10.3.1 The basic model.

a) Draw a spring (k) mass (m) system in a configuration where the spring is stretched.

b) On the drawing indicate the variable x.

c) Draw a free-body diagram of the mass.

d) Write the equation of linear momentum balance for the mass.

e) Rearrange the momentum balance equation to get the harmonic oscillator equation in standard form.

f) Write the general solution to the harmonic oscillator equation in two different ways (one as a sum of a sine and cosine function and one as a phase shifted sine or cosine function).

g) What is the natural frequency of this system?

h) What is the period?

i) What is the frequency (or circular frequency)?

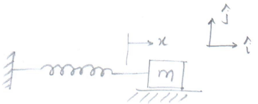
j) Find the solution for the special case that the mass is released from rest at $x(0) = x_0$.

- give the analytic expression.
- plot the position vs time for at least one whole cycle of motion.
- with the same time scale, plot velocity vs time (what is the peak velocity).
- with the same time scale, plot both the potential and kinetic energies vs time.

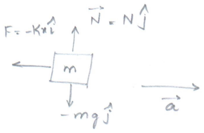
k) Find the solution for the special case that the mass is launched at v_0 from the rest position (just the analytic form, no need to repeat all the parts just above).

9.3.1

(a, b)



c) F.B.D of Mass



d) Equation of linear momentum balance

$$ma\hat{i} = -kx\hat{i} - mg\hat{j} + N\hat{j} \quad \text{——} \quad (i)$$

e) $(i)\cdot\hat{i} \Rightarrow ma = -Kx$

$$m\ddot{x} = -Kx$$

$$\ddot{x} + \frac{K}{m}x = 0$$

$$\Rightarrow \ddot{x} + \omega^2 x = 0 \quad , \quad \omega = \sqrt{\frac{K}{m}}$$

Equation of harmonic oscillator

f) General solution

i) $x = A\sin\omega t + B\cos\omega t$, where $\omega = \sqrt{\frac{K}{m}}$

ii) $x = C\sin(\omega t + \phi)$, $\omega = \sqrt{\frac{K}{m}}$

g) Natural frequency $\omega_1 = \sqrt{\frac{K}{m}}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

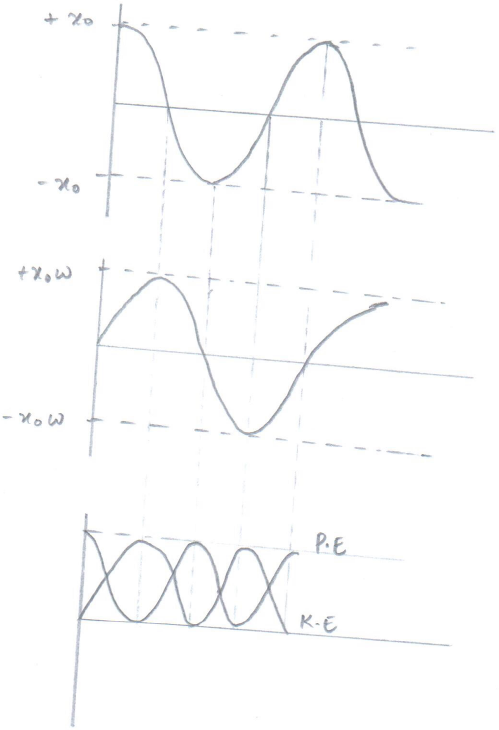
h) Period $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}$

i) Frequency, $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{K}{m}}$

j) solution for $x(0) = x_0$

$\dot{x}(0) = 0$

$x = x_0 \cos\omega t$



$P.E = \frac{1}{2}Kx^2$

$K.E = \frac{1}{2}mv^2$

K) $\dot{x}(0) = v_0$

$x_0 = x_0$

$\Big\} \Rightarrow x = A\sin\omega t + B\cos\omega t$

$B = x_0, \quad A = \frac{v_0}{\omega}$

$x = \frac{v_0}{\omega}\sin\omega t + x_0\cos\omega t$

10.3.2 Does the function $x = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$ satisfy the harmonic oscillator equation $\ddot{x} + \lambda^2 x = 0$ for any, possibly special, values of C_1 and C_2? Show that it does or does not.

9.45

$$x = C_1 e^{\lambda t} + C_2 \bar{e}^{\lambda t}$$

$$\dot{x} = \lambda C_1 e^{\lambda t} - \lambda C_2 \bar{e}^{\lambda t}$$

$$\ddot{x} = \lambda^2 C_1 e^{\lambda t} + \lambda^2 C_2 \bar{e}^{\lambda t}$$

$$= \lambda^2 (x)$$

$$\underline{\ddot{x} - \lambda^2 x = 0.}$$

It doesn't satisfy the harmonic oscillator equation

$$\ddot{x} + \lambda^2 x \underline{\underline{= 0}}.$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.3.3 Given that $\ddot{x} = -cx$, with $c = 1/\mathrm{s}^2$, $x(0) = 1\,\mathrm{m}$, and $\dot{x}(0) = 0$ find:

a) $x(\pi\,\mathrm{s}) =$?

b) $\dot{x}(\pi\,\mathrm{s}) =$?

<u>9.46</u>

$\ddot{x} = -cx.$

$x = A\sin\sqrt{c}t + B\cos\sqrt{c}t$

$\dot{x} = A\sqrt{c}\cos\sqrt{c}t - B\sqrt{c}\sin\sqrt{c}t.$

$x(0) = 1 \quad , \dot{x}(0) = 0$

~~$x(\pi s) \Rightarrow$~~

$\Rightarrow \quad 1 = (A\sin\sqrt{c}t + B\cos\sqrt{c}t)\big|_{t=0}$

$\Rightarrow \quad 0 = A\sqrt{c} \Rightarrow A = 0$

$\Rightarrow . \quad x = \cos\sqrt{c}t$

$\therefore \quad x(\pi s) = \cos\left(\left|\frac{1}{s}\right|\right)t = \pm 1. \quad = \pm 1 . \checkmark$

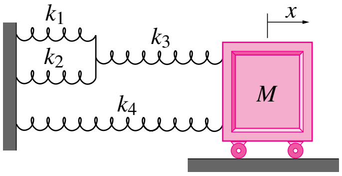
$\dot{x}(\pi s) = -\left|\frac{1}{s}\right|\sin(\sqrt{c}t) \underline{\underline{= 0}}. \quad \checkmark$

10.3.5 A spring and mass system is shown in the figure.

a) First, as a review, let k_1, k_2, and k_3 equal zero and k_4 be non-zero. What is the natural frequency of this system?

b) Now, let all the springs have non-zero stiffness. What is the stiffness of a single spring equivalent to the combination of k_1, k_2, k_3, k_4?

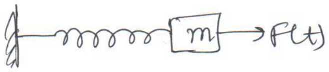
What is the frequency of oscillation of mass M?

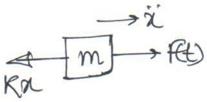


Problem 10.5

9.51.

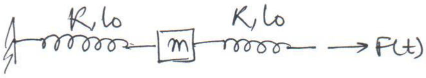
a)





$\Rightarrow m\ddot{x} + K(x) = F(t)$ — ①

(b)

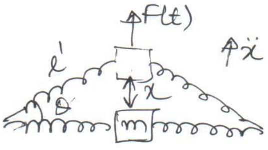


The above system is equivalent to two spring in series attached to mass m.

$\therefore K_{eq} = \frac{K_1 K_2}{K_1 + K_2}$.

Eqn of motion can be governed by $\boxed{\ddot{x} + \frac{K_{eq}}{m} \cdot x = 0}$

(c)



$\tan\theta = x/l_0$, $\sin\theta = x/l' = \theta$ (for small 'θ')

~~$m\ddot{x}+2($~~ [illegible]

~~$m\ddot{x} = -2(l'-l_0)K\theta + F(t)$~~

$\Rightarrow$ ~~$(m\ddot{x}+2($~~ $m\ddot{x} = -2\left(\sqrt{l_0^2+x^2} - l_0\right) K\sin\theta + F(t)$

$\Rightarrow m\ddot{x} + 2K\left(x - \frac{l_0 x}{(l_0^2+x^2)^{1/2}}\right) = F(t)$.

Ans

10.3.6 Mass m hangs from a spring with constant k and which has the length l_0 when it is relaxed (i.e., when no mass is attached). It only moves vertically.

a) Draw a Free-Body Diagram of the mass.

b) Write the equation of linear momentum balance.

c) Reduce this equation to a standard differential equation in x, the position x of the mass.

d) Verify that one solution is that $x(t)$ is constant at $x = l_0 + mg/k$.

