10.2 Energy methods in 1D

Energy is an important concept in science and engineering. Energy is also a kind of currency in human trade. Energy is also a concept that is somewhat bigger than can be defined inside classical mechanics, when we look at, say, the chemical energy cost of various mechanical tasks.

For a student learning mechanics, energy is first a method, or trick, for solving some simple problems of the type assigned in elementary courses like this one. As problems become more difficult (have more degrees of freedom or include, say, more-than-just-constant friction) energy becomes less useful as a problem solving technique. However, in more advanced mechanics, energy gets a central role again: energy is the central concept in some advanced ways to write equations of motion and for some methods of understanding stability.

Power, work, kinetic energy and potential energy

Before we get to the facts and theorems, we start with some definitions. Here are four words. We will use these definitions, or generalizations of them, throughout dynamics.

Power. The power of a force F is its product with the velocity v of the point on which it is acting,

$$\text{Power} = P = Fv.$$

This is the 1D version of the more general $P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}$ which we will use once we go on to 2D and 3D dynamics. In full generality, power is a scalar (not a vector). The common units for power are watts ($1\,\text{W} = \text{N m/s} = \text{J/s}$), kilowatts ($1\text{KW} = 10^3\ \text{W}$), lbf ft/s (no special name) and horsepower ($1\,\text{hp} \equiv 550\,\text{lbf ft/s} \approx 745.7 \approx 746$) ① watts).

① In fourteen hundred and ninety-two,
Columbus sailed the ocean blue,
And that's twice the watts
in a horsepower too.

$$\begin{array}{r} 746 \\ \underline{\times 2} \\ 1492 \end{array}$$

Example:
A 5 N force acting on a particle moving 3 m/s has a power of
$P = Fv = (5\,\text{N})(3\,\text{m/s}) = 15\,\text{N m/s} = 15\,\text{W} \approx 0.02\,\text{hp}$.

Work. The work W of a force is most easily defined incrementally (ΔW) for small motions Δx of a particle; motions so small that variations in force can be neglected and the force viewed as constant,

$$\text{increment of work} = \Delta W = F\Delta x.$$

This is a special 1D reduction of the more general $\Delta W = \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{r}}$. Even in 2D and 3D work is a scalar. Often we want to know the work for larger (non-infinitesimal) displacements. We do this by adding up the increments. Using sloppy calculus (implicitly taking the limit of a Riemann sum):

$W = \sum \Delta W = \int dW = \int F\,dx$, which we can write more definitely as

$$\text{Work} = W = \int_{x_0}^{x} F(x')\,dx'.$$

which is the 1D version of the more general $W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$. Common units of work are Joules ($1\,\mathrm{J} = 1\,\mathrm{N\,m} = 1\,\mathrm{kg\,m^2/\,s^2}$), foot-pounds ($= 1\,\mathrm{ft\,lbf}$) and kilo-watt hours ($= 3.6 \cdot 10^6\,\mathrm{J}$).

Example:
The force $F = F_0 \sin(cx)$ pushes a mass from $x_0 = 0\,\mathrm{m}$ to $x_1 = \pi\,\mathrm{m}$ where $F_0 = 7\,\mathrm{N}$ and $c = 1/\,\mathrm{m}$. Then

$$W = \int_{x_0}^{x_1} F\,dx = \int_0^{\pi\,\mathrm{m}} F_0 \sin(cx)\,dx = -(F_0/c)\cos(cx)|_0^{\pi\,\mathrm{m}} = 14\,\mathrm{N{\cdot}m}$$

Kinetic energy. The kinetic energy quantifies the motion a little differently than momentum does. In kinetic energy high speed gets extra credit (v^2 instead of just v). Further, for kinetic energy we don't worry about which way a particle moves. The kinetic energy E_K of a particle in 1D is

$$\text{kinetic energy} = E_\mathrm{K} = \frac{1}{2}mv^2$$

In two and three dimensions the formula above applies for one particle (taking $v = |\vec{\boldsymbol{v}}|$). For a collection of particles E_K is defined as a sum of E_K for each particle separately. In full generality kinetic energy is a scalar. The units of work (force×distance) and of kinetic energy (mass×speed^2) are the same (mass×distance^2/ time^2) and so are the common measures, namely Joules, foot-pounds and kilo-watt hours.

Example:
A 3 kg mass moving at a speed of 4 m/s has a kinetic energy of $E_\mathrm{K} = mv^2/2 = (3\,\mathrm{kg})(4\,\mathrm{m/s})^2/2 = 24\,\mathrm{kg\,m^2/\,s^2} = 24\,\mathrm{J}$.

Potential energy. This is the most abstract of the definitions. The potential energy E_P associated with a force F is defined as that function of x with these properties

$$E_\mathrm{P}(x) = -\int_{x_0}^{x} F(x')\,dx' \qquad \text{and} \qquad F(x) = -\frac{d}{dx}E_\mathrm{P}(x)$$

which people write more indefinitely as $E_\mathrm{P} = -\int F\,dx$ and $F = -E_\mathrm{P}{}'$. In two and three dimensions the concept of potential energy is more subtle still, being defined by a path integral which may or may not be sensible. But it is still a scalar.

Example:
The force $F = c/x^2$ is associated with the potential energy
$E_\mathrm{P} = -\int F\,dx = c/x + C_0$.

The datum for potential energy. Potential energy always has an undetermined, and generally irrelevant, integration constant. The integration constant is irrelevant because usually we care about *changes* in energy. So in the example above we could set $C_0 = 0$ and write $E_\mathrm{P} = -\int F\,dx = c/x$. In general we define the *datum* for potential energy as that position where we set the potential energy to zero.

- For near-earth gravity the datum is usually set at the height of the ground (so that $E_\mathrm{P} = mgh$), a launch point, or of a conspicuous physical point (say the hinge of a pendulum).
- For inverse-square gravity the datum is usually set at $r = \infty$ so that formulas are most simple.
- For springs that datum is usually set at the position where the spring is 'relaxed' (un-stretched and at its rest-length), again simplifying the terms in energy equations.

Potential energy is a shortcut for calculating work. From the definition of potential energy we can calculate work of a force in moving a particle from one place to another as:

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{\mathrm{P}_2} - E_{\mathrm{P}_1}\right).$$

Of course you need to know, or find, $E_\mathrm{P}(x)$ first in order to use this shortcut.

Example:
The work of $F = c/x^2$ in moving a mass from x_1 to x_2 is

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{\mathrm{P}_2} - E_{\mathrm{P}_1}\right) = c/x_1 - c/x_2$$

Where we used that $E_\mathrm{P} = c/x$ has the needed property that $F = -\dfrac{d}{dx}E_\mathrm{P}$.

Why all this new language? All of the words above are defined in terms of position, velocity and force. So anything we say about power, work and kinetic and potential energies we could say already using x, v and F. More particularly, we already have two ways of quantifying the motion of a particle, v and $L = mv$. Why do we need a third, $E_\mathrm{K} = mv^2/2$? The answer is this, to simplify the solution of some problems. Various facts and theorems are simpler if commonly appearing groups of terms are given names. And all of the definitions above are common groups.

Then, luckily, some of them turn out to be more general than just 1D particle mechanics.

The new vocabulary makes thinking easier. Various so-called 'one degree of freedom' problems can be solved by noting that energy is conserved. And features of solutions of more-complex problems can be extracted or checked by making sure that energy balance comes out right.

Power and work

The simplest relation between the quantities we have defined above is that between Power and work:

$$W = \int F\,dx = \int Fv\frac{dx}{v} = \int Fv\,dt$$

or more definitely

$$W = \int_{x_0}^{x_1} F\,dx = \int_{t_0}^{t_1} Fv\,dt = \int_{t_0}^{t_1} P\,dt$$

Example: Integrate power to get work.
If the power of a force acting on a particle is $P = P_0(ct^2)$ where $P_0 = 10\,\mathrm{W}$ and $c = 3/\,\mathrm{s}^2$ then over 3 seconds the work done by the force is:

$$\begin{aligned} W &= \int_{t_0}^{t_1} P\,dt = \int_{t_0}^{t_1} P_0(ct^2)\,dt = P_0 ct^3/3\Big|_{t_0}^{t_1} = (10\,\mathrm{W})(3/\,\mathrm{s}^2)t^3/3\Big|_0^{3\,\mathrm{s}} \\ &= 270\,\mathrm{W\,s} = 270\,\mathrm{J} \end{aligned}$$

Power and rate-of-change of kinetic energy

On the inside cover the third basic law of mechanics is energy balance. Energy balance takes a number of different forms, depending on context. The power balance equation from the front cover and simplified for a particle is

$$P = \dot{E}_{\mathrm{K}},$$

where, recall, $P = Fv$ is the power of the applied force F. The derivation of this result from $F = ma$ for a particle is simple enough, and is good to know. First note the following result from using the chain of differentiation:

$$\frac{d}{dt}\left(v^2\right) = 2v\frac{dv}{dt} = 2v\dot{v} = 2va.$$

When we need to call on this simple kinematics (calculus) result it usually comes to us the other way around. So what you should remember is this formula, one of the basic tricks of the trade:

$$va = \frac{d}{dt}\left(\frac{v^2}{2}\right).$$

Multiplying both sides by m and substituting in $F = ma$ we get our 1D power balance equation:

$$Fv = \frac{d}{dt}\left(\frac{mv^2}{2}\right).$$

The power of a given force depends on the speed of the object to which it is applied. When a finite non-zero force is applied to a stationary object the power of the force is zero and so is the rate of change of kinetic energy. If the object accelerates, its speed is increasing, but when the speed is zero, $\dot{E}_\mathrm{K} = 0$.

Example:
A constant force F is applied to an initially stationary mass m starting at $t = 0$. Then $v = Ft/m$, $E_\mathrm{K} = mv^2/2 = F^2t^2/(2m)$ and $P = Fv = F^2t/m$. Note that $\dot{E}_\mathrm{K} = P$ and both are zero at $t = 0$.

Work is change of kinetic energy

Integrating the power balance equation in time we get

$$\int P\,dt = \int \dot{E}_\mathrm{K}\,dt = \Delta E_\mathrm{K} \tag{10.16}$$

More definitely, and also using the work integral, we have that the work of the net force on a particle is the change of its kinetic energy:

$$\int_{t_1}^{t_2} \underbrace{Fv}_{P}\,dt = \int_{x_1}^{x_2} F\,dx = E_{\mathrm{K}_2} - E_{\mathrm{K}_1}$$

Once we remember that

work is change in kinetic energy,

we can use it without deriving it every time from $F = ma$ or from more general energy balance equations.

Example:
A force applied to a particle m varies sinusoidally with position according to $F = F_0 \cos(cx)$. At $x = 0$ the particle has speed $v = v_0$. Then

$$
\begin{aligned}
W = \Delta E_{\mathrm{K}} \quad &\Rightarrow \quad \int_0^x F(x')\,dx' = \Delta\left(mv^2/2\right) \\
&\Rightarrow \quad F_0 \sin(cx)/c = mv^2/2 - mv_0^2/2 \\
\text{so} \quad & v = \sqrt{v_0^2 + 2F_0\sin(cx)/(mc)}
\end{aligned}
$$

The above example illustrates three points you should remember:

- The work-energy equations always leave the *sign* of the velocity unknown. You can see this because the derivation involves v^2. You can also see it in formulas you get for velocity. They involve a square root, and thus, implicitly a $\pm$. Whether one, the other or both roots are relevant depends on reasoning that lies outside the energy equation itself.
- The work-energy equations can generate formulas that, in certain situations, are nonsense: If the initial speed v_0 is not high enough the particle will not get very far. In particular if $v_0^2 < 2F_0/(mc)$ the inside of the square root will be negative for some x and the "answer" will be imaginary. These are values of x that the particle will never reach.
- Here we have apparently solved for something about the motion of a particle. And we have, partially. But to find the $x(t)$ we would have to integrate again. And that next integral is hard. That is, energy balance lets us solve for some aspects of the motion, namely speed vs position, without ever needing to know in detail how position varies with time.

Conservation of energy

Many people leave high-school physics loving conservation of energy. It makes certain special homework problems easy. In the real world the principle is also useful for building intuition, and sometimes also for problem solving. In 1D particle mechanics energy conservation is a theorem②.

②Be forewarned, generally we cannot think of energy conservation as necessarily applicable nor, if applicable, as derivable from the equations of mechanics.

Recall that if a particle is acted on by a force that varies with position, $F = F(x)$, then we can define a potential energy $E_{\mathrm{P}} = -\int F\,dx$ and that the work done by the force when the particle moves from x_1 to x_2 is

$$-(E_{\mathrm{P}_2} - E_{\mathrm{P}_1}) = -\Delta E_{\mathrm{P}}.$$

That is, the decrease in E_{P} is the amount of work that the force does. Or, in other words, E_{P} represents a potential to do work. Because work causes an increase in kinetic energy, E_{P} is called the *potential energy* of the force field. Now we can compare this result with the work-energy equation 10.16 to find that

$$-\Delta E_\mathrm{P} = \Delta E_\mathrm{K} \qquad \Rightarrow \qquad 0 = \Delta \underbrace{(E_\mathrm{P} + E_\mathrm{K})}_{E_\mathrm{T}}.$$

The *total energy* E_T doesn't change ($\Delta E_\mathrm{T} = 0$) and thus is a constant. In other words,

> as a particle moves in the presence of a force field with a potential energy, the total energy $E_\mathrm{T} = E_\mathrm{K} + E_\mathrm{P}$ is constant.

This fact goes by the name of *conservation of energy*.

