of the period

$$f = \frac{1}{T} = \frac{\lambda}{2\pi} = \frac{\sqrt{(k/m)}}{2\pi}.$$

Typically, frequency f is measured in cycles per second or *Hertz* and the angular frequency λ in radians per second. A computer or watch quartz timing crystal has mechanical vibrations at a frequency of millions of cycles per second, some molecules about a million times faster than that. On the other extreme, the free vibrations of the whole earth have frequencies of thousandths of a cycle per second (i.e. thousands of seconds per cycle). The slowest vibration mode of the earth has a period of about 54 minutes⑤. An oscillation is called 'fast' if its frequency or angular frequency is high and it is called 'slow' if the period is long.

⑤Why does the earth oscillate? First because it can. It has both mass and an 'elastic restoring force'. The elastic restoring force comes from a combination of two things: 1) the elasticity of rock and 2) the self-gravitation of the earth trying to bundle itself into a ball. What gets the earth started oscillating? Mostly big earthquakes.

Amplitude. The amplitude of the sine wave that results from the addition of the *sine* function and the *cosine* function is given by the square root of the sum of the squares of the two amplitudes. That is, the amplitude of the resulting sine wave is $\sqrt{A^2 + B^2}$. Another way of describing this sum is through the trigonometric identity:

$$A\cos(\lambda\, t) + B\sin(\lambda\, t) = R\cos(\lambda\, t - \phi), \tag{10.27}$$

where $R = \sqrt{A^2 + B^2}$ and $\tan\phi = B/A$ (see box 10.2 on page 6).

Note that an oscillating particle could have a high velocity even if the vibrations are 'slow', so long as the amplitude is high enough. And it could have a small velocity even if the oscillations are 'fast', if the amplitude is low enough. In common usage 'speed' of oscillations has to do with frequency, not particle velocity.

Initial conditions determine the constants A and B

The constants A and B in equation 10.26 could have any value. Or, equivalently, the amplitude R and phase ϕ in equation 10.27 could be anything. These are determined by the way motion is started, the *initial conditions*. The following two special initial conditions are worth getting a feel for.

Release from rest. The simplest motion is *release from rest*, meaning the initial velocity of the mass is zero. We find the motion from the general solution

$$x(t) = A\cos(\sqrt{(k/m)}\,t) + B\sin(\sqrt{(k/m)}\,t).$$

At $t=0$, this general solution has to agree with the initial condition that $x(0)=x_0$ and the initial velocity is $v(0)=v_0=0$. In this case

$$x(0)=x_0 \quad \text{and} \quad v(0)=0 \quad \Rightarrow \quad A=x_0 \quad \text{and} \quad B=0.$$

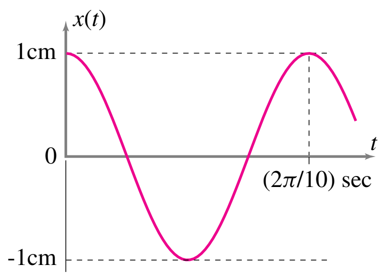


Figure 10.24: The position of a mass as a function of time if $k=50\,\mathrm{N/m}$, $m=0.5\,\mathrm{kg}$, $x(0)=1\,\mathrm{cm}$ and $v(0)=0$.

Example: Release from rest

The mass in fig. 10.20 is 0.5 kg, the spring constant is $k=50\,\mathrm{N/m}$, and the initial displacement is 1 cm, then

$$x(0) = A\underbrace{\cos(0)}_{1} + B\underbrace{\sin(0)}_{0} = A \qquad \Rightarrow \qquad A = 1\,\mathrm{cm}.$$

The initial velocity must also match, so

$$v(t)=\dot{x}(t) = -A\sqrt{(k/m)}\sin(\sqrt{(k/m)}\,t) + B\sqrt{(k/m)}\cos(\sqrt{(k/m)}\,t).$$

10.2 $A\cos(\lambda t) + B\sin(\lambda t) = R\cos(\lambda t - \phi)$ derivation and visualization

Here we show that a *cosine* function and a *sine* function add to a new sine wave. By sine wave we mean a function whose shape is the same as the sine function, though it may be displaced along the time axis. For example $\cos t$ and $\cos(t - const)$ are both sine waves.

