

10.2 Energy methods in 1D

Energy is an important concept in science and engineering. Energy is also a kind of currency in human trade. Energy is also a concept that is somewhat bigger than can be defined inside classical mechanics, when we look at, say, the chemical energy cost of various mechanical tasks.

For a student learning mechanics, energy is first a method, or trick, for solving some simple problems of the type assigned in elementary courses like this one. As problems become more difficult (have more degrees of freedom or include, say, more-than-just-constant friction) energy becomes less useful as a problem solving technique. However, in more advanced mechanics, energy gets a central role again: energy is the central concept in some advanced ways to write equations of motion and for some methods of understanding stability.

Power, work, kinetic energy and potential energy

Before we get to the facts and theorems, we start with some definitions. Here are four words. We will use these definitions, or generalizations of them, throughout dynamics.

Power. The power of a force F is its product with the velocity v of the point on which it is acting,

$$\text{Power} = P = Fv.$$

This is the 1D version of the more general $P = \vec{F} \cdot \vec{v}$ which we will use once we go on to 2D and 3D dynamics. In full generality, power is a scalar (not a vector). The common units for power are watts ($1 \text{ W} = \text{N m/s} = \text{J/s}$), kilowatts ($1 \text{ KW} = 10^3 \text{ W}$), lbf ft/s (no special name) and horsepower ($1 \text{ hp} \equiv 550 \text{ lbf ft/s} \approx 745.7 \approx 746$ ① watts).

Example:

A 5 N force acting on a particle moving 3 m/s has a power of
 $P = Fv = (5 \text{ N})(3 \text{ m/s}) = 15 \text{ N m/s} = 15 \text{ W} \approx 0.02 \text{ hp}.$

Work. The work W of a force is most easily defined incrementally (ΔW) for small motions Δx of a particle; motions so small that variations in force can be neglected and the force viewed as constant,

$$\text{increment of work} = \Delta W = F\Delta x.$$

This is a special 1D reduction of the more general $\Delta W = \vec{F} \cdot \Delta \vec{r}$. Even in 2D and 3D work is a scalar. Often we want to know the work for larger (non-infinitesimal) displacements. We do this by adding up the increments. Using sloppy calculus (implicitly taking the limit of a Riemann sum):

$$W = \sum \Delta W = \int dW = \int F dx, \text{ which we can write more definitely as}$$

①

In fourteen hundred and ninety-two,
 Columbus sailed the ocean blue,
 And that's twice the watts
 in a horsepower too.

$$\begin{array}{r} 746 \\ \times 2 \\ \hline 1492 \end{array}$$

$$\text{Work} = W = \int_{x_0}^x F(x') dx'.$$

which is the 1D version of the more general $W = \int \vec{F} \cdot d\vec{r}$. Common units of work are Joules ($1 \text{ J} = 1 \text{ N m} = 1 \text{ kg m}^2/\text{s}^2$), foot-pounds ($= 1 \text{ ft lbf}$) and kilo-watt hours ($= 3.6 \cdot 10^6 \text{ J}$).

Example:

The force $F = F_0 \sin(cx)$ pushes a mass from $x_0 = 0 \text{ m}$ to $x_1 = \pi \text{ m}$ where $F_0 = 7 \text{ N}$ and $c = 1/\text{m}$. Then

$$W = \int_{x_0}^{x_1} F dx = \int_0^{\pi \text{ m}} F_0 \sin(cx) dx = -(F_0/c) \cos(cx)|_0^{\pi \text{ m}} = 14 \text{ N}\cdot\text{m}$$

Kinetic energy. The kinetic energy quantifies the motion a little differently than momentum does. In kinetic energy high speed gets extra credit (v^2 instead of just v). Further, for kinetic energy we don't worry about which way a particle moves. The kinetic energy E_K of a particle in 1D is

$$\text{kinetic energy} = E_K = \frac{1}{2}mv^2$$

In two and three dimensions the formula above applies for one particle (taking $v = |\vec{v}|$). For a collection of particles E_K is defined as a sum of E_K for each particle separately. In full generality kinetic energy is a scalar. The units of work (force $\times$ distance) and of kinetic energy (mass $\times$ speed²) are the same (mass $\times$ distance²/time²) and so are the common measures, namely Joules, foot-pounds and kilo-watt hours.

Example:

A 3 kg mass moving at a speed of 4 m/s has a kinetic energy of $E_K = mv^2/2 = (3 \text{ kg})(4 \text{ m/s})^2/2 = 24 \text{ kg m}^2/\text{s}^2 = 24 \text{ J}$.

Potential energy. This is the most abstract of the definitions. The potential energy E_P associated with a force F is defined as that function of x with these properties

$$E_P(x) = - \int_{x_0}^x F(x') dx' \quad \text{and} \quad F(x) = -\frac{d}{dx}E_P(x)$$

which people write more indefinitely as $E_P = -\int F dx$ and $F = -E_P'$. In two and three dimensions the concept of potential energy is more subtle still, being defined by a path integral which may or may not be sensible. But it is still a scalar.

Example:

The force $F = c/x^2$ is associated with the potential energy
 $E_p = -\int F dx = c/x + C_0$.

The datum for potential energy. Potential energy always has an undetermined, and generally irrelevant, integration constant. The integration constant is irrelevant because usually we care about *changes* in energy. So in the example above we could set $C_0 = 0$ and write $E_p = -\int F dx = c/x$. In general we define the *datum* for potential energy as that position where we set the potential energy to zero.

- For near-earth gravity the datum is usually set at the height of the ground (so that $E_p = mgh$), a launch point, or of a conspicuous physical point (say the hinge of a pendulum).
- For inverse-square gravity the datum is usually set at $r = \infty$ so that formulas are most simple.
- For springs that datum is usually set at the position where the spring is ‘relaxed’ (un-stretched and at its rest-length), again simplifying the terms in energy equations.

Potential energy is a shortcut for calculating work. From the definition of potential energy we can calculate work of a force in moving a particle from one place to another as:

$$\text{work} = \int_{x_1}^{x_2} F(x') dx' = -(E_{p2} - E_{p1}).$$

Of course you need to know, or find, $E_p(x)$ first in order to use this shortcut.

Example:

The work of $F = c/x^2$ in moving a mass from x_1 to x_2 is

$$\text{work} = \int_{x_1}^{x_2} F(x') dx' = -(E_{p2} - E_{p1}) = c/x_1 - c/x_2$$

Where we used that $E_p = c/x$ has the needed property that $F = -\frac{d}{dx}E_p$.

Why all this new language? All of the words above are defined in terms of position, velocity and force. So anything we say about power, work and kinetic and potential energies we could say already using x , v and F . More particularly, we already have two ways of quantifying the motion of a particle, v and $L = mv$. Why do we need a third, $E_K = mv^2/2$? The answer is this, to simplify the solution of some problems. Various facts and theorems are simpler if commonly appearing groups of terms are given names. And all of the definitions above are common groups.

Then, luckily, some of them turn out to be more general than just 1D particle mechanics.

