10.2 Energy methods in 1D

Energy is an important concept in science and engineering. Energy is also a kind of currency in human trade. Energy is also a concept that is somewhat bigger than can be defined inside classical mechanics, when we look at, say, the chemical energy cost of various mechanical tasks.

For a student learning mechanics, energy is first a method, or trick, for solving some simple problems of the type assigned in elementary courses like this one. As problems become more difficult (have more degrees of freedom or include, say, more-than-just-constant friction) energy becomes less useful as a problem solving technique. However, in more advanced mechanics, energy gets a central role again: energy is the central concept in some advanced ways to write equations of motion and for some methods of understanding stability.

Power, work, kinetic energy and potential energy

Before we get to the facts and theorems, we start with some definitions. Here are four words. We will use these definitions, or generalizations of them, throughout dynamics.

Power. The power of a force F is its product with the velocity v of the point on which it is acting,

$$\text{Power} = P = Fv.$$

This is the 1D version of the more general $P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$ which we will use once we go on to 2D and 3D dynamics. In full generality, power is a scalar (not a vector). The common units for power are watts ($1\,\text{W} = \text{N m/s} = \text{J/s}$), kilowatts ($1\text{KW} = 10^3\ \text{W}$), lbf ft/s (no special name) and horsepower ($1\,\text{hp} \equiv 550\,\text{lbf ft/s} \approx 745.7 \approx 746$ ① watts).

① In fourteen hundred and ninety-two,
Columbus sailed the ocean blue,
And that's twice the watts
in a horsepower too.

$$\begin{array}{r} 746 \\ \underline{\times 2} \\ 1492 \end{array}$$

Example:
A 5 N force acting on a particle moving 3 m/s has a power of
$P = Fv = (5\,\text{N})(3\,\text{m/s}) = 15\,\text{N m/s} = 15\,\text{W} \approx 0.02\,\text{hp}$.

Work. The work W of a force is most easily defined incrementally (ΔW) for small motions Δx of a particle; motions so small that variations in force can be neglected and the force viewed as constant,

$$\text{increment of work} = \Delta W = F\Delta x.$$

This is a special 1D reduction of the more general $\Delta W = \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{r}}$. Even in 2D and 3D work is a scalar. Often we want to know the work for larger (non-infinitesimal) displacements. We do this by adding up the increments. Using sloppy calculus (implicitly taking the limit of a Riemann sum):

$W = \sum \Delta W = \int dW = \int F\,dx$, which we can write more definitely as

$$\text{Work} = W = \int_{x_0}^{x} F(x')\,dx'.$$

which is the 1D version of the more general $W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$. Common units of work are Joules ($1\,\text{J} = 1\,\text{N m} = 1\,\text{kg m}^2/\,\text{s}^2$), foot-pounds ($= 1\,\text{ft lbf}$) and kilo-watt hours ($= 3.6 \cdot 10^6\,\text{J}$).

Example:

The force $F = F_0 \sin(cx)$ pushes a mass from $x_0 = 0\,\text{m}$ to $x_1 = \pi\,\text{m}$ where $F_0 = 7\,\text{N}$ and $c = 1/\,\text{m}$. Then

$$W = \int_{x_0}^{x_1} F\,dx = \int_0^{\pi\,\text{m}} F_0 \sin(cx)\,dx = -(F_0/c)\cos(cx)|_0^{\pi\,\text{m}} = 14\,\text{N·m}$$

Kinetic energy. The kinetic energy quantifies the motion a little differently than momentum does. In kinetic energy high speed gets extra credit (v^2 instead of just v). Further, for kinetic energy we don't worry about which way a particle moves. The kinetic energy E_K of a particle in 1D is

$$\text{kinetic energy} = E_\text{K} = \frac{1}{2}mv^2$$

In two and three dimensions the formula above applies for one particle (taking $v = |\vec{\boldsymbol{v}}|$). For a collection of particles E_K is defined as a sum of E_K for each particle separately. In full generality kinetic energy is a scalar. The units of work (force×distance) and of kinetic energy (mass×speed^2) are the same (mass×distance^2/ time^2) and so are the common measures, namely Joules, foot-pounds and kilo-watt hours.

Example:

A 3 kg mass moving at a speed of 4 m/s has a kinetic energy of $E_\text{K} = mv^2/2 = (3\,\text{kg})(4\,\text{m/s})^2/2 = 24\,\text{kg m}^2/\,\text{s}^2 = 24\,\text{J}$.