The trig identity approach. The quickest approach is to start with the function $f(t) = R\cos(\lambda t - \phi)$ and use the trig identity for cosines angle addition:

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Thus:

$$\begin{aligned} R\cos(\lambda t - \phi) &= R\cos\lambda t\cos\phi + R\sin\lambda t\sin\phi \\ &= A\cos\lambda t + B\sin\lambda t. \end{aligned}$$

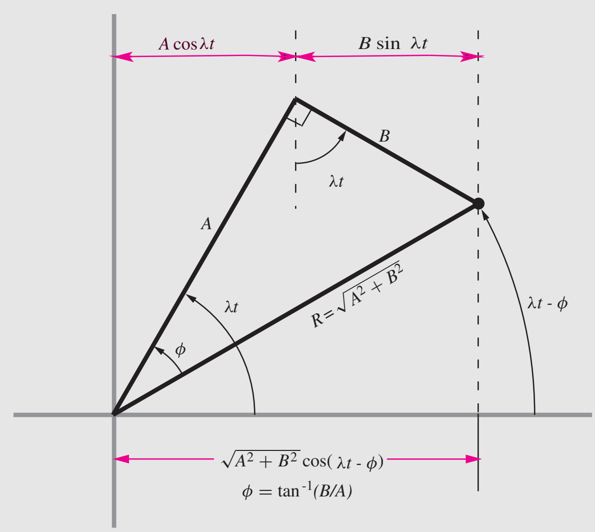
We can run the reasoning from right to left and set $A = R\cos\phi$ and $B = R\sin\phi$ and then solve for R and ϕ in terms of A and B. Thus demonstrating the title of this box. If you have trouble remembering the trig identity, you can derive it using the picture to the right. Trigonometry is the one subject in which circular reasoning is good.

The geometric approach. Consider the line segment A spinning in circles about the origin at rate λ; that is, the angle the segment makes with the positive x axis is λt. The projection of that segment onto the x axis is $A\cos(\lambda t)$, a sine wave. Now consider the segment labeled B in the figure, glued at a right angle to A. The length of its projection on the x-axis is $B\sin(\lambda t)$. So, the sum of these two projections is $A\cos(\lambda t) + B\sin(\lambda t)$. The two segments A and B make up a right triangle with diagonal $R = \sqrt{A^2 + B^2}$.

The projection or 'shadow' of R on the x axis is the same as the sum of the shadows of A and B. The angle it makes with the x axis is $\lambda t - \phi$ where one can see from the triangle drawn that $\phi = \arctan(B/A)$. So, by adding the shadow lengths, we see

$$A\cos(\lambda t) + B\sin(\lambda t) = \sqrt{A^2 + B^2}\cos(\lambda t - \phi)\,.$$

The function $f(t) = R\cos(\lambda t - \phi)$ is a sine wave. In particular it is the cosine function with a maximum at $\lambda t - \phi$.



Evaluating at $t = 0$ and matching with the initial condition $v_0 = 0$ ⑥

$$v(0) = -A\sqrt{(k/m)}\underbrace{\sin(0)}_{0} + B\sqrt{(k/m)}\underbrace{\cos(0)}_{1} = B\sqrt{(k/m)} \quad \Rightarrow \quad B = 0.$$

Substituting in $k = 50\,\mathrm{N/m}$ and $m = 0.5\,\mathrm{kg}$, we get

$$x(t) = 1 \cdot \cos\left(\sqrt{\underbrace{\left(\frac{50\,\mathrm{N/m}}{0.5\,\mathrm{kg}}\right)}_{100\,\mathrm{s}^{-2}}}\; t\right) \mathrm{cm} = 1\cos(10t/\mathrm{s})\,\mathrm{cm}$$

which is plotted in fig. 10.24.