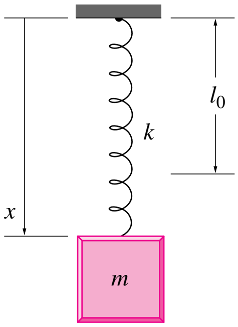
e) What is the meaning of that solution? (That is, describe in words what is going on.)

f) Define a new variable $\hat{x} = x - (l_0 + mg/k)$. Substitute $x = \hat{x} + (l_0 + mg/k)$ into your differential equation and note that the equation is simpler in terms of the variable $\hat{x}$.

g) Assume that the mass is released from an an initial position of $x = D$. What is the motion of the mass?

h) What is the period of oscillation of this oscillating mass?

i) Why might this solution not make physical sense for a long, soft spring if the initial stretch is large. In other words, what is wrong with this solution if $D > \ell_0 + 2mg/k$?



Problem 10.6

Answer:

(b) $mg - k(x - \ell_0) = m\ddot{x}$

(c) $\ddot{x} + \frac{k}{m}x = g + \frac{k\ell_0}{m}$

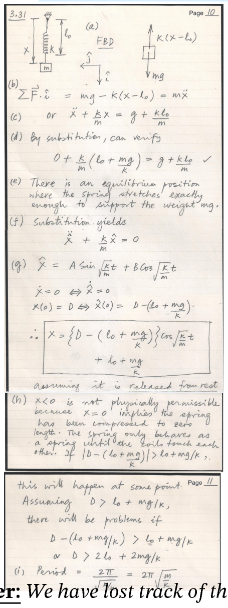
(e) This solution is the static equilibrium position; i.e., when the mass is hanging at rest, its weight is exactly balanced by the upwards force of the spring at this constant position x.

(f) $\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$

(g) $x(t) = \left[D - \left(\ell_0 + \frac{mg}{k}\right)\right] \cos\sqrt{\frac{k}{m}}\, t + \left(\ell_0 + \frac{mg}{k}\right)$

(h) period$=2\pi\sqrt{\frac{m}{k}}$.

(i) If the initial position D is more than $\ell_0 + 2mg/k$, then the spring is in compression for part of the motion. A floppy spring would buckle when in compression.



(b) $\sum \vec{F}\cdot\hat{i} = mg - K(x-l_0) = m\ddot{x}$

(c) or $\ddot{x} + \frac{K}{m}x = g + \frac{Kl_0}{m}$

(d) By substitution, can verify

$0 + \frac{K}{m}\left(l_0 + \frac{mg}{K}\right) = g + \frac{Kl_0}{m}$ ✓

(e) There is an equilibrium position where the spring stretches exactly enough to support the weight mg.

(f) Substitution yields

$\ddot{\hat{x}} + \frac{K}{m}\hat{x} = 0$

(g) $\hat{x} = A \sin\sqrt{\frac{K}{m}}t + B \cos\sqrt{\frac{K}{m}}t$

$\dot{x} = 0 \Leftrightarrow \dot{\hat{x}} = 0$

$x(0) = D \Leftrightarrow \hat{x}(0) = D - \left(l_0 + \frac{mg}{K}\right)$

$\therefore \boxed{x = \left\{D - \left(l_0 + \frac{mg}{K}\right)\right\} \cos\sqrt{\frac{K}{m}}t + l_0 + \frac{mg}{K}}$

assuming it is released from rest

(h) $x<0$ is not physically permissible because $x=0$ implies the spring has been compressed to zero length. The spring only behaves as a spring until the coils touch each other. If $\left|D - \left(l_0 + \frac{mg}{K}\right)\right| > l_0 + mg/K$,.

this will happen at some point. Page 11

Assuming $D > l_0 + mg/K$,

there will be problems if

$D - (l_0 + mg/K) > l_0 + mg/K$

or $D > 2l_0 + 2mg/K$

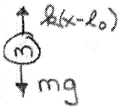
(i) Period $= \frac{2\pi}{\sqrt{K/m}} = 2\pi\sqrt{\frac{m}{K}}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

9.53

Mass m hanging from spring of constant k, ℓ_0

a) Free body diagram:



b) $\Sigma F_x = ma \Rightarrow mg - k(x-\ell_0) = ma$

c) $m\ddot{x} + k(x-\ell_0) = mg \Rightarrow \ddot{x} + \frac{k}{m}(x-\ell_0) = g$

d) Check $x(t) = \ell_0 + \frac{mg}{k}$, $\ddot{x} = 0$

$\Rightarrow 0 + \frac{k}{m}(\ell_0 + \frac{mg}{k} - \ell_0) = \frac{k}{m}\frac{mg}{k} = g$ ✓

e) This is the deformed equilibrium position of the system under gravity load.

f) Let $\hat{x} = x - (\ell_0 + \frac{mg}{k}) \Rightarrow x = \hat{x} + (\ell_0 + \frac{mg}{k})$

$\therefore$ we have $\ddot{\hat{x}} + \frac{k}{m}(\hat{x} + \ell_0 - \ell_0 + \frac{mg}{k}) = g$

OR $\ddot{\hat{x}} + \frac{k}{m}(\hat{x} + \frac{mg}{k}) = g \Rightarrow \ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$

g) We solve $\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$ to get $\hat{x} = C_1 \sin(t\sqrt{\frac{k}{m}}) + C_2\cos(t\sqrt{\frac{k}{m}})$

$\Rightarrow$ Initial conditions: $\hat{x}(0) = D$, $\dot{\hat{x}}(0) = 0$

$\dot{x} = C_1\sqrt{\frac{k}{m}}\cos(t\sqrt{\frac{k}{m}}) - C_2\sqrt{\frac{k}{m}}\sin(t\sqrt{\frac{k}{m}})$

$\dot{x}(0) = 0 = C_1\sqrt{\frac{k}{m}} \therefore C_1 = 0$

$x(0) = C_2 = D \Rightarrow \boxed{\hat{x}(t) = D\cos(t\sqrt{k/m})}$

h) Period $= \frac{2\pi}{\omega}$, where $\omega = \sqrt{k/m}$ $\therefore$ $\boxed{T = 2\pi\sqrt{m/k}}$

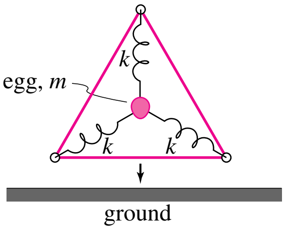
i) This would not make sense because the spring would want to oscillate upwards past the top of the spring (at its support).

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.3.7 One egg-drop contestant used a structure which held the egg (mass m) at the center with rubber bands. Consider the 2D model shown. The springs are linear with spring constants k. After falling a height h the frame hits the ground on a flat edge. Assume small motions (deflection ≪ side-length) and that the springs do not buckle.

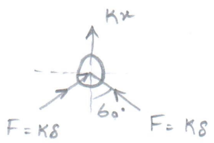
a) what is the vibration frequency after impact?

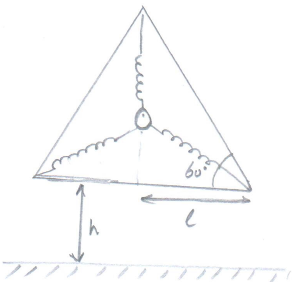
b) What is the maximum vertical deflection of the egg (relative to its equilibrium position)?



Problem 10.7

9.3.7





Equation of Motion

$$m\ddot{x} = -Kx - 2\left[K \frac{x}{2} \cos 60^\circ\right]$$

$$m\ddot{x} = -Kx - \frac{Kx}{2}$$

$$m\ddot{x} = -\frac{3}{2}Kx$$

Vibration Frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{3K}{2m}}$$

b) Maximum vertical deflection of the Egg

$$\frac{1}{2} m \dot{x}_0^2 = \frac{1}{2} K \cdot \frac{3}{2} \cdot x^2$$

$$\frac{1}{2} \cdot K \cdot \frac{3}{2} x^2 = mgh$$

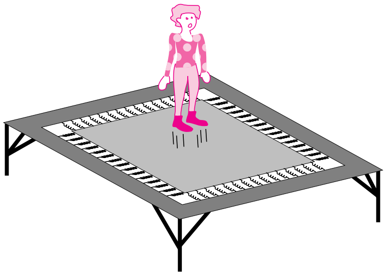
$$x = \sqrt{\frac{2mgh}{\frac{3K}{2}}} = \sqrt{\frac{4mgh}{3K}}$$

10.3.8 A person jumps on a trampoline. The trampoline is modeled as having an effective vertical undamped linear spring with stiffness $k = 200\,\text{lbf}/\text{ft}$. The person is modeled as a rigid mass $m = 150\,\text{lbm}$. $g = 32.2\,\text{ft}/\text{s}^2$.

a) What is the period of motion if the person's motion is so small that her feet never leave the trampoline?

b) What is the maximum amplitude of motion (amplitude of the sine wave) for which her feet never leave the trampoline?

c) (harder) If she repeatedly jumps so that her feet clear the trampoline by a height $h = 5\,\text{ft}$, what is the period of this motion (note, the contact time is *not* exactly half of a vibration period)? [Hint, a neat graph of height vs time will help.]

