10.1 Force and motion in 1D

Now we focus on special problems in which one particle moves on a straight line. With motion in only one direction, the kinematics is simple. It's essentially a rehash of freshman calculus. Even in 1D, vectors can be useful because of their help with signs. But vectors are not really needed and we will not be zealous in their use (in this one chapter).

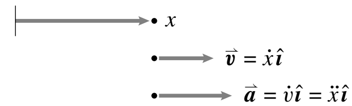

Figure 10.1: One-dimensional position, velocity, and acceleration. Positive values for $x, \dot{x}$ and $\ddot{x}$ describe positions, velocities and accelerations in the positive x direction.

Position, velocity, and acceleration in one dimension

We can call the direction of motion the $\hat{\imath}$ direction and the position of the particle, the distance of a particle in the $+\hat{\imath}$ direction from a reference point, x (see fig. 10.1). The particle might have some spacial extent, so to be precise we can define x as the x coordinate of the particle's center-of-mass. We can write the position $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ as

$$
\begin{array}{llllll}
 & \text{position} & \vec{r} = x\hat{\imath} & & & = x\hat{\imath}, \\
 & \text{velocity} & \vec{v} = v\hat{\imath} & = \dfrac{dx}{dt}\hat{\imath} & & = \dot{x}\hat{\imath}, \\
\text{and} & \text{acceleration} & \vec{a} = a\hat{\imath} & = \dfrac{dv}{dt}\hat{\imath} & = \dfrac{d^2x}{dt^2}\hat{\imath} & = \ddot{x}\hat{\imath}.
\end{array}
\tag{10.1}
$$

Figure 10.2 shows example graphs of $x(t)$ and $v(t)$ versus time.

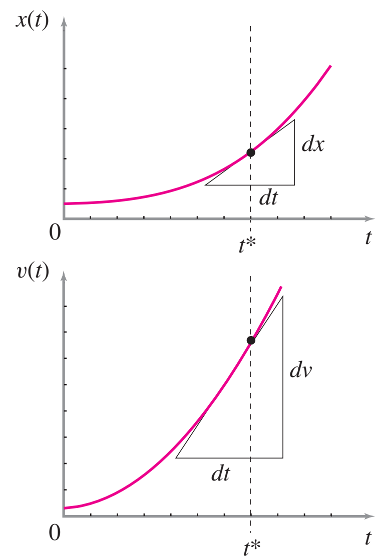

Figure 10.2: Graphs of $x(t)$ and $v(t) = \frac{dx}{dt}$ versus time. The slope of the position curve dx/dt at t^* is $v(t^*)$. And the slope of the velocity curve dv/dt at t^* is $a(t^*)$.

Signs. Vectors help with signs. When not using vectors we will take v and a to be positive if they have the same direction as increasing x (or y or whatever coordinate describes position). Even though we pedantically declare that 'velocity is a vector' and 'acceleration is a vector', we will loosely use the words 'velocity' and 'acceleration' to stand for scalars using these sign conventions.

Example: Position, velocity, and acceleration in one dimension
If position is given as

$$
\begin{array}{rl}
 & x(t) = 3e^{4t/\mathrm{s}}\,\mathrm{m} \\
\text{then} & v(t) = dx/dt = 12e^{4t/\mathrm{s}}\,\mathrm{m/s} \\
\text{and} & a(t) = dv/dt = 48e^{4t/\mathrm{s}}\,\mathrm{m/s^2}.
\end{array}
$$

So at, say, time $t = 2\,\mathrm{s}$ the acceleration is

$$a\big|_{t=2\,\mathrm{s}} = 48e^{4\cdot 2\,\mathrm{s}/\mathrm{s}}(\mathrm{m/s^2}) = 48 \cdot e^8\,\mathrm{m/s^2} \approx 1.43 \cdot 10^5\,\mathrm{m/s^2}.$$

Fussing with units. In this example above we used 1/s as part of the argument of the exponential function. Thus when the exponential $e^{4t/\mathrm{s}}$ is differentiated with respect to time t the 1/s is grouped with the 4 as the coefficient of t in the exponential. So the factor 4/s comes out front according to the chain rule of differentiation. Treating units as quantities, manipulated like all others makes the answer come out with the right units also. Note in the last line that, when the dimensional quantity (2 s) is substituted in for the variable t, the units cancel. For more on units see appendix 4.