⑥ It is tempting, but wrong, to evaluate $x(t)$ at $t = 0$ and then differentiate to get $v(0)$. Why wrong? Because $x(0)$ is just a number, differentiating it would always give zero, even when the initial velocity is not zero.

Initially at motion. The case where the initial position is zero but there is some initial velocity is similar. How could this happen? Say we start paying attention just after a still mass has been hit by a hammer.

Example:
Again use $m = 0.5\,\mathrm{kg}$ and $k = 50\,\mathrm{N/m}$. But now use $x(0) = 0$ and $v(0) = 10\,\mathrm{cm/s}$. Following the same procedure we get

$$x(t) = B\sin(\sqrt{(k/m)}\,t)$$

with $B\sqrt{(k/m)} = 10\,\mathrm{cm/s} \quad \Rightarrow \quad B = 1\,\mathrm{cm}.$
The motion, $x(t) = (1\,\mathrm{cm}) \cdot \sin\left(\frac{0.1t}{s}\right)$, is shown in fig. 10.25.

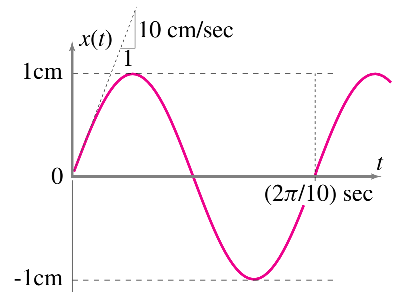


Figure 10.25: The position of a mass as a function of time if $k = 50\,\mathrm{N/m}$, $m = 0.5\,\mathrm{kg}$, $x(0) = 0$ and $v(0) = 10\ \mathrm{cm/s}$.

Energy

The harmonic oscillator conserves energy, as one can check with either analytical or numerical solutions. Conversely, we could start with the idea of energy conservation to find the governing differential equations.

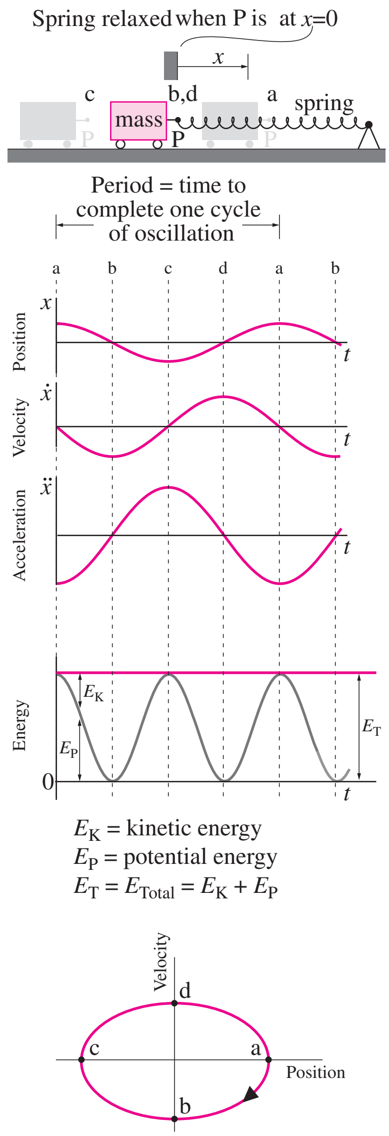
Conservation of energy

The harmonic oscillator is friction free. So the total mechanical energy, the sum of the kinetic energy $E_\mathrm{K} = \frac{1}{2}mv^2$ and the potential energy (from eqn. (10.19)) $E_\mathrm{P} = \frac{1}{2}k(\Delta L)^2$, is constant in time.

$$E_\mathrm{T} = E_\mathrm{K} + E_\mathrm{P} = \text{constant}.$$

Energy oscillations.