Potential energy. This is the most abstract of the definitions. The potential energy E_P associated with a force F is defined as that function of x with these properties

$$E_\text{P}(x) = -\int_{x_0}^{x} F(x')\,dx' \qquad \text{and} \qquad F(x) = -\frac{d}{dx}E_\text{P}(x)$$

which people write more indefinitely as $E_\text{P} = -\int F\,dx$ and $F = -E_\text{P}{}'$. In two and three dimensions the concept of potential energy is more subtle still, being defined by a path integral which may or may not be sensible. But it is still a scalar.

Example:
The force $F = c/x^2$ is associated with the potential energy
$E_\mathrm{P} = -\int F\,dx = c/x + C_0$.

The datum for potential energy. Potential energy always has an undetermined, and generally irrelevant, integration constant. The integration constant is irrelevant because usually we care about *changes* in energy. So in the example above we could set $C_0 = 0$ and write $E_\mathrm{P} = -\int F\,dx = c/x$. In general we define the *datum* for potential energy as that position where we set the potential energy to zero.

- For near-earth gravity the datum is usually set at the height of the ground (so that $E_\mathrm{P} = mgh$), a launch point, or of a conspicuous physical point (say the hinge of a pendulum).
- For inverse-square gravity the datum is usually set at $r = \infty$ so that formulas are most simple.
- For springs that datum is usually set at the position where the spring is 'relaxed' (un-stretched and at its rest-length), again simplifying the terms in energy equations.

Potential energy is a shortcut for calculating work. From the definition of potential energy we can calculate work of a force in moving a particle from one place to another as:

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{\mathrm{P}_2} - E_{\mathrm{P}_1}\right).$$

Of course you need to know, or find, $E_\mathrm{P}(x)$ first in order to use this shortcut.

Example:
The work of $F = c/x^2$ in moving a mass from x_1 to x_2 is

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{\mathrm{P}_2} - E_{\mathrm{P}_1}\right) = c/x_1 - c/x_2$$

Where we used that $E_\mathrm{P} = c/x$ has the needed property that $F = -\dfrac{d}{dx}E_\mathrm{P}$.

Why all this new language? All of the words above are defined in terms of position, velocity and force. So anything we say about power, work and kinetic and potential energies we could say already using x, v and F. More particularly, we already have two ways of quantifying the motion of a particle, v and $L = mv$. Why do we need a third, $E_\mathrm{K} = mv^2/2$? The answer is this, to simplify the solution of some problems. Various facts and theorems are simpler if commonly appearing groups of terms are given names. And all of the definitions above are common groups.

Then, luckily, some of them turn out to be more general than just 1D particle mechanics.

The new vocabulary makes thinking easier. Various so-called 'one degree of freedom' problems can be solved by noting that energy is conserved. And features of solutions of more-complex problems can be extracted or checked by making sure that energy balance comes out right.

Power and work

The simplest relation between the quantities we have defined above is that between Power and work:

$$W = \int F\,dx = \int F v \frac{dx}{v} = \int F v\,dt$$

or more definitely

$$W = \int_{x_0}^{x_1} F\,dx = \int_{t_0}^{t_1} F v\,dt = \int_{t_0}^{t_1} P\,dt$$

Example: Integrate power to get work.
If the power of a force acting on a particle is $P = P_0(ct^2)$ where $P_0 = 10\,\mathrm{W}$ and $c = 3/\,\mathrm{s}^2$ then over 3 seconds the work done by the force is:

$$W = \int_{t_0}^{t_1} P\,dt = \int_{t_0}^{t_1} P_0(ct^2)\,dt = P_0 c t^3/3\Big|_{t_0}^{t_1} = (10\,\mathrm{W})(3/\,\mathrm{s}^2)t^3/3\Big|_0^{3\,\mathrm{s}}$$
$$= 270\,\mathrm{W\,s} = 270\,\mathrm{J}$$