The new vocabulary makes thinking easier. Various so-called ‘one degree of freedom’ problems can be solved by noting that energy is conserved. And features of solutions of more-complex problems can be extracted or checked by making sure that energy balance comes out right.

Power and work

The simplest relation between the quantities we have defined above is that between Power and work:

$$W = \int F dx = \int Fv \frac{dx}{v} = \int Fv dt$$

or more definitely

$$W = \int_{x_0}^{x_1} F dx = \int_{t_0}^{t_1} Fv dt = \int_{t_0}^{t_1} P dt$$

Example: Integrate power to get work.

If the power of a force acting on a particle is $P = P_0(ct^2)$ where $P_0 = 10 \text{ W}$ and $c = 3/\text{s}^2$ then over 3 seconds the work done by the force is:

$$\begin{aligned} W &= \int_{t_0}^{t_1} P dt = \int_{t_0}^{t_1} P_0(ct^2) dt = P_0 ct^3/3 \Big|_{t_0}^{t_1} = (10 \text{ W})(3/\text{s}^2)t^3/3 \Big|_0^{3\text{s}} \\ &= 270 \text{ W s} = 270 \text{ J} \end{aligned}$$

Power and rate-of-change of kinetic energy

On the inside cover the third basic law of mechanics is energy balance. Energy balance takes a number of different forms, depending on context. The power balance equation from the front cover and simplified for a particle is

$$P = \dot{E}_K,$$

where, recall, $P = Fv$ is the power of the applied force F . The derivation of this result from $F = ma$ for a particle is simple enough, and is good to know. First note the following result from using the chain of differentiation:

$$\frac{d}{dt}(v^2) = 2v \frac{dv}{dt} = 2v\dot{v} = 2va.$$

When we need to call on this simple kinematics (calculus) result it usually comes to us the other way around. So what you should remember is this formula, one of the basic tricks of the trade:

$$va = \frac{d}{dt} \left(\frac{v^2}{2} \right).$$

Multiplying both sides by m and substituting in $F = ma$ we get our 1D power balance equation:

$$Fv = \frac{d}{dt} \left(\frac{mv^2}{2} \right).$$

The power of a given force depends on the speed of the object to which it is applied. When a finite non-zero force is applied to a stationary object the power of the force is zero and so is the rate of change of kinetic energy. If the object accelerates, its speed is increasing, but when the speed is zero, $\dot{E}_K = 0$.

Example:

A constant force F is applied to an initially stationary mass m starting at $t = 0$. Then $v = Ft/m$, $E_K = mv^2/2 = F^2t^2/(2m)$ and $P = Fv = F^2t/m$. Note that $\dot{E}_K = P$ and both are zero at $t = 0$.

Work is change of kinetic energy

Integrating the power balance equation in time we get

$$\int P dt = \int \dot{E}_K dt = \Delta E_K \quad (10.16)$$

More definitely, and also using the work integral, we have that the work of the net force on a particle is the change of its kinetic energy:

$$\int_{t_1}^{t_2} \underbrace{Fv}_P dt = \int_{x_1}^{x_2} F dx = E_{K2} - E_{K1}$$

Once we remember that

work is change in kinetic energy,

we can use it without deriving it every time from $F = ma$ or from more general energy balance equations.

Example:

A force applied to a particle m varies sinusoidally with position according to $F = F_0 \cos(cx)$. At $x = 0$ the particle has speed $v = v_0$. Then

$$\begin{aligned} W = \Delta E_K &\Rightarrow \int_0^x F(x') dx' = \Delta (mv^2/2) \\ &\Rightarrow F_0 \sin(cx)/c = mv^2/2 - mv_0^2/2 \\ \text{so } v &= \sqrt{v_0^2 + 2F_0 \sin(cx)/(mc)} \end{aligned}$$

The above example illustrates three points you should remember:

- The work-energy equations always leave the *sign* of the velocity unknown. You can see this because the derivation involves v^2 . You can also see it in formulas you get for velocity. They involve a square root, and thus, implicitly a $\pm$. Whether one, the other or both roots are relevant depends on reasoning that lies outside the energy equation itself.
- The work-energy equations can generate formulas that, in certain situations, are nonsense: If the initial speed v_0 is not high enough the particle will not get very far. In particular if $v_0^2 < 2F_0/(mc)$ the inside of the square root will be negative for some x and the “answer” will be imaginary. These are values of x that the particle will never reach.
- Here we have apparently solved for something about the motion of a particle. And we have, partially. But to find the $x(t)$ we would have to integrate again. And that next integral is hard. That is, energy balance lets us solve for some aspects of the motion, namely speed vs position, without ever needing to know in detail how position varies with time.

Conservation of energy

Many people leave high-school physics loving conservation of energy. It makes certain special homework problems easy. In the real world the principle is also useful for building intuition, and sometimes also for problem solving. In 1D particle mechanics energy conservation is a theorem^②.

^②Be forewarned, generally we cannot think of energy conservation as necessarily applicable nor, if applicable, as derivable from the equations of mechanics.

Recall that if a particle is acted on by a force that varies with position, $F = F(x)$, then we can define a potential energy $E_p = -\int F dx$ and that the work done by the force when the particle moves from x_1 to x_2 is

$$-(E_{p2} - E_{p1}) = -\Delta E_p.$$

That is, the decrease in E_p is the amount of work that the force does. Or, in other words, E_p represents a potential to do work. Because work causes an increase in kinetic energy, E_p is called the *potential energy* of the force field. Now we can compare this result with the work-energy

equation 10.16 to find that

$$-\Delta E_P = \Delta E_K \quad \Rightarrow \quad 0 = \underbrace{\Delta(E_P + E_K)}_{E_T}.$$

The *total energy* E_T doesn't change ($\Delta E_T = 0$) and thus is a constant. In other words,

as a particle moves in the presence of a force field with a potential energy, the total energy $E_T = E_K + E_P$ is constant.

This fact goes by the name of *conservation of energy*.

Example: Falling ball

Consider the ball in the free-body diagram 10.13. If we define gravitational potential energy as minus the work gravity does on a ball while it is lifted from the ground, then

$$E_P = - \int_0^y (-mg) dy' = mgy = mgh.$$

For vertical motion

$$E_K = \frac{1}{2} m \dot{y}^2.$$

So conservation of energy says that in free fall:

$$\text{Constant} = E_P + E_K = mgy + m\dot{y}^2/2$$

which you could also derive directly from $m\ddot{y} = -mg$.

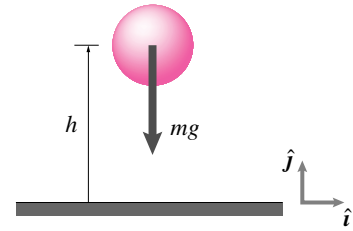


Figure 10.13: Free-body diagram of a falling ball, assuming gravity is the only significant external force acting on the ball.

Using conservation of energy to find equations of motion. On the one hand conservation of energy sometimes gives us a (partial) solution to a mechanics problem. On the other, we can use conservation of energy to find the “equations of motion”. The basic strategy is to take the derivative of the conservation of energy equation.