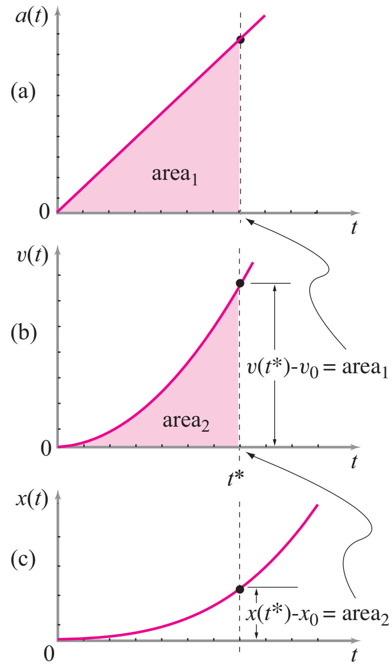

Figure 10.3: One-dimensional kinematics of a particle: (a) is a graph of the acceleration of a particle $a(t)$; (b) is a graph of the particle velocity $v(t)$ and the integral of $a(t)$ from $t_0 = 0$ to t^*, the shaded area under the acceleration curve; (c) is the position of the particle $x(t)$ and the integral of $v(t)$ from $t_0 = 0$ to t^*, the shaded area under the velocity curve.

1D kinematics ⇔ calculus. One-dimensional kinematics problems can include almost all of the skills in elementary calculus. In kinematics you are often given position, velocity or acceleration as function of time and you have to differentiate or integrate to find one of the other quantities. For example, if you are given the velocity $v(t)$ as a function of time and are asked to find the acceleration $a(t)$, you have to differentiate. If instead you were asked to find the position $x(t)$, you would calculate an integral (see fig. 10.3). Using the fundamental theorem of calculus, we get the integral versions of the relations between position, velocity, and acceleration (see fig. 10.3).

$$x(t) \;=\; x_0 + \int_{t_o}^{t} v(\tau)\,d\tau \qquad \text{with} \qquad x_0 = x(t_0), \text{ and}$$

$$v(t) \;=\; v_0 + \int_{t_o}^{t} a(\tau)\,d\tau \qquad \text{with} \qquad v_0 = v(t_0).$$

With indefinite notation, these equations can also be written as:

$$\begin{aligned} x &= \textstyle\int v\,dt \\ v &= \textstyle\int a\,dt. \end{aligned}$$

If acceleration is given as a function of time, then position is found by integrating twice.

1D kinematics, bicycles and calculus. To put it another way, almost every calculus question could be phrased as a question about a bicycle speedometer①. With a bicycle speedometer (which includes a distance-measuring odometer) you can read your speed and distance travelled as functions of time. And given one of those two functions you could find the other using calculus. Acceleration is also of interest, but few bicycle speedometers also measure acceleration.

① To cover the range of calculus problems you need to be a very good rider, however, and be able to ride frontwards, backwards, at zero speed and infinitely fast.

Differential equations

A *differential equation* is an equation that involves derivatives. Thus the equation relating position to velocity is

$$\frac{dx}{dt} = v \qquad \text{or, more explicitly} \qquad \frac{dx(t)}{dt} = v(t),$$

is a differential equation. An *ordinary differential equation* (ODE) is an equation that contains some terms that are ordinary derivatives (as opposed to partial derivatives and partial differential equations which we don't use in this book).

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Example: Calculating a derivative solves an ODE
Given that the height of an elevator as a function of time on its 5 seconds long 3 meter trip from the first to second floor is

$$y(t) = (3\,\mathrm{m})\frac{\left(1 - \cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}$$

we can solve the differential equation $v = \frac{dy}{dt}$ by differentiating to get

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[(3\,\mathrm{m})\frac{\left(1 - \cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}\right] = \frac{3\pi}{10}\sin\left(\frac{\pi t}{5\,\mathrm{s}}\right)\ \mathrm{m/s}$$

Note: this would be a harsh elevator because of the jump in the acceleration (not calculated above) at the start and stop.

A little less trivial is the case when you want to find a function when you are given the derivative.

Example: Integration solves a simple ODE
Assume that you start at home ($x = 0$) and, over about 30 seconds, you accelerate towards a steady-state speed of $4\,\mathrm{m/s}$ according to (see fig. 10.4)

$$v(t) = 4(1 - e^{-t/(30\,\mathrm{s})})\ \mathrm{m/s}.$$

Your ride lasts 1000 seconds. We can find how far you go by solving

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0.$$

This is simply solved by integration. Say, after 1000 seconds

$$\begin{aligned}
x(t = 1000\,\mathrm{s}) &= \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} 4(1 - e^{-t/(30\,\mathrm{s})})(\mathrm{m/s})\,dt \\
&= \left(4t + (120\,\mathrm{s})e^{-t/(30\,\mathrm{s})}\right)\Big|_0^{1000\,\mathrm{s}}\ \mathrm{m/s} \\
&= \left((4\cdot 1000\,\mathrm{s} + (120\,\mathrm{s})e^{-100/3}) - (0 + (120\,\mathrm{s})e^{0})\right)\ \mathrm{m/s} \\
&= \left(4000 - 120 + 120e^{-100/3}\right)\ \mathrm{m} \\
&\approx 3880\,\mathrm{m} \qquad \text{(to within an angstrom or so)}
\end{aligned}$$

This is only 120 m less than if the whole trip was travelled at a steady $4\,\mathrm{m/s}$ (then $x = 4\,\mathrm{m/s} \times 1000\,\mathrm{s} = 4000\,\mathrm{m}$).

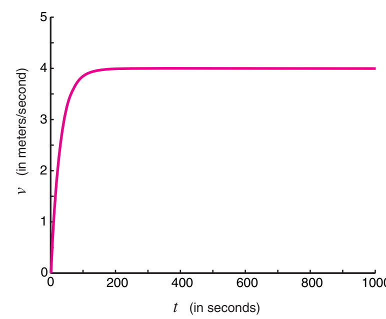

Figure 10.4: Velocity (speed) vs time for a 1000 s bike trip. The rider increases speed to exponentially approach 4 m/s.

Unlike the integral above, many integrals cannot be evaluated by hand (analytically).

Example: An ODE that leads to an intractable integral
Assume now that

$$v(t) = \frac{4t}{t + e^{-t/(30\,\mathrm{s})}\,\mathrm{s}}\ \mathrm{m}.$$

Again we have a bike trip where you start at zero speed and approach a steady speed of $4\,\mathrm{m/s}$. So your position as a function of time should be similar. Following the last example, we have

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0$$

with the given $v(t)$. The integral for position is then

$$x(t=1000\,\mathrm{s}) = \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} \frac{4t}{t+e^{-t/(30\,\mathrm{s})}\,\mathrm{s}}\,\mathrm{m}\,dt \qquad (10.2)$$

$$= \ldots$$

which is the kind of thing you have nightmares about seeing on an exam. You couldn't solve this integral if your life depended on it. No one could. There is no formula for $x(t)$ that solves the differential equation, unless you regard eqn. (10.2) as a formula. In days of old they would say 'the problem has been reduced to quadrature' meaning that the remaining work was evaluating an integral②, even if they didn't know how to evaluate it exactly.

②Literally *quadrature* means finding a square (a 'quad') with area equal to the area under a given curve. The phrase 'reduced to quadrature' is used more generally to mean integration, even if the integration is of several variables.

Just because a differential equation can't be solved analytically with pencil and paper doesn't mean it can't be solved numerically. Most often the setup for numerical solution is not that difficult. Note that for numerical solution you either need dimensionless calculations, or at least need all variables to be in consistent units.

Example: Numerical solution of ODE
One of many ways to evaluate the integral of the above example numerically is by the following pseudo code.

```
ODE = { xdot = 4 t / (t+e^(-t/30)) }
IC  = { x(0) = 0  }
solve ODE with IC and evaluate at t=1000
```

The result is $x \approx 3988\,\mathrm{m}$ which is also, as expected because of the similarity with the previous example, only slightly shy of the steady-speed approximation of 4000 m. ③

③**Numerical error vs real difference.** When you notice such small differences (12m out of 4000m) based on computer calculation you need to question whether the difference is something real in the problem or, rather, is due to numerical errors. Two ways to check are with so-called convergence tests and by using your canned package's error estimate. We checked that the numerical integration is accurate to about 10^{-3}m, less than the 1m resolution that we printed (no need to type lots of digits with little information). Thus the $\approx$ 12m difference between the constant v solution and the solution where v approaches a constant, is a real difference. The startup is genuinely quicker in the second example (12 m behind constant v versus 120 m behind constant v), and not a numerical artifact.

More differential equations.

As mentioned, because dynamics equations contain derivatives they are all differential equations. A catalogue of the simplest differential equations and their solutions is given in box 3.1 on page 203.

The equations of dynamics

We want to understand kinetics (mechanics, dynamics), not just kinematics. The subject of mechanics is held up by the three pillars of material properties, geometry, and 'Newton's laws' (see page 28). Here we begin to flesh out the 'Newton's laws' pillar beyond statics (the first 8 chapters of this book), using kinematics (we just started with that above) to the third pillar, dynamics.

Linear momentum balance

For a particle moving in the x direction the velocity and acceleration are $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. Thus the linear momentum and its rate of change are

$$\begin{aligned}\vec{\boldsymbol{L}} &\equiv \sum m_i \vec{\boldsymbol{v}}_i &= m\vec{\boldsymbol{v}} = mv\hat{\boldsymbol{\imath}}, \text{ and}\\ \dot{\vec{\boldsymbol{L}}} &\equiv \sum m_i \vec{\boldsymbol{a}}_i &= m\vec{\boldsymbol{a}} = ma\hat{\boldsymbol{\imath}}.\end{aligned}$$

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Using any of the free-body diagrams in fig. 10.5, where $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$, the equation of linear momentum balance, ④ eqn. I from the front inside cover, or equation 12.1 reduces to:

$$F\hat{\boldsymbol{\imath}} = ma\hat{\boldsymbol{\imath}} \tag{10.3}$$

which in scalar form is the central subject of this section.

$$F = ma$$

In scalar form, F is the net force to the right and a is the acceleration to the right. For the equation $F = ma$ to have content each of the terms must have some meaning in other contexts. And, at least intuitively, each does (see box 10.1 on page 9).

Force. The force F could come from a spring, or a fluid or from your hand pushing the thing to the right or left, or any combination of these things. The most general case we want to consider here is that the force is determined by the position and velocity of the particle as well as the present time. Thus

$$F = f(x, v, t). \tag{10.4}$$

What do we mean 'determined by'? We mean that we have an independent way of knowing the force from its position, velocity and time, even without thinking yet about the linear momentum balance equation $F = ma$. Special cases would be, say,

$$\begin{array}{lclll}
F &=& f(t) & = F_0 \sin(\beta t) & \text{an oscillating load,} \\
F &=& mg & & \text{the force of earth's gravity} \\
F &=& f(v) & = -cv & \text{a linear viscous drag,} \\
F &=& f(x) & = -kx & \text{a linear spring, and} \\
F &=& f(x,v,t) & = -kx - cv + F_0 \sin(\beta t) & \text{a combination of forces.}
\end{array}$$

All elementary 1D particle mechanics problems can be reduced to the solution of this pair of coupled first order differential equations,

$$\begin{aligned}
\frac{dv}{dt} &= \underbrace{f(x,v,t)/m}_{a(t)} \quad (a) \\
\frac{dx}{dt} &= v(t) \qquad\qquad\; (b)
\end{aligned} \tag{10.5}$$

where the function $f(x, v, t)$ is given and $x(t)$ and $v(t)$ are to be found.

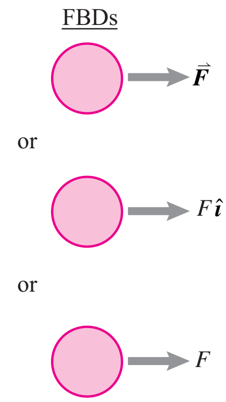

Figure 10.5: Three possible free-body diagrams for 1D particle motion. It is assumed that $\vec{\boldsymbol{F}}$ represents the sum of all forces acting on the particle. Note, even though the inertial term ma is of special interest, it does not show on the FBD. See the free-body diagram Chapter 3 for the prescription about this. See box 15.2 on page 730 for a further warning about d'Alembert 'forces'.

④**Angular momentum?** We skip thinking about angular momentum balance in this section. Why? If we pick an origin on the line of travel, all terms on both sides of all angular momentum balance equations are zero, and the equation $0 = 0$ tells us nothing new.

Know a solution when you see one. How can you tell if candidate functions solve a differential equation? First you can tell that the initial conditions are satisfied by evaluating the expressions at $t = 0$. To check that the differential equations are satisfied, you plug the candidate solutions into the equation and see that an identity results. Differential equations are satisfied when the unknown functions therein are replaced with specific functions that make the equations correct.

Viscous drag. If the only applied force is a viscous drag, $F = -cv$ (see fig. 10.6), then linear momentum balance ($F = ma$) would be $-cv = ma$ and Eqns. 10.5 are

$$\begin{aligned}\frac{dv}{dt} &= -cv/m \\ \frac{dx}{dt} &= v\end{aligned}$$

where c and m are constants and $x(t)$ and $v(t)$ are yet to be determined functions of time. Because the force only slows the particle there will be no motion, unless the particle was given some initial velocity. In general, you need to specify the initial position and velocity to find a solution. So we complete the problem statement by specifying initial conditions

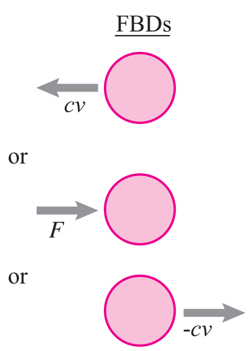

Figure 10.6: Three acceptable free-body diagrams showing the viscous drag on a particle moving to the right at speed v. The second FBD depends on being told on the side, say here, that $F = -cv$.

Example: Slowing with viscous drag
Find $x(t)$ given that

$$x(0) = x_0 \quad \text{and} \quad v(0) = v_0$$

where x_0 and v_0 are given constants. Before worrying about how to solve such equations, you should know how to recognize a solution. The following two functions, assume for now that they fell from the sky, solve the differential equations.

$$\begin{aligned} v(t) &= v_0 e^{-ct/m}, \qquad \text{and} \\ x(t) &= x_0 + m v_0 (1 - e^{-ct/m})/c \end{aligned}$$

Plugging this presumed solution for $v(t)$ into $\dot{v} = -cv/m$ gives, and this is what we want, $0 = 0$. And similarly, when the presumed solution for $x(t)$ is plugged in to $\dot{x} = v$ you also get the 'satisfying' result that $0 = 0$.

Replacing the unknown functions $v(t)$ and $x(t)$ with the given formulas gives an identity. Thus the given formulas satisfy (or solve) the differential equations. Just like the case of integration (or equivalently the solution for x of the ODE $\dot{x} = v(t)$), one often cannot find formulas for the solutions of differential equations.

Example: A dynamics problem with no pencil and paper solution
Consider the following case which models a particle in a sinusoidal force field with a second applied force that oscillates in time. Using the dimensional constants c, d, F_0, β, and m,

$$\begin{aligned}\frac{dv}{dt} &= \left(c \sin(x/d) + F_0 \sin(\beta t)\right)/m \\ \frac{dx}{dt} &= v\end{aligned}$$

with initial conditions $x(0) = 0$ and $v(0) = 0$.

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There is no known formula for $x(t)$ that solves this ODE.

Just writing the ordinary differential equations and initial conditions is analogous to setting up an integral in freshman calculus. The solution is reduced to quadrature. Because numerical solution of sets of ordinary differential equations is a standard part of all modern computation packages you are in some sense done when you get this far. A computer can finish up for you.

Some special cases in 1D mechanics

There are various special cases of eqn. (10.5) which have simple solutions.

Example: The simplest dynamics problem.
1D, particle, no force. Formally working out the details,

$$\begin{array}{llrcl}
F = ma \text{ with } F = 0 & \Rightarrow & 0 & = & ma \\
\text{definition of } a & \Rightarrow & \dot{v} & = & 0 \\
\text{integrating} & \Rightarrow & v & = & v_0 \quad (= \text{any constant}) \\
\text{integrating again} & \Rightarrow & x & = & x_0 + v_0 t
\end{array}$$

In a sense we have thus derived Newton's first law, 'an object in motion tends to stay in motion unless acted upon by a force'.

Constant force. Another simple case is constant force F which leads to constant acceleration $a = F/m$. Using calculus you should know well by now, you get the following formulas:

$$\begin{aligned}
a = const \Rightarrow \quad x &= x_0 + v_0 t + at^2/2 \\
a = const \Rightarrow \quad v &= v_0 + at \\
a = const \Rightarrow \quad v &= \pm\sqrt{v_0^2 + 2a(x - x_0)}.
\end{aligned}$$

These are much seen in high school physics because, by permuting what is given and what is unknown, one can make up 100 homework problems that can be solved with these formulas and without calculus.

Force given as a function of time. Say F is given as $F = F(t)$. This general case shows up when some kind of motor force is controlled by a human or computer to vary in time is some predetermined manner.

$$\begin{array}{llrcl}
F = ma \text{ with } F = F(t) & \Rightarrow & F(t) & = & ma \\
\text{definition of } a & \Rightarrow & \dot{v} & = & F(t)/m \\
\text{integrating} & \Rightarrow & v & = & v_0 + \frac{1}{m}\int_0^t F(\tau)\, d\tau
\end{array}$$

And we have to integrate once again to get position.

Example: Ramping up the acceleration at the start
If you get a car going by gradually depressing the 'accelerator' so that its acceleration increases linearly with time, we have

$$
\begin{array}{rrclclcl}
 & a & = & ct & & \text{(take } t = 0 \text{ at the start)} & & \\
\Rightarrow & v(t) & = & \int_0^t a d\tau + v_0 & = & \int_0^t c\tau d\tau & = & ct^2/2 \\
 & & & & & \text{(since } v_0 = 0\text{)} & & \\
\Rightarrow & x(t) & = & \int_0^t v d\tau + x_0 & = & \int_0^t (c\tau^2/2) d\tau & = & ct^3/6 \\
 & & & & & \text{(since } x_0 = 0\text{).} & &
\end{array}
$$

The distance the car travels is proportional to the cube of the time that has passed from dead stop.

The overall subject of 'vibrations' is in some sense about what happens when something is shaken. We can think of 'shaking' as applying a force which varies sinusoidally in time.

Example: Force varies sinusoidally in time.
Assume a 1 kg mass starts from rest and has a force of $F = 2\cos(2\pi t/\mathrm{s})\,\mathrm{N}$ applied. That's a force that oscillates once per second with an amplitude of 2 N. What is the position at $t = 10\,\mathrm{s}$?

$$
\begin{array}{llrcl}
F = ma \text{ with } F = F(t) & \Rightarrow & 2\cos(2\pi t/\mathrm{s})\,\mathrm{N} & = & m\dot{v} \\
\text{integrating} & \Rightarrow & v & = & v_0 + \frac{1}{m}\int_0^t 2\cos(2\pi t/\mathrm{s})\,\mathrm{N}\,d\tau \\
\text{using freshman calculus} & \Rightarrow & v & = & v_0 + \frac{1}{\pi m}\sin(2\pi t/\mathrm{s})\,\mathrm{N\,s} \\
\text{IC: } v(0) = 0 \Rightarrow v_0 = 0 & \Rightarrow & v & = & \frac{1}{\pi m}\sin(2\pi t/\mathrm{s})\,\mathrm{N\,s} \\
\text{integrate again, using } \dot{x} = v & \Rightarrow & x & = & x_0 - \frac{1}{2\pi^2 m}\cos(2\pi t/\mathrm{s})\,\mathrm{N\,s}^2 \\
\text{IC: } x(0) = 0 \Rightarrow x_0 = \frac{1}{2\pi^2 m} & \Rightarrow & x & = & \frac{1}{2\pi^2 m}\left(1 - \cos(2\pi t/\mathrm{s})\right)\,\mathrm{N\,s}^2
\end{array}
$$

Now we can substitute in $m = 1\,\mathrm{kg}$ and $t = 10\,\mathrm{s}$ to get $x = 0.0\,\mathrm{m}$. The algebraic cancellation of units came about naturally from substituting in the definition of a Newton $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$. We carried the units through even though the final answer was 0.

Force depends on velocity. This case is encountered when, say, an object moves through a fluid and other forces, say gravity, are negligible. Here we have

$$F = ma \quad \Rightarrow \quad F(v) = m\dot{v}.$$

This is solved by multiplying both sides by dt and dividing both sides by $F(v)$ and integrating to get

$$\int dt = m\int \frac{dv}{F(v)} \quad \Rightarrow \quad t = m\int_{v_0}^{v} \frac{dv'}{F(v')}$$

If we want to know position vs time we have to integrate once again.

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Example: The slowing of a bullet.
The main force on a bullet after it leaves the gun and before it hits its mark is from air drag. This drag is roughly proportional to the speed squared, thus

$$F = ma \quad \Rightarrow \quad -cv^2 = m\dot{v} \quad \Rightarrow \quad -c\int dt = m\int_{v_0}^{v} \frac{dv'}{v'^2}.$$

Carrying out the integrals ($\int dt = t$ and $\int v^{-2}\,dv = -v^{-1}$) we get

$$ct = m\left(\frac{1}{v} - \frac{1}{v_0}\right) \quad \Rightarrow \quad v = \frac{v_0}{cv_0 t/m + 1}$$

To get position we would integrate again to get:

$$x = \int_0^t v(t')\,dt' = \int_0^t \frac{v_0}{cv_0 t'/m + 1}\,dt' = (m/c)\ln(1 + cv_0 t/m)$$

Interestingly, according to this equation (which becomes less and less accurate as the bullet slows and gravity and eventually viscous forces become important) the bullet goes an infinite distance before stopping.

Force varies with position. This case, where $F = F(x)$ will be treated in some detail in the next section on energy.

The simplest ODEs

The simplest and most common ODEs in dynamics, and in the rest of science and engineering, are

Linear: e.g., no functions squared.

First or second order: Have only first or second derivatives, respectively, and

Constant coefficient: All multiples of the derivatives are constants, not functions of time.

The most essential of these are

$$\dot{x} = 0, \dot{x} = C, \ddot{x} = 0, \ddot{x} = C, \dot{x} = Cx, \dot{x} = -Cx, \ddot{x} = Cx \;\&\; \ddot{x} = -Cx.$$

These are discussed in appendix 3.1 on page 203.

What do the terms in $F = ma$ mean?

Some say '$F = ma$ is the definition of force'. This is a legitimate point of view. But, if we care about force for other reasons, which we do, it is not useful. For the equation $F = ma$ to be useful we need to be able to think about F, m and a independently from each other.

Mass, m**.** Now that we know about atoms (these centuries) and what they are made of (these decades) we can approximately (with about one percent accuracy) define the mass of a system by (in principle) by counting up the total number of protons and neutrons and multiplying by $1.67 \cdot 10^{-27}$ kg. That is, mass is a measure of the extent of matter. Given that we think of mass as the amount of matter, we could more accurately and more easily use a reference volume of a pure chemical substance as a reference. This way, with a good balance and some trouble, we could get an accuracy of parts per million. Officially,

mass is measured in comparison to a fancy piece of metal

locked in a box in some basement in some government building. That calibrated kilogram is accurate to about a part in 20,000,000 (See Appendix 4 starting on page 225). We can find the mass of a more complicated thing using that reference mass and a (very good) balance.

Acceleration, a**.** Because this is a course in mechanics and not in philosophy of science, we will just accept the concepts of space (x) and time (t) as given and measurable (using rulers and timers). So acceleration is operationally well defined as

$$a = d^2x/dt^2$$

with no use of $F = ma$. We have presupposed that we measure position relative to a Newtonian reference system. This definition is sensible to at least to about one part per billion for most engineering purposes (see page 33).

Force, F**.** Here is where people argue. It's easiest to define force using the deformation of solids. When one thing pushes on another, think of your little finger as caught in between. How much your finger is squeezed, as measured by how loud you yell, is a measure of force. More technically, we could look at the small amounts of deformation occurring where the bodies contact, and use the deformation as a measure of force. Or, more practically, we could interpose a calibrated material and measure its deformation. Such a chunk of material with deformation-measuring electronics is called a *load cell.* Load cells are sold by the millions (say, in bathroom scales). A load cell uses nothing about $F = ma$ to operate accurately.

One reason it is nice to think of force as having a life away from $F = ma$ is that the whole coherent and useful subject of statics, useful for design-

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ing bridges and airplane landing gear, has little or no use for $F = ma$. Alternatively, still without thinking about $F = ma$, one could define force in terms of the net effect of earth's gravitational pull on a calibrated mass at some officially-ordained location like Potsdam.

However you like to define force, the great result is that with any one of several possible independent definitions, things mostly work out. Miraculously, the same concept of force F works no matter which of these three you take as defining:

$$F = mg, \tag{10.6}$$
$$F = kx, \text{or} \tag{10.7}$$
$$F = ma. \tag{10.8}$$

Pick your favorite as fundamental and use the others with confidence.

Units of force. Most beginners prefer that forces be measured in Newtons. One Newton is $1\ \mathrm{kg\,m/s^2}$. But people not living in the SI world tend instead to use the kgf, also called a kilogram force or kp or kilopond. One kgf is the weight of a kilogram. It's about 9.8 N. In the English system the force most analogous to the Newton is called the Poundal, it's $1\ \mathrm{lbm\,ft/s^2}$ and is little used. More commonly used is the pound force, lbf. It's the weight of a pound mass and is about 32.2 Poundals. Read more about units of force in box 4.3 on page 235.