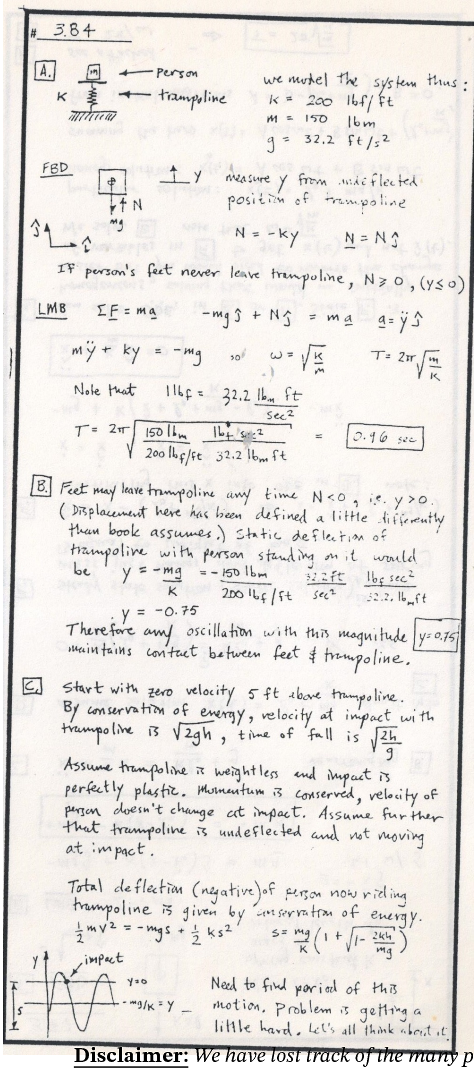


Problem 10.8: A person jumps on a trampoline.

Answer:
(a) period= $\dfrac{2\pi}{\sqrt{\frac{k}{m}}} = 0.96\,\text{s}$

(b) maximum amplitude=0.75 ft

(c) period= $2\sqrt{\dfrac{2h}{g}} + \sqrt{\dfrac{m}{k}}\left[\pi + 2\tan^{-1}\sqrt{\dfrac{mg}{2kh}}\right] \approx 1.64\,\text{s}$.



3.84

A. person
trampoline
K m

we model the system thus:
$K = 200\ \text{lbf}/\text{ft}$
$m = 150\ \text{lbm}$
$g = 32.2\ \text{ft}/\text{s}^2$

FBD y measure y from undeflected position of trampoline
N mg $\hat{j}$ $\hat{\imath}$

$N = -ky$ $\quad \underline{N} = N\hat{j}$

IF person's feet never leave trampoline, $N \geq 0$, $(y \leq 0)$

LMB $\Sigma \underline{F} = m\underline{a}$ $\quad -mg\hat{j} + N\hat{j} = m\underline{a}$ $\quad \underline{a} = \ddot{y}\hat{j}$

$m\ddot{y} + ky = -mg$ so $\omega = \sqrt{\frac{K}{m}}$ $\quad T = 2\pi\sqrt{\frac{m}{K}}$

Note that $1\,\text{lbf} = 32.2\,\frac{\text{lb}_m\ \text{ft}}{\text{sec}^2}$

$T = 2\pi\sqrt{\frac{150\,\text{lb}_m}{200\,\text{lb}_f/\text{ft}}\ \frac{\text{lb}_f\ \text{sec}^2}{32.2\ \text{lb}_m\,\text{ft}}} = \boxed{0.96\ \text{sec}}$

B. Feet may leave trampoline any time $N<0$, i.e. $y>0$. (Displacement here has been defined a little differently than book assumes.) Static deflection of trampoline with person standing on it would be $y = -\frac{mg}{K} = -\frac{150\,\text{lbm}}{200\,\text{lbf}/\text{ft}}\ \frac{32.2\,\text{ft}}{\text{sec}^2}\ \frac{\text{lb}_f\ \text{sec}^2}{32.2\,\text{lb}_m\text{ft}}$

$y = -0.75$

Therefore any oscillation with this magnitude maintains contact between feet & trampoline. $\boxed{y = 0.75}$

C. Start with zero velocity 5 ft above trampoline. By conservation of energy, velocity at impact with trampoline is $\sqrt{2gh}$, time of fall is $\sqrt{\frac{2h}{g}}$

Assume trampoline is weightless and impact is perfectly plastic. Momentum is conserved, velocity of person doesn't change at impact. Assume further that trampoline is undeflected and not moving at impact.

Total deflection (negative) of person now riding trampoline is given by conservation of energy.

$\frac{1}{2}mv^2 = -mgs + \frac{1}{2}ks^2 \qquad s = \frac{mg}{K}\left(1 + \sqrt{1 - \frac{2kh}{mg}}\right)$

y impact $y=0$ $-mg/K = y$ s

Need to find period of this motion. Problem is getting a little hard. Let's all think about it.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

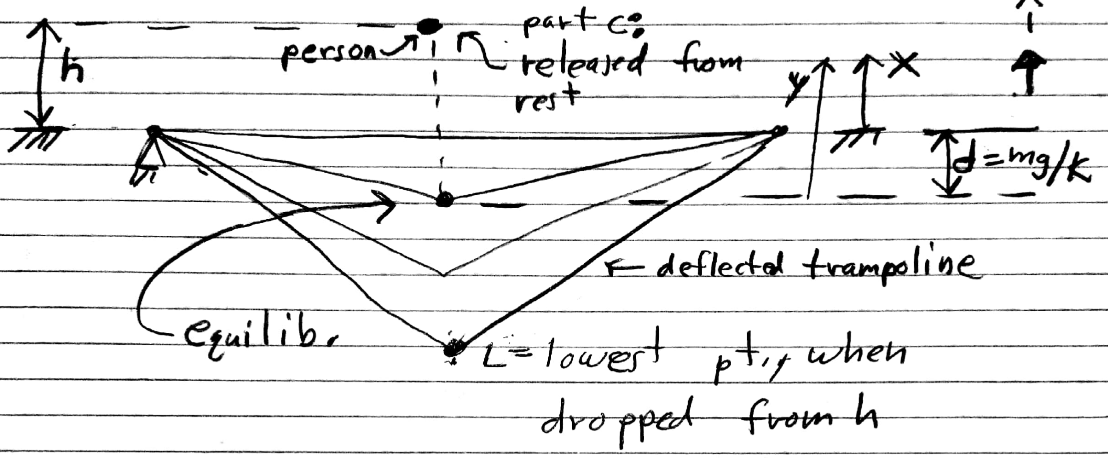
①/⑥

TRAMPOLINE SOLUTION 9.3.8

See Matlab Arithmetic on final page

$m = 150\ \text{lbm}$ $g = 32.2\ \text{ft/s}^2$

$k = 200\ \text{lbf/ft}$ $h = 5\ \text{ft}$



FBDs: flight: $\downarrow -mg\,\hat{\imath}$ for $x>0$

on tramoline: $\downarrow -mg\,\hat{\jmath}$ for $x<0$

$\downarrow T = kx$ (note: $T<0$)

$\underbrace{}$ $-T\hat{\imath} = -kx\hat{\imath}$

[Note: sign convention trampoline tension is positive even though its always compression.]

LMB: flight $(x>0)$: $-mg = m\ddot{x}$ ①

(above trampoline) $\ddot{x} = -g$

on trampoline: $-mg - kx = m\ddot{x}$ ②

$(x<0)$

→

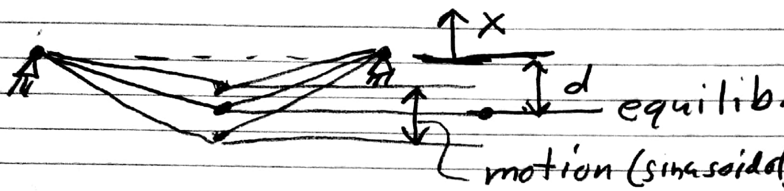
② / 6

9.3.8

a) Motion for small oscillations

$\Rightarrow x < 0$ for all t

See Matlab script.



period?