Example: Falling ball eqns. from energy.

$$\begin{aligned} E_T = \text{constant} \quad \Rightarrow \quad 0 &= \frac{d}{dt} E_T \\ &= \frac{d}{dt} (E_P + E_K) \\ &= \frac{d}{dt} (mgy + m\dot{y}^2/2) \\ &= (mg\dot{y} + m\dot{y}\ddot{y}) \\ \Rightarrow \quad m\ddot{y} &= -mg. \end{aligned}$$

We had to assume (and this is just a technical point) that $\dot{y} \neq 0$ in one of the cancellations. We have used energy balance to derive linear-momentum balance.

One can also find equations of motion starting with power balance.

$$P = \dot{E}_K$$

as derived here in detail for the case of gravity acting on a particle.

$$\begin{aligned}
 P &= \frac{d}{dt}(E_K) && \text{(Power balance)} \\
 \vec{F} \cdot \vec{v} &= \frac{d}{dt}(E_K) && \text{(Power of external force)} \\
 (-mg\hat{j}) \cdot (\dot{y}\hat{j}) &= \frac{d}{dt} \left[\frac{1}{2}mv^2 \right] && \text{(expanding terms)} \\
 -mg\dot{y} &= \frac{1}{2}m \frac{d}{dt}(\dot{y}^2) && \text{(evaluate dot product, substitute for } v) \\
 -mg\dot{y} &= \frac{1}{2}m(2\dot{y} \cdot \ddot{y}) && \text{(the chain rule)} \\
 \ddot{y} &= -g && \text{(cancel terms, switch sides),}
 \end{aligned} \tag{10.17}$$

The potential energy of a spring is $k(\Delta\ell)^2/2$. Besides near-earth gravity, which we already covered ($E_p = mgh$), the main elementary use of potential energy is for the stretch of a linear spring. Integrating $dW = Fdx$ for a linear spring with force on an object $F = -kx$, where x is the spring stretch, from the rest length, we get

$$E_p = - \int_0^x F(x') dx' = - \int_0^x -kx' dx' = \frac{1}{2}kx^2. \tag{10.18}$$

In the above example we measured x from the rest position of one end of the spring. But often the natural x coordinate will not be so nicely set up. It is safer to remember the spring's potential energy in terms of its stretch:

$$E_p = \frac{k \Delta\ell^2}{2}, \tag{10.19}$$

where we measure $\Delta\ell = \ell - \ell_0$ where ℓ_0 is the spring's rest length ($\ell_0 =$ length when the tension is zero).

Thus for a spring and mass oscillator, the subject of the next section, conservation of energy tells us that $mv^2/2 + kx^2/2 = \text{constant}$.

Is energy balance a principle or a calculation trick?

For one dimensional particle motion, momentum balance, power balance, and energy balance can each be derived from either of the others. If we take $F = ma$ as primary, energy calculations are just a convenience of notation or, in the case of the work-energy relation, a useful calculation technique (trick).

Historically, conservation of energy was first noted in particle mechanics problems. But because the position-dependent forces of springs and gravity seemed so fundamental, that they had a description as the derivative of a potential gave the energy relations the smell of something more fundamental. And so it has turned out that energy is an important topic for chemistry, thermodynamics, electrodynamics and sub-atomic physics. It's not just an analogy, it's the same energy. Thus energy is the primary

currency of exchange between, say, the superficially disparate chemical and mechanical systems.

The exchange of energy between these forms, in the context of particle mechanical models, can give the sense that we are doing the same 1D momentum based mechanics calculations when actually we are using more general energy balance equations, equations that cannot be derived from $F = ma$.

Terrestrial locomotion: Trains, cars, bicycles and animals

A free-body diagram of an accelerating car, treated as a 1D particle system, is shown in fig. 10.5 on page 484. The point represents the car, the force is the propulsion force from the wheel-ground interaction, and for now, we have neglected air friction. Without worrying about details we could say then that the power of the propulsion force is equal to the rate of change of kinetic energy.

Example: Accelerating car

An aggressive 1 ton car can accelerate with $a = 0.5g$ while going 60 mph. Neglecting friction and air resistance, the power of the propulsion force is

$$\begin{aligned}
 P &= Fv = mav = (1 \text{ ton})(0.5g)(60 \text{ mi/hr}) \\
 &= (1 \text{ ton})(0.5g)(60 \text{ mi/hr}) \underbrace{\left(\frac{2000 \text{ lbm}}{\text{ton}}\right)}_1 \underbrace{\left(\frac{5280 \text{ ft}}{\text{mi}}\right)}_1 \underbrace{\left(\frac{1 \text{ hr}}{3600 \text{ s}}\right)}_1 \underbrace{\left(\frac{1 \text{ lbf}}{g \cdot \text{lbm}}\right)}_1 \underbrace{\left(\frac{1 \text{ hp}}{550 \text{ ft} \cdot \text{lbf/s}}\right)}_1 \\
 &= (0.5 \cdot 60 \cdot 2000 \cdot 5280)/(3600 \cdot 550) \text{ hp} \approx 160 \text{ hp}
 \end{aligned}$$

Note the judicious multiple multiplications by 1 so that all units cancel but for horse-power; ton cancels ton, hr cancels hr, g cancels g and so on. The car engine needs to supply this 160 hp plus any internal transmission dissipation. And more still to cover the tire drag and air drag *etc.*

Such calculations are deceptively simple. Some apparent paradoxes:

- The propulsive force on the car comes from the interaction of the ground with the car. Are we saying that the (dead-as-a-doormat) ground supplies a power of, say, 160 hp to an accelerating car?
- The point of application of the force on the car is at the bottom of the tire. That point has no velocity. So the actual power of the ground force on the car (tire) is zero. How is that reconciled with, say, the 160 hp that we get from particle mechanics.

These are legitimate concerns which are discussed further in box 10.1 on page 12. The bottom line is that the calculation turns out, perhaps by the demands of dimensional consistency, to be useful and correct.

Drag power. The drag force of air on moving things has an effect on the energy balance. Air drag is important for cars, bicycles and animals that are moving quickly (say, running people). The air drag is proportional to

$$F_d = \frac{1}{2} \rho C_d A v^2$$

What are the proportionalities in the drag formula?

- The cross-sectional area (the area visible from directly in front) A . The bigger the area the more air has to be pushed out of the way. For a car $A \approx 2 \text{ m}^2$
- The density of air ρ . The more mass has to be pushed out of the way, the bigger the force. For rough calculations one can remember that the density of air is about one thousandth that of water $\rho_{air} \approx 1 \text{ kg/m}^3$. But the density varies in human environments from about 1.1 kg/m^3 in high-altitude (low pressure), high-temperature (gas expands when hot), humid (water vapor is lighter than air) environments up to about 1.4 kg/m^3 in low-lying cold dry places.
- The relative speed squared v^2 . The faster you are moving the more air per unit time you must displace, and each bit of air gets displaced with a bigger speed. Typical highway speeds are about $v = 30 \text{ m/s}$ ($\approx 67 \text{ mph}$) and a typical human walking speed is about $v = 1 \text{ m/s}$ (about 10% over $\approx 2 \text{ mph}$).
- A shape coefficient, sometimes called a drag coefficient C or C_d . Different shapes of the same size, can displace the air more or less as the vehicle passes through. Streamlined shapes have small C_d
- One half ($1/2$). Convention has a factor of $1/2$. This simplifies the power interpretation.

The drag power is

$$P = F_d v = \frac{1}{2} \rho C_d A v^3$$

which is a key result: increasing the speed 1% increases the power demand by 3% and doubling the speed multiplies the power demand by a factor of 8. This huge dependence of power on speed motivates smug energy-conservers to drive annoyingly slowly on highways.

Another way of writing the drag equation is

$$P = C_d \times (\text{The relative kinetic energy swept by the vehicle per unit time})$$

How's that? The volume "swept" by the vehicle per unit time is its area times speed vA . Here area A means the area seen by the wind (the area of a shadow of the object made by a light shining in the wind direction on to a plane whose normal is in the wind direction). The air mass swept per unit time is thus ρvA . The kinetic energy of the air, measured as moving relative to the vehicle, is $v^2/2$ per unit mass. Putting this together we get the swept kinetic energy of the air per unit time is $\rho A v^3/2$.

How big is the drag coefficient C_d ? When in doubt take dimensionless constants as 1 and you are usually not too far off. At one extreme is a flat plate whose normal is parallel to the wind direction. Such a plate has $C_d \approx 1.25$. On the other hand, a good airfoil has $C_d \approx 0.05$. People, animals, and bicyclists all have C_d close to 1.

Drag on cars. For the worst cars C_d is actually almost 1. For typical cars on the street $C_d \approx 0.35$. For the best high-efficiency cars on the market in 2007, $C_d \approx 0.25$. Real marketed cars may one day get drag as low as $C_d \approx 0.2$. And concept cars that are shaped like trout can have drag as low as $C_d \approx 0.1$.

The drag power of a 2 m^2 car going 30 m/s ($\approx 67 \text{ mph} = 108 \text{ km/hr}$) is about

$$\begin{aligned} P_{\text{drag}} &= \frac{1}{2} C_d \rho A v^3 \approx \frac{1}{2} \cdot 0.35 \cdot (1 \text{ kg/m}^3) \cdot (2 \text{ m}^2) \cdot (30 \text{ m/s})^3 \\ &= 9450 \text{ kg m}^2/\text{s}^3 = 9.45 \text{ KW} \approx 13 \text{ hp} \end{aligned}$$

That is, comparing with the example above, a car that needs 160 hp extra to make a zippy pass needs only 13 hp to move steadily along at a typical highway speed. For the units conversion we used $1 \text{ N} = 1 \text{ kg m/s}^2$, $1 \text{ J} = 1 \text{ N m}$, $1 \text{ W} = 1 \text{ J/s}$, and $1 \text{ KW} \approx 1.34 \text{ hp}$.

Caveat on the drag “law”. While there is some physics in the reasoning behind the drag law $F_d = \rho C_d A v^2/2$ the emphasis should be on the word “some”, the whole chaotic nature of turbulent flow is not captured. The quadratic drag law is an empirical fit. For a given shape the C_d actually depends on the surface texture. And for a given shape and texture the C_d depends on v , the v^2 doesn’t capture all of the velocity dependence. Nonetheless, the drag law is a reasonable approximation for most engineering purposes where drag is important.

Summary

There are two basic types of energy problems

- Problems where force or acceleration is given as a function of position ($a = a(x)$ or $F = F(x)$) and energy methods are basically a trick for finding $v(x)$.
- Problems where work, energy or power is of interest for its own sake because of, say, interest in engine power, dissipated energy, etc.

Of course the two problem types can also overlap.

10.1 Energetics of locomotion: using particle equations for non-particle systems

On page 9 we showed a naive locomotion power example in which we used

$$P = Fv$$

where v was the car velocity, F the thrust on the car, and P was 'the power' of the locomotion force. We pointed out two issues.

- How does it make sense for the passive ground, the source of the propulsive force, to supply power?
- The point of application of the ground force on the car is at the bottom of a wheel, a point that is not moving ($v = 0$). So how can Fv be other than zero?

The basic issue is that a car is not a particle, it has many moving parts and also some chemistry, so particle equations need to be interpreted with some care.

Particle equations are exact for non-particle systems

The most general form for linear momentum balance, as applied to a complicated system moving and deforming in complicated ways, reduces to equation $\vec{F} = m\vec{a}$. That is, so long as we interpret $\vec{F}$ to be the total force on the system, $\vec{a}$ to be the acceleration of the center of mass, and m to be the total mass of the system.

The power and energy equations in this chapter have been based on $\vec{F} = m\vec{a}$ (or their 1D scalar version $F = ma$) so apply to *any* system. But the terms P and E_K have meanings that go beyond particle mechanics. So while it is correct that (we derived it from $F = ma$),

$$Fv = \frac{d}{dt} \left(\frac{1}{2}mv^2 \right)$$

for non-particle systems it is *not correct* that Fv is the actual power of the force applied nor that $mv^2/2$ is the kinetic energy of the system.

To understand the situation depends on understanding multi-body systems where we will see that the power of a force is $\vec{F} \cdot \vec{v}_p$ where $\vec{v}_p$ is the velocity of the material point to which the force is applied; and the kinetic energy is larger than $\frac{1}{2}mv^2$ because of motion relative to the average motion. Remember to reconsider these issues when you know more.

More general energy balance equations

Without worrying about what we can derive from what, there is no doubt that for any closed system we can write the energy bal-

ance equation from the front inside cover of the book, the first law of thermodynamics, as:

$$\dot{Q} + P = \dot{E}_K + \dot{E}_P + \dot{E}_{\text{int}}$$

About the ever-shifting sign conventions, here we use $\dot{Q}$ as the heat flow *in* to the system, P is the power of external forces *on* the system, $\dot{E}_K$ and $\dot{E}_P$ are the rate of *increase* of the kinetic and potential energies of the system, and $\dot{E}_{\text{int}}$ is the rate of *increase* of internal energy. We can consider an accelerating car using this energy equation. For simplicity assume that no external forces do work on the car (the ground certainly does no work, and let's neglect air friction for now). We can also look at a car on level ground so there are no changes in gravitational potential energy. Finally, even though a car has many moving parts, the bulk of the material goes at the speed of a typical point on the body of the car. Thus the particle formula for kinetic energy is reasonably accurate. Putting this altogether we have

$$\begin{aligned} \dot{Q} + \underbrace{P}_0 &= \frac{d}{dt} \left(\frac{1}{2}mv^2 \right) + \underbrace{\dot{E}_P}_0 + \dot{E}_{\text{int}} \\ \Rightarrow \frac{d}{dt} \left(\frac{1}{2}mv^2 \right) &= -\dot{E}_{\text{int}} + \dot{Q} \end{aligned}$$

The rate of loss of chemical potential energy $-\dot{E}_{\text{int}}$ less the heat flow out $-\dot{Q}$ is what we call the power of the engine. Say chemical energy is being lost (used up) at a rate of $-\dot{E}_{\text{int}} = 40$ KW. Say the heat flow out the exhaust is $-\dot{Q} = 30$ KW. Then, with that 40 KW of fuel use and that 25% efficient engine, we would have

$$\frac{d}{dt} \left(\frac{1}{2}mv^2 \right) = -\dot{E}_{\text{int}} + \dot{Q} = 40 \text{ KW} - 30 \text{ KW} = 10 \text{ KW} \approx 13 \text{ hp.}$$

But even this is not quite right because it does not take account of the flow of gases in and out of the car. Things can be messy if you look carefully.

What's the bottom line? In the end, with some sloppiness of thought but not much inaccuracy, we are not far off thinking that the change of kinetic energy of the car has to come from some place. And that place is the work of the engine as supplied by the decrease in chemical potential energy of the fuel. When we write $P = \dot{E}_K$ for a car, the P in that equation is the force applied to the car times the velocity of the car. But that P is not the power supplied by the outside agents on the car (e.g., the passive ground). Rather it is the power of forces inside the car. Never mind that we're modeling a car as a particle with no internal structure, at least for momentum-balance purposes.

This whole situation can only be properly clarified when we look at the power of internal and external forces in multi-body systems.

SAMPLE 10.9 Which is the best bicycle helmet? Assume a bicyclist moves with speed 25 mph when her head hits a brick wall. Assume her head is rigid and that it has constant deceleration as it travels through the 2 inches of the bicycle helmet. What is the deceleration? What force is required? (Neglect force from the neck on the head.)

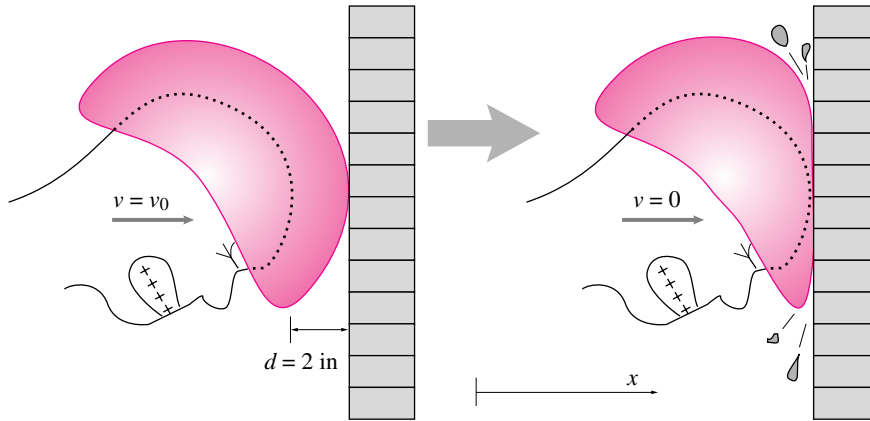


Figure 10.14:

Solution

Solution 1 – Kinematics method 1: We are given the initial speed of V_0 , a final speed of 0, and a constant acceleration a (which is negative) over a given distance of travel d . If we call t_c the time when the helmet is fully crushed,

$$\begin{aligned}
 v(t) &= v_0 + \int_0^{t_c} a(t') dt' = v_0 + at_c \\
 0 = v(t_c) &= v_0 + at_c \Rightarrow t_c = -v_0/a \quad (10.20) \\
 x(t) &= x_0 + \int_0^{t_c} v(t') dt' = 0 + \int_0^{t_c} (v_0 + at) dt \\
 d = x(t_c) &= 0 + v_0 t_c + at_c^2/2 \\
 d &= v_0 \left(\frac{-v_0}{a} \right) + a \left(\frac{v_0}{a} \right)^2 / 2 \Rightarrow d = \frac{-v_0^2}{2a} \quad (\text{using (10.20)}) \\
 \Rightarrow a &= \frac{-v_0^2}{2d} \\
 &= \frac{-(25 \text{ mph})^2}{2 \cdot (2 \text{ in})} \\
 &= \frac{-25^2}{4} \cdot \frac{\text{mi}^2}{\text{hr}^2 \cdot \text{in}} \cdot \underbrace{\left(\frac{5280 \text{ ft}}{\text{mi}} \right)^2}_1 \cdot \underbrace{\left(\frac{1 \text{ hr}}{3600 \text{ s}} \right)^2}_1 \cdot \underbrace{\left(\frac{12 \text{ in}}{\text{ft}} \right)}_1 \cdot \underbrace{\left(\frac{1 \text{ g}}{32.2 \text{ ft/s}^2} \right)}_1 \\
 &= \frac{-25}{4} \cdot \frac{5280^2}{3600^2} \cdot 12 \cdot \frac{1}{32.2} \text{ g} \\
 a &= -125 \text{ g}
 \end{aligned}$$

To stop from 25 mph in 2 inches requires an acceleration that is 125 times that of gravity.

Solution 2 – Kinematics method 2:

$$\begin{aligned}
\frac{dv}{dt} &= a \Rightarrow dv = a dt \\
\Rightarrow v dv &= a v dt \Rightarrow v dv = a \frac{dx}{dt} dt \\
\Rightarrow v dv &= a dx \\
\Rightarrow \int v dv &= \int a dx \\
\Rightarrow \Delta \frac{v^2}{2} &= ax \quad (\text{since } a = \text{constant}) \\
\Rightarrow 0 - \frac{v_0^2}{2} &= ad \Rightarrow a = \frac{-v_0^2}{2d} \quad (\text{as before})
\end{aligned}$$

Solution 3 – Quote formulas:

$$\begin{aligned}
v &= \sqrt{2ad\epsilon} \\
\Rightarrow a &= \frac{v^2}{2d} \quad \text{which is right if you know how to interpret it!}
\end{aligned}$$

Solution 4 – Work-Energy:

Constant acceleration $\Rightarrow$ constant force

$$\begin{aligned}
\text{Work in} &= \Delta E_K \\
-Fd &= 0 - \frac{mv_0^2}{2} \Rightarrow F = \frac{mv_0^2}{2d}
\end{aligned}$$

$$\text{But } \vec{F} = m\vec{a} \Rightarrow -F\hat{i} = ma\hat{i} \Rightarrow a = \frac{-F}{m}$$

$$\text{So } a = \frac{-v_0^2}{2d} \quad (\text{again})$$

Assuming a head mass of 8 lbm, the force on the head during impact is

$$|F| = \frac{mv_0^2}{2d} = ma = 8 \text{ lbm} \cdot 125g.$$

$$|F| = 1000 \text{ lbf}$$

During a collision in which an 8 lbm head decelerates from 25 mph to 0 in 2 inches, the force applied to the head is 1000 lbf.

Note 1: The way to minimize the peak acceleration when stopping from a given speed over a given distance is to have constant acceleration. The ‘best’ possible helmet, the one we assumed, causes constant deceleration. There is no helmet of any possible material with 2 in thickness that could make the deceleration for this collision less than 125g or the peak force less than 1000 lbf.

Note 2: Collisions with head decelerations of 250g or greater are often fatal. Even 125g usually causes brain injury. So, the best possible helmet does not insure against injury for fast riders hitting solid objects.

Note 3: Epidemiological evidence suggests that, on average, chances of serious brain injury are decreased by about a factor of 5 by wearing a helmet.

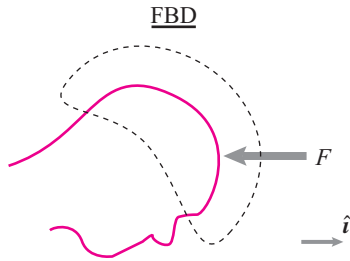


Figure 10.15: F is the force of the helmet on the moving head.

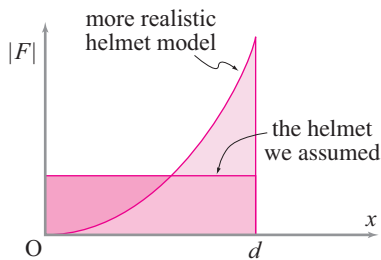


Figure 10.16

SAMPLE 10.10 Dissipated energy in viscous drag: A ball of mass $m = 1$ kg is dropped from rest from a height $h = 100$ m under gravity. The air resistance on the ball is modeled as viscous drag $F_s = cv$ where v is the speed of the ball and $c = 0.25$ kg/s is the drag coefficient. Find the energy dissipated in overcoming the air resistance during the entire flight of the ball.

Solution There are various ways in which we could calculate the energy dissipated in viscous drag. The most straightforward way is to compute the work done by the drag force on the body, $\int F_s dx$ during the entire flight. This calculation will be very easy if we knew the drag force as a function of position, that is, if we have $F_s(x)$. Unfortunately, we have $F_s \equiv F_s(v) = cv$ and we do not know v as a function of position. However, we can find the speed v as a function of time by solving the equation of motion $F = ma$ and determine the speed just before the ball hits the ground. Now, we can find the energy of the ball in two positions — just when it starts falling and just before it hits the ground. The difference between the two energies is what is lost or dissipated in overcoming the air resistance. Let 'A' denote position-1 from where the ball is dropped, i.e., $y_A = h$, and 'B' denote position-2, a hair above the ground, i.e., $y_B = 0$. Taking the ground as the datum for potential energy, we have,

$$\begin{aligned} E_A &= (E_K + E_P)_A = \frac{1}{2}m \overbrace{v_A^2}^0 + mg \overbrace{y_A}^h \\ &= mgh \\ E_B &= (E_K + E_P)_B = \frac{1}{2}m \overbrace{v_B^2}^? + mg \overbrace{y_B}^0 \\ &= \frac{1}{2}mv_B^2. \end{aligned}$$

Therefore, the energy dissipated in air resistance is

$$E_{\text{drag}} = \Delta E = E_A - E_B = mgh - \frac{1}{2}mv_B^2 \quad (10.21)$$

Now, we just need to find v_B . From the free-body diagram shown in fig. 10.17, we have,

$$\begin{aligned} m \overbrace{\ddot{y}}^v &= -mg - cv \\ \text{or } \frac{dv}{dt} &= -g - \frac{c}{m}v \\ \Rightarrow \int_0^{v(t)} \frac{dv}{\tilde{c}v + g} &= - \int_0^t d\tau \end{aligned}$$

where $\tilde{c} = \frac{c}{m}$. Thus,

$$\begin{aligned} \frac{1}{\tilde{c}} \ln(\tilde{c}v + g) \Big|_0^{v(t)} &= -t \\ \Rightarrow \ln \frac{\tilde{c}v(t) + g}{g} &= -\tilde{c}t \\ \Rightarrow v(t) &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t} - 1). \end{aligned} \quad (10.22)$$

So, we have solved for $v(t)$. Unfortunately, we cannot find v_B from this expression because we do not know what t is when the ball reaches the ground. Thus we need to first find t_B

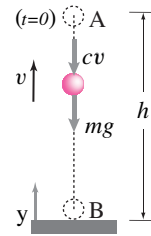


Figure 10.17: Free-body diagram of the falling ball. Note that the drag force $F_s = cv$ is shown acting downwards. This is because we have assumed v to be positive upwards and the drag force always acts in the opposite direction of the velocity.

and then substitute it in eqn. (10.22) to find v_B . From eqn. (10.22), we have,

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{g}{c}(e^{-\tilde{c}t} - 1) \\
 \Rightarrow \int_h^0 dy &= \frac{g}{c} \int_0^{t_B} (e^{-\tilde{c}t} - 1) dt \\
 \Rightarrow -h &= \frac{g}{c} \left(\frac{e^{-\tilde{c}t}}{-\tilde{c}} - t \right) \Big|_0^{t_f} \\
 &= \frac{g}{c} \left(-\frac{e^{-\tilde{c}t_f}}{\tilde{c}} - t_f + \frac{1}{\tilde{c}} \right) \\
 \text{or } \frac{\tilde{c}h}{g} &= \frac{1}{\tilde{c}}(e^{-\tilde{c}t_f} - 1) + t_f. \tag{10.23}
 \end{aligned}$$

① A transcendental equation in t is one where t appears both as an argument of a trigonometric or exponential function and elsewhere. Such equations can almost never be solved by hand in closed form.

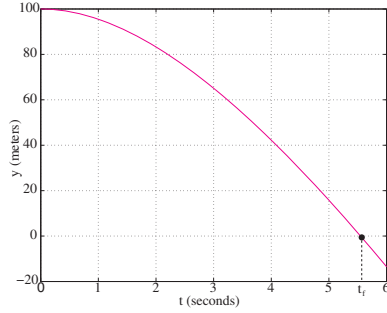


Figure 10.18: A graph of $y(t) = h - \frac{mg}{c} \left(\frac{m}{c} e^{-\frac{c}{m}t} + t - \frac{m}{c} \right)$ for determining t_f when $y = 0$. This is the same as solving eqn. (10.23) for t_f .

This turns out to be a transcendental equation ① with no simple solution for t_f . We can, however, solve it numerically (either using a computer program, or by trial and error). For the given values of m , c , and h , we solve eqn. (10.23) by trial and error (to locate zero crossing), and find that $t_f = 5.5495$ s (see fig. 10.18). Substituting $t = t_f$ in eqn. (10.22), we get

$$\begin{aligned}
 v_B &= \frac{mg}{c}(e^{-\frac{c}{m}t_f} - 1) \\
 &= \frac{1 \text{ kg} \cdot 9.81 \text{ m/s}^2}{0.25 \text{ kg/s}} (e^{-\frac{0.25 \text{ kg/s}}{1 \text{ kg}} \cdot 5.5495 \text{ s}} - 1) \\
 &= -29.44 \text{ m/s}.
 \end{aligned}$$

Note that v comes out to be negative, which is expected because we assumed v to be positive upwards. The velocity is clearly directed downwards once the ball starts falling. Now, substituting the values of m , g , h , and v_B in eqn. (10.21), we get

$$\begin{aligned}
 E_{\text{drag}} &= mgh - \frac{1}{2}mv_B^2 \\
 &= 1 \text{ kg} \cdot 9.81 \text{ m/s}^2 \cdot 100 \text{ m} - \frac{1}{2} \cdot 1 \text{ kg} \cdot (29.44 \text{ m/s})^2 \\
 &= 547.64 \text{ N}\cdot\text{m}
 \end{aligned}$$

Thus more than half of the initial energy is dissipated in air friction. If there were no viscous drag on the ball, its speed just before hitting the ground would be

$$v_B = \sqrt{2gh} = 44.29 \text{ m/s}.$$

$$E_{\text{drag}} = 547.64 \text{ N}\cdot\text{m}$$

SAMPLE 10.11 How much time does it take for a car of mass 800 kg to go from 0 mph to 60 mph, if we assume that the engine delivers a constant power P of 40 horsepower during this period. (1 horsepower = 745.7 W)

Solution

$$\begin{aligned} P &= \dot{W} \equiv \frac{dW}{dt} \\ dW &= P dt \\ W_{12} &= \int_{t_0}^{t_1} P dt = P(t_1 - t_0) = P \Delta t \\ \Delta t &= \frac{W_{12}}{P}. \end{aligned}$$

Now, from IIIa in the inside front cover,

$$\begin{aligned} W_{12} &= (E_K)_2 - (E_K)_1 \\ &= \frac{1}{2} m (v_2^2 - v_1^2) \\ &= \frac{800 \text{ kg} [(60 \text{ mph})^2 - 0]}{2} \\ &= \frac{1}{2} \cdot 800 \text{ kg} \left(60 \frac{\text{mi}}{\text{hr}} \cdot \frac{1.61 \times 10^3 \text{ m}}{1 \text{ mi}} \cdot \frac{1 \text{ hr}}{3600 \text{ s}} \right)^2 \\ &= 288.01 \times 10^3 \text{ kg} \cdot \text{m}^2 / \text{s}^2 \\ &= 288 \text{ KJoule}. \end{aligned}$$

Therefore,

$$\Delta t = \frac{288 \times 10^3 \text{ J}}{40 \times 745.7 \text{ W}} = 9.66 \text{ s}.$$

Thus it takes about 10 s to accelerate from a standstill to 60 mph.

$$\Delta t = 9.66 \text{ s}$$

Note 1: This model gives a roughly realistic answer *but* it is not a realistic model, at least at the start, at time t_0 . In the model here, the acceleration is infinite at the start (the power jumps from zero to a finite value at the start, when the velocity is zero), something the finite-friction tires would not allow.

Note 2: We have been a little sloppy in quoting the energy equation. Since there are no external forces doing work on the car, somewhat more properly we should perhaps have written

$$0 = \dot{E}_K + \dot{E}_{\text{int}} + \dot{E}_P$$

and set $-(\dot{E}_{\text{int}} + \dot{E}_P) = \text{'the engine power'}$ where the engine power is from the decrease in gasoline potential energy ($-\dot{E}_P$ is positive) less the increase in 'heat' ($\dot{E}_{\text{int}}$) from engine inefficiencies.

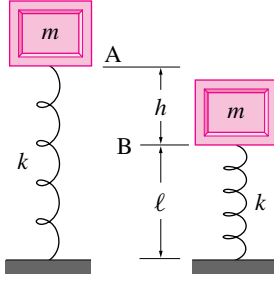


Figure 10.19

SAMPLE 10.12 Energy of a mass-spring system. A mass $m = 2 \text{ kg}$ is attached to a spring with spring constant $k = 2 \text{ kN/m}$. The relaxed (unstretched) length of the spring is $\ell = 40 \text{ cm}$. The mass is pulled up and released from rest at position A shown in Fig. 10.19. The mass falls by a distance $h = 10 \text{ cm}$ before reaching position B, which is the relaxed position of the spring. Find the speed at point B.

Solution The total energy of the mass-spring system at any instant or position consists of the energy stored in the spring and the sum of potential and kinetic energies of the mass. For potential energy of the mass, we need to select a datum where the potential energy is zero. We can select any horizontal plane to be the datum. Let the ground support level of the spring be the datum. Then, at position A,

$$\text{Energy in the spring} = \frac{1}{2}k(\text{stretch})^2 = \frac{1}{2}kh^2 \quad (\text{see eqn. (10.19), page 505})$$

$$\text{Energy of the mass} = E_K + E_P = \frac{1}{2}m \underbrace{v_A^2}_0 + mg(\ell + h) = mg(\ell + h).$$

Therefore, the total energy at position A

$$E_A = \frac{1}{2}kh^2 + mg(\ell + h).$$

Let the speed of the mass at position B be v_B . When the mass is at B, the spring is relaxed, i.e., there is no stretch in the spring. Therefore, at position B,

$$\text{Energy in the spring} = \frac{1}{2}k(\text{stretch})^2 = 0$$

$$\text{Energy of the mass} = E_K + E_P = \frac{1}{2}mv_B^2 + mg\ell,$$

and the total energy

$$E_B = \frac{1}{2}mv_B^2 + mg\ell.$$

Because the net change in the total energy of the system from position A to position B is

$$\begin{aligned} 0 &= \Delta E \\ &= E_A - E_B = \frac{1}{2}kh^2 + mg(\ell + h) - \frac{1}{2}mv_B^2 - mg\ell \\ &= \frac{1}{2}(kh^2 - mv_B^2) + mgh \\ \Rightarrow v_B^2 &= kh^2/m + 2gh \\ \Rightarrow |v_B| &= (kh^2/m + 2gh)^{1/2} \\ &= ((2000 \text{ N/m} \cdot (0.1 \text{ m})^2 / 2 \text{ kg}) + 2 \cdot 9.81 \text{ m/s}^2 \cdot 0.1 \text{ m})^{1/2} \\ &= 3.46 \text{ m/s}. \end{aligned}$$

$$|v_B| = 3.46 \text{ m/s}$$

10.2 Energy methods in 1D

Preparatory Problems

10.2.1 A mass m is at position x moving at velocity v and being acted upon by force F . For each of the quantities below:

- give the symbol used for the quantity
- describe the quantity in words
- give a formula to evaluate the quantity in terms of some or all of m, x, v and F and any other variables you may need.
- Give the standard units for the quantity in the SI system.
- Give the standard units for the quantity in the English system.

- Power
- Kinetic energy
- Work
- Potential energy

10.2.2 Write an equation relating the two words in each of these pairs. If any conditions or descriptions of the situation are needed, give them. If you know more than one equation (or form for a given equation), give all that you know. All should be given in the context of this section: 1D motion.

- work and power
- work and kinetic energy
- power and kinetic energy
- work and potential energy
- potential energy and kinetic energy

10.2.3 A force $F = F_0 \sin(ct)$ acts on a particle with mass $m = 3$ kg which has position $x = 3$ m, velocity $v = 5$ m/s at $t = 2$ s. $F_0 = 4$ N and $c = 2/s$. At $t = 2$ s evaluate (give numbers and units):

- a ,
- E_K ,
- P ,
- $\dot{E}_K$,
- the rate at which the force is doing work. *

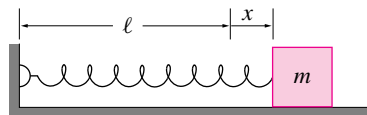
10.2.4 A force only depends on position according to $F = C_0 + C_1 x$ where C_0 and C_1 are constants. What is the work done by this force when the point to which it is applied moves from x_1 to x_2 ? Answer in terms of some or all of C_0, C_1, x_1 and x_2 .

10.2.5 Find the potential E_p associated with each of these force fields.

- $F = 0$.
- $F = F_0$ (=constant).
- $F = kx$.
- $F = A \sin(x/x_0)$.
- $F = c/x^2$.

10.2.6 Consider a spring-mass system with $m = 2$ kg and $k = 5$ N/m. The mass is pulled to the right a distance $x = x_0 = 0.5$ m from the unstretched position and released from rest. No external forces act on the mass.

- What are the initial potential and kinetic energy of the system?
- What is the potential and kinetic energy of the system as the mass passes through the static equilibrium (unstretched spring) position?
- What is the speed of the mass when it passes through the static equilibrium position?



Problem 10.2.6

10.2.7 A mass m is held in place by a spring whose restoring force is $T(x) = kx$. Derive the equation of motion of the system (that is, find the acceleration a in terms of x).

10.2.8 The peak propulsion force on a 4-wheel-drive car is about μmg where $\mu \approx 1$ for rubber on road (a bit more for fancy racing tires). Assume a car starts from rest at position zero. Answer the following questions with symbols and with numbers (using $\mu = 1$, $m = 1000$ kg, and $g = 10$ m/s²).

- What is the minimum distance required to reach $v_1 = 60$ mph?
- What is the extra distance required to get from $v_1 = 60$ mph up to $v_2 = 70$ mph?
- What is the peak power used by the engine in getting up to $v_1 = 60$ mph (assuming no dissipation and no air friction)?

10.2.9 A car (mass $m = 1000$ kg) traveling at speed $v_0 = 30$ m/s crashes into a brick wall and comes to a stop as the front end of the car compresses a distance $d = 1$ m. Answer with symbols and numbers. Assume constant deceleration during the crash. Neglect the mass of the crushing region of the car.

- What is the total energy dissipated in the crash?
- What is the force of the car on the wall?
- What is the force of the wall on the car?
- What is the deceleration of the car passengers (assuming they are strapped in and move with the bulk of the car). Answer in g's?
- Assuming an $m_p = 50$ kg person, what is the force of the seat belts on the person (answer in body weight).
- If a parent was holding a 15 kg child on his lap, what force would he need to hold on to the child through the crash (answer in N and in number of child body weights).

More-Involved Problems

10.2.10 A kid ($m = 90$ lbm) stands on a $h = 10$ ft wall and jumps down, accelerating with $g = 32$ ft/s². Upon hitting the ground with straight legs, she bends them so her body slows to a stop over a distance $d = 1$ ft. Neglect the mass of her legs. Assume constant deceleration as she brakes the fall.

- What is the total distance her body falls? *
- What is the potential energy lost?
- How much work must be absorbed by her legs?

- d) What is the force of her legs on her body? Answer in symbols, numbers and numbers of body weight (*i.e.*, find F/mg).

10.2.11 In traditional archery, when pulling an arrow back the force starts from 0 and increases approximately linearly up to the peak ‘draw force’ F_{draw} . The draw force varies from about $F_{draw} = 25$ lbf for a bow made for a small person to about $F_{draw} = 75$ lbf for a bow made for a big strong person. The distance the arrow is pulled back, the draw length ℓ_{draw} , varies from about $\ell_{draw} = 2$ ft for a small adult to about 30 inch for a big adult. An arrow has mass of about 300 grain (1 grain ≈ 64.8 milligram, so an arrow has mass of about $19.44 \approx 20$ gm $\approx 3/4$ ounce). Give all answers in symbols and numbers.

- What is the range of speeds you can expect an arrow to fly?
- What is the range of heights an arrow might go if shot straight up (it’s a bad approximation, but for this problem neglect air friction)?

10.2.12 A big person ($m = 100$ kg) jumps on a trampoline which we model as a linear spring with stiffness k . You know that the trampoline deflects $d_0 = 20$ cm under the stationary weight mg of the person (use $g = 10$ m/s²). Assume there is no dissipation and the person is jumping repeatedly a height $h = 1$ m above the unloaded surface of the trampoline. Give all answers with symbols and numbers.

- What is the stiffness k of the spring (answer in terms of some or all of m , g and d_0).
- What is the maximum deflection of the trampoline during these jumps?
- What is the peak force of the trampoline on the jumper? (answer in symbols, Newtons, and numbers of body weights).

10.2.13 For the car of problem 10.2.8 what is the average power required to reach speed v_1 ? There are two plausible ways to calculate this power:

$$\bar{P}_1 = \int_0^x P(x') dx'/x \quad \text{and}$$

$$\bar{P}_2 = \int_0^t P(t') dt'/t.$$

Use both. Do the two methods give the same answer? If so, why, and will the answers be the same for all problems? If not, why not, in what cases will the answers agree, and, when they differ, which one is right?

10.2.14 For problem 10.2.9 which answers would change, and in which way, if the deceleration was not exactly constant during the crash? That is, for which quantities would the answer be bigger, which smaller, which the same, for which would the answer depend on the nature of the non-constant acceleration?

10.2.15 The earth’s gravitational pull on a mass m is $F = -\frac{mgR^2}{r^2}$, where mg is the pull at the surface of the earth and R is the radius of the earth. Assume a ballistic rocket is shot straight up with a launch velocity of v_0 (measured in a ‘fixed’ not-rotating-with-the-earth frame). Assume the rocket goes in a straight radial line as the earth turns underneath it (relative to the surface of the earth this rocket would be launched somewhat to the West to cancel the earth’s rotation). Assume the period of active thrust is negligibly short (hence the word *ballistic*: “relating to or characteristic of the motion of objects moving under their own momentum and the force of gravity”).

- Solve for v as a function of r (and some or all of m , g , R and v_0).
- Find the maximum height the rocket reaches.
- Find the ‘escape velocity’ v_{escape} , the minimum launch speed needed for the rocket to never return.
- On one graph plot height (r or $r - R$) vs t for a v_0 just below v_{escape} and for v_0 just greater than v_{escape} . If you use numerical methods to make this plot use $g = 10$ m/s², $R = 6400$ km, and $m = 1$ kg. Make sure your axes are such that you can see a clear qualitative difference between the two cases.

10.2.16 The power available to a very strong accelerating cyclist over short periods of time (up to, say, about 1 minute)

is about 1 horsepower. Assume a rider starts from rest and uses this constant power. Assume a mass (bike + rider) of 150 lbm, a realistic drag force of $.006 \text{ lbf}/(\text{ft/s})^2 v^2$. Neglect other drag forces.

- What is the peak (steady state) speed of the cyclist?
- Using analytic or numerical methods make an accurate plot of speed vs. time.
- What is the acceleration as $t \rightarrow \infty$ in this solution?
- What is the acceleration as $t \rightarrow 0$ in your solution?
- How would you improve the model to fix the problem with the answer above?