SAMPLE 10.9 Which is the best bicycle helmet? Assume a bicyclist moves with speed 25 mph when her head hits a brick wall. Assume her head is rigid and that it has constant deceleration as it travels through the 2 inches of the bicycle helmet. What is the deceleration? What force is required? (Neglect force from the neck on the head.)

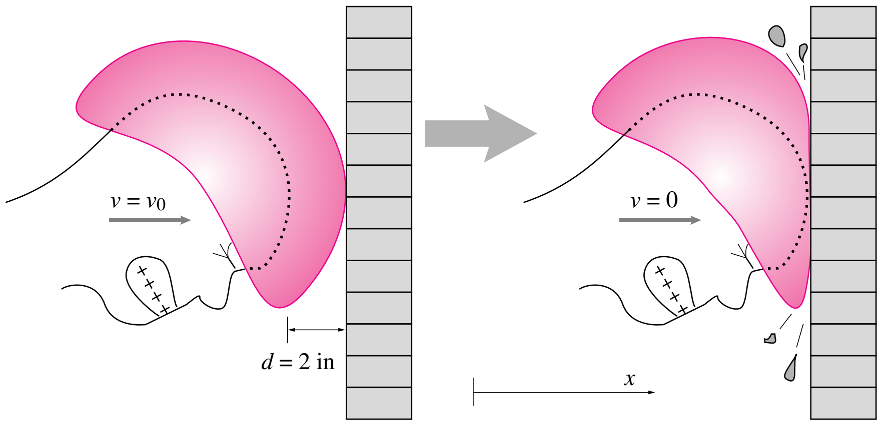

Figure 10.14:

Solution

Solution 1 – Kinematics method 1: We are given the initial speed of V_0, a final speed of 0, and a constant acceleration a (which is negative) over a given distance of travel d. If we call t_c the time when the helmet is fully crushed,

$$
\begin{aligned}
v(t) &= v_0 + \int_0^{t_c} a(t')dt' = v_0 + at_c \\
0 = v(t_c) &= v_0 + at_c \quad \Rightarrow \quad t_c = -v_0/a \qquad (10.20) \\
x(t) &= x_0 + \int_0^{t_c} v(t')dt' = 0 + \int_0^{t_c} (v_0 + at)dt \\
d = x(t_c) &= 0 + v_0 t_c + a t_c^2/2 \\
d &= v_0\left(\frac{-v_0}{a}\right) + a\left(\frac{v_0}{a}\right)^2/2 \quad \Rightarrow \quad d = \frac{-v_0^2}{2a} \qquad \text{(using (10.20))} \\
\Rightarrow \quad a &= \frac{-v_0^2}{2d} \\
&= \frac{-(25\,\text{mph})^2}{2\cdot(2\in)} \\
&= \frac{-25^2}{4}\cdot\frac{\text{mi}^2}{\text{hr}^2\cdot\in}\cdot\underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)^2}_{1}\cdot\underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)^2}_{1}\cdot\underbrace{\left(\frac{12\in}{\text{ft}}\right)}_{1}\cdot\underbrace{\left(\frac{1\text{g}}{32.2\,\text{ft/s}^2}\right)}_{1} \\
&= \frac{-25}{4}\cdot\frac{5280^2}{3600^2}\cdot 12\cdot\frac{1}{32.2}g \\
a &= -125g
\end{aligned}
$$

To stop from 25 mph in 2 inches requires an acceleration that is 125 times that of gravity.

Solution 2 – Kinematics method 2:

$$
\begin{aligned}
\frac{dv}{dt} &= a \quad\Rightarrow\quad dv = adt\\
\Rightarrow\quad vdv &= avdt \quad\Rightarrow\quad vdv = a\frac{dx}{dt}dt\\
\Rightarrow\quad vdv &= adx\\
\Rightarrow\quad \int vdv &= \int adx\\
\Rightarrow\quad \Delta\frac{v^2}{2} &= ax \qquad\qquad \text{(since } a = \text{constant)}\\
\Rightarrow\quad 0-\frac{v_0^2}{2} &= ad \quad\Rightarrow\quad a = \frac{-v_0^2}{2d} \qquad\qquad \text{(as before)}
\end{aligned}
$$

Solution 3 – Quote formulas:

$$
\begin{aligned}
\}\}v &= \sqrt{2ad\varepsilon}\\
\Rightarrow\quad a &= \frac{v^2}{2d} \qquad \text{which is right if you know how to interpret it!}
\end{aligned}
$$

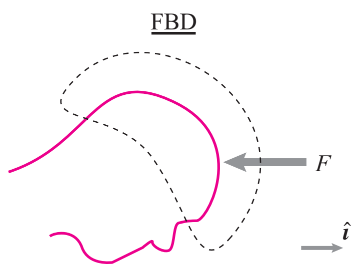

Figure 10.15: F is the force of the helmet on the moving head.