Example: Falling ball

Consider the ball in the free-body diagram 10.13. If we define gravitational potential energy as minus the work gravity does on a ball while it is lifted from the ground, then

$$E_\mathrm{P} = -\int_0^y (-mg)\, dy' = mgy = mgh.$$

For vertical motion

$$E_\mathrm{K} = \frac{1}{2} m\dot{y}^2.$$

So conservation of energy says that in free fall:

$$\text{Constant} = E_\mathrm{P} + E_\mathrm{K} = mgy + m\dot{y}^2/2$$

which you could also derive directly from $m\ddot{y} = -mg$.

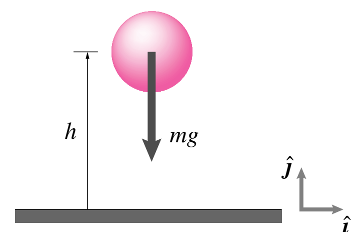


Figure 10.13: Free-body diagram of a falling ball, assuming gravity is the only significant external force acting on the ball.

Using conservation of energy to find equations of motion. On the one hand conservation of energy sometimes gives us a (partial) solution to a mechanics problem. On the other, we can use conservation of energy to find the "equations of motion". The basic strategy is to take the derivative of the conservation of energy equation.

Example: Falling ball eqns. from energy.

$$\begin{aligned} E_\mathrm{T} = \text{constant} \quad \Rightarrow \quad 0 &= \tfrac{d}{dt} E_\mathrm{T} \\ &= \tfrac{d}{dt}(E_\mathrm{P} + E_\mathrm{K}) \\ &= \tfrac{d}{dt}(mgy + m\dot{y}^2/2) \\ &= (mg\dot{y} + m\dot{y}\ddot{y}) \\ \Rightarrow \quad m\ddot{y} &= -mg. \end{aligned}$$

We had to assume (and this is just a technical point) that $\dot{y} \neq 0$ in one of the cancellations. We have used energy balance to derive linear-momentum balance.

One can also find equations of motion starting with power balance.

$$P = \dot{E}_\mathrm{K}$$

as derived here in detail for the case of gravity acting on a particle.

$$
\begin{aligned}
P &= \frac{d}{dt}\left(E_{\mathrm{K}}\right) && \text{(Power balance)}\\
\vec{\boldsymbol{F}}\cdot\vec{\boldsymbol{v}} &= \frac{d}{dt}\left(E_{\mathrm{K}}\right) && \text{(Power of external force)}\\
(-mg\hat{\boldsymbol{\jmath}})\cdot(\dot{y}\hat{\boldsymbol{\jmath}}) &= \frac{d}{dt}\left[\frac{1}{2}mv^2\right] && \text{(expanding terms)}\\
-mg\dot{y} &= \frac{1}{2}m\frac{d}{dt}\left(\dot{y}^2\right) && \text{(evaluate dot product, substitute for } v\text{)}\\
-mg\dot{y} &= \frac{1}{2}m\left(2\dot{y}\cdot\ddot{y}\right) && \text{(the chain rule)}\\
\ddot{y} &= -g && \text{(cancel terms, switch sides),}
\end{aligned}
\tag{10.17}
$$

The potential energy of a spring is $k(\Delta\ell)^2/2$. Besides near-earth gravity, which we already covered ($E_{\mathrm{P}} = mgh$), the main elementary use of potential energy is for the stretch of a linear spring. Integrating $dW = F dx$ for a linear spring with force on an object $F = -kx$, where x is the spring stretch, from the rest length, we get

$$E_{\mathrm{P}} = -\int_0^x F(x')\,dx' = -\int_0^x -kx'\,dx' = \frac{1}{2}kx^2. \tag{10.18}$$

In the above example we measured x from the rest position of one end of the spring. But often the natural x coordinate will not be so nicely set up. It is safer to remember the spring's potential energy in terms of its stretch:

$$E_{\mathrm{P}} = \frac{k\,\Delta\ell^2}{2}, \tag{10.19}$$

where we measure $\Delta\ell = \ell - \ell_0$ where ℓ_0 is the spring's rest length ($\ell_0 =$ length when the tension is zero).

Thus for a spring and mass oscillator, the subject of the next section, conservation of energy tells us that $mv^2/2 + kx^2/2 = constant$.

Is energy balance a principle or a calculation trick?

For one dimensional particle motion, momentum balance, power balance, and energy balance can each be derived from either of the others. If we take $F = ma$ as primary, energy calculations are just a convenience of notation or, in the case of the work-energy relation, a useful calculation technique (trick).

Historically, conservation of energy was first noted in particle mechanics problems. But because the position-dependent forces of springs and gravity seemed so fundamental, that they had a description as the derivative of a potential gave the energy relations the smell of something more fundamental. And so it has turned out that energy is an important topic for chemistry, thermodynamics, electrodynamics and sub-atomic physics. It's not just an analogy, it's the same energy. Thus energy is the primary currency of exchange between, say, the superficially disparate chemical and mechanical systems.

The exchange of energy between these forms, in the context of particle mechanical models, can give the sense that we are doing the same 1D momentum based mechanics calculations when actually we are using more general energy balance equations, equations that cannot be derived from $F = ma$.

Terrestrial locomotion: Trains, cars, bicycles and animals

A free-body diagram of an accelerating car, treated as a 1D particle system, is shown in fig. 10.5 on page 484. The point represents the car, the force is the propulsion force from the wheel-ground interaction, and for now, we have neglected air friction. Without worrying about details we could say then that the power of the propulsion force is equal to the rate of change of kinetic energy.

Example: Accelerating car
An aggressive 1 ton car can accelerate with $a = 0.5g$ while going 60 mph. Neglecting friction and air resistance, the power of the propulsion force is

$$\begin{aligned} P &= Fv = mav = (1\,\text{ton})(0.5g)(60\,\text{mi/hr}) \\ &= (1\,\text{ton})(0.5g)(60\,\text{mi/hr})\underbrace{\left(\frac{2000\,\text{lbm}}{\text{ton}}\right)}_{1}\underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}_{1}\underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)}_{1}\underbrace{\left(\frac{1\,\text{lbf}}{g\cdot\text{lbm}}\right)}_{1}\underbrace{\left(\frac{1\,\text{hp}}{550\,\text{ft}\cdot\text{lbf/s}}\right)}_{1} \\ &= (0.5\cdot 60\cdot 2000\cdot 5280)/(3600\cdot 550)\,\text{hp} \approx 160\,\text{hp} \end{aligned}$$

Note the judicious multiple multiplications by 1 so that all units cancel but for horse-power; ton cancels ton, hr cancels hr, g cancels g and so on. The car engine needs to supply this 160 hp plus any internal transmission dissipation. And more still to cover the tire drag and air drag *etc.*

Such calculations are deceptively simple. Some apparent paradoxes:

- The propulsive force on the car comes from the interaction of the ground with the car. Are we saying that the (dead-as-a-doormat) ground supplies a power of, say, 160 hp to an accelerating car?
- The point of application of the force on the car is at the bottom of the tire. That point has no velocity. So the actual power of the ground force on the car (tire) is zero. How is that reconciled with, say, the 160 hp that we get from particle mechanics.

These are legitimate concerns which are discussed further in box 10.1 on page 12. The bottom line is that the calculation turns out, perhaps by the demands of dimensional consistency, to be useful and correct.

Drag power. The drag force of air on moving things has an effect on the energy balance. Air drag is important for cars, bicycles and animals that are moving quickly (say, running people). The air drag is proportional to

$$F_d = \frac{1}{2}\rho C_d A v^2$$

What are the proportionalities in the drag formula?

- The cross-sectional area (the area visible from directly in front) A. The bigger the area the more air has to be pushed out of the way. For a car $A \approx 2\,\mathrm{m}^2$
- The density of air ρ. The more mass has to be pushed out of the way, the bigger the force. For rough calculations one can remember that the density of air is about one thousandth that of water $\rho_{air} \approx 1\,\mathrm{kg/\,m^3}$. But the density varies in human environments from about $1.1\,\mathrm{kg/\,m^3}$ in high-altitude (low pressure), high-temperature (gas expands when hot), humid (water vapor is lighter than air) environments up to about $1.4\,\mathrm{kg/\,m^3}$ in low-lying cold dry places.
- The relative speed squared v^2. The faster you are moving the more air per unit time you must displace, and each bit of air gets displaced with a bigger speed. Typical highway speeds are about $v = 30\,\mathrm{m/\,s}$ ($\approx 67\,\mathrm{mph}$) and a typical human walking speed is about $v = 1\,\mathrm{m/s}$ (about 10% over $\approx 2\,\mathrm{mph}$).
- A shape coefficient, sometimes called a <u>d</u>rag coefficient C or C_d. Different shapes of the same size, can displace the air more or less as the vehicle passes through. Streamlined shapes have small C_d
- One half (1/2). Convention has a factor of 1/2. This simplifies the power interpretation.

The drag power is

$$P = F_d v = \frac{1}{2}\rho C A v^3$$

which is a key result: increasing the speed 1% increases the power demand by 3% and doubling the speed multiplies the power demand by a factor of 8. This huge dependence of power on speed motivates smug energy-conservers to drive annoyingly slowly on highways.

Another way of writing the drag equation is

$$P = C_d \times (\text{ The relative kinetic energy swept by the vehicle per unit time})$$

How's that? The volume "swept" by the vehicle per unit time is its area times speed vA. Here area A means the area seen by the wind (the area of a shadow of the object made by a light shining in the wind direction on to a plane whose normal is in the wind direction). The air mass swept per unit time is thus $\rho v A$. The kinetic energy of the air, measured as moving relative to the vehicle, is $v^2/2$ per unit mass. Putting this together we get the swept kinetic energy of the air per unit time is $\rho A v^3/2$.

How big is the drag coefficient C_d? When in doubt take dimensionless constants as 1 and you are usually not too far off. At one extreme is a flat plate whose normal is parallel to the wind direction. Such a plate has $C_d \approx 1.25$. On the other hand, a good airfoil has $C_d \approx 0.05$. People, animals, and bicyclists all have C_d close to 1.

Drag on cars. For the worst cars C_d is actually almost 1. For typical cars on the street $C_d \approx 0.35$. For the best high-efficiency cars on the market in 2007, $C_d \approx 0.25$. Real marketed cars may one day get drag as low as $C_d \approx 0.2$. And concept cars that are shaped like trout can have drag as low as $C_d \approx 0.1$.

The drag power of a $2\,\mathrm{m}^2$ car going $30\,\mathrm{m/s}$ ($\approx 67\,\mathrm{mph} = 108\,\mathrm{km/hr}$) is about

$$\begin{aligned} P_{drag} &= \frac{1}{2}C\rho A v^3 \approx \frac{1}{2}\cdot 0.35\cdot(1\,\mathrm{kg/m^3})\cdot(2\,\mathrm{m^2})\cdot(30\,\mathrm{m/s})^3 \\ &= 9450\,\mathrm{kg\,m^2/s^3} = 9.45\,\mathrm{KW} \approx 13\,\mathrm{hp} \end{aligned}$$

That is, comparing with the example above, a car that needs 160 hp extra to make a zippy pass needs only 13 hp to move steadily along at a typical highway speed. For the units conversion we used $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$, $1\,\mathrm{J} = 1\,\mathrm{N\,m}$, $1\,\mathrm{W} = 1\,\mathrm{J/s}$, and $1\,\mathrm{KW} \approx 1.34\,\mathrm{hp}$.

Caveat on the drag “law”. While there is some physics in the reasoning behind the drag law $F_d = \rho C A v^2/2$ the emphasis should be on the word “some”, the whole chaotic nature of turbulent flow is not captured. The quadratic drag law is an empirical fit. For a given shape the C_d actually depends on the surface texture. And for a given shape and texture the C_d depends on v, the v^2 doesn’t capture all of the velocity dependence. Nonetheless, the drag law is a reasonable approximation for most engineering purposes where drag is important.

Summary

There are two basic types of energy problems

- Problems where force or acceleration is given as a function of position ($a = a(x)$ or $F = F(x)$) and energy methods are basically a trick for finding $v(x)$.
- Problems where work, energy or power is of interest for its own sake because of, say, interest in engine power, dissipated energy, etc.

Of course the two problem types can also overlap.

10.1 Energetics of locomotion: using particle equations for non-particle systems

On page 9 we showed a naive locomotion power example in which we used

$$P = Fv$$

where v was the car velocity, F the thrust on the car, and P was 'the power' of the locomotion force. We pointed out two issues.

- How does it make sense for the passive ground, the source of the propulsive force, to supply power?
- The point of application of the ground force on the car is at the bottom of a wheel, a point that is not moving ($v = 0$). So how can Fv be other than zero?

The basic issue is that a car is not a particle, it has many moving parts and also some chemistry, so particle equations need to be interpreted with some care.

Particle equations are exact for non-particle systems

The most general form for linear momentum balance, as applied to a complicated system moving and deforming in complicated ways, reduces to equation $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. That is, so long as we interpret $\vec{\boldsymbol{F}}$ to be the total force on the system, $\vec{\boldsymbol{a}}$ to be the acceleration of the center of mass, and m to be the total mass of the system.

The power and energy equations in this chapter have been based on $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ (or their 1D scalar version $F = ma$) so apply to *any* system. But the terms P and E_K have meanings that go beyond particle mechanics. So while it is correct that (we derived it from $F = ma$),

$$Fv = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

> for non-particle systems it is *not correct* that Fv is the actual power of the force applied nor that $mv^2/2$ is the kinetic energy of the system.

To understand the situation depends on understanding multi-body systems where we will see that the power of a force is $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_P$ where $\vec{\boldsymbol{v}}_P$ is the velocity of the material point to which the force is applied; and the kinetic energy is larger than $\frac{1}{2}mv^2$ because of motion relative to the average motion. Remember to reconsider these issues when you know more.

More general energy balance equations

Without worrying about what we can derive from what, there is no doubt that for any closed system we can write the energy balance equation from the front inside cover of the book, the first law of thermodynamics, as:

$$\dot{Q} + P = \dot{E}_\mathrm{K} + \dot{E}_\mathrm{P} + \dot{E}_\mathrm{int}.$$

About the ever-shifting sign conventions, here we use $\dot{Q}$ as the heat flow *in* to the system, P is the power of external forces *on* the system, $\dot{E}_\mathrm{K}$ and $\dot{E}_\mathrm{P}$ are the rate of *increase* of the kinetic and potential energies of the system, and $\dot{E}_\mathrm{int}$ is the rate of *increase* of internal energy. We can consider an accelerating car using this energy equation. For simplicity assume that no external forces do work on the car (the ground certainly does no work, and let's neglect air friction for now). We can also look at a car on level ground so there are no changes in gravitational potential energy. Finally, even though a car has many moving parts, the bulk of the material goes at the speed of a typical point on the body of the car. Thus the particle formula for kinetic energy is reasonably accurate. Putting this altogether we have

$$\begin{aligned}\dot{Q} + \underbrace{P}_{0} &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \underbrace{E_\mathrm{P}}_{0} + \dot{E}_\mathrm{int}\\ \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2\right) &= -\dot{E}_\mathrm{int} + \dot{Q}\end{aligned}$$

The rate of loss of chemical potential energy $-\dot{E}_\mathrm{int}$ less the heat flow out $-\dot{Q}$ is what we call the power of the engine. Say chemical energy is being lost (used up) at a rate of $-\dot{E}_\mathrm{int}$=40 KW. Say the heat flow out the exhaust is $-\dot{Q} = 30$ KW. Then, with that 40 KW of fuel use and that 25% efficient engine, we would have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\dot{E}_\mathrm{int} + \dot{Q} = 40\,\mathrm{KW} - 30\,\mathrm{KW} = 10\,\mathrm{KW} \approx 13\,\mathrm{hp}.$$

But even this is not quite right because it does not take account of the flow of gases in and out of the car. Things can be messy if you look carefully.

What's the bottom line? In the end, with some sloppiness of thought but not much inaccuracy, we are not far off thinking that the change of kinetic energy of the car has to come from some place. And that place is the work of the engine as supplied by the decrease in chemical potential energy of the fuel. When we write $P = \dot{E}_\mathrm{K}$ for a car, the P in that equation is the force applied to the car times the velocity of the car. But that P is not the power supplied by the outside agents on the car (e.g., the passive ground). Rather it is the power of forces inside the car. Never mind that we're modeling a car as a particle with no internal structure, at least for momentum-balance purposes.

This whole situation can only be properly clarified when we look at the power of internal and external forces in multi-body systems.

SAMPLE 10.9 Which is the best bicycle helmet? Assume a bicyclist moves with speed 25 mph when her head hits a brick wall. Assume her head is rigid and that it has constant deceleration as it travels through the 2 inches of the bicycle helmet. What is the deceleration? What force is required? (Neglect force from the neck on the head.)

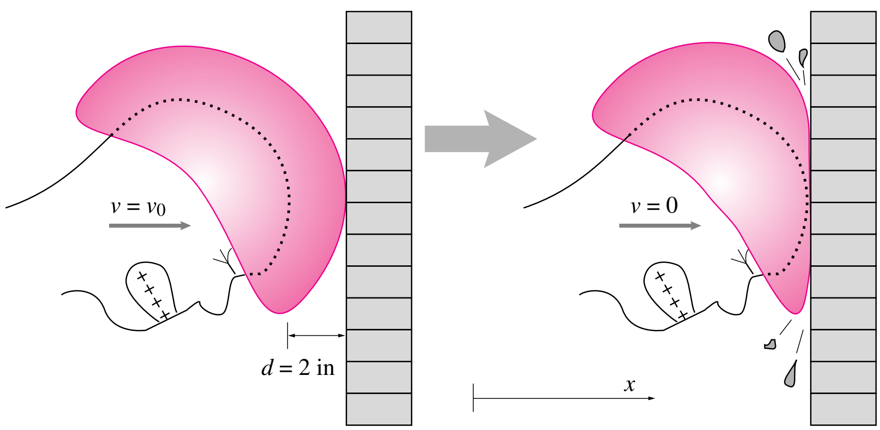


Figure 10.14:

Solution

Solution 1 – Kinematics method 1: We are given the initial speed of V_0, a final speed of 0, and a constant acceleration a (which is negative) over a given distance of travel d. If we call t_c the time when the helmet is fully crushed,

$$
\begin{aligned}
v(t) &= v_0 + \int_0^{t_c} a(t')dt' = v_0 + at_c \\
0 = v(t_c) &= v_0 + at_c \quad \Rightarrow \quad t_c = -v_0/a \qquad (10.20)\\
x(t) &= x_0 + \int_0^{t_c} v(t')dt' = 0 + \int_0^{t_c} (v_0 + at)dt \\
d = x(t_c) &= 0 + v_0 t_c + at_c^2/2 \\
d &= v_0\left(\frac{-v_0}{a}\right) + a\left(\frac{v_0}{a}\right)^2/2 \quad \Rightarrow \quad d = \frac{-v_0^2}{2a} \qquad \text{(using (10.20))} \\
\Rightarrow \quad a &= \frac{-v_0^2}{2d} \\
&= \frac{-(25\,\mathrm{mph})^2}{2\cdot(2\,\in)} \\
&= \frac{-25^2}{4}\cdot\frac{\mathrm{mi}^2}{\mathrm{hr}^2\cdot\in}\cdot\underbrace{\left(\frac{5280\,\mathrm{ft}}{\mathrm{mi}}\right)^2}_{1}\cdot\underbrace{\left(\frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right)^2}_{1}\cdot\underbrace{\left(\frac{12\,\in}{\mathrm{ft}}\right)}_{1}\cdot\underbrace{\left(\frac{1\mathrm{g}}{32.2\,\mathrm{ft/s^2}}\right)}_{1} \\
&= \frac{-25}{4}\cdot\frac{5280^2}{3600^2}\cdot 12\cdot\frac{1}{32.2}g \\
a &= -125g
\end{aligned}
$$

To stop from 25 mph in 2 inches requires an acceleration that is 125 times that of gravity.

Solution 2 – Kinematics method 2:

$$\begin{aligned}
\frac{dv}{dt} &= a \quad \Rightarrow \quad dv = adt \\
\Rightarrow \quad vdv &= avdt \quad \Rightarrow \quad vdv = a\frac{dx}{dt}dt \\
\Rightarrow \quad vdv &= adx \\
\Rightarrow \quad \int vdv &= \int adx \\
\Rightarrow \quad \Delta\frac{v^2}{2} &= ax \qquad \text{(since } a = \text{constant)} \\
\Rightarrow \quad 0 - \frac{v_0^2}{2} &= ad \quad \Rightarrow \quad a = \frac{-v_0^2}{2d} \qquad \text{(as before)}
\end{aligned}$$

Solution 3 – Quote formulas:

$$\begin{aligned}
\}\}v &= \sqrt{2ad\varepsilon} \\
\Rightarrow \quad a &= \frac{v^2}{2d} \qquad \text{which is right if you know how to interpret it!}
\end{aligned}$$

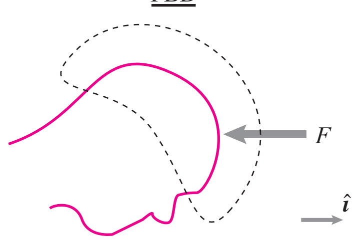


Figure 10.15: F is the force of the helmet on the moving head.

Solution 4 – Work-Energy:
Constant acceleration $\Rightarrow$ constant force

$$\begin{aligned}
\text{Work in} &= \Delta E_{\mathrm{K}} \\
-Fd &= 0 - \frac{mv_0^2}{2} \quad \Rightarrow \quad F = \frac{mv_0^2}{2d} \\
\text{But } \vec{F} = m\vec{a} \quad &\Rightarrow \quad -F\hat{\imath} = ma\hat{\imath} \Rightarrow a = \frac{-F}{m} \\
\text{So } a &= \frac{-v_0^2}{2d} \qquad \text{(again)}
\end{aligned}$$

Assuming a head mass of 8 lbm, the force on the head during impact is

$$|F| = \frac{mv_0^2}{2d} = ma = 8\,\text{lbm} \cdot 125g.$$

$$\boxed{|F| = 1000\,\text{lbf}}$$

During a collision in which an 8 lbm head decelerates from 25 mph to 0 in 2 inches, the force applied to the head is 1000 lbf.

Note 1: The way to minimize the peak acceleration when stopping from a given speed over a given distance is to have constant acceleration. The ‘best’ possible helmet, the one we assumed, causes constant deceleration. There is no helmet of any possible material with 2 ∈ thickness that could make the deceleration for this collision less than 125g or the peak force less than 1000 lbf.

Note 2: Collisions with head decelerations of 250g or greater are often fatal. Even 125g usually causes brain injury. So, the best possible helmet does not insure against injury for fast riders hitting solid objects.

Note 3: Epidemiological evidence suggests that, on average, chances of serious brain injury are decreased by about a factor of 5 by wearing a helmet.

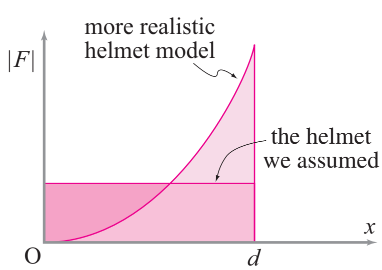


Figure 10.16

SAMPLE 10.10 Dissipated energy in viscous drag: A ball of mass $m = 1\,\text{kg}$ is dropped from rest from a height $h = 100\,\text{m}$ under gravity. The air resistance on the ball is modeled as viscous drag $F_s = cv$ where v is the speed of the ball and $c = 0.25\,\text{kg/ s}$ is the drag coefficient. Find the energy dissipated in overcoming the air resistance during the entire flight of the ball.

Solution There are various ways in which we could calculate the energy dissipated in viscous drag. The most straightforward way is to compute the work done by the drag force on the body, $\int F_s dx$ during the entire flight. This calculation will be very easy if we knew the drag force as a function of position, that is, if we have $F_s(x)$. Unfortunately, we have $F_s \equiv F_s(v) = cv$ and we do not know v as a function of position. However, we can find the speed v as a function of time by solving the equation of motion $F = ma$ and determine the speed just before the ball hits the ground. Now, we can find the energy of the ball in two positions — just when it starts falling and just before it hits the ground. The difference between the two energies is what is lost or dissipated in overcoming the air resistance. Let 'A' denote position-1 from where the ball is dropped, *i.e.*, $y_A = h$, and 'B' denote position-2, a hair above the ground, *i.e.*, $y_B = 0$. Taking the ground as the datum for potential energy, we have,

$$
\begin{aligned}
E_A &= (E_K + E_P)_A = \frac{1}{2} m \overbrace{v_A^2}^{0} + mg \overbrace{y_A}^{h} \\
&= mgh \\
E_B &= (E_K + E_P)_B = \frac{1}{2} m \overbrace{v_B^2}^{?} + mg \overbrace{y_B}^{0} \\
&= \frac{1}{2} m v_B^2 .
\end{aligned}
$$

Therefore, the energy dissipated in air resistance is

$$
E_{\text{drag}} = \Delta E = E_A - E_B = mgh - \frac{1}{2} m v_B^2 \tag{10.21}
$$

Now, we just need to find v_B. From the free-body diagram shown in fig. 10.17, we have,

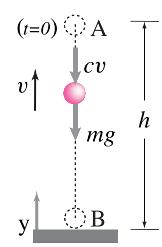


Figure 10.17: Free-body diagram of the falling ball. Note that the drag force $F_s = cv$ is shown acting downwards. This is because we have assumed v to be positive upwards and the drag force always acts in the opposite direction of the velocity.

$$
\begin{aligned}
m \overbrace{\ddot{y}}^{\dot{v}} &= -mg - cv \\
\text{or} \quad \frac{dv}{dt} &= -g - \frac{c}{m} v \\
\Rightarrow \quad \int_0^{v(t)} \frac{dv}{\tilde{c}v + g} &= -\int_0^t d\tau
\end{aligned}
$$

where $\tilde{c} = \frac{c}{m}$. Thus,

$$
\begin{aligned}
\frac{1}{\tilde{c}} \ln(\tilde{c}v + g)\Big|_0^{v(t)} &= -t \\
\Rightarrow \quad \ln \frac{\tilde{c}v(t) + g}{g} &= -\tilde{c}t \\
\Rightarrow \quad v(t) &= \frac{g}{\tilde{c}} (e^{-\tilde{c}t} - 1).
\end{aligned} \tag{10.22}
$$

So, we have solved for $v(t)$. Unfortunately, we cannot find v_B from this expression because we do not know what t is when the ball reaches the ground. Thus we need to first find t_B

and then substitute it in eqn. (10.22) to find v_B. From eqn. (10.22), we have,

$$\begin{aligned}
\frac{dy}{dt} &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t} - 1)\\
\Rightarrow \quad \int_h^0 dy &= \frac{g}{\tilde{c}}\int_0^{t_B}(e^{-\tilde{c}t} - 1)dt\\
\Rightarrow \quad -h &= \frac{g}{\tilde{c}}\left(\frac{e^{-\tilde{c}t}}{-\tilde{c}} - t\right)\Bigg|_0^{t_f}\\
&= \frac{g}{\tilde{c}}\left(-\frac{e^{-\tilde{c}t_f}}{\tilde{c}} - t_f + \frac{1}{\tilde{c}}\right)\\
\text{or} \quad \frac{\tilde{c}h}{g} &= \frac{1}{\tilde{c}}(e^{-\tilde{c}t_f} - 1) + t_f.
\end{aligned} \tag{10.23}$$

This turns out to be a transcendental equation (1) with no simple solution for t_f. We can, however, solve it numerically (either using a computer program, or by trial and error). For the given values of m, c, and h, we solve eqn. (10.23) by trial and error (to locate zero crossing), and find that $t_f = 5.5495\,\mathrm{s}$ (see fig. 10.18). Substituting $t = t_f$ in eqn. (10.22), we get

$$\begin{aligned}
v_B &= \frac{mg}{c}(e^{-\frac{c}{m}t_f} - 1)\\
&= \frac{1\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}}{0.25\,\mathrm{kg/\,s}}(e^{-\frac{0.25\,\mathrm{kg/\,s}}{1\,\mathrm{kg}}\cdot 5.5495\,\mathrm{s}} - 1)\\
&= -29.44\,\mathrm{m/s}.
\end{aligned}$$

Note that v comes out to be negative, which is expected because we assumed v to be positive upwards. The velocity is clearly directed downwards once the ball starts falling. Now, substituting the values of m, g, h, and v_B in eqn. (10.21), we get

$$\begin{aligned}
E_{\mathrm{drag}} &= mgh - \frac{1}{2}mv_B^2\\
&= 1\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot 100\,\mathrm{m} - \frac{1}{2}\cdot 1\,\mathrm{kg}\cdot(29.44\,\mathrm{m/s})^2\\
&= 547.64\,\mathrm{N{\cdot}m}
\end{aligned}$$

Thus more than half of the initial energy is dissipated in air friction. If there were no viscous drag on the ball, its speed just before hitting the ground would be

$$v_B = \sqrt{2gh} = 44.29\,\mathrm{m/s}.$$

$E_{\mathrm{drag}} = 547.64\,\mathrm{N{\cdot}m}$

(1) A transcendental equation in t is one where t appears both as an argument of a trigonometric or exponential function and elsewhere. Such equations can almost never be solved by hand in closed form.

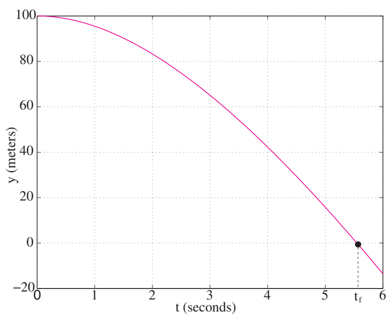


Figure 10.18: A graph of $y(t) = h - \frac{mg}{c}\left(\frac{m}{c}e^{-\frac{c}{m}t} + t - \frac{m}{c}\right)$ for determining t_f when $y = 0$. This is the same as solving eqn. (10.23) for t_f.

SAMPLE 10.11 How much time does it take for a car of mass 800 kg to go from 0 mph to 60 mph, if we assume that the engine delivers a constant power P of 40 horsepower during this period. (1 horsepower = 745.7 W)

Solution

$$\begin{aligned} P &= \dot{W} \equiv \frac{dW}{dt} \\ dW &= P dt \\ W_{12} &= \int_{t_0}^{t_1} P dt = P(t_1 - t_0) = P\Delta t \\ \Delta t &= \frac{W_{12}}{P}. \end{aligned}$$

Now, from IIIa in the inside front cover,

$$\begin{aligned} W_{12} &= (E_\mathrm{K})_2 - (E_\mathrm{K})_1 \\ &= \frac{1}{2} m(v_2^2 - v_1^2) \\ &= \frac{800\,\mathrm{kg}[(60\,\mathrm{mph})^2 - 0]}{2} \\ &= \frac{1}{2} \cdot 800\,\mathrm{kg} \left(60 \frac{\mathrm{mi}}{\mathrm{hr}} \cdot \frac{1.61 \times 10^3\,\mathrm{m}}{1\,\mathrm{mi}} \cdot \frac{1\,\mathrm{hr}}{3600\,\mathrm{s}}\right)^2 \\ &= 288.01 \times 10^3\ \mathrm{kg} \cdot\ \mathrm{m} \cdot\ \mathrm{m/s^2} \\ &= 288\ \mathrm{KJoule}. \end{aligned}$$

Therefore,

$$\Delta t = \frac{288 \times 10^3\,\mathrm{J}}{40 \times 745.7\,\mathrm{W}} = 9.66\,\mathrm{s}.$$

Thus it takes about 10 s to accelerate from a standstill to 60 mph.

$$\boxed{\Delta t = 9.66\,\mathrm{s}}$$

Note 1: This model gives a roughly realistic answer *but* it is not a realistic model, at least at the start, at time t_0. In the model here, the acceleration is infinite at the start (the power jumps from zero to a finite value at the start, when the velocity is zero), something the finite-friction tires would not allow.

Note 2: We have been a little sloppy in quoting the energy equation. Since there are no external forces doing work on the car, somewhat more properly we should perhaps have written

$$0 = \dot{E}_\mathrm{K} + \dot{E}_\mathrm{int} + \dot{E}_\mathrm{P}$$

and set $-(\dot{E}_\mathrm{int} + \dot{E}_\mathrm{P})$ = 'the engine power' where the engine power is from the decrease in gasoline potential energy ($-\dot{E}_\mathrm{P}$ is positive) less the increase in 'heat' ($\dot{E}_\mathrm{int}$) from engine inefficiencies.

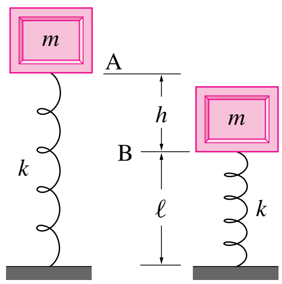


Figure 10.19

SAMPLE 10.12 **Energy of a mass-spring system.** A mass $m = 2\,\mathrm{kg}$ is attached to a spring with spring constant $k = 2\,\mathrm{kN/m}$. The relaxed (unstretched) length of the spring is $\ell = 40\,\mathrm{cm}$. The mass is pulled up and released from rest at position A shown in Fig. 10.19. The mass falls by a distance $h = 10\,\mathrm{cm}$ before reaching position B, which is the relaxed position of the spring. Find the speed at point B.

Solution The total energy of the mass-spring system at any instant or position consists of the energy stored in the spring and the sum of potential and kinetic energies of the mass. For potential energy of the mass, we need to select a datum where the potential energy is zero. We can select any horizontal plane to be the datum. Let the ground support level of the spring be the datum. Then, at position A,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = \frac{1}{2}kh^2 \quad (\text{see eqn. (10.19), page 505})\\ \text{Energy of the mass} &= E_\mathrm{K} + E_\mathrm{P} = \frac{1}{2}m\underbrace{v_A^2}_{0} + mg(\ell+h) = mg(\ell+h).\end{aligned}$$

Therefore, the total energy at position A

$$E_A = \frac{1}{2}kh^2 + mg(\ell+h).$$

Let the speed of the mass at position B be v_B. When the mass is at B, the spring is relaxed, i.e., there is no stretch in the spring. Therefore, at position B,

$$\begin{aligned}\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = 0\\ \text{Energy of the mass} &= E_\mathrm{K} + E_\mathrm{P} = \frac{1}{2}mv_B^2 + mg\ell,\end{aligned}$$

and the total energy

$$E_B = \frac{1}{2}mv_B^2 + mg\ell.$$

Because the net change in the total energy of the system from position A to position B is

$$\begin{aligned}0 &= \Delta E\\ &= E_A - E_B = \frac{1}{2}kh^2 + mg(\ell+h) - \frac{1}{2}mv_B^2 - mg\ell\\ &= \frac{1}{2}(kh^2 - mv_B^2) + mgh\\ \Rightarrow\quad v_B^2 &= kh^2/m + 2gh\\ \Rightarrow\quad |v_B| &= \left(kh^2/m + 2gh\right)^{1/2}\\ &= \left((2000\,\mathrm{N/m}\cdot(0.1\,\mathrm{m})^2/2\,\mathrm{kg}) + 2\cdot 9.81\,\mathrm{m/s^2}\cdot 0.1\,\mathrm{m}\right)^{1/2}\\ &= 3.46\,\mathrm{m/s}.\end{aligned}$$

$|v_B| = 3.46\,\mathrm{m/s}$

10.2 Energy methods in 1D

Preparatory Problems

10.2.1 A mass m is at position x moving at velocity v and being acted upon by force F. For each of the quantities below:

i give the symbol used for the quantity

ii describe the quantity in words

iii give a formula to evaluate the quantity in terms of some or all of m, x, v and F and any other variables you may need.

iv Give the standard units for the quantity in the SI system.

v Give the standard units for the quantity in the English system.

a) Power
b) Kinetic energy
c) Work
d) Potential energy

10.2.2 Write an equation relating the two words in each of these pairs. If any conditions or descriptions of the situation are needed, give them. If you know more than one equation (or form for a given equation), give all that you know. All should be given in the context of this section: 1D motion.

a) work and power
b) work and kinetic energy
c) power and kinetic energy
d) work and potential energy
e) potential energy and kinetic energy

10.2.3 A force $F = F_0 \sin(ct)$ acts on a particle with mass $m = 3\,\mathrm{kg}$ which has position $x = 3\,\mathrm{m}$, velocity $v = 5\,\mathrm{m/s}$ at $t = 2\,\mathrm{s}$. $F_0 = 4\,\mathrm{N}$ and $c = 2/\,\mathrm{s}$. At $t = 2\,\mathrm{s}$ evaluate (give numbers and units):

a) a,
b) E_K,
c) P,
d) $\dot{E}_\mathrm{K}$,
e) the rate at which the force is doing work. *

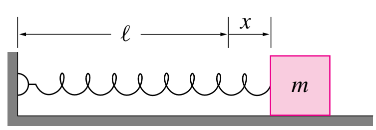
10.2.4 A force only depends on position according to $F = C_0 + C_1 x$ where C_0 and C_1 are constants. What is the work done by this force when the point to which it is applied moves from x_1 to x_2? Answer in terms of some or all of C_0, C_1, x_1 and x_2.

10.2.5 Find the potential E_P associated with each of these force fields.

a) $F = 0$.
b) $F = F_0$ (=constant).
c) $F = kx$.
d) $F = A\sin(x/x_0)$.
e) $F = c/x^2$.

10.2.6 Consider a spring-mass system with $m = 2\,\mathrm{kg}$ and $k = 5\,\mathrm{N/m}$. The mass is pulled to the right a distance $x = x_0 = 0.5\,\mathrm{m}$ from the unstretched position and released from rest. No external forces act on the mass.

a) What are the initial potential and kinetic energy of the system?
b) What is the potential and kinetic energy of the system as the mass passes through the static equilibrium (unstretched spring) position?
c) What is the speed of the mass when it passes through the static equilibrium position?



Problem 10.2.6

10.2.7 A mass m is held in place by a spring whose restoring force is $T(x) = kx$. Derive the equation of motion of the system (that is, find the acceleration a in terms of x).

10.2.8 The peak propulsion force on a 4-wheel-drive car is about μmg where $\mu \approx 1$ for rubber on road (a bit more for fancy racing tires). Assume a car starts from rest at position zero. Answer the following questions with symbols and with numbers (using $\mu = 1$, $m = 1000\,\mathrm{kg}$, and $g = 10\,\mathrm{m/s^2}$).

a) What is the minimum distance required to reach $v_1 = 60\,\mathrm{mph}$?
b) What is the extra distance required to get from $v_1 = 60\,\mathrm{mph}$ up to $v_2 = 70\,\mathrm{mph}$?
c) What is the peak power used by the engine in getting up to $v_1 = 60\,\mathrm{mph}$ (assuming no dissipation and no air friction)?

10.2.9 A car (mass $m = 1000\,\mathrm{kg}$) traveling at speed $v_0 = 30\,\mathrm{m/s}$ crashes into a brick wall and comes to a stop as the front end of the car compresses a distance $d = 1\,\mathrm{m}$. Answer with symbols and numbers. Assume constant deceleration during the crash. Neglect the mass of the crushing region of the car.

a) What is the total energy dissipated in the crash?
b) What is the force of the car on the wall?
c) What is the force of the wall on the car?
d) What is the deceleration of the car passengers (assuming they are strapped in and move with the bulk of the car). Answer in g's?
e) Assuming an $m_p = 50\,\mathrm{kg}$ person, what is the force of the seat belts on the person (answer in body weight).
f) If a parent was holding a 15 kg child on his lap, what force would he need to hold on to the child through the crash (answer in N and in number of child body weights).

More-Involved Problems

10.2.10 A kid ($m = 90\,\mathrm{lbm}$) stands on a $h = 10\,\mathrm{ft}$ wall and jumps down, accelerating with $g = 32\,\mathrm{ft/s^2}$. Upon hitting the ground with straight legs, she bends them so her body slows to a stop over a distance $d = 1\,\mathrm{ft}$. Neglect the mass of her legs. Assume constant deceleration as she brakes the fall.

a) What is the total distance her body falls? *
b) What is the potential energy lost?
c) How much work must be absorbed by her legs?

d) What is the force of her legs on her body? Answer in symbols, numbers and numbers of body weight (*i.e.,* find F/mg).

10.2.11 In traditional archery, when pulling an arrow back the force starts from 0 and increases approximately linearly up to the peak 'draw force' F_{draw}. The draw force varies from about $F_{draw} = 25\,\text{lbf}$ for a bow made for a small person to about $F_{draw} = 75\,\text{lbf}$ for a bow made for a big strong person. The distance the arrow is pulled back, the draw length ℓ_{draw}, varies from about $\ell_{draw} = 2\,\text{ft}$ for a small adult to about 30 inch for a big adult. An arrow has mass of about 300 grain (1 grain $\approx$ 64.8milli gm, so an arrow has mass of about 19.44 $\approx$ 20 gm $\approx$ 3/4 ounce). Give all answers in symbols and numbers.

a) What is the range of speeds you can expect an arrow to fly?
b) What is the range of heights an arrow might go if shot straight up (it's a bad approximation, but for this problem neglect air friction)?

10.2.12 A big person ($m = 100\,\text{kg}$) jumps on a trampoline which we model as a linear spring with stiffness k. You know that the trampoline deflects $d_0 = 20\,\text{cm}$ under the stationary weight mg of the person (use $g = 10\,\text{m/s}^2$). Assume there is no dissipation and the person is jumping repeatedly a height $h = 1\,\text{m}$ above the unloaded surface of the trampoline. Give all answers with symbols and numbers.

a) What is the stiffness k of the spring (answer in terms of some or all of m, g and d_0).
b) What is the maximum deflection of the trampoline during these jumps?
c) What is the peak force of the trampoline on the jumper? (answer in symbols, Newtons, and numbers of body weights).

10.2.13 For the car of problem 10.2.8 what is the average power required to reach speed v_1? There are two plausible ways to calculate this power:

$$\bar{P}_1 = \int_0^x P(x')\,dx'/x \quad \text{and}$$
$$\bar{P}_2 = \int_0^t P(t')\,dt'/t\,.$$

Use both. Do the two methods give the same answer? If so, why, and will the answers be the same for all problems? If not, why not, in what cases will the answers agree, and, when they differ, which one is right?

10.2.14 For problem 10.2.9 which answers would change, and in which way, if the deceleration was not exactly constant during the crash? That is, for which quantities would the answer be bigger, which smaller, which the same, for which would the answer depend on the nature of the non-constant acceleration?

10.2.15 The earth's gravitational pull on a mass m is $F = -\frac{mgR^2}{r^2}$, where mg is the pull at the surface of the earth and R is the radius of the earth. Assume a ballistic rocket is shot straight up with a launch velocity of v_0 (measured in a 'fixed' not-rotating-with-the-earth frame). Assume the rocket goes in a straight radial line as the earth turns underneath it (relative to the surface of the earth this rocket would be launched somewhat to the West to cancel the earth's rotation). Assume the period of active thrust is negligibly short (hence the word *ballistic*: "relating to or characteristic of the motion of objects moving under their own momentum and the force of gravity").

a) Solve for v as a function of r (and some or all of m, g, R and v_0).
b) Find the maximum height the rocket reaches.
c) Find the 'escape velocity' v_{escape}, the minimum launch speed needed for the rocket to never return.
d) On one graph plot height (r or $r - R$) vs t for a v_0 just below v_{escape} and for v_0 just greater than v_{escape}. If you use numerical methods to make this plot use $g = 10\,\text{m/s}^2$, $R = 6400\,\text{km}$, and $m = 1\,\text{kg}$. Make sure your axes are such that you can see a clear qualitative difference between the two cases.

10.2.16 The power available to a very strong accelerating cyclist over short periods of time (up to, say, about 1 minute) is about 1 horsepower. Assume a rider starts from rest and uses this constant power. Assume a mass (bike + rider) of 150 lbm, a realistic drag force of .006 lbf/(ft/s)$^2v^2$. Neglect other drag forces.

a) What is the peak (steady state) speed of the cyclist?
b) Using analytic or numerical methods make an accurate plot of speed vs. time.
c) What is the acceleration as $t \to \infty$ in this solution?
d) What is the acceleration as $t \to 0$ in your solution?
e) How would you improve the model to fix the problem with the answer above?

10.2.3 A force $F = F_0 \sin(ct)$ acts on a particle with mass $m = 3\,\text{kg}$ which has position $x = 3\,\text{m}$, velocity $v = 5\,\text{m/s}$ at $t = 2\,\text{s}$. $F_0 = 4\,\text{N}$ and $c = 2/\text{s}$. At $t = 2\,\text{s}$ evaluate (give numbers and units):

a) a,

b) E_K,

c) P,

d) $\dot{E}_K$,

e) the rate at which the force is doing work.

Answer:
(a) $-1.008\,\text{m/s}$, (b) 37.5 Joules, (c), (d) and (e) -15.12 Watts

9.2.3

$F = F_0 \sin(ct)$

$m = 3\text{kg}$

$\dot{x}\big|_{t=2} = 5\,m/s$, $x\big|_{t=2} = 3m$, $c = 2/s$

a) $m\ddot{x} = F = F_0 \sin ct$

$x = -\frac{F_0}{mc^2} \sin ct + At + B$

$a = \ddot{x} = \frac{F_0 \sin ct}{m} = \frac{4 \sin(4)}{3} = \frac{4(-0.756)}{3}$

$a = -1.008\,m/s$

a) $E_K = \frac{1}{2} m v^2 = \frac{1}{2} \cdot m \cdot (5)^2 = \frac{75}{2}$ Joules

$= 37.5$ Joules

b) $P = F \cdot V = 4(-0.756)5 = -15.12$ Watts

c) $\dot{E}_K = \frac{d}{dt}\left(\frac{1}{2} m \dot{x}^2\right) = \frac{1}{2} m \dot{x} \times 2 \times \ddot{x} = m\dot{x}\ddot{x} = -3 \times 5 \times 1.008$

$= -15.1$ Watts

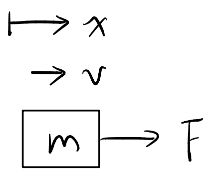
e) Rate at which Force is doing work

$W_F = F \cdot V = -15.12$ watts

MAE 2030 HW2 P6 Solution

Duan Li

Given :



1D problem

$F = F_0 \sin(ct)$, $F_0 = 4\,N$, $c = 2\,s^{-1}$

$m = 3\,kg$

@ $t = 2\,s$, $x = 3\,m$, $v = 5\,m/s$

Find : @ $t = 2\,s$

a) a acceleration

b) E_k kinetic energy

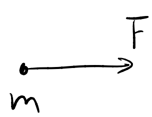
c) P power of the force

d) $\dot{E}_k$

e) the rate at which the force is doing the work

Soln :

a) FBD :



LMB : $m\ddot{x} = F$

Let $t' = 2\,s$

$$a(t') = \ddot{x}(t') = \frac{1}{m} F(t') = \frac{1}{m} F_0 \sin(ct')$$

$$= \frac{1}{3\,kg} 4N \sin(2\,s^{-1} \cdot 2s) = \boxed{-1.01\ m/s^2}$$

b) $$E_k(t') = \frac{1}{2} m v(t)^2 = \frac{1}{2}\, 3\,kg \cdot (5\,m/s)^2 = \boxed{37.5\ J}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

c) $P(t') = F(t')\, v(t') = 4\text{N} \sin(2\text{s}^{-1} \cdot 2\text{s}) \cdot 5 \text{ m/s}$

$= \boxed{-15.14 \text{ W}}$

d) Power balance eqn :

$\dot{E}_k(t') = P(t') = \boxed{-15.14 \text{ W}}$

e) $\frac{dW}{dt} = \frac{F\,dx}{dt} = Fv = P$

$\left.\frac{dW}{dt}\right|_{t'} = P(t') = \boxed{-15.14 \text{ W}}$

10.2.4 A force only depends on position according to $F = C_0 + C_1 x$ where C_0 and C_1 are constants. What is the work done by this force when the point to which it is applied moves from x_1 to x_2? Answer in terms of some or all of C_0, C_1, x_1 and x_2.

9.31.

$F = C_0 + C_1 x$

workdone, $W = \int_{x_1}^{x_2} F\,dx$

$= \int_{x_1}^{x_2} (C_0 + C_1 x)\,dx = \left[C_0 x + \frac{C_1 x^2}{2} \right]_{x_1}^{x_2}$

$\Rightarrow W = C_0 (x_2 - x_1) + \frac{C_1}{2} [x_2^2 - x_1^2]$

$\Rightarrow W = (x_2 - x_1) \left[C_0 + \frac{1}{2} C_1 (x_2 + x_1) \right]$ Ans

10.2.5 Find the potential E_P associated with each of these force fields.

a) $F = 0$.

b) $F = F_0$ (=constant).

c) $F = kx$.

d) $F = A\sin(x/x_0)$.

e) $F = c/x^2$.

9.32.

$$F = -\frac{dE_p}{dx}.$$

a) $F = 0$, $\frac{dE_p}{dx} = 0 \Rightarrow \boxed{E_p = C = \text{constt.}}$

b) $F = F_o$, $\int dE_p = \int F_o\,dx$

$$\Rightarrow \boxed{E_p = F_o x + C}$$

c) $F = Kx$.

$$\Rightarrow \int dE_p = \int Kx\,dx$$

$$\Rightarrow \boxed{E_p = \frac{1}{2}Kx^2 + C.}$$

d) $F = A\sin(x/x_0)$

$$\Rightarrow \boxed{E_p = -Ax_0\cos(x/x_0) + C.}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.2.7 A mass m is held in place by a spring whose restoring force is $T(x) = kx$. Derive the equation of motion of the system (that is, find the acceleration a in terms of x).

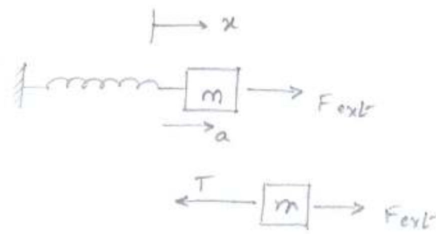
9.2.7

Equation of motion

$m\ddot{x} = F_{ext} - T$

$m\ddot{x} = F_{ext} - Kx$

$m\ddot{x} + Kx = F_{ext}$



10.2.8 The peak propulsion force on a 4-wheel-drive car is about μmg where $\mu \approx 1$ for rubber on road (a bit more for fancy racing tires). Assume a car starts from rest at position zero. Answer the following questions with symbols and with numbers (using $\mu = 1$, $m = 1000\,\text{kg}$, and $g = 10\,\text{m/s}^2$).

a) What is the minimum distance required to reach $v_1 = 60\,\text{mph}$?

b) What is the extra distance required to get from $v_1 = 60\,\text{mph}$ up to $v_2 = 70\,\text{mph}$?

c) What is the peak power used by the engine in getting up to $v_1 = 60\,\text{mph}$ (assuming no dissipation and no air friction)?

9.35

$F_P = \mu m g$

$= 1 \times 1000 \times 10\,\text{N}.$

$\therefore$ acceleration, $a = 10\,\text{ms}^{-1}$.

Since motion is constant acceleration.

$v = u + at$ — (1)

$v^2 = u^2 + 2as$ — (2)

& $s = ut + \frac{1}{2}at^2$ (3) can be used.

here $u = 0$.

(a) $\therefore$ from (2)

$s = \dfrac{v^2}{2a}$.

$v = 60\,\text{mph} = \dfrac{60 \times 1.609 \times 10^3}{60 \times 60}\,\text{ms}^{-1} = 26.8166\,\text{ms}^{-1}$.

$\therefore S_1 = 35.9566\,\text{m}$. Ans

(b) from (1) $\Rightarrow$ $v_2 = 70\,\text{mph} = 31.286\,\text{ms}^{-1}$.

$\therefore$ $31.286 = 26.8166 + 10t$

$\Rightarrow t = 0.44695\,\text{sec}$. Ans

(c) ~~Forward~~ $P = \int F \cdot v\,dt$

peak power used by engine in getting up to $v_1 = 60\,\text{mph}$

10.2.9 A car (mass $m = 1000\,\text{kg}$) traveling at speed $v_0 = 30\,\text{m/s}$ crashes into a brick wall and comes to a stop as the front end of the car compresses a distance $d = 1\,\text{m}$. Answer with symbols and numbers. Assume constant deceleration during the crash. Neglect the mass of the crushing region of the car.

a) What is the total energy dissipated in the crash?

b) What is the force of the car on the wall?

c) What is the force of the wall on the car?

d) What is the deceleration of the car passengers (assuming they are strapped in and move with the bulk of the car). Answer in g's?

e) Assuming an $m_p = 50\,\text{kg}$ person, what is the force of the seat belts on the person (answer in body weight).

f) If a parent was holding a 15 kg child on his lap, what force would he need to hold on to the child through the crash (answer in N and in number of child body weights).

9.36

$m = 1000\,\text{Kg}$.

$v_0 = 30\,\text{ms}^{-1}$.

$d = 1\,\text{m}$.

a). Total energy dissipated in crash

$= (KE)_{\text{before crash}} - (KE)_{\text{after crash}}$.

(neglecting change of potential energy).

$= \frac{1}{2} \times 1000 \times 30^2 - 0$

$= 450\,\text{KJ}$ Ans

(b) Since constant accel^n is assumed during crash.

F = Constt.

$F \times 1 = 450\,\text{KJ}$

$\Rightarrow F = 450\,\text{KN}$. Ans.

(c) Newton Second law $\rightarrow$ force of the wall on car $= 450\,\text{KN}$.

(d) $F = 450\,\text{KN}$

$\therefore a = \dfrac{450 \times 10^3}{10^3} = 450\,\text{m/s}^2$.

$\boxed{a = 45.871 g.}$ $\quad g = 9.81\,\text{ms}^{-2}$.

Ans

(e) Force of the seat belt on the person

$= 50 \times 450\,\text{N}$

$= 2293.577$ times body weight.

(f) $F = 450 \times 15\,\text{N}$. If parent is held stationary by belt.

$= 6.750\,\text{KN} = 6750\,\text{N}$. Ans

$= 450$ times child body weight.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.2.10 A kid ($m = 90\,\text{lbm}$) stands on a $h = 10\,\text{ft}$ wall and jumps down, accelerating with $g = 32\,\text{ft/s}^2$. Upon hitting the ground with straight legs, she bends them so her body slows to a stop over a distance $d = 1\,\text{ft}$. Neglect the mass of her legs. Assume constant deceleration as she brakes the fall.

a) What is the total distance her body falls?

b) What is the potential energy lost?

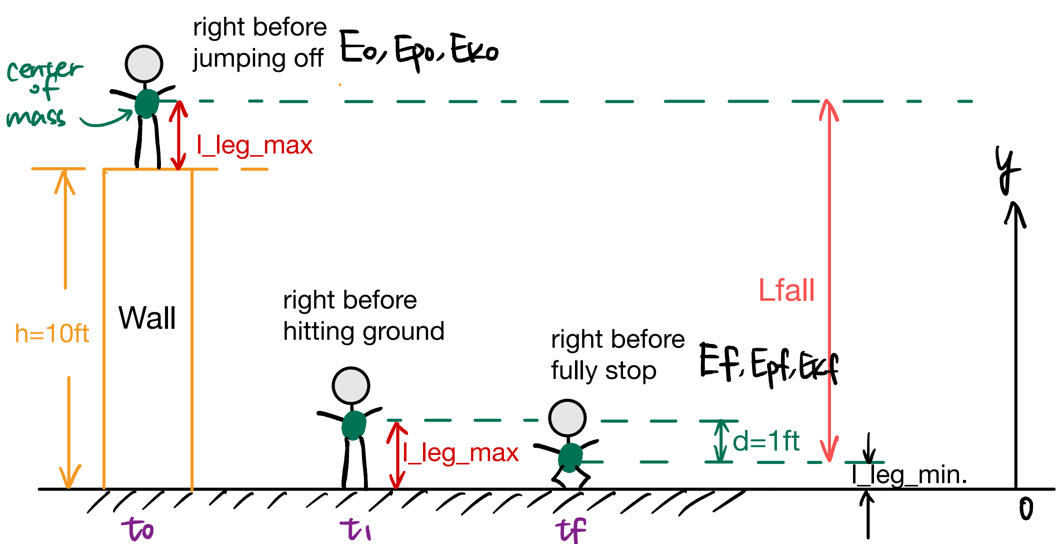
c) How much work must be absorbed by her legs?

d) What is the force of her legs on her body? Answer in symbols, numbers and numbers of body weight (*i.e.*, find F/mg).

Answer:
11 ft

Katherine Cao

Lfall



a)

$$L_{fall} = l_{legmax} + h - l_{legmin}$$
$$= l_{legmax} + h - (l_{legmax} - d)$$
$$= \boxed{h+d} = 10\text{ft} + 1\text{ft}$$
$$= \boxed{11\text{ft}}$$

(also given in HW hints)

The total distance center of mass fell equals the height of the wall h plus the distance of bending d

$$\Rightarrow L_{fall} = h + d$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

b) Change in potential energy

= final potential energy − initial potential energy

$$\Rightarrow \Delta E_p = E_{pf} - E_{p0} = mg(-L_{fall})$$

$$= -mg(h+d) = -990 \text{ lbf-ft}$$

Potential energy lost = negative of change in potential energy

See P232~233 of RuinaPratap (or read more of Chapter 4.1) for unit conversions

Thus, $E_{p\,lost} = -\Delta E_p = \boxed{mg(h+d)}$

$$= (90\text{ lbm})(32\text{ ft}/s^2)(11\text{ ft}) = 31680 \text{ pdl-ft}$$

$$= \boxed{990 \text{ lbf-ft}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

c) comparing the kid's state right before jumping off the wall and right before fully stop, velocity of the kid is zero for both states

$$E_0 = E_{p0} + \cancel{E_{k0}}^{\,0} = E_{p0}$$
$$E_f = E_{pf} + \cancel{E_{kf}}^{\,0} = E_{pf}$$

Change in ~~total~~ energy
= final total energy − initial ~~total~~ energy

$$\Delta E = E_f - E_0 = E_{pf} - E_{p0} = \Delta E_p$$

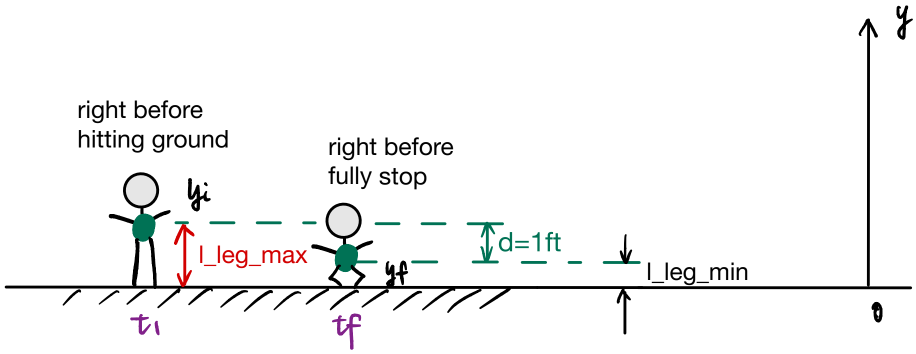
Total energy lost
= negative of change in total energy

$$\Rightarrow \quad E_{lost} = -\Delta E = -\Delta E_p$$

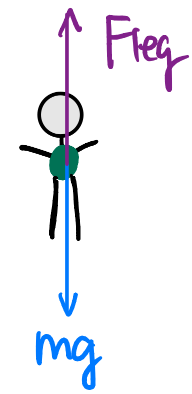
from b) $\Delta E_p = -mg(h+d) = -990 \text{ lbf-ft}$

Work absorbed by leg should equal ~~total~~ energy lost of center of mass of the kid

$$\boxed{W_{absorb} = E_{lost} = mg(h+d) = 990 \text{ lbf-ft}}$$



FBD for t_i to t_f



$y_i = l_{legmax}$ $\quad y_f = l_{legmin} = l_{legmax} - d$

d) Work done by leg on center of mass should equal energy change of center of mass:

$$W_{leg} = \Delta E \quad \text{from c)} \quad \Delta E = \Delta E_p = -mg(h+d)$$

With constant deceleration, using newton's 2nd law

$$\Sigma F = F_{leg} - mg = ma$$

$$F_{leg} = m(g+a) \quad m, g, a \text{ all constant value}$$

$\Rightarrow$ F_{leg} is constant

$$W_{leg} = \int_{y_i}^{y_f} F_{leg}\,dy = F_{leg}\, y \Big|_{y_i = l_{legmax}}^{y_f = l_{legmax} - d}$$

$$= F_{leg}(y_f - y_i) = F_{leg}(l_{legmax} - d - l_{legmax})$$

$$= -F_{leg}\, d = \Delta E = -mg(h+d)$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$$\Rightarrow F_{leg} = \frac{mg(h+d)}{d} = \boxed{\frac{(h+d)}{d} mg}$$

$$= \left(\frac{10\,ft + 1\,ft}{1\,ft}\right)(90\,lbm)(32\,ft/s^2) = 31680\ pdl = \boxed{990\ lbf}$$

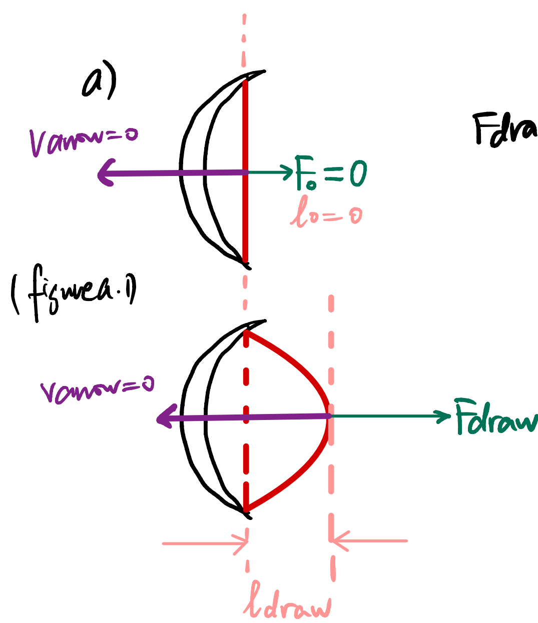
10.2.11 In traditional archery, when pulling an arrow back the force starts from 0 and increases approximately linearly up to the peak 'draw force' F_{draw}. The draw force varies from about F_{draw} = 25 lbf for a bow made for a small person to about F_{draw} = 75 lbf for a bow made for a big strong person. The distance the arrow is pulled back, the draw length ℓ_{draw}, varies from about ℓ_{draw} = 2 ft for a small adult to about 30 inch for a big adult. An arrow has mass of about 300 grain (1 grain ≈ 64.8milli gm, so an arrow has mass of about 19.44 ≈ 20 gm ≈ 3/4 ounce). Give all answers in symbols and numbers.

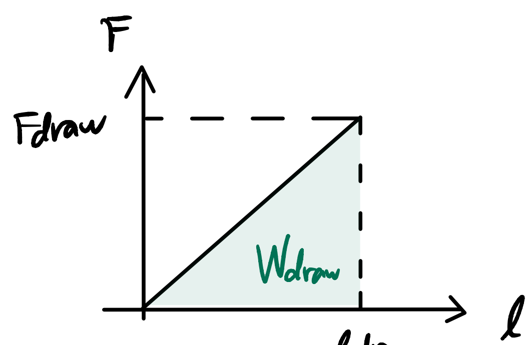
a) What is the range of speeds you can expect an arrow to fly?

b) What is the range of heights an arrow might go if shot straight up (it's a bad approximation, but for this problem neglect air friction)?

M_{arrow} = 3/4 ounce ≈ 0.047 lbm

a)





(figure a.2)

The total energy of the bow-arrow system at any time consists of the kinetic energy of the arrow and the potential energy stored in the bow's string. The change in total energy of the system is due to Work done by exteral force, in this case, Wdraw done by draw force.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

As stated in the prompt, draw force approximately linearly increase from 0 to peak Fdraw over a length of ldraw

$\Rightarrow F = \text{Fdraw}\, l / \text{ldraw}$

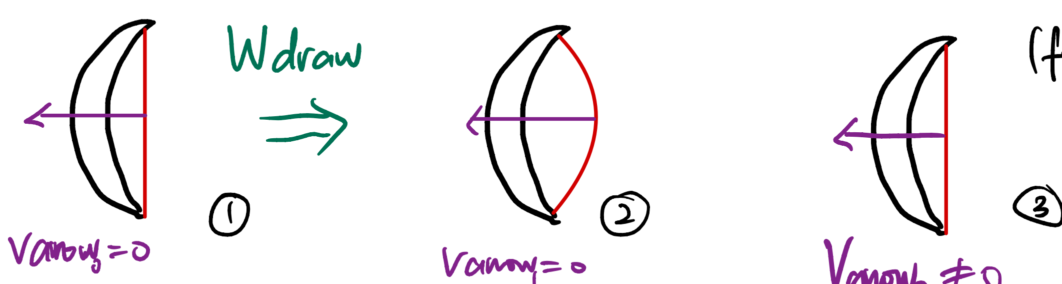
Thus,

$$W_{draw} = \int_{l_0=0}^{l_{draw}} F\,dl = \int_{l_0=0}^{l_{draw}} \frac{F_{draw}\, l}{l_{draw}}\, dl$$

(or look at the area of figure a.2)

$$= \left.\frac{F_{draw}\, l^2}{2\, l_{draw}}\right|_{l_0}^{l_{draw}} = \frac{1}{2} F_{draw}\, l_{draw}$$

The above Wdraw is the work done by the draw force to the bow-arrow system during the process of pulling the arrow back. Looking at the process of "releasing" the arrow, there is no eternal force doing work to the system, thus there is no change when considering the total energy of the bow-arrow system.



(figure a.3)

$E_0 = E_{arrow_0} + E_{string_0} = 0 + 0 = 0$

$E_1 = E_{arrow_1} + E_{string_1} = E_0 + W_{draw}$

$E_2 = E_{arrow_2} + E_{string_2} = E_1$

As shown in figure a.3

comparing state ① and ③, the difference in total energy of the bow-arrow system equals work done by external force W_{draw}. Since bow string restored to resting length, W_{draw} is fully converted to kinetic energy of the arrow E_{arrow_2} if neglect energy loss throughout the process.

Thus,

$$W_{draw} = \frac{1}{2} F_{draw} \ell_{draw} = E_{arrow_2} = \frac{1}{2} M_{arrow} V_{arrow_2}^2$$

$$\Rightarrow V_{arrow_2} = \sqrt{\frac{F_{draw} \ell_{draw}}{M_{arrow}}}$$

The upper limit of V_{arrow_2} set by bow for a big, strong person and the lower limit set by bow for a small person.

$$V_{arrow_2\,max} = \sqrt{\frac{F_{draw\,big}\,l_{big}}{M_{arrow}}} = \sqrt{\frac{75\text{ lbf } 30\text{ in}\left(\frac{1\text{ ft}}{12\text{ in}}\right)}{0.047\text{ lbm}}}$$

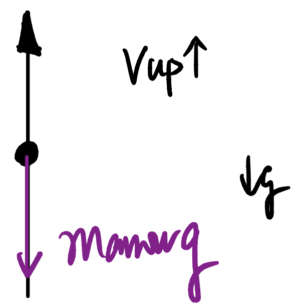
$$= \sqrt{\frac{(75\text{ lbf})(32.2\frac{\text{lbm}}{\text{lbf}}\text{ ft/s}^2)\ 2.5\text{ ft}}{0.047\text{ lbm}}} \simeq 358\text{ ft/s}$$

$$V_{arrow_2\,min} = \sqrt{\frac{F_{draw\,small}\,l_{small}}{M_{arrow}}} = \sqrt{\frac{25\text{ lbf}\cdot 2\text{ ft}}{0.047\text{ lbm}}}$$

$$= \sqrt{\frac{(25\text{ lbf})(32.2\frac{\text{lbm}}{\text{lbf}}\cdot\text{ ft/s}^2)\ 2\text{ ft}}{0.047\text{ lbm}}} \simeq 185\text{ ft/s}$$

$$\boxed{\text{Thus}\quad 185\text{ ft/s} \le V_{arrow_2} \le 358\text{ ft/s}}$$

b) arrow



If shot straight up, assuming no energy loss due to drag on arrow due to air or other reasons the ~~total~~ energy of the arrow is conserved, meaning the sum of the arrow's kinetic energy and gravitational potential energy does not change.

$$E_{arrow} = E_{k\,arrow} + E_{p\,arrow}$$

$$= E_{k\,arrow\_release} + E_{p\,arrow_0} = E_{k\,arrow\text{-}top} + E_{p\,arrow\text{-}top}$$

$$= \frac{1}{2} M_{arrow} V_{arrow}^2 + M_{arrow} g(h - h_0)$$

$$= \frac{1}{2} M_{arrow} V_{arrow_2}^2 + mg(h_0 - h_0)$$

$$= \frac{1}{2} M_{arrow} V_{arrow\,top}^2 + mg(h_{top} - h_0)$$

$$\Rightarrow \frac{1}{2} m_{arrow} v_{arrow_2}^2 + 0 = 0 + m_{arrow} g \Delta h$$

$$\Rightarrow \Delta h = \frac{\frac{1}{2} m_{arrow} v_{arrow_2}^2}{m_{arrow} g}$$

using range of v_{arrow_2} in part a

Δh_{max} set by bow for big strong person.

Δh_{min} set by bow for small person

$$\Delta h_{max} = \frac{\frac{1}{2} m_{arrow} v_{arrow_2\,max}^2}{m_{arrow} g}$$

$$= \frac{\frac{1}{2}(0.047\,lbm)(358\,ft/s)^2}{(0.047\,lbm)(32.2\,ft/s^2)}$$

$$\approx 1990\,ft$$

$$\Delta h_{min} = \frac{\frac{1}{2} m_{arrow} v_{arrow_2\,min}^2}{m_{arrow} g}$$

$$= \frac{\frac{1}{2}(0.047\,lbm)(185\,ft/s)^2}{(0.047\,lbm)(32.2\,ft/s^2)}$$

$$\approx 531\,ft$$

$$\Rightarrow \boxed{531\,ft \le \Delta h \le 1990\,ft}$$

10.2.12 A big person ($m = 100\,\text{kg}$) jumps on a trampoline which we model as a linear spring with stiffness k. You know that the trampoline deflects $d_0 = 20\,\text{cm}$ under the stationary weight mg of the person (use $g = 10\,\text{m/s}^2$). Assume there is no dissipation and the person is jumping repeatedly a height $h = 1\,\text{m}$ above the unloaded surface of the trampoline. Give all answers with symbols and numbers.

a) What is the stiffness k of the spring (answer in terms of some or all of m, g and d_0).

b) What is the maximum deflection of the trampoline during these jumps?

c) What is the peak force of the trampoline on the jumper? (answer in symbols, Newtons, and numbers of body weights).

9.39.

$m = 100\,\text{kg}$

$d_0 = 20\,\text{cm} = 0.2\,\text{m}$ (under stationary weight mg).

(a) Stiffness ~~k~~ of the spring,

$$K = \frac{mg}{d_0} = \frac{100 \times 9.81}{0.2} = 4905\,\text{N/m}$$

(b) Initial (K.E + PE) of system, when person at height 1m.

$$= 0 + mg \times 1$$

$$= 981\,\text{J}$$

Final (KE + PE) of system, when spring is having maximum compression

$$= 0 + \frac{1}{2} K d_{max}^2 - mg d_{max}.$$

Since Assuming there is no dissipation.

$$-mg d_{max} + \frac{1}{2} K d_{max}^2 = 981\,\text{J}$$

$$\Rightarrow \frac{1}{2} \times 4905\, d_{max}^2 - 981\, d_{max} - 981 = 0$$

$$\Rightarrow 2452.5\, d_{max}^2 - 981\, d_{max} - 981 = 0.$$

$$\Rightarrow d_{max} = 0.8633\,\text{m}, -0.46332\,\text{m}.$$

$$\boxed{d_{max} = +0.8633\,\text{m}}$$ Ans

(c)

$$F_{peak} = K d_{max}.$$

$$= 4905 \times 0.8633$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.2.13 For the car of problem 10.2.8 what is the average power required to reach speed v_1? There are two plausible ways to calculate this power:

$$\bar{P}_1 = \int_0^x P(x')\,dx'/x \qquad \text{and}$$

$$\bar{P}_2 = \int_0^t P(t')\,dt'/t\,.$$

Use both. Do the two methods give the same answer? If so, why, and will the answers be the same for all problems? If not, why not, in what cases will the answers agree, and, when they differ, which one is right?

9.2.13

$F_p = \mu m g \simeq mg$

$m = 1000\ \text{kg}$

$\mu = 1$

$g = 10\ \text{m/s}^2$

$v_1 = 60\ \text{mph}$

$\Rightarrow \quad a = \frac{mg}{m} = g$

for $v_1 = 60\ \text{mph} = \frac{60 \times 1.6093 \times 10^3}{60 \times 60}\ \frac{\text{m}}{\text{sec}}$

$= 26.8\ \text{m/sec}$

at any point $\quad v = gt\ ; \quad x = \frac{1}{2} g t^2\ ; \quad v = \sqrt{2gx}$

$F = mg;$

Instantaneous power

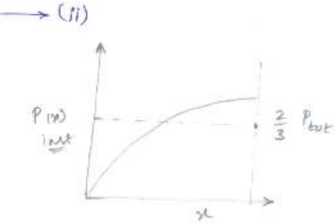
$P = F \cdot v = mgv$

$P(x) = mg\sqrt{2gx} = mg^2 t$

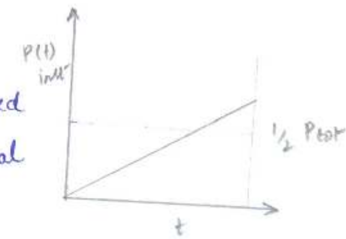
$\bar{P}_1 = \frac{1}{x}\int_0^x mg\sqrt{2gx'}\,dx' = \frac{mg\sqrt{2g}\ x^{3/2}}{x\,(3/2)} = \frac{2}{3} mg\sqrt{2gx} = \frac{2}{3} mg^2 t \longrightarrow \text{(i)}$

$\bar{P}_2 = \frac{1}{t}\int_0^t mg^2 t'\,dt' = \frac{mg^2}{2} t \longrightarrow \text{(ii)}$

The two methods give diffent powers
Power averged over distance travelled and
since $P \propto \sqrt{x}$, this avg power is more
than the time averaged power $\bar{P}(t)\,\Delta t$



If acceleration is zero and input force is balanced by some dissipative force, say friction, then the input power averaged over distance or time will be same and equal to instantaneous power during the ride.



when the differ the time averaged power is right because $\bar{P}(t)\cdot t$ then represents total input energy;

ie $\bar{P}(t) = \dfrac{\Delta E}{\Delta t}$

10.2.14 For problem 10.2.9 which answers would change, and in which way, if the deceleration was not exactly constant during the crash? That is, for which quantities would the answer be bigger, which smaller, which the same, for which would the answer depend on the nature of the non-constant acceleration?

9.2.14

If deacceleration is not constant, all answer will change except for part (a) ie the total energy dissipated.

answer to (b), (c), (d), (e) and (f) will depend on instantaneous deacceleration.

10.2.15 The earth's gravitational pull on a mass m is $F = -\frac{mgR^2}{r^2}$, where mg is the pull at the surface of the earth and R is the radius of the earth. Assume a ballistic rocket is shot straight up with a launch velocity of v_0 (measured in a 'fixed' not-rotating-with-the-earth frame). Assume the rocket goes in a straight radial line as the earth turns underneath it (relative to the surface of the earth this rocket would be launched somewhat to the West to cancel the earth's rotation). Assume the period of active thrust is negligibly short (hence the word *ballistic*: "relating to or characteristic of the motion of objects moving under their own momentum and the force of gravity").

a) Solve for v as a function of r (and some or all of m, g, R and v_0).

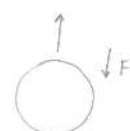
b) Find the maximum height the rocket reaches.

c) Find the 'escape velocity' v_{escape}, the minimum launch speed needed for the rocket to never return.

d) On one graph plot height (r or $r-R$) vs t for a v_0 just below v_{escape} and for v_0 just greater than v_{escape}. If you use numerical methods to make this plot use $g = 10\,\mathrm{m/s^2}$, $R = 6400\,\mathrm{km}$, and $m = 1\,\mathrm{kg}$. Make sure your axes are such that you can see a clear qualitative difference between the two cases.

9.2.15

a) $F = -\frac{mgR^2}{r^2}$



$$m\ddot{y} = -\frac{mgR^2}{(y+R)^2}$$

where y is the altitude of rocket

$$m\left(\frac{d}{dt}v\right) = -\frac{mgR^2}{(g+R)^2}$$

$$mv\frac{dv}{dy} = -\frac{mgR^2}{(y+R)^2}$$

$$\Rightarrow \int_{v_0}^{v} mv\,dv = -\int_0^h \frac{mgR^2}{(y+R)^2}\,dy$$

$$\frac{1}{2}m(v^2-v_0) = \frac{mgR^2}{(y+R)}\Big|_0^h = \left(\frac{mgR^2}{h+R}\right) - mgR$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 - \frac{mgR^2h}{R(R+h)}$$

$$v^2 = v_0^2 - \frac{2gRh}{R+h}$$

$$v = \sqrt{v_0^2 - \frac{2gRh}{R+h}}$$

b) Maximum height

$$0 = v_0^2 - \frac{2gRh_{max}}{R+h_{max}} \Rightarrow \frac{2gR}{v_0^2} = 1 + \frac{R}{h_{max}}$$

$$h_{max} = \frac{R}{\frac{2gR}{v_0^2} - 1} = \frac{Rv_0^2}{2gR - v_0^2}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

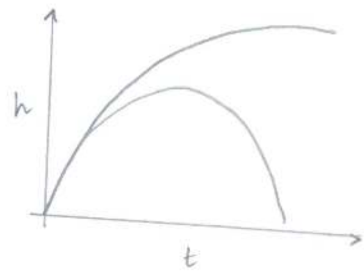
c) For v_0 = Velocity of Escape

$h_{max} \longrightarrow \infty$

$\Rightarrow 2gR - v_0^2 \longrightarrow 0$

$V_{escape} = \sqrt{2gR}$

d) $\dfrac{dh}{dt} = \sqrt{v_0^2 - \dfrac{2gRh}{R+h}}$



10.2.16 The power available to a very strong accelerating cyclist over short periods of time (up to, say, about 1 minute) is about 1 horsepower. Assume a rider starts from rest and uses this constant power. Assume a mass (bike + rider) of 150 lbm, a realistic drag force of $.006\,\mathrm{lbf}/(\mathrm{ft/s})^2 v^2$. Neglect other drag forces.

a) What is the peak (steady state) speed of the cyclist?

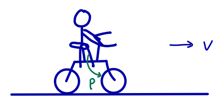
b) Using analytic or numerical methods make an accurate plot of speed vs. time.

c) What is the acceleration as $t \to \infty$ in this solution?

d) What is the acceleration as $t \to 0$ in your solution?

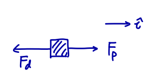
e) How would you improve the model to fix the problem with the answer above?

HW3 P10 Teaya Yang



Given: $P = 1\text{ HP} = 550\text{ lbf ft/s}$ $g = 1\text{ lbf/lbm} = 32.2\text{ ft/s}^2$
$V_0 = 0\text{ m/s}$
$m = 150\text{ lbm}$
$c = 0.006\text{ lbf s}^2/\text{ft}^2$

FBD:



* P = power supplied from the person's body chemistry to the mechanical energy of the system

a) Find V at steady state.

At steady state: $\Sigma \dot{E}_{in} = \Sigma \dot{E}_{out}$

Power Balance: $\Sigma\,\text{Power} = \dot{E}_K$

$P - P_d = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$

$P - F_d V = m V a$

$\frac{P}{V} - F_d = ma$ ①

LMB: $F_p \hat{\imath} - F_d \hat{\imath} = ma\,\hat{\imath}$

$F_p - F_d = ma$ ②

① and ②: $F_d = \frac{P}{V}$

Plugging this back into ② gives ODE: $\frac{P}{V} - cV^2 = m\ddot{x}$

At steady state, V = const. so $a = 0$

$0 = \frac{P}{V} - cV^2$

$V = \left(\frac{P}{c}\right)^{\frac{1}{3}} = \left(\frac{550\,\cancel{\text{lbf}}\,\text{ft/s}}{0.006\,\cancel{\text{lbf}}\,\text{s}^2/\text{ft}^2}\right)^{1/3} = \boxed{45.1\text{ ft/s}} = \boxed{30.8\text{ mph}}$

b) Plot $v(t)$

ODE: $m\ddot{x} = \frac{P}{V} - cV^2$

Integrated numerically in Matlab with ODE2 (Mid point method)

* V_0 had to be set to a small number to make sure $\dot{V}$ can be evaluated at $t=0$.

Plot is on next page.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

approximate
steady state
value

Terminal velocity reaches 45.1 ft/s as predicted by analytical results.

acceleration approaches
∞ at $t \to 0$

acceleration approaches
0 as $t \to \infty$

Since the difference in $a(t)$ is too significant between smaller and larger t, the axes are set to only show a section of the plot for visibility.

c) $a(t \to \infty) = ?$

Acceleration reaches 0 as $t \to \infty$. This is okay since we reach a terminal velocity at some point with the same power input.

d) $a(t \to 0) = ?$

Acceleration approaches ∞ as $t \to 0$. The largest acceleration due to friction given $\mu_{max} = 1$ is $a = \frac{\mu m g}{m} = \mu g = 32.2\ \text{ft/s}$, so an acceleration greater than this value does not make sense.

This disagreement betwee our solution and real life system indicates that our model fails at small t, see part e) for possible improvement on model.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

* See Matlab code generating the above plots below, part e) follows after.

Contents


```
% MAE 2030 HW3 P10 Sprig 2021
% Numerically integrates for the velocity of cyclist over time
% ODE2 function taken from Michelle DeSouza
```

Solving

Parameters

```
p.m = 150; % [lbm]
p.P = 550; % [lbf*ft/s]
p.c = 0.006; % [lbf*s^2/ft^2]

% IC
x0 = 0;
v0 = .0001; % Small number so that zdot can be evaluated at t=0
z0 = [x0 v0]';

% Time
tend = 1500; % Final time
tspan = [0 tend];

% Equation
eq = @(t,z) rhs(t,z,p); % Make rhs an anonymous function

% Set up options
options.RelTol = 1e-5; % Given accuracy
options.maxtime = 600; % Give up if accuracy not achieved in this many seconds.

% Solve
[tarray,zarray] = ODE2(eq,tspan,z0,options);
```

Plotting

```
% Plot v(t)
xsol = zarray(:,1); vsol = zarray(:,2);
figure(1)
hold on
box on
plot(tarray,vsol,'b-')
xlabel('t(s)')
ylabel('v(ft/s)')
title('Velocity of Cyclist')

% Plot a(t)
asol = (p.P./vsol - p.c*vsol.^2)/p.m;
figure(2)
hold on
box on
plot(tarray,asol,'b-') % Eliminating first points
```

```
axis([0 2 0 35])
xlabel('t(s)')
ylabel('a(ft/s)')
title('Acceleration of Cyclist')
```

RHS Equation

```
function zdot = rhs(t,z,p)
% Unpack variables
x = z(1);
v = z(2);

% Unpack parameters
m = p.m;
P = p.P;
c = p.c;

% Equation
xdot = v;
vdot = (P/v - c*v^2)/m;

zdot = [xdot vdot]';

end
```

Published with MATLAB® R2020b

```
function [tarray zarray] = ODE2(eqn, tspan, z0, options)
%Solve any ODE with midpoint method
%From Michelle's solution to P8

tstart = tspan(1);
tend = tspan(2);

%unpack options
RelTol = options.RelTol; % max allowable relative error
maxtime = options.maxtime; % max computer time used

tic; % start keeping track of time

% The number of points in progressively more accurate solutions
nsols = 30; %number of solutions
npoints_array = 5*2.^[1:nsols]; % 10, 20, 30, 80, 160 ... 5*2^30.
%Initialize array of solutions at the final time
zendarray = zeros(nsols,length(z0)); % array of the solutions at tend

% Outer loop, progressively more accurate solutions,
% Each time through the loop is a new solution.
    for j = 1:nsols % iteration loop
    npoints = npoints_array(j); %number of points in this solution
    %disp(['Number of points this iteration is: 'num2str(npoints)])

    % Initialize various arrays
    tarray = linspace(tstart, tend, npoints); %Soln at these times
    zarray = zeros(npoints, length(z0)); % Claim memory for zarray
    z = z0; % z is present value of z, used in loop below
    zarray(1,:) = z'; % The first row is the initial condition z0.
    %%%%%%%%%%%%%%%%%%
        for i = 1:npoints-1 % Euler's method is in this 6-line loop
        ti = tarray(i);
        h = tarray(i+1) - tarray(i);
        zdoti = eqn(ti,z); %evaluate midpoint zdot!

        %evaluate midpoint zdot
        tmid = ti + h/2; %calculate time at half step
        zmid = z + zdoti*(h/2); %calculate z at half step
        zdotmid = eqn(tmid,zmid); %calculate zdot at half step

        %the next lines are the same as Euler's method
        znew = z + zdotmid*h; % use midpoint zdot to take a step
        zarray(i+1,:) = z'; % add a new row to the solution array
        z = znew; % the new value of z becomes the present value

            if toc>maxtime % Timeout if it is taking too long
            disp([' *** Max time has passed: '
            num2str(maxtime) ' seconds. ***'])
            disp([' (Solution up to timeout has been saved.' ...
            ' Increase maxtime or RelTol or both.)'])
            tarray = tarray(1:i); % the time so far
            zarray = zarray(1:i,:); % save the solution so far
            break % breaks out of Euler method loop
            end % of max time check if

        end % of Euler's method

    %%%%%%%%%%%%%%%%%%
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

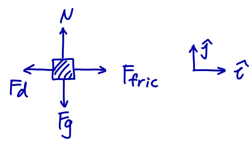
```
    if toc>maxtime % Timeout if it is taking too long
        break % breaks out of iterative solution loop
    end % check max time if

    zendarray(j,:) = zarray(end,:); % add to list of solutions the last line of most recent solution

        if j>1 % compare solutions once you have more than one solution
        % More or less arbitrary measure of error
        error = (zendarray(j,:) - zendarray(j-1,:)); % differences between this solution and the previous
        error = sum(abs(error)); % sum all the differences as a measure of error
        error = error/sum(zendarray(j,:)); %Compare error to value of z
        %disp([' Error this iteration is: ' num2str(error)])
        %disp([' Time spent solving so far is: ' num2str(toc)' seconds.'])

            if error < RelTol
                disp(['Success! RelTol of ' num2str(RelTol) ' achieved using Midpoint method.'])
                break; % exits the outer loop
            end % of error if

        end % of error check if

    end % of iteration loop

end % of function
```

Published with MATLAB® R2020b

e) What can we do to change the model?

(From Prof. Ruina)

Using the maximum friction force between the bike and the ground as the initial bounded force that can be exerted on the bike, our model can be modified as follows:



$$\vec{F}_{fric} = \mu m g \, \hat{\imath}$$

Maximum friction coefficient between bicycle tire and asphalt road: $\mu = 1$

We can use this force until the power becomes $P = 1\,hp$.

$$P_{fric} = \mu m g v$$

ODE for low velocity: $m\ddot{x} = \mu m g - c v^2$ (I)

ODE for high velocity: $m\ddot{x} = \frac{P}{v} - c v^2$ (II) (same as before)

So use (I) until $P_{fric} = P$ and then use (II) till t_{end}.

This is done in Matlab.

Plot showing the power that we use:

So, highlighted part would be what make up our new model. The power would be constant except the initial part.

e) Results :

* The same v(t) vs. t graph is shown with 2 different time axes below to show the comparison between different times.

more rapid increase in old model

Same steady state velocity

The model with constant power causes velocity to increase more rapidly initially.

The velocities converge and reach the same steady state level for $t \to \infty$.

* The same a(t) plot is shown in two different time axes below.

infinite acceleration

constant acceleration

decreasing acceleration

accelerations converge

Initially, the acceleration of the constant power model starts at infinity and decreases rapidly, whereas the constant force model ensures a constant acceleration.

Acceleration of the new model starts at a finite value due to the constant friction force, then it decreases and approaches 0 at $t \to \infty$.

* More plots on next page.

constant power

same constant power

increasing power

This plot shows the comparison between the power variations over time of the two models. While power is constant in the old model, the new model we are using has an initially rapidly increasing power and it becomes constant once P=1hp is reached.

infinite force requred

finite maximum force initially

output force approaches constant

This is only the force due to outputted power (not including drag), so unlike acceleration this force approaches a constant. In the original model, an infinite force is required at $t=0$ to produce the target output power, while in the new model, the cyclist starts at a maximum force and the amount of force required to output the same power decreases after a few seconds due to the increasing velocity.

See next page for Matlab script for new model.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

Contents


```
% MAE 2030 HW3 P10 Sprig 2021
% Cyclist with maximum friction force, drag force, and constant
% power(eventually)
% Numerically integrates for the velocity of cyclist over time
% Solved using midpoint method wrtitten by Michelle DeSouza
```

Solve the problem first using constant friction

```
% Parameters
p.m = 150; % [lbm]
p.P = 550; % [lbf*ft/s}
p.c = 0.006; % [lbf*s^2/ft^2]
p.g = 1; % [lbf/lbm]
p.mu = 1;

% IC
x0 = 0;
v0 = 0; % It's okay to have 0 initial velocity now!
z0 = [x0 v0]';

% Time
tend = 10; % time in which we search for when power becomes > 1hp
tspan = [0 tend];

% Equation
eq1 = @(t,z) rhs1(t,z,p); % Make rhs1 an anonymous function

% Set up options
options.RelTol = 1e-5; % Given accuracy
options.maxtime = 600; % Give up if accuracy not achieved in this many seconds.

% Solve
[tarray1,zarray1] = ODE2(eq1,tspan,z0,options);
```

Extract points with power below 1hp from first solution

```
% Calculate all Pfric using solved velocities
x1 = zarray1(:,1); v1 = zarray1(:,2);
Pfric = v1*p.mu*p.m*p.g;
Pgiven = p.P*ones(length(tarray1),1);

% Find element of Pfric closest to 1hp
[~,idx] = min(abs(Pfric-p.P));

% Plot What we did just to be clear
```

```
figure(1)
hold on
box on
plot(tarray1,Pfric,'b-') % plot Pfric
plot(tarray1,Pgiven,'r--'); % Plot constant P
legend('Power from friction','Constant power given')
xlabel('t(s)')
ylabel('Power(lbf*ft/s)')
title('Acceleration of Cyclist')

% Then trim everything to this length
x1 = x1(1:idx); v1 = v1(1:idx);
tarray1 = tarray1(1:idx);
Pfric = Pfric(1:idx);
a1 = (p.mu*p.m*p.g - p.c*v1.^2)/p.m; % Calculate acceleration
```

Integrating with constant power

```
% Now the last points from before become ICs
x0 = x1(end); v0 = v1(end);
z0 = [x0 v0]';

% Time
tend = 1500;
tspan = [0 tend];

% Equation
eq2 = @(t,z) rhs2(t,z,p); % Make rhs2 an anonymous function

% Set up options
options.RelTol = 1e-5; % Given accuracy
options.maxtime = 600; % Give up if accuracy not achieved in this many seconds.

% Solve
[tarray2,zarray2] = ODE2(eq2,tspan,z0,options);
```

Put two parts together

```
tarray2 = tarray2 + tarray1(end); % make tarray2 start at end of tarray1
t = [tarray1,tarray2]; % put tarrays together
x2 = zarray2(:,1); v2 = zarray2(:,2);
x = [x1;x2]; v = [v1;v2]; % put x and v together
narrays = length(x); % length of all the arrays in final solution
P = [Pfric;p.P*ones(narrays-idx,1)]; % put power together
F = P./v; % Driving force
a2 = (p.P./v2 - p.c*v2.^2)/p.m; % acceleration from second part
a = [a1;a2]; % put acceleration together
```

Solve Original Model like in HW3P10 to Compare

IC

```
x0 = 0;
v0 = .0001; % Small number so that zdot can be evaluated at t=0
z0 = [x0 v0]';
```

```
% Time
tend = 1500; % Final time
tspan = [0 tend];

% Equation
eq = @(t,z) rhs2(t,z,p); % Make rhs an anonymous function

% Set up options
options.RelTol = 1e-5; % Given accuracy
options.maxtime = 600; % Give up if accuracy not achieved in this many seconds.

% Solve
[t2,z2] = ODE2(eq,tspan,z0,options);
v2 = z2(:,2);
```

Make various plots to show final solution

Plot v(t)

```
figure(2)
hold on
box on
plot(t,v,'b-')
plot(t2,v2,'r--')
% axis([0 1500 0 50])
axis([0 100 0 30]) %zoomed in view
xlabel('t(s)')
ylabel('v(ft/s)')
title('Velocity of Cyclist')
legend('New Model','Original Model')
```

Plot a(t)

```
a2 = (p.P./v2 - p.c*v2.^2)/p.m;
figure(3)
hold on
box on
plot(t,a,'b-')
plot(t2,a2,'r--')
legend('New Model','Original Model')
axis([0 20 0 5])
xlabel('t(s)')
ylabel('a(ft/s^2)')
title('Acceleration of Cyclist')
```

Plot power

```
P2 = p.P*ones(length(t2),1);
figure(4)
hold on
box on
plot(t,P,'b-')
plot(t2,P2,'r--')
legend('New Model','Original Model')
axis([0 25 0 600])
```

```
xlabel('t(s)')
ylabel('P(lbf*ft/s)')
title('Power on Bike')
```

Plot Force

```
F2 = P2./v2;
figure(5)
hold on
box on
plot(t,F,'b-')
plot(t2,F2,'r--')
legend('New Model','Original Model')
axis([0 35 0 200])
xlabel('t(s)')
ylabel('F(lbf)')
title('Drving Force on Bike')
```

RHS Equations

```
% RHS with constant friction force
function zdot = rhs1(t,z,p)

% Unpack variables
x = z(1);
v = z(2);

% Unpack parameters
m = p.m;
c = p.c;
mu = p.mu;
g = p.g;

% Equations
xdot = v;
vdot = (mu*m*g - c*v^2)/m;

zdot = [xdot vdot]';

end

% RHS with constant power
function zdot = rhs2(t,z,p)

% Unpack variables
x = z(1);
v = z(2);

% Unpack parameters
m = p.m;
c = p.c;
P = p.P;

% Equations
xdot = v;
vdot = (P/v - c*v^2)/m;

zdot = [xdot vdot]';
```

```
end
```

Published with MATLAB® R2020b