Power and rate-of-change of kinetic energy

On the inside cover the third basic law of mechanics is energy balance. Energy balance takes a number of different forms, depending on context. The power balance equation from the front cover and simplified for a particle is

$$P = \dot{E}_\mathrm{K},$$

where, recall, $P = Fv$ is the power of the applied force F. The derivation of this result from $F = ma$ for a particle is simple enough, and is good to know. First note the following result from using the chain of differentiation:

$$\frac{d}{dt}\left(v^2\right) = 2v\frac{dv}{dt} = 2v\dot{v} = 2va.$$

When we need to call on this simple kinematics (calculus) result it usually comes to us the other way around. So what you should remember is this formula, one of the basic tricks of the trade:

$$va = \frac{d}{dt}\left(\frac{v^2}{2}\right).$$

Multiplying both sides by m and substituting in $F = ma$ we get our 1D power balance equation:

$$Fv = \frac{d}{dt}\left(\frac{mv^2}{2}\right).$$

The power of a given force depends on the speed of the object to which it is applied. When a finite non-zero force is applied to a stationary object the power of the force is zero and so is the rate of change of kinetic energy. If the object accelerates, its speed is increasing, but when the speed is zero, $\dot{E}_K = 0$.

Example:
A constant force F is applied to an initially stationary mass m starting at $t = 0$. Then $v = Ft/m$, $E_K = mv^2/2 = F^2t^2/(2m)$ and $P = Fv = F^2t/m$. Note that $\dot{E}_K = P$ and both are zero at $t = 0$.

Work is change of kinetic energy

Integrating the power balance equation in time we get

$$\int P\,dt = \int \dot{E}_K\,dt = \Delta E_K \tag{10.16}$$

More definitely, and also using the work integral, we have that the work of the net force on a particle is the change of its kinetic energy:

$$\int_{t_1}^{t_2} \underbrace{Fv}_{P}\,dt = \int_{x_1}^{x_2} F\,dx = E_{K_2} - E_{K_1}$$

Once we remember that

work is change in kinetic energy,

we can use it without deriving it every time from $F = ma$ or from more general energy balance equations.

Example:
A force applied to a particle m varies sinusoidally with position according to $F = F_0 \cos(cx)$. At $x = 0$ the particle has speed $v = v_0$. Then

$$W = \Delta E_\mathrm{K} \quad \Rightarrow \quad \int_0^x F(x')\,dx' = \Delta\left(mv^2/2\right)$$
$$\Rightarrow \quad F_0 \sin(cx)/c = mv^2/2 - mv_0^2/2$$
$$\text{so} \qquad v = \sqrt{v_0^2 + 2F_0 \sin(cx)/(mc)}$$

The above example illustrates three points you should remember:

- The work-energy equations always leave the *sign* of the velocity unknown. You can see this because the derivation involves v^2. You can also see it in formulas you get for velocity. They involve a square root, and thus, implicitly a $\pm$. Whether one, the other or both roots are relevant depends on reasoning that lies outside the energy equation itself.
- The work-energy equations can generate formulas that, in certain situations, are nonsense: If the initial speed v_0 is not high enough the particle will not get very far. In particular if $v_0^2 < 2F_0/(mc)$ the inside of the square root will be negative for some x and the "answer" will be imaginary. These are values of x that the particle will never reach.
- Here we have apparently solved for something about the motion of a particle. And we have, partially. But to find the $x(t)$ we would have to integrate again. And that next integral is hard. That is, energy balance lets us solve for some aspects of the motion, namely speed vs position, without ever needing to know in detail how position varies with time.

Conservation of energy

Many people leave high-school physics loving conservation of energy. It makes certain special homework problems easy. In the real world the principle is also useful for building intuition, and sometimes also for problem solving. In 1D particle mechanics energy conservation is a theorem②.

②Be forewarned, generally we cannot think of energy conservation as necessarily applicable nor, if applicable, as derivable from the equations of mechanics.

Recall that if a particle is acted on by a force that varies with position, $F = F(x)$, then we can define a potential energy $E_\mathrm{P} = -\int F\,dx$ and that the work done by the force when the particle moves from x_1 to x_2 is

$$-(E_{\mathrm{P}_2} - E_{\mathrm{P}_1}) = -\Delta E_\mathrm{P}.$$

That is, the decrease in E_P is the amount of work that the force does. Or, in other words, E_P represents a potential to do work. Because work causes an increase in kinetic energy, E_P is called the *potential energy* of the force field. Now we can compare this result with the work-energy equation 10.16 to find that

$$-\Delta E_{\mathrm{P}} = \Delta E_{\mathrm{K}} \qquad \Rightarrow \qquad 0 = \Delta \underbrace{(E_{\mathrm{P}} + E_{\mathrm{K}})}_{E_{\mathrm{T}}}.$$

The *total energy* E_{T} doesn't change ($\Delta E_{\mathrm{T}} = 0$) and thus is a constant. In other words,

> as a particle moves in the presence of a force field with a potential energy, the total energy $E_{\mathrm{T}} = E_{\mathrm{K}} + E_{\mathrm{P}}$ is constant.