$\omega t_p = 2\pi$

$t_p = 2\pi/\omega$

$t_p = 2\pi\sqrt{m/k}$

$= 2\pi\sqrt{\frac{150\,lbm}{200\,lbf/ft}}$

$= 2\pi\sqrt{\frac{150/32.2}{200}}\,s$

$= \boxed{.9595}$

see matlab

② $\Rightarrow$ $m\ddot{x} + kx = -mg$

$$\Rightarrow x = \underbrace{A\cos(\underbrace{\omega}_{\sqrt{k/m}} t)}_{x_h} \underbrace{- mg/k}_{x_p}$$

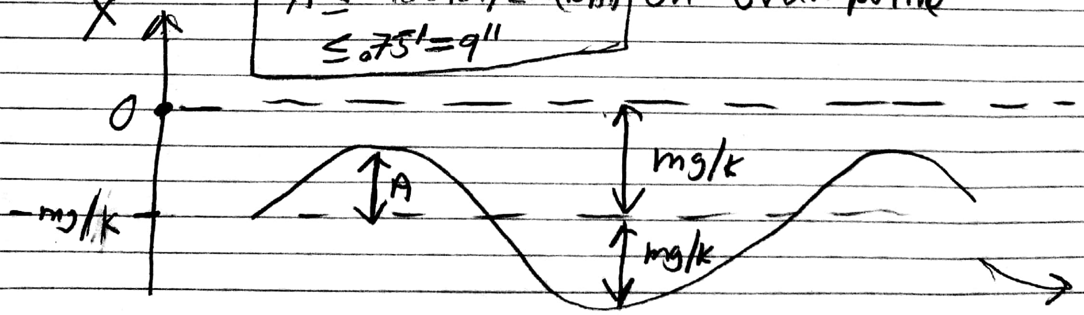
x_h = homog. soln

x_p = partic. soln.

So familiar that we can eyeball this soln.

could also have $\sin(\omega t)$, but we don't care about phase

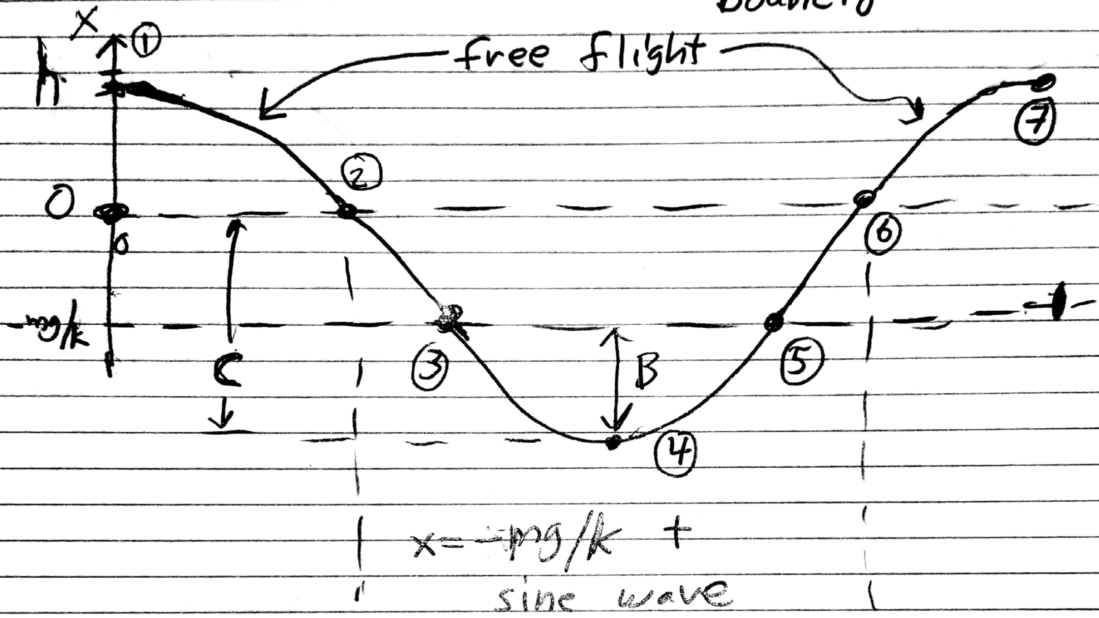
b) $\boxed{A \le mg/k \quad A \le 150\,lbf/200\,(lbf/ft) \quad \le .75' = 9''}$ to keep oscillations on trampoline



3/6

9.3.8 (cont'd)

c) (See fig. on first page). Period of bouncing



①→② & ⑥→⑦ $\ddot{x} = -g \Rightarrow \frac{1}{2} g t_{12}^2 = h$

$x(0) = h$

$t_{12} = \sqrt{2h/g}$

$t_{67} = t_{12}$

③→⑤ half a sine wave $\Rightarrow t_{35} = t_p/2$

$= 2\pi\sqrt{m/k}$

→

④/⑥

9.3.8 c cont'd

Now, the hard part: t_{23} & t_{56}

First, find amplitude of sine wave.

Conservation of energy:

$$E_1 = E_4 \qquad E_{k4} = E_{k1} = 0$$

$$mgh = mg(-mg/k - B) + k\underbrace{(mg/k + B)}_{C}{}^2/2$$

$$\frac{kC^2}{2} - mgC - mgh = 0$$

$$\Rightarrow \quad C = \frac{mg \pm \sqrt{(mg)^2 + 2mghk}}{k}$$

positive root for pt. w/ x<0 $\Rightarrow C = \frac{mg}{k}\left(1 + \sqrt{1 + 2hk/mg}\right)$

$$B = C - mg/k$$

$$= \frac{mg}{k}\sqrt{1 + 2hk/mg}$$

<u>Checks</u>: $h \to 0 = B = mg/k$ = max. amp. osc. for contact ✓

Cons. of Energy for big bounce of small mass;

$h = $ big

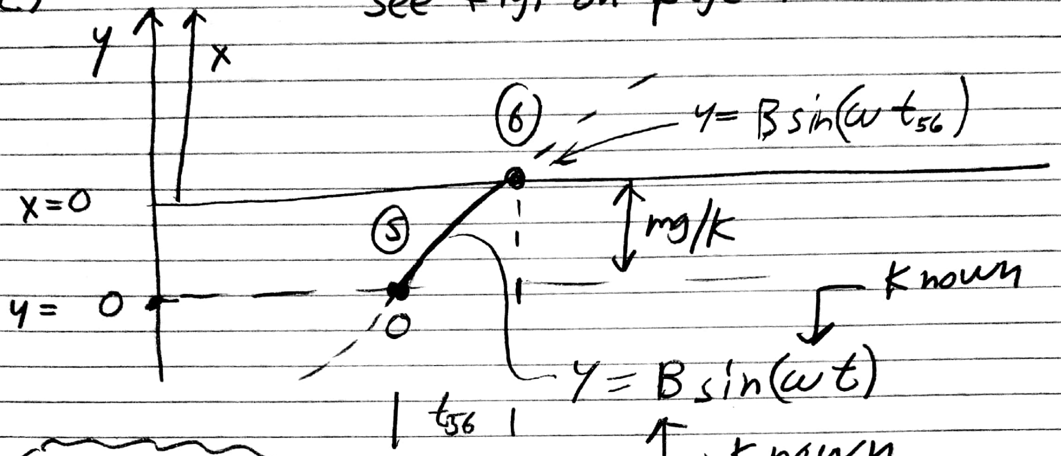
$k = $ big

$\Rightarrow \quad B \approx 2\sqrt{\frac{mgh}{k}} \Rightarrow \frac{Bk^2}{2} = mgh$ ✓

⑤⑥

9.3.8 (cont'd)

c) see fig. on page 1



Alternative finding of t_{56} & B is on next page.

$$\frac{mg}{K} = B\sin(\omega t_{56})$$

$$\Rightarrow t_{56} = \sin^{-1}\left(\frac{mg}{BK}\right)/\omega$$

$$t_{23} = t_{56}$$

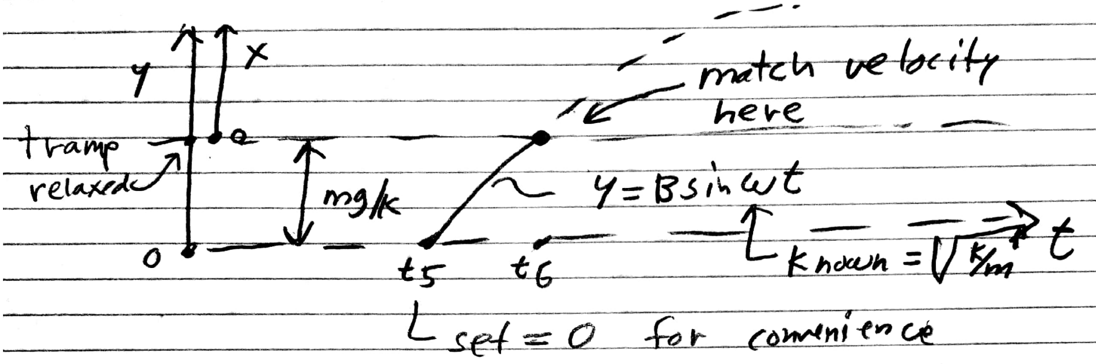
$$t_{total} = \frac{t_p}{2} + 2t_{12} + t_{23} + t_{56}$$

$$\boxed{t_{TOT} = 1.676 \text{ s}}$$

see Matlab on next page

5b/6

Alternative way to find t_{56} and B = amplitude of sine wave



Vel: V(sine wave at t_6) = V(flight at t_6)

$$\Rightarrow B\omega\cos(t_{56}\omega) = \sqrt{2gh} \quad ①$$

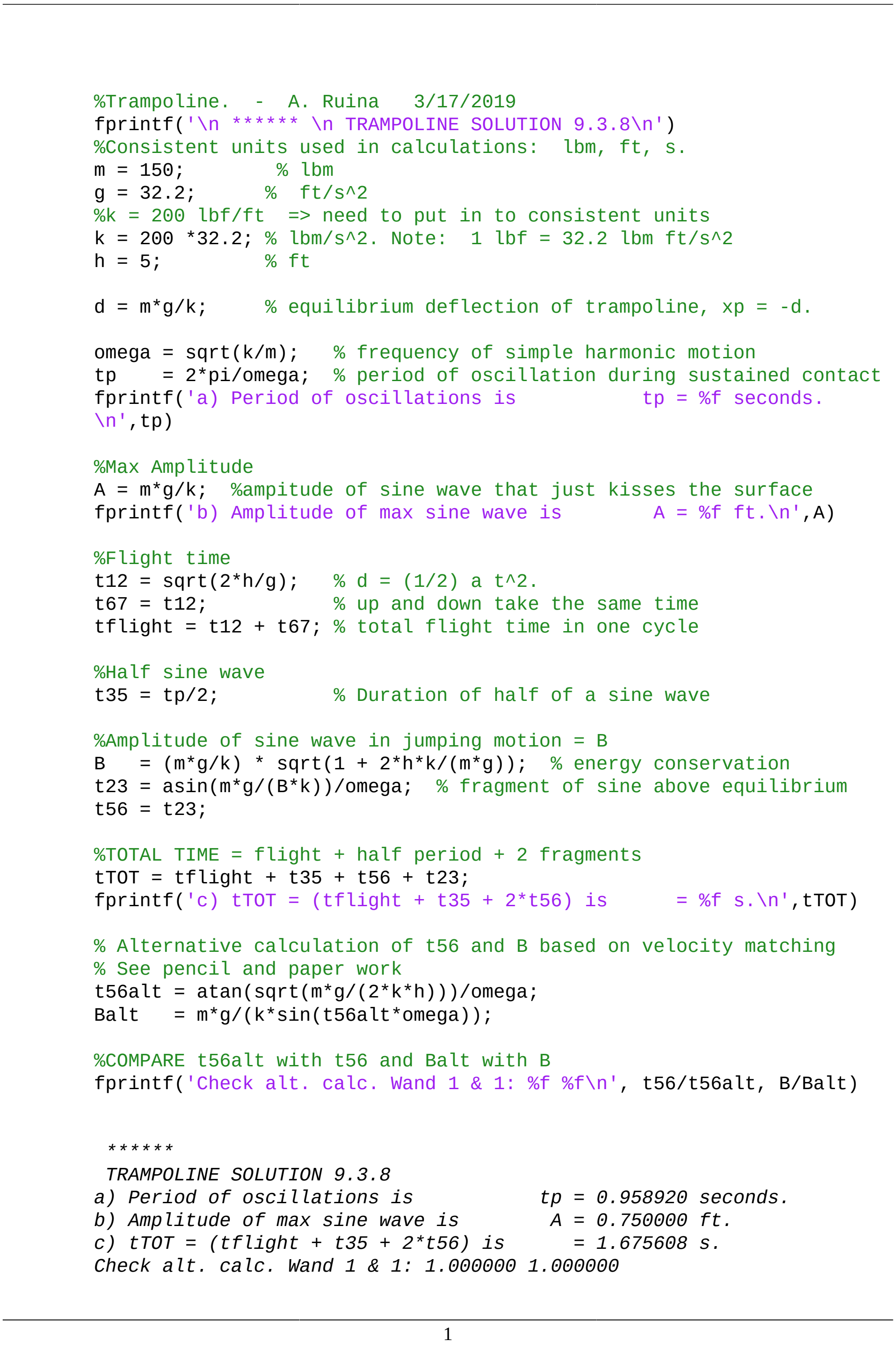
pos: $$B\sin(t_{56}\omega) = mg/k \quad ②$$

$$① \,\&\, ② \Rightarrow \tan(t_{56}\omega) = \frac{(mg/k)\cdot\omega}{\sqrt{2gh}} = \frac{mg/k}{\sqrt{2ghm/k}}$$

$$t_{56} = \operatorname{atan}\left(\sqrt{\frac{mg}{2kh}}\right)\Big/\sqrt{\frac{k}{m}}$$

$$B = \frac{mg/k}{\sin(t_{56}\omega)}$$

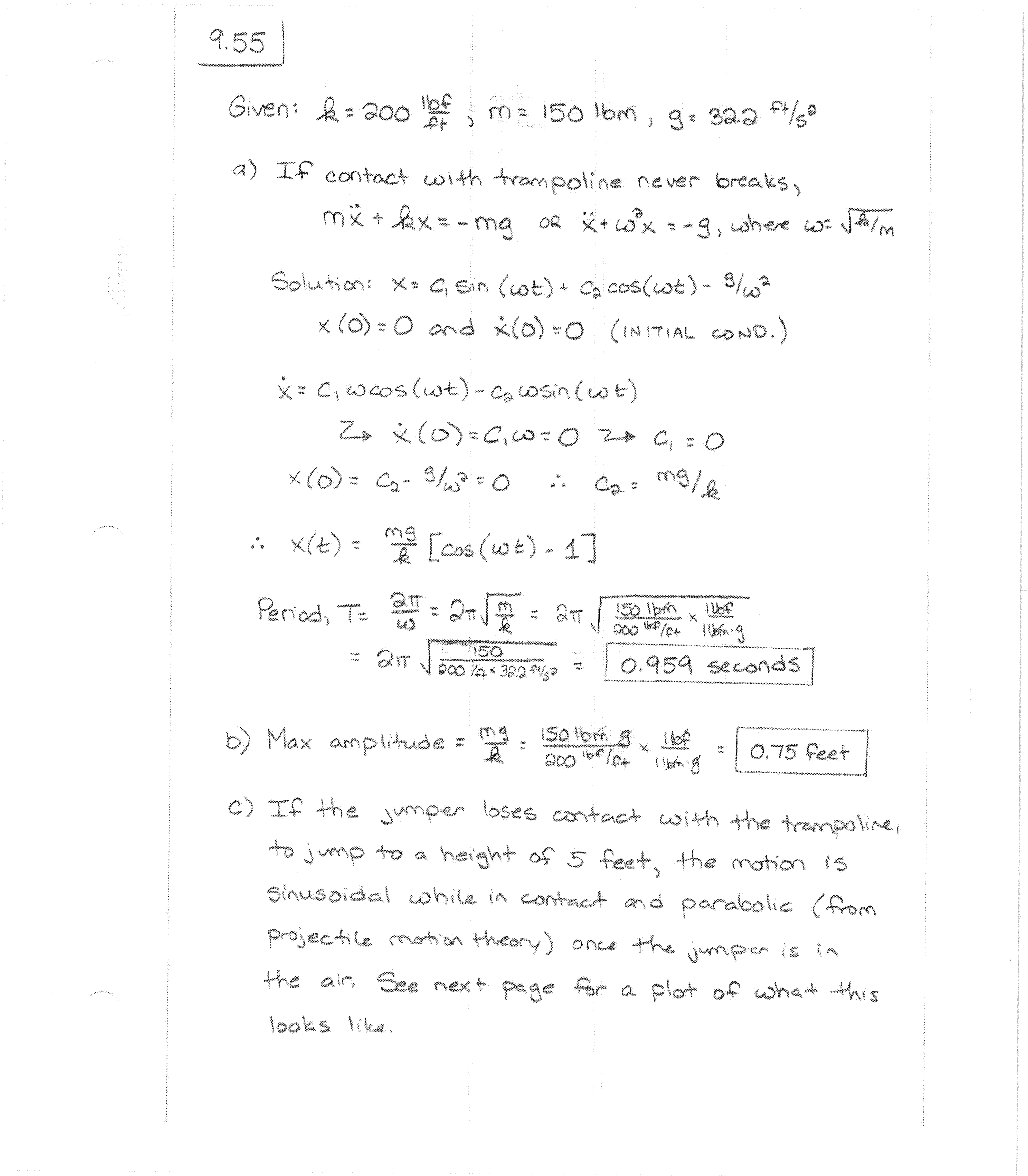
see Matlab, agrees w/ previous →



```
%Trampoline.  -  A. Ruina   3/17/2019
fprintf('\n ****** \n TRAMPOLINE SOLUTION 9.3.8\n')
%Consistent units used in calculations:  lbm, ft, s.
m = 150;        % lbm
g = 32.2;      %  ft/s^2
%k = 200 lbf/ft  => need to put in to consistent units
k = 200 *32.2; % lbm/s^2. Note:  1 lbf = 32.2 lbm ft/s^2
h = 5;         % ft

d = m*g/k;     % equilibrium deflection of trampoline, xp = -d.

omega = sqrt(k/m);   % frequency of simple harmonic motion
tp    = 2*pi/omega;  % period of oscillation during sustained contact
fprintf('a) Period of oscillations is           tp = %f seconds.
\n',tp)

%Max Amplitude
A = m*g/k;  %ampitude of sine wave that just kisses the surface
fprintf('b) Amplitude of max sine wave is        A = %f ft.\n',A)

%Flight time
t12 = sqrt(2*h/g);   % d = (1/2) a t^2.
t67 = t12;           % up and down take the same time
tflight = t12 + t67; % total flight time in one cycle

%Half sine wave
t35 = tp/2;          % Duration of half of a sine wave

%Amplitude of sine wave in jumping motion = B
B   = (m*g/k) * sqrt(1 + 2*h*k/(m*g));  % energy conservation
t23 = asin(m*g/(B*k))/omega;  % fragment of sine above equilibrium
t56 = t23;

%TOTAL TIME = flight + half period + 2 fragments
tTOT = tflight + t35 + t56 + t23;
fprintf('c) tTOT = (tflight + t35 + 2*t56) is      = %f s.\n',tTOT)

% Alternative calculation of t56 and B based on velocity matching
% See pencil and paper work
t56alt = atan(sqrt(m*g/(2*k*h)))/omega;
Balt   = m*g/(k*sin(t56alt*omega));

%COMPARE t56alt with t56 and Balt with B
fprintf('Check alt. calc. Wand 1 & 1: %f %f\n', t56/t56alt, B/Balt)
```

```
 ******
 TRAMPOLINE SOLUTION 9.3.8
a) Period of oscillations is           tp = 0.958920 seconds.
b) Amplitude of max sine wave is        A = 0.750000 ft.
c) tTOT = (tflight + t35 + 2*t56) is      = 1.675608 s.
Check alt. calc. Wand 1 & 1: 1.000000 1.000000
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*



9.55

Given: $k = 200\ \frac{\text{lbf}}{\text{ft}}$, $m = 150$ lbm, $g = 32.2\ \text{ft/s}^2$

a) If contact with trampoline never breaks,

$m\ddot{x} + kx = -mg$ OR $\ddot{x} + \omega^2 x = -g$, where $\omega = \sqrt{k/m}$

Solution: $x = C_1 \sin(\omega t) + C_2 \cos(\omega t) - g/\omega^2$

$x(0) = 0$ and $\dot{x}(0) = 0$ (INITIAL COND.)

$\dot{x} = C_1 \omega \cos(\omega t) - C_2 \omega \sin(\omega t)$

$\Rightarrow \dot{x}(0) = C_1 \omega = 0 \Rightarrow C_1 = 0$

$x(0) = C_2 - g/\omega^2 = 0 \quad \therefore C_2 = mg/k$

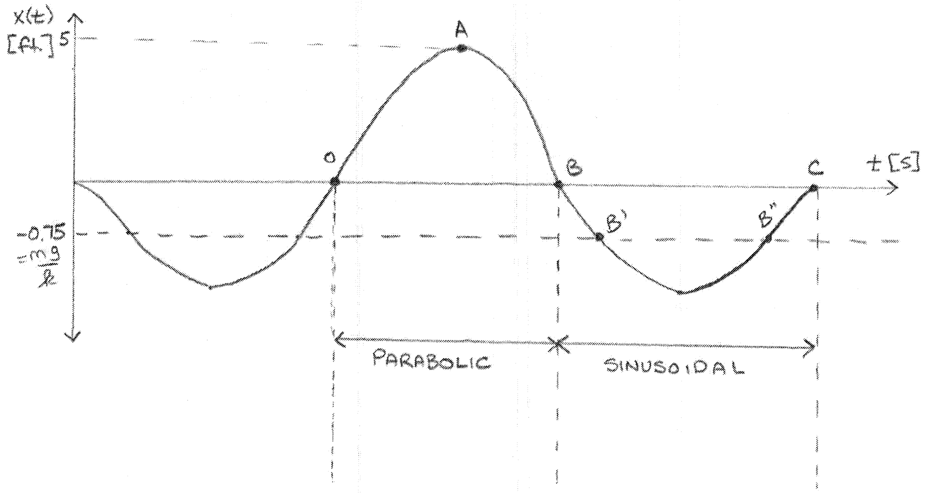
$\therefore \quad x(t) = \frac{mg}{k}[\cos(\omega t) - 1]$

$$\text{Period, } T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{150\text{ lbm}}{200\text{ lbf/ft}} \times \frac{1\text{ lbf}}{1\text{ lbm}\cdot g}}$$

$$= 2\pi\sqrt{\frac{150}{200\ 1/\text{ft} \times 32.2\ \text{ft/s}^2}} = \boxed{0.959\text{ seconds}}$$

b) Max amplitude $= \frac{mg}{k} = \frac{150\text{ lbm } g}{200\text{ lbf/ft}} \times \frac{1\text{ lbf}}{1\text{ lbm}\cdot g} = \boxed{0.75\text{ feet}}$

c) If the jumper loses contact with the trampoline, to jump to a height of 5 feet, the motion is sinusoidal while in contact and parabolic (from projectile motion theory) once the jumper is in the air. See next page for a plot of what this looks like.



To analyze, assume we begin at point A, a height of 5 feet above the trampoline. $(x=5, \dot{x}=0)$

From A → B, $x(t) = \frac{-1}{2}gt^2 + h_0 = 5 - \frac{1}{2}gt^2$

$\therefore$ $x(t) = 0$ when $5 - \frac{1}{2}gt^2 = 0$, or $t = \sqrt{\frac{2(5\text{ ft})}{32.2\text{ ft/s}^2}}$

$t = 0.5573$ seconds

$\therefore$ Total time from O to B is $2t = 1.115$ seconds

@ B, $x = 0$ and $\dot{x} = -gt/_{t=0.5573} = -17.945\text{ ft/s} = \dot{x}_0$

We use this as an initial condition to define a new sine wave, setting point B as $t=0$.

$$x(t) = c_1 \sin\left(\sqrt{\tfrac{k}{m}}t\right) + c_2 \cos\left(\sqrt{\tfrac{k}{m}}t\right) - \frac{gm}{k}$$

$$x(0) = 0 = C_1(0) + C_2 - \frac{mg}{k} \quad \therefore \quad C_2 = \frac{mg}{k}$$

$$\dot{x}(0) = \dot{x}_0 = C_1\sqrt{\frac{k}{m}}\cos(0) - C_2\sqrt{\frac{k}{m}}\sin(0)$$

$$\therefore \quad C_1\sqrt{\frac{k}{m}} = \dot{x}_0 \quad \text{OR} \quad C_1 = \dot{x}_0\sqrt{\frac{m}{k}}$$

$$\therefore \quad x(t) = \dot{x}_0\sqrt{\frac{m}{k}}\sin\left(\sqrt{\frac{k}{m}}t\right) + \frac{mg}{k}\cos\left(\sqrt{\frac{k}{m}}t\right) - \frac{mg}{k}$$

We want to find when this expression $= -\frac{mg}{k}$

$\Rightarrow$ This is point B' on previous graph

Using Matlab or a calculator to solve,

$$t = -\tan^{-1}\left(\frac{g}{\dot{x}_0}\sqrt{\frac{m}{k}}\right) \Big/ \sqrt{\frac{k}{m}}$$

$$= -\tan^{-1}\left(\frac{32.2\ \text{ft}/\text{s}^2}{-17.945\ \text{ft}/\text{s}}\sqrt{\frac{150\ \text{lbm}}{200\ \text{lbf}/\text{ft} \times 32.2\ \text{ft}/\text{s}^2}}\right) \Big/ \sqrt{\frac{200 \times 32.2}{150}} = 0.04079\ \text{s}$$

$\therefore$ Distance from B to B' = 0.04079 s

We know distance from B' to B'' $= \frac{T}{2} = \frac{0.959\ \text{s}}{2}$

$= 0.4795$ seconds

From B'' to C is the same as B to B' = 0.04079 s

$\therefore$ Total period $T = 1.115\ \text{s} + 2(0.0408\ \text{s}) + 0.4795\ \text{s}$

$$= \boxed{1.676\ \text{seconds}}$$

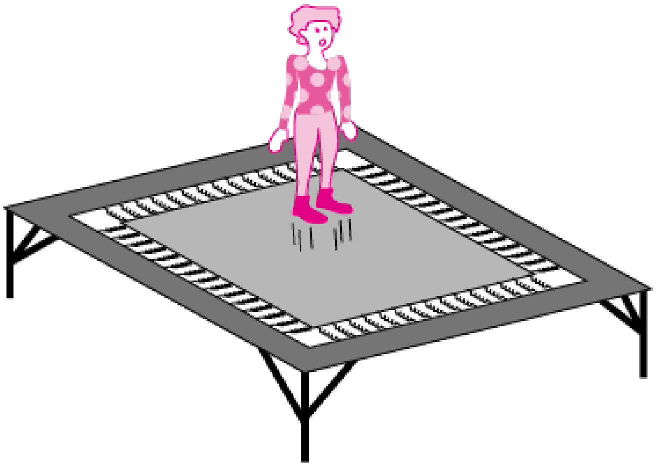
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.3.8 A person jumps on a trampoline. The trampoline is modeled as having an effective vertical undamped linear spring with stiffness $k = 200\,\text{lbf}/\,\text{ft}$. The person is modeled as a rigid mass $m = 150\,\text{lbm}$. $g = 32.2\,\text{ft}/\text{s}^2$.|

a) What is the period of motion if the person's motion is so small that her feet never leave the trampoline? *

b) What is the maximum amplitude of motion (amplitude of the sine wave) for which her feet never leave the trampoline? *

c) (harder) If she repeatedly jumps so that her feet clear the trampoline by a height $h = 5\,\text{ft}$, what is the period of this motion (note, the contact time is *not* exactly half of a vibration period)? [Hint, a neat graph of height vs time will help.] *

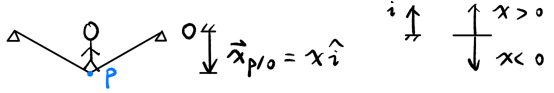


MAE 2030 HW3 p11 Solution

Duan Li

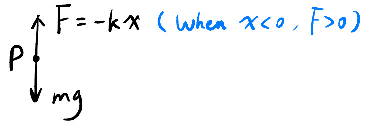
Given: $k = 200$ lbf/ft , $m = 150$ lbm , $g = 32.2\ ft/s^2 = \frac{1\ lbf}{1\ lbm}$

Coordinates:

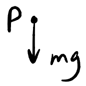


FBD's

① Contact with trampoline



② In flight



a) Given: Person always on trampoline. Find period.

Use contact FBD ①

LMB: $m\ddot{\vec{x}} = \sum\vec{F}$

$m\ddot{x} = -kx - mg$ ③

Solve ODE ③

$x = x_h + x_p$

Guess $x_p = \text{constant} \Rightarrow m\overset{0}{\cancel{\ddot{x}_p}} = -kx_p - mg$

$x_p = -\frac{mg}{k}$

x_h is general solution of $m\ddot{x}_h = -kx_h$

Guess $x = e^{rt} \Rightarrow$

$mr^2 = -k \rightarrow r_{1,2} = \pm i\omega$, $\omega = \sqrt{\frac{k}{m}}$

$x_h = A'e^{r_1t} + B'e^{r_2t} = A\sin(\omega t) + B\cos(\omega t)$

$x = x_h + x_p = A\sin(\omega t) + B\cos(\omega t) - \frac{mg}{k}$

No need to solve for A, B for now

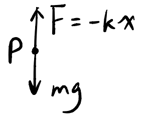
$$w = \sqrt{\frac{k}{m}} = \sqrt{\frac{200\ \cancel{lbf}/ft}{150\ \cancel{lbm}}\ \underbrace{\frac{1\ \cancel{lbm}\cdot g}{1\ \cancel{lbf}}}_{=1\ (\text{unit conversion})}} = \sqrt{\frac{200\ /ft}{150}\ \frac{32.2\ ft/s^2}{1}} = 6.552\ rad/s$$

Period of motion

$$T = \frac{2\pi}{w} = \frac{2\pi}{6.552\ rad/s} = \boxed{0.959\ s}$$

b) Given : Person always on trampoline . Find max amplitude.

Use contact FBD ①



@ Equilibrium position x_{eq} , $\ddot{x}\big|_{x_{eq}} = 0$, $\dot{x}\big|_{x_{eq}} = 0$

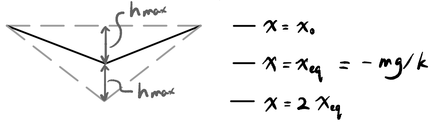
Solve equation ③ to get $x_{eq} = -\frac{mg}{k} = x_P$

The harmonic oscillation is symmetric about the equilibrium position , so the trampoline has equal travel distance on each side.

We assume* the trampoline does not go beyond the $x=0$ line , so the maximum amplitude

$$h_{max} = 0 - x_{eq} = \frac{mg}{k} = \boxed{0.750\ ft}$$

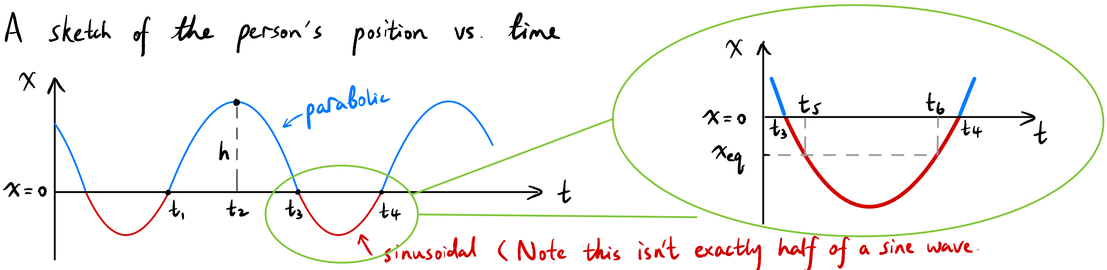
A sketch of the trampoline's motion at max amplitude



* Watch videos of trampolines. (Model trampoline as massless spring. Shoes don't apply tension. When tension = 0 , $kx = 0$, $x = 0$.)

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

c) A sketch of the person's position vs. time



sinusoidal (Note this isn't exactly half of a sine wave. So $t_4 - t_3 \neq \frac{1}{2}T$ from part (a).)

Need to find the period T'

$$T' = t_4 - t_1 = (t_3 - t_1) + (t_4 - t_3) = 2(t_3 - t_2) + (t_4 - t_3)$$

For the blue curve, the person's in flight. Use flight FBD ② $P \downarrow mg$