As the mass moves, energy is exchanged back and forth between kinetic and potential energies. At the extremes in the displacement, where the spring is most stretched, the potential energy is at a maximum and the kinetic energy is zero. When the mass passes through the center position the spring is relaxed, the potential energy is at a minimum (zero) and the mass is at its peak speed, and the kinetic energy is at its peak.



Cross plot or phase plane portrait

Figure 10.26: **Harmonic oscillator.** At $t = 0$ the mass is released from rest at a. The spring is relaxed at $x = 0$ (points b and d). Some things to note: The acceleration curve is proportional to the negative of the displacement curve; The displacement is at a maximum or minimum when the velocity is zero; The velocity is at a maximum or minimum when the displacement is zero; The kinetic and potential energy fluctuate at twice the frequency as the position; The motion is an ellipse in the cross plot of velocity *vs.* position.

Motion details. Let's assume the block in fig. 10.26 is released from rest at $x = x_A > 0$. The mass begins to move to the left and the spring does positive work on the mass since the motion and the force are in the same direction. After the block passes through the rest point $x = O$, it does work on the spring until it comes to rest at its left extreme. The spring then commences to do work on the block again as the block gains kinetic energy in its rightward motion. The block then passes through the rest position and does work on the spring until its kinetic energy is all used up and it is back in its rest position.

Note that the potential and kinetic energy each have two local maxima and minima for each oscillation of the mass, thus their plots are sine-waves with twice the frequency of the basic oscillation.

Example: Check.
Using the special case where the motion starts from rest (i.e., $x(t) = A\cos(\sqrt{k/m}\,t)$), we can check that the total energy really is constant.

$$\begin{aligned}
E_T &= E_P + E_K \\
&= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\
&= \frac{1}{2}k(\underbrace{A\cos(\sqrt{k/m}t)}_{x})^2 + \frac{1}{2}m(\underbrace{A\sqrt{k/m}\sin(\sqrt{k/m}t)}_{v})^2 \\
&= \frac{1}{2}kA^2\underbrace{\{\cos^2(\sqrt{k/m}t) + \sin^2(\sqrt{k/m}t)\}}_{1} \\
&= \frac{1}{2}kA^2 = \quad \text{initial energy in spring}
\end{aligned}$$

which does not change with time.

Using energy to derive the oscillator equation

Conversely we can start with energy balance and derive the equations of motion. Starting from E_T = constant, we get

$$\begin{aligned}
0 &= \frac{d}{dt}E_T = \frac{d}{dt}(E_P + E_K) \\
&= \frac{d}{dt}(\frac{1}{2}kx^2 + \frac{1}{2}mv^2) \\
&= kx\underbrace{\dot{x}}_{v} + mv\underbrace{\dot{v}}_{a} \\
&= kx\cancel{v} + m\cancel{v}\underbrace{a}_{\ddot{x}} \\
0 &= kx + m\ddot{x}
\end{aligned}$$

which is the harmonic oscillator equation. A technical defect of this derivation is that it does not apply at the instants when $v = 0$ (that is, $0 \cdot x = 0 \cdot y$ does not imply that $x = y$). Thus, technically, from this derivation we only know the differential equation holds for those times when $v \neq 0$. Nonetheless, it gives the right equation for all times.

Similarly, power balance also leads to the harmonic oscillator equation. Referring to the FBD in fig. 10.20, the equation of power balance for the block during its motion after release is:

$$\underbrace{P}_{\text{Power in}} = \underbrace{\dot{E}_K}_{\text{Rate of change of kinetic energy}}$$

$$\begin{aligned}
\vec{\boldsymbol{F}}_{spring} \cdot \vec{\boldsymbol{v}}_A &= \frac{d}{dt}\left(\frac{1}{2}mv_A^2\right) \\
-kx_A\hat{\boldsymbol{\imath}} \cdot \dot{x}_A\hat{\boldsymbol{\imath}} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}_A^2\right) \\
-kx_A\dot{x}_A &= m\dot{x}_A\ddot{x}_A\,.
\end{aligned}$$