Solution 4 – Work-Energy:
Constant acceleration $\Rightarrow$ constant force

$$
\begin{aligned}
\text{Work in} &= \Delta E_{\mathrm{K}}\\
-Fd &= 0-\frac{mv_0^2}{2} \quad\Rightarrow\quad F = \frac{mv_0^2}{2d}\\
\text{But } \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \quad &\Rightarrow\quad -F\hat{\boldsymbol{\imath}} = ma\hat{\boldsymbol{\imath}} \Rightarrow a = \frac{-F}{m}\\
\text{So } a &= \frac{-v_0^2}{2d} \qquad\qquad \text{(again)}
\end{aligned}
$$

Assuming a head mass of 8 lbm, the force on the head during impact is

$$
|F| = \frac{mv_0^2}{2d} \quad = \quad ma \quad = \quad 8\,\text{lbm}\cdot 125g.
$$

$$\boxed{|F| = 1000\,\text{lbf}}$$

During a collision in which an 8 lbm head decelerates from 25 mph to 0 in 2 inches, the force applied to the head is 1000 lbf.

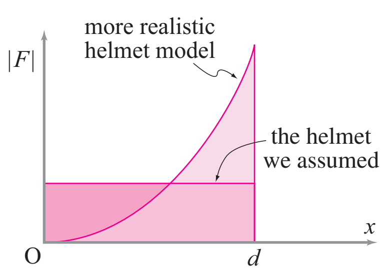

Figure 10.16

Note 1: The way to minimize the peak acceleration when stopping from a given speed over a given distance is to have constant acceleration. The 'best' possible helmet, the one we assumed, causes constant deceleration. There is no helmet of any possible material with 2 ∈ thickness that could make the deceleration for this collision less than 125g or the peak force less than 1000 lbf.

Note 2: Collisions with head decelerations of 250g or greater are often fatal. Even 125g usually causes brain injury. So, the best possible helmet does not insure against injury for fast riders hitting solid objects.

Note 3: Epidemiological evidence suggests that, on average, chances of serious brain injury are decreased by about a factor of 5 by wearing a helmet.

SAMPLE 10.10 Dissipated energy in viscous drag: A ball of mass $m = 1$ kg is dropped from rest from a height $h = 100$ m under gravity. The air resistance on the ball is modeled as viscous drag $F_s = cv$ where v is the speed of the ball and $c = 0.25$ kg/ s is the drag coefficient. Find the energy dissipated in overcoming the air resistance during the entire flight of the ball.

Solution There are various ways in which we could calculate the energy dissipated in viscous drag. The most straightforward way is to compute the work done by the drag force on the body, $\int F_s dx$ during the entire flight. This calculation will be very easy if we knew the drag force as a function of position, that is, if we have $F_s(x)$. Unfortunately, we have $F_s \equiv F_s(v) = cv$ and we do not know v as a function of position. However, we can find the speed v as a function of time by solving the equation of motion $F = ma$ and determine the speed just before the ball hits the ground. Now, we can find the energy of the ball in two positions — just when it starts falling and just before it hits the ground. The difference between the two energies is what is lost or dissipated in overcoming the air resistance. Let 'A' denote position-1 from where the ball is dropped, *i.e.*, $y_A = h$, and 'B' denote position-2, a hair above the ground, *i.e.*, $y_B = 0$. Taking the ground as the datum for potential energy, we have,

$$\begin{aligned}
E_A &= (E_K + E_P)_A = \frac{1}{2}m\overbrace{v_A^2}^{0} + mg\overbrace{y_A}^{h} \\
&= mgh \\
E_B &= (E_K + E_P)_B = \frac{1}{2}m\overbrace{v_B^2}^{?} + mg\overbrace{y_B}^{0} \\
&= \frac{1}{2}mv_B^2.
\end{aligned}$$

Therefore, the energy dissipated in air resistance is

$$E_{\text{drag}} = \Delta E = E_A - E_B = mgh - \frac{1}{2}mv_B^2 \tag{10.21}$$

Now, we just need to find v_B. From the free-body diagram shown in fig. 10.17, we have,

$$\begin{aligned}
m\overbrace{\ddot{y}}^{\dot{v}} &= -mg - cv \\
\text{or} \quad \frac{dv}{dt} &= -g - \frac{c}{m}v \\
\Rightarrow \quad \int_0^{v(t)} \frac{dv}{\tilde{c}v + g} &= -\int_0^t d\tau
\end{aligned}$$

where $\tilde{c} = \frac{c}{m}$. Thus,

$$\begin{aligned}
\frac{1}{\tilde{c}} \ln(\tilde{c}v + g)\Big|_0^{v(t)} &= -t \\
\Rightarrow \quad \ln\frac{\tilde{c}v(t) + g}{g} &= -\tilde{c}t \\
\Rightarrow \quad v(t) &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t} - 1).
\end{aligned} \tag{10.22}$$

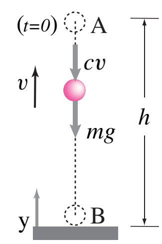

Figure 10.17: Free-body diagram of the falling ball. Note that the drag force $F_s = cv$ is shown acting downwards. This is because we have assumed v to be positive upwards and the drag force always acts in the opposite direction of the velocity.

So, we have solved for $v(t)$. Unfortunately, we cannot find v_B from this expression because we do not know what t is when the ball reaches the ground. Thus we need to first find t_B

and then substitute it in eqn. (10.22) to find v_B. From eqn. (10.22), we have,

$$
\begin{aligned}
\frac{dy}{dt} &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t}-1) \\
\Rightarrow \quad \int_h^0 dy &= \frac{g}{\tilde{c}}\int_0^{t_B}(e^{-\tilde{c}t}-1)dt \\
\Rightarrow \quad -h &= \frac{g}{\tilde{c}}\left(\frac{e^{-\tilde{c}t}}{-\tilde{c}}-t\right)\Bigg|_0^{t_f} \\
&= \frac{g}{\tilde{c}}\left(-\frac{e^{-\tilde{c}t_f}}{\tilde{c}}-t_f+\frac{1}{\tilde{c}}\right) \\
\text{or} \quad \frac{\tilde{c}h}{g} &= \frac{1}{\tilde{c}}(e^{-\tilde{c}t_f}-1)+t_f. \qquad (10.23)
\end{aligned}
$$

This turns out to be a transcendental equation①with no simple solution for t_f. We can, however, solve it numerically (either using a computer program, or by trial and error). For the given values of m, c, and h, we solve eqn. (10.23) by trial and error (to locate zero crossing), and find that $t_f = 5.5495\,\mathrm{s}$ (see fig. 10.18). Substituting $t = t_f$ in eqn. (10.22), we get

①A transcendental equation in t is one where t appears both as an argument of a trigonometric or exponential function and elsewhere. Such equations can almost never be solved by hand in closed form.

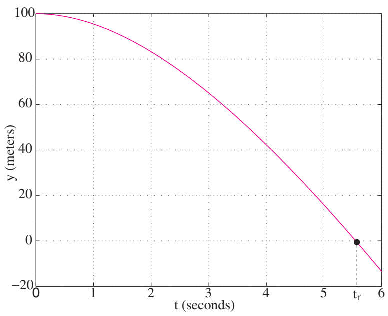

Figure 10.18: A graph of $y(t) = h - \frac{mg}{c}\left(\frac{m}{c}e^{-\frac{c}{m}t}+t-\frac{m}{c}\right)$ for determining t_f when $y = 0$. This is the same as solving eqn. (10.23) for t_f.

$$
\begin{aligned}
v_B &= \frac{mg}{c}(e^{-\frac{c}{m}t_f}-1) \\
&= \frac{1\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}}{0.25\,\mathrm{kg/\,s}}(e^{-\frac{0.25\,\mathrm{kg/\,s}}{1\,\mathrm{kg}}\cdot 5.5495\,\mathrm{s}}-1) \\
&= -29.44\,\mathrm{m/s}.
\end{aligned}
$$

Note that v comes out to be negative, which is expected because we assumed v to be positive upwards. The velocity is clearly directed downwards once the ball starts falling. Now, substituting the values of m, g, h, and v_B in eqn. (10.21), we get

$$
\begin{aligned}
E_{\mathrm{drag}} &= mgh - \frac{1}{2}mv_B^2 \\
&= 1\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot 100\,\mathrm{m} - \frac{1}{2}\cdot 1\,\mathrm{kg}\cdot(29.44\,\mathrm{m/s})^2 \\
&= 547.64\,\mathrm{N{\cdot}m}
\end{aligned}
$$

Thus more than half of the initial energy is dissipated in air friction. If there were no viscous drag on the ball, its speed just before hitting the ground would be

$$v_B = \sqrt{2gh} = 44.29\,\mathrm{m/s}.$$

$$\boxed{E_{\mathrm{drag}} = 547.64\,\mathrm{N{\cdot}m}}$$

SAMPLE 10.11 How much time does it take for a car of mass 800 kg to go from 0 mph to 60 mph, if we assume that the engine delivers a constant power P of 40 horsepower during this period. (1 horsepower = 745.7 W)

Solution

$$\begin{aligned} P &= \dot{W} \equiv \frac{dW}{dt} \\ dW &= P dt \\ W_{12} &= \int_{t_0}^{t_1} P dt = P(t_1 - t_0) = P\Delta t \\ \Delta t &= \frac{W_{12}}{P}. \end{aligned}$$

Now, from IIIa in the inside front cover,

$$\begin{aligned} W_{12} &= (E_K)_2 - (E_K)_1 \\ &= \frac{1}{2} m(v_2^2 - v_1^2) \\ &= \frac{800\,\text{kg}[(60\,\text{mph})^2 - 0]}{2} \\ &= \frac{1}{2} \cdot 800\,\text{kg} \left(60 \frac{\text{mi}}{\text{hr}} \cdot \frac{1.61 \times 10^3\,\text{m}}{1\,\text{mi}} \cdot \frac{1\,\text{hr}}{3600\,\text{s}}\right)^2 \\ &= 288.01 \times 10^3\ \text{kg} \cdot\ \text{m} \cdot\ \text{m/s}^2 \\ &= 288\ \text{KJoule}. \end{aligned}$$

Therefore,

$$\Delta t = \frac{288 \times 10^3\,\text{J}}{40 \times 745.7\,\text{W}} = 9.66\,\text{s}.$$

Thus it takes about 10 s to accelerate from a standstill to 60 mph.

$$\boxed{\Delta t = 9.66\,\text{s}}$$

Note 1: This model gives a roughly realistic answer *but* it is not a realistic model, at least at the start, at time t_0. In the model here, the acceleration is infinite at the start (the power jumps from zero to a finite value at the start, when the velocity is zero), something the finite-friction tires would not allow.

Note 2: We have been a little sloppy in quoting the energy equation. Since there are no external forces doing work on the car, somewhat more properly we should perhaps have written

$$0 = \dot{E}_K + \dot{E}_{int} + \dot{E}_P$$

and set $-(\dot{E}_{int} + \dot{E}_P)$ = 'the engine power' where the engine power is from the decrease in gasoline potential energy ($-\dot{E}_P$ is positive) less the increase in 'heat' ($\dot{E}_{int}$) from engine inefficiencies.

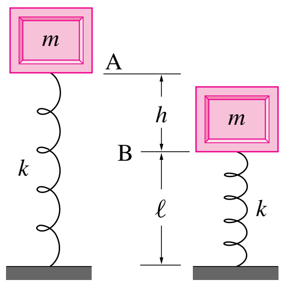

Figure 10.19

SAMPLE 10.12 **Energy of a mass-spring system.** A mass $m = 2\,\text{kg}$ is attached to a spring with spring constant $k = 2\,\text{kN/m}$. The relaxed (unstretched) length of the spring is $\ell = 40\,\text{cm}$. The mass is pulled up and released from rest at position A shown in Fig. 10.19. The mass falls by a distance $h = 10\,\text{cm}$ before reaching position B, which is the relaxed position of the spring. Find the speed at point B.

Solution The total energy of the mass-spring system at any instant or position consists of the energy stored in the spring and the sum of potential and kinetic energies of the mass. For potential energy of the mass, we need to select a datum where the potential energy is zero. We can select any horizontal plane to be the datum. Let the ground support level of the spring be the datum. Then, at position A,

$$\begin{aligned} \text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = \frac{1}{2}kh^2 \quad \text{(see eqn. (10.19), page 505)} \\ \text{Energy of the mass} &= E_K + E_P = \frac{1}{2}m\underbrace{v_A^2}_{0} + mg(\ell+h) = mg(\ell+h). \end{aligned}$$

Therefore, the total energy at position A

$$E_A = \frac{1}{2}kh^2 + mg(\ell+h).$$

Let the speed of the mass at position B be v_B. When the mass is at B, the spring is relaxed, i.e., there is no stretch in the spring. Therefore, at position B,

$$\begin{aligned} \text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = 0 \\ \text{Energy of the mass} &= E_K + E_P = \frac{1}{2}mv_B^2 + mg\ell, \end{aligned}$$

and the total energy

$$E_B = \frac{1}{2}mv_B^2 + mg\ell.$$

Because the net change in the total energy of the system from position A to position B is

$$\begin{aligned} 0 &= \Delta E \\ &= E_A - E_B = \frac{1}{2}kh^2 + mg(\ell+h) - \frac{1}{2}mv_B^2 - mg\ell \\ &= \frac{1}{2}(kh^2 - mv_B^2) + mgh \\ \Rightarrow \quad v_B^2 &= kh^2/m + 2gh \\ \Rightarrow \quad |v_B| &= \left(kh^2/m + 2gh\right)^{1/2} \\ &= \left((2000\,\text{N/m}\cdot(0.1\,\text{m})^2/2\,\text{kg}) + 2\cdot 9.81\,\text{m/s}^2\cdot 0.1\,\text{m}\right)^{1/2} \\ &= 3.46\,\text{m/s}. \end{aligned}$$

$$\boxed{|v_B| = 3.46\,\text{m/s}}$$