SAMPLE 10.1 **Time derivatives:** The position of a particle varies with time as $\vec{\boldsymbol{r}}(t) = (C_1 t + C_2 t^2)\hat{\boldsymbol{\imath}}$, where $C_1 = 4\,\text{m/s}$ and $C_2 = 2\,\text{m/s}^2$.

1. Find the velocity and acceleration of the particle as functions of time.
2. Sketch the position, velocity, and acceleration of the particle against time from $t = 0$ to $t = 5\,\text{s}$.
3. Find the position, velocity, and acceleration of the particle at $t = 2\,\text{s}$.

Solution

1. We are given the position of the particle as a function of time. We need to find the velocity (time derivative of position) and the acceleration (time derivative of velocity).

$$\vec{\boldsymbol{r}} = (C_1 t + C_2 t^2)\hat{\boldsymbol{\imath}} = (4\,\text{m/s}\,t + 2\,\text{m/s}^2\,t^2)\hat{\boldsymbol{\imath}} \tag{10.9}$$

$$\vec{\boldsymbol{v}} \equiv \frac{d\vec{\boldsymbol{r}}}{dt} = \frac{d}{dt}(C_1 t + C_2 t^2)\hat{\boldsymbol{\imath}}$$

$$= (C_1 + 2C_2 t)\hat{\boldsymbol{\imath}} = (4\,\text{m/s} + 4\,\text{m/s}^2\;t)\hat{\boldsymbol{\imath}} \tag{10.10}$$

$$\vec{\boldsymbol{a}} \equiv \frac{d\vec{\boldsymbol{v}}}{dt} = \frac{d}{dt}(C_1 + 2C_2 t)\hat{\boldsymbol{\imath}}$$

$$= 2C_2\hat{\boldsymbol{\imath}} = (4\,\text{m/s}^2)\hat{\boldsymbol{\imath}} \tag{10.11}$$

$$\boxed{\vec{\boldsymbol{v}} = (4\,\text{m/s} + 4\,\text{m/s}^2\;t)\hat{\boldsymbol{\imath}},\;\vec{\boldsymbol{a}} = (4\,\text{m/s}^2)\hat{\boldsymbol{\imath}}.}$$

Thus, we find that the velocity is a linear function of time and the acceleration is time-independent (a constant).

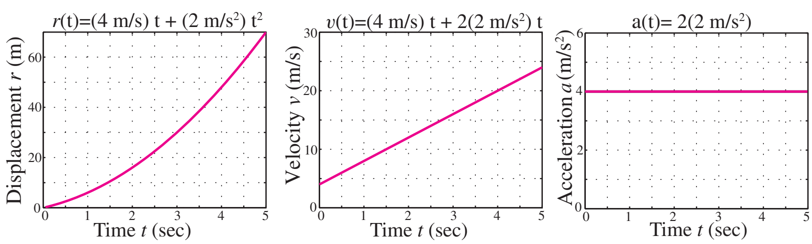

Figure 10.7:

2. We plot eqns. (10.9, 10.10, and 10.11) against time by taking 100 points between $t = 0$ and $t = 5\,\text{s}$, and evaluating $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ at those points. The plots are shown in fig. 10.7.
3. We can find the position, velocity, and acceleration at $t = 2\,\text{s}$ by evaluating their expressions at the given time instant:

$$\vec{\boldsymbol{r}}|_{t=2\,\text{s}} = [(4\,\text{m/s})\cdot(2\,\text{s}) + (2\,\text{m/s}^2)\cdot(2\,\text{s})^2]\hat{\boldsymbol{\imath}}$$

$$= (16\,\text{m})\hat{\boldsymbol{\imath}}$$

$$\vec{\boldsymbol{v}}|_{t=2\,\text{s}} = [(4\,\text{m/s}) + (4\,\text{m/s}^2)\cdot(2\,\text{s})]\hat{\boldsymbol{\imath}}$$

$$= (12\,\text{m/s})\hat{\boldsymbol{\imath}}$$

$$\vec{\boldsymbol{a}}|_{t=2\,\text{s}} = (4\,\text{m/s}^2)\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{a}}(\text{for all t})$$

$$\boxed{\text{At } t = 2\,\text{s},\;\vec{\boldsymbol{r}} = (16\,\text{m})\hat{\boldsymbol{\imath}},\;\vec{\boldsymbol{v}} = (12\,\text{m/s})\hat{\boldsymbol{\imath}},\;\vec{\boldsymbol{a}} = (4\,\text{m/s}^2)\hat{\boldsymbol{\imath}}.}$$

SAMPLE 10.2 Math review: Solving simple differential equations. For the following differential equations, find the solution for the given initial conditions.

1. $\dfrac{dv}{dt} = a,\ v(t=0) = v_0$, where a is a constant.
2. $\dfrac{d^2x}{dt^2} = a,\ x(t=0) = x_0,\ \dot{x}(t=0) = \dot{x}_0$, where a is a constant.

Solution

1.

$$\begin{aligned} \frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a\,dt \\ \text{or} \qquad \int dv &= \int a\,dt = a\int dt \\ \text{or} \qquad v &= at + C, \quad \text{where C is a constant of integration} \end{aligned}$$

Now, substituting the initial condition into the solution,

$$v(t=0) \equiv v_0 = a\cdot 0 + C \quad \Rightarrow \quad C = v_0.$$

Therefore,

$$v = at + v_0.$$

$$\boxed{v = v_0 + at}$$

Alternatively, we can use definite integrals:

$$\int_{v_0}^{v} dv = \int_0^t a\,dt \quad \Rightarrow \quad v - v_0 = at \quad \Rightarrow \quad v = v_0 + at.$$

2. This is a second order differential equation in x. We can solve this equation by first writing it as a first order differential equation in $v \equiv dx/dt$, solving for v by integration, and then solving again for x in the same manner.

$$\begin{aligned} \frac{d^2x}{dt^2} &= a \quad \text{or} \quad \frac{dv}{dt} = a \\ \text{or} \qquad \int dv &= \int a\,dt \\ \Rightarrow \quad v \equiv \dot{x} &= at + C_1 && (10.12) \\ \text{but, } v \equiv \frac{dx}{dt}, \quad \Rightarrow \quad \int dx &= \int at\,dt + \int C_1\,dt \\ \text{or} \qquad x &= \frac{1}{2}at^2 + C_1 t + C_2, && (10.13) \end{aligned}$$

where C_1 and C_2 are constants of integration. Substituting the initial condition for $\dot{x}$ in Eqn. (10.12), we get

$$\dot{x}(t=0) = \dot{x}_0 = a\cdot 0 + C_1 \quad \Rightarrow \quad C_1 = \dot{x}_0.$$

Similarly, substituting the initial condition for x in Eqn. (10.13), we get

$$x(t=0) = x_0 = \frac{1}{2}a\cdot 0 + \dot{x}_0\cdot 0 + C_2 \quad \Rightarrow \quad C_2 = x_0.$$

Therefore,

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}at^2.$$

$$\boxed{x(t) = x_0 + \dot{x}_0 t + \tfrac{1}{2}at^2}$$

SAMPLE 10.3 **Constant speed motion:** A ship cruises at a constant speed of 15 knots (15 nautical miles per hour) due Northeast. It passes a lighthouse at 8:30 am. The next lighthouse is approximately 35 nautical miles straight ahead. At what time does the ship pass the next lighthouse?

Solution We are given the distance s and the speed of travel v. We need to find how long it takes to travel the given distance.

$$\begin{aligned} s &= vt \\ \Rightarrow \quad t &= \frac{s}{v} = \frac{35\text{nautical miles}}{15(\text{nautical miles})/\text{hour}} = 2.33\text{ hrs.} \end{aligned}$$

Now, the time at $t = 0$ is 8:30 am. Therefore, the time after 2.33 hrs (2 hours 20 minutes) will be 10:50 am.

$$\boxed{10 : 50\text{ am}}$$

SAMPLE 10.4 **Constant velocity motion:** A particle travels with constant velocity $\vec{\boldsymbol{v}} = 5\,\text{m/s}\hat{\boldsymbol{\imath}}$. The initial position of the particle is $\vec{\boldsymbol{r}}_0 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$. Find the position of the particle at $t = 3\,\text{s}$.

Solution Here, we are given the velocity, *i.e.*, the time derivative of position:

$$\vec{\boldsymbol{v}} \equiv \frac{d\vec{\boldsymbol{r}}}{dt} = v_0\hat{\boldsymbol{\imath}}, \qquad \text{where } v_0 = 5\,\text{m/s}.$$

We need to find $\vec{\boldsymbol{r}}$ at $t = 3\,\text{s}$, given that $\vec{\boldsymbol{r}}$ at $t = 0$ is $\vec{\boldsymbol{r}}_0$.

$$\begin{aligned} d\vec{\boldsymbol{r}} &= v_0\hat{\boldsymbol{\imath}}dt \\ \Rightarrow \quad \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}(t)} d\vec{\boldsymbol{r}} &= \int_0^t v_0\hat{\boldsymbol{\imath}}dt = v_0\hat{\boldsymbol{\imath}}\int_0^t dt \\ \vec{\boldsymbol{r}}(t) - \vec{\boldsymbol{r}}_0 &= v_0\hat{\boldsymbol{\imath}}t \\ \vec{\boldsymbol{r}}(t) &= \vec{\boldsymbol{r}}_0 + v_0t\hat{\boldsymbol{\imath}} \\ \vec{\boldsymbol{r}}(3\,\text{s}) &= (2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}) + (5\,\text{m/s})\cdot(3\,\text{s})\hat{\boldsymbol{\imath}} \\ &= 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{r}} = 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}}$$

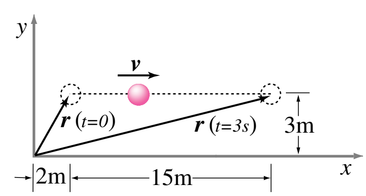

Figure 10.8

Comments: We could solve this problem more compactly by working with scalars or components. It is given that the velocity is constant and is only in the x-direction. Therefore, the y-component of particle position will remain the same, *i.e.*, $r_y = r_{0_y} = 3\,\text{m}$, and $r_x = r_{0_x} + v_xt = 2\,\text{m} + (5\,\text{m/s})\cdot(3\,\text{s}) = 17\,\text{m}$. Thus, $\vec{\boldsymbol{r}}(3\,\text{s}) = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} = 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$.

SAMPLE 10.5 Constant acceleration: A 0.5 kg mass starts from rest and attains a speed of 20 m/s$\hat{\imath}$ in 4 s. Assuming that the mass accelerates at a constant rate, find the force acting on the mass.

Solution Here, we are given the initial velocity $\vec{v}(0) = \vec{0}$ and the final velocity $\vec{v}$ after $t = 4$ s. We have to find the force acting on the mass. The net force on a particle is given by $\vec{F} = m\vec{a}$. Thus, we need to find the acceleration $\vec{a}$ of the mass to calculate the force acting on it. Now, the velocity of a particle under constant acceleration is given by

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t.$$

Therefore, we can find the acceleration $\vec{a}$ as

$$\begin{aligned} \vec{a} &= \frac{\vec{v}(t) - \vec{v}(0)}{t} \\ &= \frac{20\,\mathrm{m/s}\hat{\imath} - \vec{0}}{4\,\mathrm{s}} \\ &= 5\,\mathrm{m/s^2}\hat{\imath}. \end{aligned}$$

The force on the particle is

$$\vec{F} = m\vec{a} = (0.5\,\mathrm{kg}) \cdot (5\,\mathrm{m/s^2}\hat{\imath}) = 2.5\,\mathrm{N}\hat{\imath}.$$

$$\boxed{\vec{F} = 2.5\,\mathrm{N}\hat{\imath}}$$

SAMPLE 10.6 Time of travel for a given distance: A ball of mass 200 gm falls freely under gravity from a height of 50 m. Find the time taken to fall through a distance of 30 m, given that the acceleration due to gravity $g = 10\,\mathrm{m/s^2}$.

Solution The entire motion is in one dimension — the vertical direction. We can, therefore, use scalar equations for distance, velocity, and acceleration. Let y denote the distance travelled by the ball. Let us measure y vertically downwards, starting from the height at which the ball starts falling (see fig. 10.9). Under constant acceleration g, we can write the distance travelled as

$$y(t) = y_0 + v_0 t + \frac{1}{2}gt^2.$$

Note that at $t = 0$, $y_0 = 0$ and $v_0 = 0$. We are given that at some instant t (that we need to find) $y = 30$ m. Thus,

$$\begin{aligned} y &= \frac{1}{2}gt^2 \\ t &= \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 30\,\mathrm{m}}{10\,\mathrm{m/s^2}}} = 2.45\,\mathrm{s}. \end{aligned}$$

$$\boxed{t = 2.45\,\mathrm{s}}$$

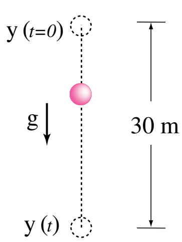

Figure 10.9

SAMPLE 10.7 **Time varying acceleration:** A force $F(t) = F_0 \sin\lambda t$ acts on an initially still cart of mass m in a particular direction. Find the speed and the distance travelled by the cart as functions of time. Plot the acceleration, the speed and the displacement of the cart against time for $0 \le t \le \pi$ s, assuming $\lambda = 1/$ s. What are the speed and the displacement of the cart at $t = \pi$ s if $F_0 = 1$ N and $m = 1$ kg?

Solution We are given the applied force and the mass of the cart. Therefore, we know the acceleration ($a = F/m$). Thus,

$$\begin{aligned} a \equiv \frac{dv}{dt} &= \frac{F_0}{m}\sin\lambda t \\ \Rightarrow \quad dv &= a_0 \sin\lambda t dt \end{aligned}$$

where $a_0 = F_0/m$. Hence,

$$\begin{aligned} \int_0^{v(t)} dv &= \int_0^t a_0 \sin\lambda\tau d\tau \\ \Rightarrow \quad v(t) &= -\frac{a_0}{\lambda}(\cos\lambda t - 1) \\ &= \frac{a_0}{\lambda}(1 - \cos\lambda t). \end{aligned}$$

Since the speed $v = \frac{dx}{dt}$, we have,

$$\begin{aligned} dx &= \frac{a_0}{\lambda}(1 - \cos\lambda t)dt \\ \int_0^{x(t)} dx &= \int_0^t \frac{a_0}{\lambda}(1 - \cos\lambda\tau)d\tau \\ \Rightarrow \quad x(t) &= \frac{a_0}{\lambda}\left(t - \frac{1}{\lambda}\sin\lambda t\right). \end{aligned}$$

$$\boxed{v(t) = \frac{F_0/m}{\lambda}(1 - \cos\lambda t), \quad x(t) = \frac{F_0/m}{\lambda}\left(t - \frac{1}{\lambda}\sin\lambda t\right)}$$

Substituting $a_0 = F_0/m = 1\,\mathrm{N}/1\,\mathrm{kg} = 1\,\mathrm{m/s^2}$, $\lambda = 1/\,\mathrm{s}$ and $t = \pi$ s in the expressions for v and x above, we find the speed and the displacement (distance travelled by the cart) at $t = \pi$ seconds as follows.

$$\begin{aligned} v(t = \pi\,\mathrm{s}) &= \frac{1\,\mathrm{m/s^2}}{1/\,\mathrm{s}}(1 - \cos\pi) \\ &= 2\,\mathrm{m/s}, \\ x(t = \pi\,\mathrm{s}) &= 1\,\mathrm{m/s}\left(\pi\,\mathrm{s} - \frac{1}{1/\,\mathrm{s}}\sin(\frac{1}{\mathrm{s}}\cdot\pi\,\mathrm{s})\right) \\ &= \pi\,\mathrm{m}. \end{aligned}$$

$$\boxed{\text{At } t = \pi\,\mathrm{s}, \quad v = 2\,\mathrm{m/s}, \quad x = \pi\,\mathrm{m}}$$

The graph of $a(t)$, $v(t)$, and $x(t)$ are shown in fig. 10.10 for $0 \le t \le \pi$ s assuming the given values of m, λ, and F_0. Note the behavior of $v(t)$ and $x(t)$ close to $t = 0$. Since the cart starts from rest, the speed builds up slowly, and the displacement builds up even more slowly because the speed is very low in the beginning.