Dividing both sides by $\dot{x}_A$ (assuming it is not zero), we again get our friend,

$$-kx_A = m\ddot{x}_A \qquad \text{or} \qquad m\ddot{x}_A + kx_A = 0.$$

Dealing with offsets

Consider a mass hanging from a spring, as in fig. 10.27. If we assume statics, we can find the equilibrium stretch of the spring as mg/k. For dynamics we can keep track of the position of a hanging mass at least 3 ways.

$$\begin{aligned}
&y && = \text{distance down from the ceiling,} \\
&z = y - \ell_0 && = \text{distance the spring is stretched, or} \\
&x = z - mg/k = y - \ell_0 - mg/k && = \text{distance below static equilib}
\end{aligned}$$

At the equilibrium position $y = \ell_0 + mg/k, z = mg/k$ and $x = 0$. For x, y and z we can define velocity v and acceleration a as (down is positive)

$$v = \dot{y} = \dot{z} = \dot{x} \quad \text{and} \quad a = \ddot{y} = \ddot{z} = \ddot{x}.$$

Using the free-body diagram shown we can write $\sum F_i = ma$ for each of these cases as

$$\begin{aligned}
m\ddot{y} = -k\Delta\ell + mg \quad &\Rightarrow \quad m\ddot{y} = -k\underbrace{(y - \ell_0)}_{\Delta\ell} \quad +mg \\
m\ddot{z} = -k\Delta\ell + mg \quad &\Rightarrow \quad m\ddot{z} = -k\underbrace{z}_{\Delta\ell} \quad +mg \\
m\ddot{y} = -k\Delta\ell + mg \quad &\Rightarrow \quad m\ddot{x} = -k\underbrace{(x + mg/k)}_{\Delta\ell} \quad +mg, \qquad (10.28)
\end{aligned}$$

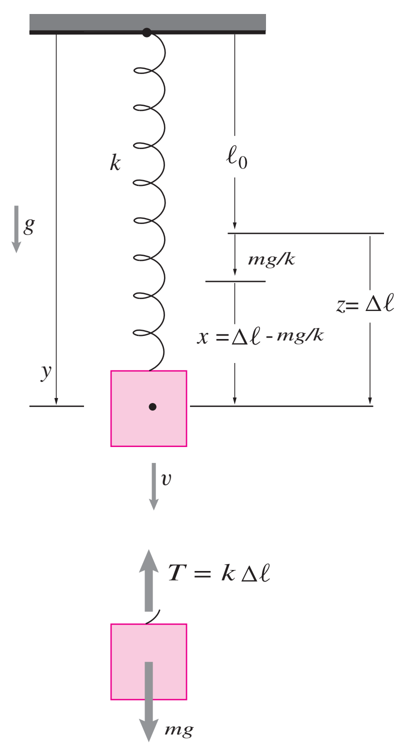


Figure 10.27: Spring and mass with gravity. The position can be measured by x, y or z.

where in each of the three cases we had to solve for $\Delta\ell$ in terms of x,y and z respectively.

$$\begin{aligned} m\ddot{y} + ky &= k\ell_0 + mg \\ m\ddot{z} + kz &= mg \\ m\ddot{x} + kx &= 0. \end{aligned} \tag{10.29}$$

All three equations 10.29 must describe the same motion. But they are three different differential equations with three different solutions. From a course in differential equations you know you can write the solutions as

General solution = general <u>h</u>omogeneous solution + any <u>p</u>articular solution

$$\begin{aligned} y(t) &= \underbrace{A\cos\sqrt{(k/m)}t + B\sin\sqrt{(k/m)}t}_{y_h} + \underbrace{\ell_0 + mg/k}_{y_p} \\ z(t) &= \underbrace{A\cos\sqrt{(k/m)}t + B\sin\sqrt{(k/m)}t}_{z_h} + \underbrace{mg/k}_{z_p} \\ x(t) &= \underbrace{A\cos\sqrt{(k/m)}t + B\sin\sqrt{(k/m)}t}_{x_h} \end{aligned} \tag{10.30}$$

That is, x has simple sinusoidal motions but y and z have sine waves plus a constant. This gives us two basic choices for finding the motion:

1. Try to simplify the ODE by picking the coordinate that makes the math the easiest (in this case$x$), or
2. Pick any convenient coordinate and use the math of homogeneous and particular solutions to find the motions.

See more discussion of constant forcing in section 11.2 on page 577.

Numerical solution

Most numerical differential equation solvers depend on writing the equations in first order form. We do this by defining $v = \dot{x}$. Thus

$$m\ddot{x} + kx = 0 \quad \Rightarrow \quad m\dot{v} + kx = 0.$$

Combining the definition of v with the differential equation we get the set of two coupled first order equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x \end{aligned} \tag{10.31}$$

We can think of this as

$$\dot{z} = f(z)$$

where z is the list of two numbers $z(1) = x$ and $z(2) = v$ so

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{k}{m}z_1 \end{bmatrix}.$$

which is a form easy to use with any numerical ODE solver.

Energy check. Note that there is no mention of energy in setting up the numerical solution. Yet we claim that the solution conserves energy. What does this mean?

After we find the numerical solution we have numerical values for x at a sequence of times and for v at a sequence of times. We can use these lists to calculate

$$E_{\mathrm{T}} = kx^2/2 + mv^2/2.$$

at all of these times. The claim is that this list of numbers is a list of the same number, again and again. That is, a property of the solution is that it 'conserves' energy, meaning that it finds that the total energy is a constant.

This observation is hugely useful in numerical calculations. Why? Because we know a priori that energy should be conserved. If the numerical solution does not give this constancy of energy it tells us we have made a mistake. How constant? Of course there are numerical errors, so we expect the energy to only be constant to the same accuracy as the numerical solution (typical errors are one part per thousand or less).

If it's wrong then it's wrong. Isn't the energy check merely saying: 'If the solution gives the wrong answer we know that the solution is wrong.'? Actually, it's a bit more than that because the idea of energy conservation transcends the details of the actual motion. In this case, for example, we can check energy conservation without ever knowing that the position and velocity oscillate exactly as sine waves. In more complicated problems we might never know formulas for the motions. Yet we might know we have energy conservation. So, even not knowing anything about the solution in detail, we can use energy conservation to check, for example, a numerical solution. If a numerical solution doesn't conserve energy, when the system conserves energy, then we can't trust anything about the numerical solution.

10.3 Finding solutions to the harmonic oscillator equation

In the same way that you know, say, that $a^2 + b^2 = c^2$ for a right triangle, you should know that

$$\ddot{x} + \lambda^2 x = 0 \qquad \Rightarrow \qquad \text{a sine wave.}$$

Knowing these does not depend on knowing how to derive them. But should you be both insecure and math-inclined here are some derivations of the latter.

Note that the methods below still involve guesses of sorts. That is how it is for differential equations, most of them no-one can solve. And the ones that people can solve they solve by guessing.

Method I: separation of variables

Start with

$$\ddot{x} + \lambda^2 x = 0 \tag{10.32}$$

and multiply both sides by $\dot{x}$ (that is, guess that this is a good thing to try) to get

$$\dot{x}\ddot{x} + \lambda^2 x\dot{x} = 0.$$

Notice that

$$\frac{d}{dt}\frac{\dot{x}^2}{2} = \dot{x}\ddot{x} \qquad \text{and} \qquad \frac{d}{dt}\frac{x^2}{2} = x\dot{x}$$

so

$$\frac{d}{dt}\left\{m\dot{x}^2 + kx^2\right\} = 0.$$

The expression in braces is thus a constant, call it C^2 to show that it is positive,

$$\dot{x}^2/\lambda^2 + x^2 = C^2.$$

We could also have jumped to this result by conservation of energy

$$\begin{aligned} & E_\mathrm{K} + E_\mathrm{P} = E_\mathrm{T} \\ \Rightarrow \quad & m\frac{\dot{x}^2}{2} + k\frac{x^2}{2} = E_\mathrm{T} \\ \Rightarrow \quad & \dot{x}^2/\lambda^2 + x^2 = 2E_\mathrm{T}/k = C^2. \end{aligned}$$

Now we separate variables to write

$$dx/\sqrt{C^2 - x^2} = \lambda dt.$$

Then integrate. How? Either substitute $x = C\sin\theta$, or guess the solution, or look up the integral on your symbolic calculator, a symbolic math program or the internet. The result is

$$\begin{aligned} \cos^{-1}(x/C) &= \lambda t - c_2 \\ \text{so} \qquad x &= C\cos(\lambda t - c_2) \end{aligned}$$

where C and c_2 are arbitrary constants. That's one form of the general solution of the harmonic oscillator equation. You can plug it back into eqn. (10.25) and see that you get satisfaction, that is $0 = 0$, for all values of C, c_2 and t. The other standard form

$$x = A\cos\lambda t + B\sin\lambda t$$

follows from the reasoning in box 10.2 on page 6. This is a solution for any value of the constants A and B.

Method II: complex variables

For all linear constant-coefficient homogeneous ordinary differential equations, including eqn. (10.32), a great guess, just one term, is that

$$x(t) = e^{\alpha t}.$$

Plugging this guess in and cancelling the common non-zero factor $e^{\alpha t}$ from each term gives

$$\alpha^2 + \lambda^2 = 0.$$

So the guess implies that

$$\alpha = \pm\sqrt{-\lambda^2}.$$

That's the square root of a negative number. So

$$\alpha = \pm\lambda i \qquad \text{where} \qquad i = \sqrt{-1}.$$

Our guess has given us two complex solutions:

$$x_1(t) = e^{i\lambda t} \qquad \text{and} \qquad x_2(t) = e^{-i\lambda t}.$$

You can check that both of these satisfy eqn. (10.32). We can multiply both solutions by arbitrary constants and add them and check that we still have a solution, namely the two term complex solution

$$x(t) = Ce^{i\lambda t} + De^{-i\lambda t}.$$

We then use the Euler formula (The'most remarkable ... astounding ... jewel' - Richard Feynman) that relates exponentials to sine waves

$$e^{i\lambda t} = \cos\lambda t + i\sin\lambda t.$$

We get a solution with 4 terms: Further, C and D can be complex so $C = C_1 + C_2 i$ and $D = D_1 + D_2 i$ so

$$x(t) = \quad (C_1 + iC_2)[\cos(\ \lambda t) + i\sin(\ \lambda t)] \tag{10.33}$$

$$+ \quad (D_1 + iD_2)[\cos(-\lambda t) + i\sin(-\lambda t)]. \tag{10.34}$$

valid for any values of the constants C_1, C_2, D_1 and D_2. Multiplying this out we get, from our simple one-term guess that $x = e^{\alpha t}$, 8 additive terms. By using the simplifying rules that $\sin -\lambda t = -\sin\lambda t$ and $\cos -\lambda t = \cos\lambda t$ these collapse to four terms. Down here in the simple real world we only care about the real part of this solution (or we simply pick $C_1 = D_1$ and $C_2 = D_2$ to cancel the imaginary terms). We are then left with the two terms

$$x(t) = \underbrace{(C_1 + D_1)}_{A}\cos\lambda t + \underbrace{(-C_2 - D_2)}_{B}\sin\lambda t.$$

which, after defining the new constants $A = C_1 + D_1$ and $B = -C_2 - D_2$, is our general (real) solution to the harmonic oscillator equation. Using box 10.2 on page 6 we can write this as one sine wave,

$$x(t) = C\cos(\lambda t - \phi).$$

We are stuck with λ but C and $\phi = c_2$ are arbitrary constants. One simple guess has given us $1 \to 2 \to 4 \to 8 \to 4 \to 2 \to 1$ solution.