This fact goes by the name of *conservation of energy*.

Example: Falling ball
Consider the ball in the free-body diagram 10.13. If we define gravitational potential energy as minus the work gravity does on a ball while it is lifted from the ground, then

$$E_{\mathrm{P}} = -\int_0^y (-mg)\,dy' = mgy = mgh.$$

For vertical motion

$$E_{\mathrm{K}} = \frac{1}{2} m\dot{y}^2.$$

So conservation of energy says that in free fall:

$$\text{Constant} = E_{\mathrm{P}} + E_{\mathrm{K}} = mgy + m\dot{y}^2/2$$

which you could also derive directly from $m\ddot{y} = -mg$.

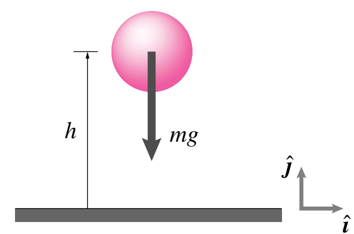

Figure 10.13: Free-body diagram of a falling ball, assuming gravity is the only significant external force acting on the ball.

Using conservation of energy to find equations of motion. On the one hand conservation of energy sometimes gives us a (partial) solution to a mechanics problem. On the other, we can use conservation of energy to find the "equations of motion". The basic strategy is to take the derivative of the conservation of energy equation.

Example: Falling ball eqns. from energy.

$$\begin{aligned}
E_{\mathrm{T}} = \text{constant} \qquad \Rightarrow \qquad 0 &= \tfrac{d}{dt} E_{\mathrm{T}} \\
&= \tfrac{d}{dt}(E_{\mathrm{P}} + E_{\mathrm{K}}) \\
&= \tfrac{d}{dt}(mgy + m\dot{y}^2/2) \\
&= (mg\dot{y} + m\dot{y}\ddot{y}) \\
\Rightarrow \qquad m\ddot{y} &= -mg.
\end{aligned}$$

We had to assume (and this is just a technical point) that $\dot{y} \neq 0$ in one of the cancellations. We have used energy balance to derive linear-momentum balance.

One can also find equations of motion starting with power balance.

$$P = \dot{E}_{\mathrm{K}}$$

as derived here in detail for the case of gravity acting on a particle.

$$
\begin{aligned}
P &= \frac{d}{dt}\left(E_{\mathrm{K}}\right) && \text{(Power balance)}\\
\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} &= \frac{d}{dt}\left(E_{\mathrm{K}}\right) && \text{(Power of external force)}\\
(-mg\,\hat{\boldsymbol{\jmath}}) \cdot (\dot{y}\,\hat{\boldsymbol{\jmath}}) &= \frac{d}{dt}\left[\frac{1}{2}mv^2\right] && \text{(expanding terms)}\\
-mg\dot{y} &= \frac{1}{2}m\frac{d}{dt}\left(\dot{y}^2\right) && \text{(evaluate dot product, substitute for } v\text{)}\\
-mg\dot{y} &= \frac{1}{2}m\left(2\dot{y}\cdot\ddot{y}\right) && \text{(the chain rule)}\\
\ddot{y} &= -g && \text{(cancel terms, switch sides),}
\end{aligned}
\tag{10.17}
$$

The potential energy of a spring is $k(\Delta\ell)^2/2$**.** Besides near-earth gravity, which we already covered ($E_{\mathrm{P}} = mgh$), the main elementary use of potential energy is for the stretch of a linear spring. Integrating $dW = F dx$ for a linear spring with force on an object $F = -kx$, where x is the spring stretch, from the rest length, we get

$$
E_{\mathrm{P}} = -\int_0^x F(x')\,dx' = -\int_0^x -kx'\,dx' = \frac{1}{2}kx^2. \tag{10.18}
$$

In the above example we measured x from the rest position of one end of the spring. But often the natural x coordinate will not be so nicely set up. It is safer to remember the spring's potential energy in terms of its stretch:

$$
E_{\mathrm{P}} = \frac{k\,\Delta\ell^2}{2}, \tag{10.19}
$$

where we measure $\Delta\ell = \ell - \ell_0$ where ℓ_0 is the spring's rest length ($\ell_0 =$ length when the tension is zero).

Thus for a spring and mass oscillator, the subject of the next section, conservation of energy tells us that $mv^2/2 + kx^2/2 = constant$.

Is energy balance a principle or a calculation trick?

For one dimensional particle motion, momentum balance, power balance, and energy balance can each be derived from either of the others. If we take $F = ma$ as primary, energy calculations are just a convenience of notation or, in the case of the work-energy relation, a useful calculation technique (trick).

Historically, conservation of energy was first noted in particle mechanics problems. But because the position-dependent forces of springs and gravity seemed so fundamental, that they had a description as the derivative of a potential gave the energy relations the smell of something more fundamental. And so it has turned out that energy is an important topic for chemistry, thermodynamics, electrodynamics and sub-atomic physics. It's not just an analogy, it's the same energy. Thus energy is the primary currency of exchange between, say, the superficially disparate chemical and mechanical systems.

The exchange of energy between these forms, in the context of particle mechanical models, can give the sense that we are doing the same 1D momentum based mechanics calculations when actually we are using more general energy balance equations, equations that cannot be derived from $F = ma$.

Terrestrial locomotion: Trains, cars, bicycles and animals

A free-body diagram of an accelerating car, treated as a 1D particle system, is shown in fig. 10.5 on page 484. The point represents the car, the force is the propulsion force from the wheel-ground interaction, and for now, we have neglected air friction. Without worrying about details we could say then that the power of the propulsion force is equal to the rate of change of kinetic energy.

Example: Accelerating car
An aggressive 1 ton car can accelerate with $a = 0.5g$ while going 60 mph. Neglecting friction and air resistance, the power of the propulsion force is

$$\begin{aligned}
P &= Fv = mav = (1\,\text{ton})(0.5g)(60\,\text{mi}/\,\text{hr}) \\
&= (1\,\text{ton})(0.5g)(60\,\text{mi}/\,\text{hr})\underbrace{\left(\frac{2000\,\text{lbm}}{\text{ton}}\right)}_{1}\underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}_{1}\underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)}_{1}\underbrace{\left(\frac{1\,\text{lbf}}{g\cdot\text{lbm}}\right)}_{1}\underbrace{\left(\frac{1\,\text{hp}}{550\,\text{ft}\cdot\text{lbf}/\,\text{s}}\right)}_{1} \\
&= (0.5\cdot 60\cdot 2000\cdot 5280)/(3600\cdot 550)\,\text{hp} \approx 160\,\text{hp}
\end{aligned}$$

Note the judicious multiple multiplications by 1 so that all units cancel but for horse-power; ton cancels ton, hr cancels hr, g cancels g and so on. The car engine needs to supply this 160 hp plus any internal transmission dissipation. And more still to cover the tire drag and air drag *etc.*

Such calculations are deceptively simple. Some apparent paradoxes:

- The propulsive force on the car comes from the interaction of the ground with the car. Are we saying that the (dead-as-a-doormat) ground supplies a power of, say, 160 hp to an accelerating car?
- The point of application of the force on the car is at the bottom of the tire. That point has no velocity. So the actual power of the ground force on the car (tire) is zero. How is that reconciled with, say, the 160 hp that we get from particle mechanics.

These are legitimate concerns which are discussed further in box 10.1 on page 12. The bottom line is that the calculation turns out, perhaps by the demands of dimensional consistency, to be useful and correct.

Drag power. The drag force of air on moving things has an effect on the energy balance. Air drag is important for cars, bicycles and animals that are moving quickly (say, running people). The air drag is proportional to

$$F_d = \frac{1}{2}\rho C_d A v^2$$

What are the proportionalities in the drag formula?

- The cross-sectional area (the area visible from directly in front) A. The bigger the area the more air has to be pushed out of the way. For a car $A \approx 2\,\mathrm{m}^2$
- The density of air ρ. The more mass has to be pushed out of the way, the bigger the force. For rough calculations one can remember that the density of air is about one thousandth that of water $\rho_{air} \approx 1\,\mathrm{kg/\,m^3}$. But the density varies in human environments from about $1.1\,\mathrm{kg/\,m^3}$ in high-altitude (low pressure), high-temperature (gas expands when hot), humid (water vapor is lighter than air) environments up to about $1.4\,\mathrm{kg/\,m^3}$ in low-lying cold dry places.
- The relative speed squared v^2. The faster you are moving the more air per unit time you must displace, and each bit of air gets displaced with a bigger speed. Typical highway speeds are about $v = 30\,\mathrm{m/\,s}$ (≈ 67 mph) and a typical human walking speed is about $v = 1\,\mathrm{m/s}$ (about 10% over ≈ 2 mph).
- A shape coefficient, sometimes called a d̲rag coefficient C or C_d. Different shapes of the same size, can displace the air more or less as the vehicle passes through. Streamlined shapes have small C_d
- One half (1/2). Convention has a factor of 1/2. This simplifies the power interpretation.

The drag power is

$$P = F_d v = \frac{1}{2}\rho C A v^3$$

which is a key result: increasing the speed 1% increases the power demand by 3% and doubling the speed multiplies the power demand by a factor of 8. This huge dependence of power on speed motivates smug energy-conservers to drive annoyingly slowly on highways.

Another way of writing the drag equation is

$$P = C_d \times (\text{ The relative kinetic energy swept by the vehicle per unit time})$$

How's that? The volume "swept" by the vehicle per unit time is its area times speed vA. Here area A means the area seen by the wind (the area of a shadow of the object made by a light shining in the wind direction on to a plane whose normal is in the wind direction). The air mass swept per unit time is thus $\rho v A$. The kinetic energy of the air, measured as moving relative to the vehicle, is $v^2/2$ per unit mass. Putting this together we get the swept kinetic energy of the air per unit time is $\rho A v^3/2$.

How big is the drag coefficient C_d? When in doubt take dimensionless constants as 1 and you are usually not too far off. At one extreme is a flat plate whose normal is parallel to the wind direction. Such a plate has $C_d \approx 1.25$. On the other hand, a good airfoil has $C_d \approx 0.05$. People, animals, and bicyclists all have C_d close to 1.

Drag on cars. For the worst cars C_d is actually almost 1. For typical cars on the street $C_d \approx 0.35$. For the best high-efficiency cars on the market in 2007, $C_d \approx 0.25$. Real marketed cars may one day get drag as low as $C_d \approx 0.2$. And concept cars that are shaped like trout can have drag as low as $C_d \approx 0.1$.

The drag power of a $2\,\mathrm{m}^2$ car going $30\,\mathrm{m/s}$ ($\approx 67\,\mathrm{mph} = 108\,\mathrm{km/hr}$) is about

$$\begin{aligned} P_{drag} &= \frac{1}{2}C\rho A v^3 \approx \frac{1}{2}\cdot 0.35\cdot(1\,\mathrm{kg/m^3})\cdot(2\,\mathrm{m^2})\cdot(30\,\mathrm{m/s})^3 \\ &= 9450\,\mathrm{kg\,m^2/s^3} = 9.45\,\mathrm{KW} \approx 13\,\mathrm{hp} \end{aligned}$$

That is, comparing with the example above, a car that needs 160 hp extra to make a zippy pass needs only 13 hp to move steadily along at a typical highway speed. For the units conversion we used $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$, $1\,\mathrm{J} = 1\,\mathrm{N\,m}$, $1\,\mathrm{W} = 1\,\mathrm{J/s}$, and $1\,\mathrm{KW} \approx 1.34\,\mathrm{hp}$.

Caveat on the drag "law". While there is some physics in the reasoning behind the drag law $F_d = \rho C A v^2/2$ the emphasis should be on the word "some", the whole chaotic nature of turbulent flow is not captured. The quadratic drag law is an empirical fit. For a given shape the C_d actually depends on the surface texture. And for a given shape and texture the C_d depends on v, the v^2 doesn't capture all of the velocity dependence. Nonetheless, the drag law is a reasonable approximation for most engineering purposes where drag is important.

Summary

There are two basic types of energy problems

- Problems where force or acceleration is given as a function of position ($a = a(x)$ or $F = F(x)$) and energy methods are basically a trick for finding $v(x)$.
- Problems where work, energy or power is of interest for its own sake because of, say, interest in engine power, dissipated energy, etc.

Of course the two problem types can also overlap.

10.1 Energetics of locomotion: using particle equations for non-particle systems

On page 9 we showed a naive locomotion power example in which we used

$$P = Fv$$

where v was the car velocity, F the thrust on the car, and P was 'the power' of the locomotion force. We pointed out two issues.

- How does it make sense for the passive ground, the source of the propulsive force, to supply power?
- The point of application of the ground force on the car is at the bottom of a wheel, a point that is not moving ($v = 0$). So how can Fv be other than zero?

The basic issue is that a car is not a particle, it has many moving parts and also some chemistry, so particle equations need to be interpreted with some care.

Particle equations are exact for non-particle systems

The most general form for linear momentum balance, as applied to a complicated system moving and deforming in complicated ways, reduces to equation $\vec{F} = m\vec{a}$. That is, so long as we interpret $\vec{F}$ to be the total force on the system, $\vec{a}$ to be the acceleration of the center of mass, and m to be the total mass of the system.

The power and energy equations in this chapter have been based on $\vec{F} = m\vec{a}$ (or their 1D scalar version $F = ma$) so apply to *any* system. But the terms P and E_K have meanings that go beyond particle mechanics. So while it is correct that (we derived it from $F = ma$),

$$Fv = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

for non-particle systems it is *not correct* that Fv is the actual power of the force applied nor that $mv^2/2$ is the kinetic energy of the system.

To understand the situation depends on understanding multi-body systems where we will see that the power of a force is $\vec{F} \cdot \vec{v}_P$ where $\vec{v}_P$ is the velocity of the material point to which the force is applied; and the kinetic energy is larger than $\frac{1}{2}mv^2$ because of motion relative to the average motion. Remember to reconsider these issues when you know more.

More general energy balance equations

Without worrying about what we can derive from what, there is no doubt that for any closed system we can write the energy balance equation from the front inside cover of the book, the first law of thermodynamics, as:

$$\dot{Q} + P = \dot{E}_K + \dot{E}_P + \dot{E}_{int}.$$

About the ever-shifting sign conventions, here we use $\dot{Q}$ as the heat flow *in* to the system, P is the power of external forces *on* the system, $\dot{E}_K$ and $\dot{E}_P$ are the rate of *increase* of the kinetic and potential energies of the system, and $\dot{E}_{int}$ is the rate of *increase* of internal energy. We can consider an accelerating car using this energy equation. For simplicity assume that no external forces do work on the car (the ground certainly does no work, and let's neglect air friction for now). We can also look at a car on level ground so there are no changes in gravitational potential energy. Finally, even though a car has many moving parts, the bulk of the material goes at the speed of a typical point on the body of the car. Thus the particle formula for kinetic energy is reasonably accurate. Putting this altogether we have

$$\dot{Q} + \underbrace{P}_{0} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \underbrace{E_P}_{0} + \dot{E}_{int}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\dot{E}_{int} + \dot{Q}$$

The rate of loss of chemical potential energy $-\dot{E}_{int}$ less the heat flow out $-\dot{Q}$ is what we call the power of the engine. Say chemical energy is being lost (used up) at a rate of $-\dot{E}_{int}$=40 KW. Say the heat flow out the exhaust is $-\dot{Q} = 30$ KW. Then, with that 40 KW of fuel use and that 25% efficient engine, we would have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\dot{E}_{int} + \dot{Q} = 40\,\mathrm{KW} - 30\,\mathrm{KW} = 10\,\mathrm{KW} \approx 13\,\mathrm{hp}.$$

But even this is not quite right because it does not take account of the flow of gases in and out of the car. Things can be messy if you look carefully.

What's the bottom line? In the end, with some sloppiness of thought but not much inaccuracy, we are not far off thinking that the change of kinetic energy of the car has to come from some place. And that place is the work of the engine as supplied by the decrease in chemical potential energy of the fuel. When we write $P = \dot{E}_K$ for a car, the P in that equation is the force applied to the car times the velocity of the car. But that P is not the power supplied by the outside agents on the car (e.g., the passive ground). Rather it is the power of forces inside the car. Never mind that we're modeling a car as a particle with no internal structure, at least for momentum-balance purposes.

This whole situation can only be properly clarified when we look at the power of internal and external forces in multi-body systems.