Note that we are solving for the relative time $(t_3 - t_2)$, so the absolute time t_2, t_3 doesn't matter.

Shift the t-axis so that $t_2 = 0$ to simplify the calculation

Initial condition $x(t_2 = 0) = h$

LMB: $m\ddot{x} = -mg$, $x(t_2 = 0) = h = 5\ \text{ft}$, $\dot{x}(t_2 = 0) = 0$

Solve the ODE

$$\ddot{x} = -g$$

$$\dot{x} = \int -g\, dt = -gt + C_1, \quad \dot{x}(0) = C_1 = 0$$

$$\dot{x} = -gt$$

$$x = \int -gt\, dt = -\tfrac{1}{2}gt^2 + C_2, \quad x(0) = C_2 = h$$

$$x = -\tfrac{1}{2}gt^2 + h$$

Solve $x(t_3) = 0 \rightarrow t_3 = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \cdot 5\ \text{ft}}{32.2\ \text{ft/s}^2}} = 0.5573\ \text{s}$

So $t_3 - t_2 = \underline{0.5573\ \text{s}}$

Also calculate the end condition of the blue curve. This will be the initial condition of the red curve.

Method 1. Conservation of energy

$$mgh = \frac{1}{2} m v(t_3)^2$$

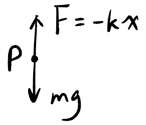
$$|v(t_3)| = \sqrt{2gh} \quad , \quad v(t_3) = -\sqrt{2gh} = -17.945 \text{ ft/s}$$

Method 2. Use the ODE solution above

$$\dot{x}(t_3) = -g t_3 = -32.2 \text{ ft/s}^2 \cdot 0.5573 \text{ s} = -17.945 \text{ ft/s}$$

For the red curve, the person's in contact with the trampoline

Use contact FBD ①



From (a) we have

$$m\ddot{x} = -kx - mg \quad , \quad x(t_3) = 0 \quad , \quad \dot{x}(t_3) = -17.945 \text{ ft/s} = \dot{x}_0$$

$$x = A\sin(\omega t) + B\cos(\omega t) - \frac{mg}{k} , \quad \omega = \sqrt{\frac{k}{m}}$$

Now need to solve for A, B

Once again, shift the t-axis so that $t_3 = 0$ to simplify the calculation

Initial condition:

$$\dot{x}(t_3 = 0) = \omega A\cos(\omega t) + \omega B\sin(\omega t) = \omega A = \dot{x}_0 \rightarrow A = \frac{\dot{x}_0}{\omega}$$

$$x(t_3 = 0) = A\sin(\omega t) + B\cos(\omega t) - \frac{mg}{k} = B - \frac{mg}{k} = 0 \rightarrow B = \frac{mg}{k}$$

$$x = \frac{\dot{x}_0}{\omega}\sin(\omega t) + \frac{mg}{k}\cos(\omega t) - \frac{mg}{k} \quad ④$$

Method 1. Matlab

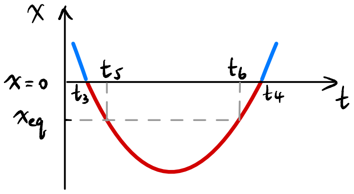
Use Matlab to solve $x(t_4) = 0 \rightarrow t_4 = 0.5610 \text{ s}$

Matlab code attached.

$[t_3, t_4] = \text{solve}(\underbrace{x(t) = 0}_{\text{from eqn ④}}, t)$

So $t_4 - t_3 = \underline{0.5610 \text{ s}}$

Method 2. Pencil, paper & calculator (Use $t_3 = 0$ for this part)



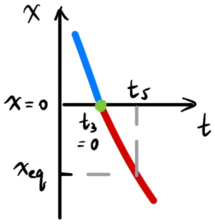
Again, the harmonic oscillation is symmetric about the equilibrium position x_{eq}.

So $t_6 - t_5 = \frac{T}{2} = \underline{0.4795 \text{ s}}$

and $t_5 - t_3 = t_4 - t_6$

@ t_5, $x = \frac{\dot{x}_0}{\omega}\sin(\omega t_5) + \frac{mg}{k}\cos(\omega t_5) - \frac{mg}{k} = x_{eq} = -\frac{mg}{k}$

Solve the equation



$$\frac{\dot{x}_0}{\omega}\sin(\omega t_5) + \frac{mg}{k}\cos(\omega t_5) = 0$$

$$\frac{\dot{x}_0}{\omega}\sin(\omega t_5) = -\frac{mg}{k}\cos(\omega t_5)$$

$$\tan(\omega t_5) = \frac{\sin(\omega t_5)}{\cos(\omega t_5)} = -\frac{mg\omega}{k\dot{x}_0}$$

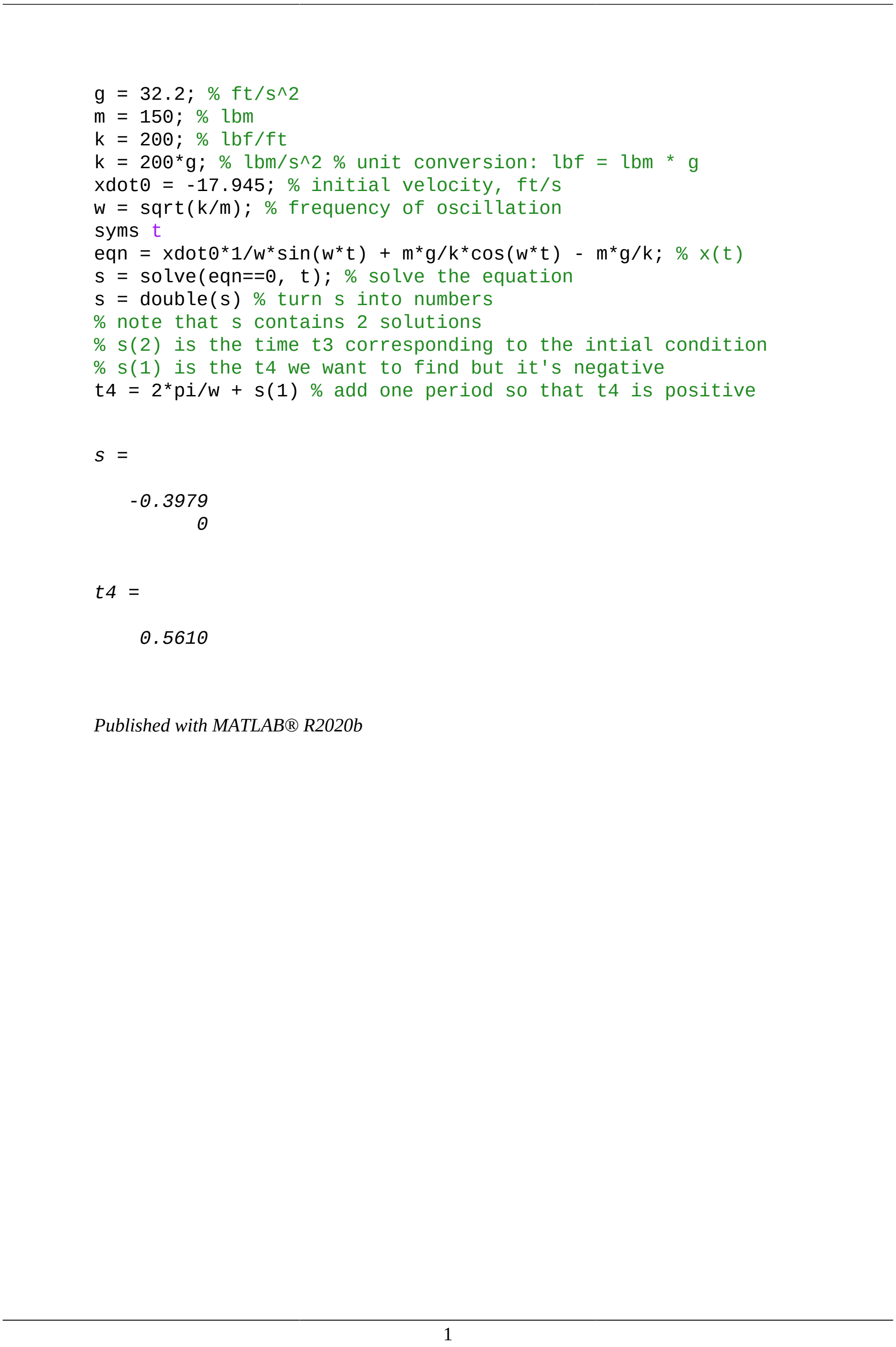
$$t_5 = \frac{1}{\omega}\tan^{-1}\left(-\frac{mg\omega}{k\dot{x}_0}\right) = \frac{1}{6.552 \text{ rad/s}}\tan^{-1}\left(-\frac{150\ \cancel{\text{lbm}}\cdot\cancel{g}\cdot 6.552 \text{ rad/s}}{200\ \cancel{\text{lbf}}/\text{ft}\cdot(-17.945 \text{ ft/s})}\underbrace{\frac{1\ \cancel{\text{lbf}}}{1\ \cancel{\text{lbm}\cdot g}}}_{=1 \text{ (unit conversion)}}\right)$$

$$= 0.0408 \text{ s}$$

So $t_5 - t_3 = \underline{0.0408 \text{ s}}$

$t_4 - t_3 = (t_5 - t_3) + (t_6 - t_5) + (t_4 - t_6) = 2(t_5 - t_3) + (t_6 - t_5) = \underline{0.5610 \text{ s}}$

The period of motion $T' = 2(t_3 - t_2) + (t_4 - t_3) = \boxed{1.676 \text{ s}}$



```
g = 32.2; % ft/s^2
m = 150; % lbm
k = 200; % lbf/ft
k = 200*g; % lbm/s^2 % unit conversion: lbf = lbm * g
xdot0 = -17.945; % initial velocity, ft/s
w = sqrt(k/m); % frequency of oscillation
syms t
eqn = xdot0*1/w*sin(w*t) + m*g/k*cos(w*t) - m*g/k; % x(t)
s = solve(eqn==0, t); % solve the equation
s = double(s) % turn s into numbers
% note that s contains 2 solutions
% s(2) is the time t3 corresponding to the intial condition
% s(1) is the t4 we want to find but it's negative
t4 = 2*pi/w + s(1) % add one period so that t4 is positive
```

```
s =

   -0.3979
         0


t4 =

    0.5610
```

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