SAMPLE 10.13 A block of mass $m = 20\,\text{kg}$ is attached to two identical springs each with spring constant $k = 1\,\text{kN/m}$. The block slides on a horizontal surface without any friction.

1. Find the equation of motion of the block.
2. What is the oscillation frequency of the block?
3. How much time does the block take to go back and forth 10 times?

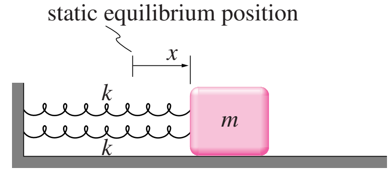

Figure 10.28

Solution

1. The free-body diagram of the block is shown in Figure 10.29. The linear momentum balance, $\sum \vec{F} = m\vec{a}$, for the block gives

$$-2kx\hat{\imath} + (N - mg)\hat{\jmath} = m\vec{a}$$

Dotting both sides with $\hat{\imath}$ we have,

$$\begin{aligned} -2kx = ma_x &= m\ddot{x} && (10.35)\\ \text{or}\quad m\ddot{x} + 2kx &= 0 && (10.36)\\ \text{or}\quad \ddot{x} + \frac{2k}{m}x &= 0. && (10.37)\end{aligned}$$

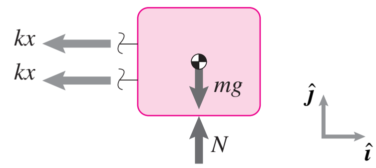

Figure 10.29

$$\boxed{\ddot{x} + \frac{2k}{m}x = 0}$$

2. Comparing Eqn. (10.37) with the standard harmonic oscillator equation, $\ddot{x} + \lambda^2 x = 0$, where λ is the oscillation frequency, we get

$$\begin{aligned} \lambda^2 &= \frac{2k}{m}\\ \Rightarrow\quad \lambda &= \sqrt{\frac{2k}{m}}\\ &= \sqrt{\frac{2\cdot(1\,\text{kN/m})}{20\,\text{kg}}}\\ &= 10\,\text{rad/s}.\end{aligned}$$

$$\boxed{\lambda = 10\,\text{rad/s}}$$

3. Time period of oscillation $T = \frac{2\pi}{\lambda} = \frac{2\pi}{10\,\text{rad/s}} = \frac{\pi}{5}\,\text{s}$. Since the time period represents the time the mass takes to go back and forth just once, the time it takes to go back and forth 10 times (*i.e.*, to complete 10 cycles of motion) is

$$t = 10T = 10\cdot\frac{\pi}{5}\,\text{s} = 2\pi\,\text{s}.$$

$$\boxed{t = 2\pi\,\text{s}}$$

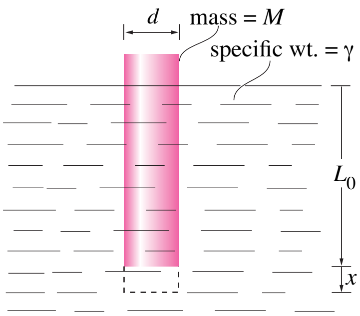

Figure 10.30

SAMPLE 10.14 Simple harmonic motion of a buoy. A cylinder of cross sectional area A and mass M is in static equilibrium inside a fluid of specific weight γ when L_o length of the cylinder is submerged in the fluid. From this position, the cylinder is pushed down vertically by a small amount x and let go. Assume that the only forces acting on the cylinder are gravity and the buoyant force and assume that the buoy's motion is purely vertical. Derive the equation of motion of the cylinder using Linear Momentum Balance. What is the period of oscillation of the cylinder?

Solution The free-body diagram of the cylinder is shown in Fig. 10.31 where F_B represents the buoyant force (see the hydrostatics chapter starting on 460). Before the cylinder is pushed down by x, the linear momentum balance of the cylinder gives

$$F_B - Mg = M\underbrace{a}_{0} = 0 \qquad \Rightarrow \qquad F_B = Mg$$

Now $F_B =$ (volume of the displaced fluid)$\cdot$ (its specific weight) $= AL_o\gamma$. Thus,

$$AL_o\gamma = Mg. \tag{10.38}$$

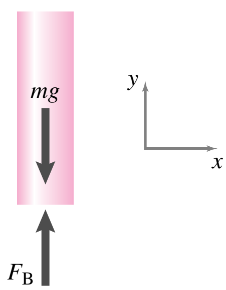

Figure 10.31:

Now, when the cylinder is pushed down by an amount x,

$$F_B' = \text{ new buoyant force } = (L_o + x)A\gamma.$$

Therefore, from LMB we get

$$\begin{array}{rrcl} & F_B' - Mg & = & -M\ddot{x} \\ \text{or} & (L_o + x)A\gamma - Mg & = & -M\ddot{x} \\ \text{or} & M\ddot{x} + A\gamma x & = & \overbrace{-AL_o\gamma + Mg}^{=0 \text{ from (10.38).}} \\ \text{or} & M\ddot{x} + A\gamma x & = & 0 \\ \text{or} & \ddot{x} + \dfrac{A\gamma}{M}x & = & 0. \end{array}$$

$$\boxed{\ddot{x} + \frac{A\gamma}{M}x = 0}$$

Comparing this equation with the standard simple harmonic equation (*e.g.*, eqn.(g), in the box on ODE's on page 203).

$$\text{The circular frequency } \lambda = \sqrt{\frac{A\gamma}{M}},$$

$$\text{Therefore,} \qquad \text{the period of oscillation } T = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{M}{A\gamma}}$$

.

$$\boxed{T = 2\pi\sqrt{\frac{M}{A\gamma}}}$$

Comments: This calculation inaccurately uses fluid statics to calculate the dynamics of a buoy; the pressure used in this calculation assumes fluid statics when actually the fluid is moving. One common partial correction is to use 'added mass' to account for fluid that moves more-or-less with the cylinder. The added mass is usually something like one-half the mass of the displaced fluid, that is one half the mass of the bouy. Another missing effect is the fluid damping. This would be added as a drag force proportional to the velocity or the velocity squared.

SAMPLE 10.15 A spring-mass system executes simple harmonic motion: $x(t) = A\cos(\lambda t - \phi)$. The system starts with initial conditions $x(0) = 25\,\text{mm}$ and $\dot{x}(0) = 160\,\text{mm/s}$ and oscillates at the rate of 2 cycles/sec.

1. Find the time period of oscillation and the oscillation frequency λ.
2. Find the amplitude of oscillation A and the phase angle ϕ.
3. Find the displacement, velocity, and acceleration of the mass at $t = 1.5\,\text{s}$.
4. Find the maximum speed and acceleration of the system.
5. Draw an accurate plot of displacement *vs*. time of the system and label all relevant quantities. What does ϕ signify in this plot?

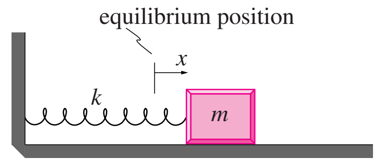

Figure 10.32

Solution

1. We are given $f = 2$ Hz. Therefore, the time period of oscillation is

$$T = \frac{1}{f} = \frac{1}{2\,\text{Hz}} = 0.5\,\text{s},$$

and the oscillation frequency $\lambda = 2\pi f = 4\pi\,\text{rad/s}$.

$$T = 0.5\,\text{s}, \quad \lambda = 4\pi\,\text{rad/s}.$$

2. The displacement $x(t)$ of the mass is given by

$$x(t) = A\cos(\lambda t - \phi).$$

Therefore the velocity (actually the speed) is

$$\dot{x}(t) = -A\lambda\sin(\lambda t - \phi)$$

At $t = 0$, we have

$$x(0) = A\cos(-\phi) = A\cos\phi \qquad (10.39)$$
$$\dot{x}(0) = -A\lambda\sin(-\phi) = A\lambda\sin\phi \qquad (10.40)$$

By squaring Eqn (10.39) and adding it to the square of [Eqn (10.40) divided by λ], we get

$$\begin{aligned} A^2\cos^2\phi + \frac{A^2\lambda^2\sin^2\phi}{\lambda^2} &= A^2 = x^2(0) + \frac{\dot{x}^2(0)}{\lambda^2} \\ \Rightarrow \quad A &= \sqrt{(25\,\text{mm})^2 + \frac{(160\,\text{mm/s})^2}{(4\pi\,\text{rad/s})^2}} \\ &= 28.06\,\text{mm}. \end{aligned}$$