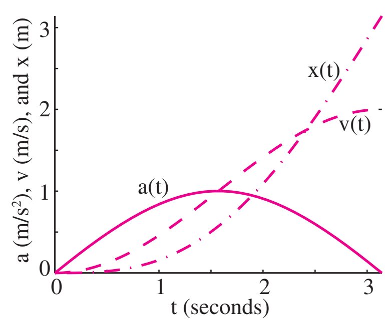

Figure 10.10: Plots of acceleration $a(t)$, speed $v(t)$ and displacement $x(t)$ as functions of time: $a(t) = a_0 \sin\lambda t$, $v(t) = \frac{a_0}{\lambda}(1 - \cos\lambda t)$, and $x(t) = \frac{a_0}{\lambda}(t - \frac{1}{\lambda}\sin\lambda t)$ where $a_0 = 1\,\mathrm{m/s^2}$ and $\lambda = 1/\,\mathrm{s}$.

SAMPLE 10.8 Numerical integration of ODE's:

1. Write the second order linear nonhomogeneous differential equation, $\ddot{x} + c\dot{x} + kx = a_0 \sin \omega t$, as a set of first order equations that can be used for numerical integration.
2. Write the second order nonlinear homogeneous differential equation, $\ddot{x} + c\dot{x}^2 + kx^3 = 0$, as a set of first order equations that can be used for numerical integration.
3. Solve the nonlinear equation given in (b) by numerical integration taking $c = 0.05$, $k = 1$, $x(0) = 0$, and $\dot{x}(0) = 0.1$. Compare this solution with that of the linear equation in (a) by setting $a_0 = 0$ and taking other values to be the same as for (b).

Solution

1.

$$\begin{array}{lrcl} \text{If we let} & \dot{x} & = & y, \\ \text{then} & \dot{y} & = & \ddot{x} = -c\dot{x} - kx + a_0 \sin \omega t \\ & & = & -cy - kx + a_0 \sin \omega t \end{array}$$

$$\text{or} \quad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ a_0 \sin \omega t \end{Bmatrix}. \qquad (10.14)$$

Equation (10.14) is written in matrix form to show that it is a set of *linear* first-order ODE's. In this case linearity means that the dependent variables only appear linearly, not as powers *etc.*

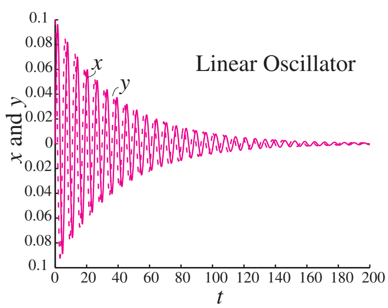

Figure 10.11: Numerical solution of the linear ODE $\ddot{x} + c\dot{x} + kx = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

2.

$$\begin{array}{lrcl} \text{If} & \dot{x} & = & y \\ \text{then} & \dot{y} & = & \ddot{x} = c\dot{x}^2 - kx^3 = -cy^2 - kx^3 \end{array}$$

$$\text{or} \quad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -cy^2 - kx^3 \end{Bmatrix}. \qquad (10.15)$$

Equation (10.15) is a set of *nonlinear* first order ODE's. It cannot be arranged as Eqn. 10.14 because of the nonlinearity in x and y. It is, however, in an appropriate form for numerical integration.

3. Now we solve the set of first order equations obtained in (b) using a numerical ODE solver with the following pseudocode.

```
ODEs = {xdot = y,  ydot = -c y^2 - k x^3}
IC   = {x(0) = 0, y(0) = 0.1}
Set  k=1, c=0.05
Solve ODEs with IC for t=0 to t=200
Plot x(t) and y(t)
```

The plot obtained from numerical integration using a Runge-Kutta based integrator is shown in fig. 10.12. A similar program used for the equation in (a) with $a_0 = 0$ gives the plot shown in fig. 10.11. The two plots show how a simple nonlinearity changes the response drastically.

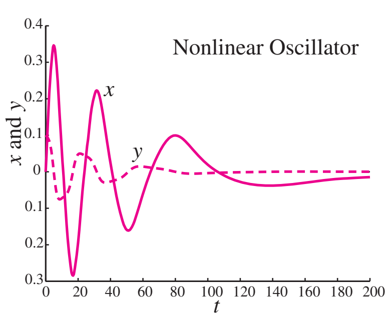

Figure 10.12: Numerical solution of the nonlinear ODE $\ddot{x} + c\dot{x}^2 + kx^3 = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.