Substituting the value of A in Eqn (10.39), we get

$$\begin{aligned} \phi &= \cos^{-1}\frac{x(0)}{A} \\ &= \cos^{-1}\frac{25\,\text{mm}}{28.06\,\text{mm}} \\ &= 0.471\,\text{rad} \approx 27^\circ. \end{aligned}$$

$$A = 28.06\,\text{mm}. \quad \phi = 0.471\,\text{rad}.$$

3. The displacement, velocity, and acceleration of the mass at any time t can now be calculated as follows

$$
\begin{aligned}
x(t) &= A\cos(\lambda t - \phi) \\
\Rightarrow \quad x(1.5\,\mathrm{s}) &= 28.06\,\mathrm{mm}\cdot\cos(6\pi - 0.471) \\
&= 25\,\mathrm{mm}.
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}(t) &= -A\lambda\sin(\lambda t - \phi) \\
\Rightarrow \quad \dot{x}(1.5\,\mathrm{s}) &= 28.06\,\mathrm{mm}\cdot(4\pi\,\mathrm{rad/s})\cdot\sin(6\pi - 0.471) \\
&= 160\,\mathrm{mm/\,s}.
\end{aligned}
$$

$$
\begin{aligned}
\ddot{x}(t) &= -A\lambda^2\cos(\lambda t - \phi) \\
\Rightarrow \quad \ddot{x}(1.5\,\mathrm{s}) &= 28.06\,\mathrm{mm}\cdot(4\pi\,\mathrm{rad/s})^2\cdot\cos(6\pi - 0.471) \\
&= -3.95\times10^3\,\mathrm{mm/\,s^2} \\
&= -3.95\,\mathrm{m/\,s^2}.
\end{aligned}
$$

① We can find the displacement and velocity at $t = 1.5\,\mathrm{s}$ without any differentiation. Note that the system completes 2 cycles in 1 second, implying that it will complete 3 cycles in 1.5 seconds. Therefore, at $t = 1.5\,\mathrm{s}$, it has the same displacement and velocity as it had at $t = 0\,\mathrm{s}$.

①

$x(1.5\,\mathrm{s}) = 25\,\mathrm{mm}.\quad \dot{x}(1.5\,\mathrm{s}) = 160\,\mathrm{mm/\,s}.\quad \ddot{x}(1.5\,\mathrm{s}) = -3.93\,\mathrm{m/\,s^2}.$

4. Maximum speed:

$$|\dot{x}_{\max}| \quad = \quad A\lambda = (28.06\,\mathrm{mm})\cdot(4\pi\,\mathrm{rad/s}) = 0.35\,\mathrm{m/s}.$$

Maximum acceleration:

$$|\ddot{x}_{\max}| \quad = \quad A\lambda^2 = (28.06\,\mathrm{mm})\cdot(4\pi\,\mathrm{rad/s})^2 = 4.43\,\mathrm{m/s^2}.$$

$|\dot{x}_{\max}| = 0.35\,\mathrm{m/s},\quad |\ddot{x}_{\max}| = 4.43\,\mathrm{m/s^2}.$

5. The plot of $x(t)$ versus t is shown in Fig. 10.33. The phase angle ϕ represents the shift in $\cos(\lambda t)$ to the right by an amount $\frac{\phi}{\lambda}$.

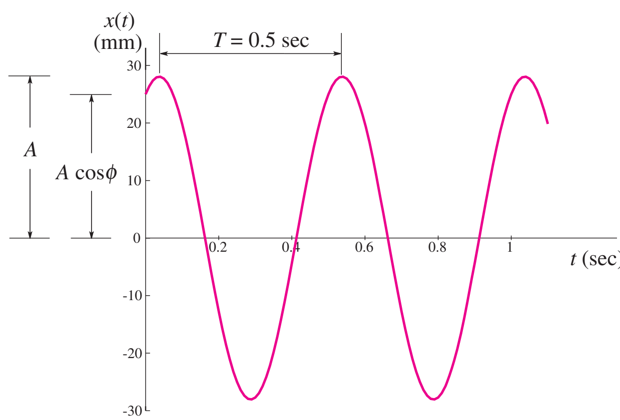

Figure 10.33: