Introduction to

STATICS
and
DYNAMICS

Andy Ruina and **Rudra Pratap**

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Reference Tables: The front and back tables concisely summarize much of the text material.

Summary of Mechanics

0) The laws of mechanics apply to any collection of material or 'body.' This body could be the overall system of study or any part of it. In the equations below, the forces and moments are those that show on a free body diagram. Interacting bodies cause equal and opposite forces and moments on each other.

I) Linear Momentum Balance (LMB)/Force Balance

Equation of Motion	$\boxed{\sum \vec{F}_i = \dot{\vec{L}}}$	The total force on a body is equal to its rate of change of linear momentum.	(I)
Impulse-momentum (integrating in time)	$\int_{t_1}^{t_2} \sum \vec{F}_i \cdot dt = \Delta \vec{L}$	Net impulse is equal to the change in momentum.	(Ia)
Conservation of momentum (if $\sum \vec{F}_i = \vec{0}$)	$\dot{\vec{L}} = \vec{0} \quad \Rightarrow$ $\Delta \vec{L} = \vec{L}_2 - \vec{L}_1 = \vec{0}$	When there is no net force the linear momentum does not change.	(Ib)
Statics (if $\dot{\vec{L}}$ is negligible)	$\sum \vec{F}_i = \vec{0}$	If the inertial terms are zero the net force on system is zero.	(Ic)

II) Angular Momentum Balance (AMB)/Moment Balance

Equation of motion	$\boxed{\sum \vec{M}_C = \dot{\vec{H}}_C}$	The sum of moments is equal to the rate of change of angular momentum.	(II)
Impulse-momentum (angular) (integrating in time)	$\int_{t_1}^{t_2} \sum \vec{M}_C dt = \Delta \vec{H}_C$	The net angular impulse is equal to the change in angular momentum.	(IIa)
Conservation of angular momentum (if $\sum \vec{M}_C = \vec{0}$)	$\dot{\vec{H}}_C = \vec{0} \quad \Rightarrow$ $\Delta \vec{H}_C = \vec{H}_{C2} - \vec{H}_{C1} = \vec{0}$	If there is no net moment about point C then the angular momentum about point C does not change.	(IIb)
Statics (if $\dot{\vec{H}}_C$ is negligible)	$\sum \vec{M}_C = \vec{0}$	If the inertial terms are zero then the total moment on the system is zero.	(IIc)

III) Power Balance (1st law of thermodynamics)

Equation of motion	$\dot{Q} + P = \underbrace{\dot{E}_K + \dot{E}_P + \dot{E}_{int}}_{\dot{E}}$	Heat flow plus mechanical power into a system is equal to its change in energy (kinetic + potential + internal).	(III)
for finite time	$\int_{t_1}^{t_2} \dot{Q} dt + \int_{t_1}^{t_2} P dt = \Delta E$	The net energy flow going in is equal to the net change in energy.	(IIIa)
Conservation of Energy (if $\dot{Q} = P = 0$)	$\dot{E} = 0 \quad \Rightarrow$ $\Delta E = E_2 - E_1 = 0$	If no energy flows into a system, then its energy does not change.	(IIIb)
Statics (if $\dot{E}_K$ is negligible)	$\dot{Q} + P = \dot{E}_P + \dot{E}_{int}$	If there is no change of kinetic energy then the change of potential and internal energy is due to mechanical work and heat flow.	(IIIc)
Pure Mechanics (if heat flow and dissipation are negligible)	$\boxed{P = \dot{E}_K + \dot{E}_P}$	In a system well modeled as purely mechanical the change of kinetic and potential energy is due to mechanical work on the system.	(IIId)

Some definitions

(Also see the index and back tables)

$\vec{r}$ or $\vec{x}$	Position	*e.g.,* $\vec{r}_i \equiv \vec{r}_{i/O}$ is the position of a point i relative to the origin, O.
$\vec{v} \equiv \frac{d\vec{r}}{dt}$	Velocity	*e.g.,* $\vec{v}_i \equiv \vec{v}_{i/O}$ is the velocity of a point i relative to O, measured in a non-rotating reference frame.
$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$	Acceleration	*e.g.,* $\vec{a}_i \equiv \vec{a}_{i/O}$ is the acceleration of a point i relative to O, measured in a Newtonian frame.
$\vec{F}$	Force	*e.g.,* the force on A from B is $F_{\text{A from B}}$.
$\vec{M}$ or $\vec{M}_C = \vec{M}_{/C}$	Moment or Torque	*e.g.,* the moment of a collection of forces about point C.
$\vec{\omega}$	Angular velocity	A measure of rotational velocity of a rigid object. $\vec{\omega}_{\mathcal{B}}$ = angular velocity of rigid object $\mathcal{B}$.
$\vec{\alpha} \equiv \dot{\vec{\omega}}$	Angular acceleration	A measure of rotational acceleration of a rigid object.
$\vec{L} \equiv \begin{cases} \sum m_i \vec{v}_i & \text{discrete} \\ \int \vec{v}\, dm & \text{continuous} \end{cases}$ $= m_{\text{tot}}\vec{v}_{\text{cm}}$	Linear momentum	A measure of a system's net translational rate (weighted by mass).
$\dot{\vec{L}} \equiv \begin{cases} \sum m_i \vec{a}_i & \text{discrete} \\ \int \vec{a}\, dm & \text{continuous} \end{cases}$ $= m_{\text{tot}}\vec{a}_{\text{cm}}$	Rate of change of linear momentum	The aspect of motion that balances the net force on a system.
$\vec{H}_{/C} \equiv \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{v}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{v}\, dm & \text{continuous} \end{cases}$	Angular momentum about point C	A measure of the rotational rate of a system about a point C (weighted by mass and distance from C).
$\dot{\vec{H}}_{/C} \equiv \begin{cases} \sum \vec{r}_{i/C} \times m_i \vec{a}_i & \text{discrete} \\ \int \vec{r}_{/C} \times \vec{a}\, dm & \text{continuous} \end{cases}$	Rate of change of angular momentum about point C	The aspect of motion that balances the net torque on a system about a point C.
$E_K \equiv \begin{cases} \frac{1}{2}\sum m_i v_i^2 & \text{discrete} \\ \frac{1}{2}\int v^2\, dm & \text{continuous} \end{cases}$	Kinetic energy	A *scalar* measure of net system motion.
E_{int} = (heat-like terms)	Internal energy	The non-kinetic non-potential part of a system's total energy.
$P \equiv \sum \vec{F}_i \cdot \vec{v}_i + \sum \vec{M}_i \cdot \vec{\omega}_i$	Power of forces and torques	The mechanical energy flow into a system. Also, $P \equiv \dot{W}$, rate of work.
$[\boldsymbol{I}^{\text{cm}}] \equiv \begin{bmatrix} I_{xx}^{cm} & I_{xy}^{cm} & I_{xz}^{cm} \\ I_{xy}^{cm} & I_{yy}^{cm} & I_{yz}^{cm} \\ I_{xz}^{cm} & I_{yz}^{cm} & I_{zz}^{cm} \end{bmatrix}$	Moment of inertia matrix about center of mass (cm)	A measure of the mass distribution in a rigid object.

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Brief Contents

Detailed Contents

the formula which correctly calculates the heuristically motivated quantities of moment and moment about an axis.

A free-body diagram is a sketch of the system to which you will apply the laws of mechanics, and all the non-negligible external forces and couples which act on it. The diagram indicates what material is in the system. The diagram shows what is, and what is not, known about the forces. Generally there is a force or moment component associated with any connection that causes or prevents a motion. Conversely, there is no force or moment component associated with motions that are freely allowed. Mechanics reasoning entirely rests on free body diagrams. Many student errors in problem solving are due to problems with their free body diagrams, so we give tips about how to avoid various common free-body diagram mistakes.

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that can be obtained from analysis are discussed, as are numerical solutions.

Here is a second approach to the kinematics of particle motion. Now, instead of using constant base vectors, we use time-varying base vectors. The discussion of polar coordinates started in Chapter 13 is completed here. Path coordinates, where one base vector is parallel to the velocity and the others orthogonal to that, are introduced. The challenging kinematics topic of relative motion is introduced in two stages: first using rotating base vectors connected to a moving rigid object and then using the more abstract notation associated with frame-dependent differentiation and the famous "five term acceleration formula."

The dynamics of particles and rigid objects is studied using the relative-motion kinematics ideas from chapter 15. This is the capstone chapter for a two-dimensional dynamics course. After this chapter a good student should be able to navigate through and use most of the skills in the concept map inside the back cover.

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Preface

General issues about content, level, organization, style and motivation. Study advice starts on page 14.

This is an engineering statics and dynamics text. It is both an introduction, aimed primarily at middle-level engineering students, and a reference. The book emphasizes use of vectors, free-body diagrams, momentum and energy balance and computation. More importantly, perhaps, the book is meant to help build an intuition for mechanics.

Prerequisite and co-requisite skills. We assume you start with some math skills.

- **Freshman calculus**. Readers are assumed to have facility with the basic geometry, algebra, trigonometry, differentiation and integration used in elementary calculus. Some of these topics are briefly reviewed in this book, but not as *ab initio* tutorials.
- This books shows how to set-up algebraic and differential equations for computer solution. You need to know, or be simultaneously learning, **a computer language** or package which can solve sets of linear algebraic equations, numerically integrate simple ordinary differential equations and make decent plots.

You may have had exposure to other useful subjects detailed foreknowledge of which this book does not assume.

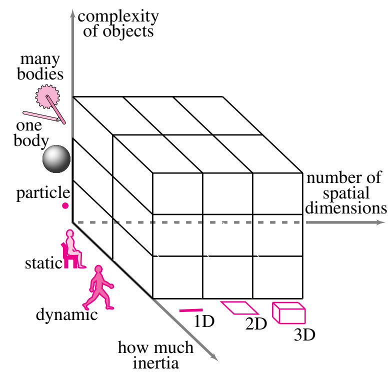
- Completion of **freshman physics** may help but is not needed.
- Vector topics, especially **dot and cross products**, are introduced here from scratch in the context of mechanics.
- A background in **linear algebra** wouldn't hurt, but the reduction of linear equations to matrix form is taught here. A key fact from linear algebra, also presented here, is that linear algebraic equations are usually easy to solve on a computer.
- A course in **differential equations** would also add perspective. But the basic concepts of differential equations are presented here as needed.

Organization

Mechanics could be subdivided into statics *vs* dynamics, particle *vs* rigid object *vs* many objects ('multi-object'), and 1 *vs* 2 *vs* 3 spatial dimensions (1D, 2D & 3D). Thus a mechanics table of contents might have one chunk of text for each of the $2 \times 3 \times 3 = 18$ combinations:

I. Statics
- A. particle
  - ∗ 1D, 2D, 3D
- B. rigid object
  - ∗ 1D, 2D, 3D
- C. many objects
  - ∗ 1D, 2D, 3D

II. Dynamics
- A. particle
  - ∗ 1D, 2D, 3D
- B. rigid object
  - ∗ 1D, 2D, 3D
- C. many objects
  - ∗ 1D, 2D, 3D



However, these $2 \times 3 \times 3 = 18$ chunks vary greatly in difficulty; 1D statics is low-level high school material and 3D multi-object dynamics is difficult graduate material. Further, the chunks use various overlapping concepts and skills. So it is not sensible to organize a book into 18 corresponding chapters. Nonetheless, some vestiges of the scheme above are used in all books, and the general flow of this book is from the bottom back left corner of the box in the figure, towards the diagonal opposite. The details of the organization, as visible in the annotated table of contents on the previous pages, has evolved through trial and error, review and revision, and many semesters of student testing.

The first eight chapters cover the basics of statics and the rest of the book covers the basics of engineering dynamics. Relatively harder topics, which might be skipped in quicker or less-advanced courses, are identifiable by chapter, section or subsection titles like "three-dimensional" or "advanced".

Coverage for courses. The sections have been divided so that the homework problems selected from one section are usually about half of a typical weekly homework assignment. The theory and examples from one section might be adequately covered in about one lecture, plus or minus.

A leisurely one semester statics course, or a more fast-paced half-semester prelude to strength of materials should use chapters 1-8, excluding topics of less interest. A typical one semester dynamics course will cover most of of chapters 9-16, reviewing chapters 1-3 at the start. A lower-level one-semester statics and dynamics course can cover the less advanced parts of chapters 1-6 and 9-14. An advanced full-year statics and dynamics course could cover most of the book. That is, the statics portion of the book fits easily in a semester and the whole of the dynamics portion in a bit more than a semester. Chapters 15-16 can also be used as a start for a second advanced dynamics course. A student who has learned the statics part of this book is well-prepared for using statics in engineering practice, for learning Strength of Materials and for going on to Dynamics. A student who has learned the dynamics portion is well prepared to go on to learn Vibrations, Systems Dynamics or more advanced Multi-object Dynamics.

Organization and formatting

Each subject is covered in various ways.

- Every section starts with **descriptive text** and short *examples* motivating and describing the theory;
- More detailed explanations of the **theory** are in boxes interspersed in the text. For example, one box explains the common derivation of angular momentum balance from $\vec{F} = m\vec{a}$ (page 1004), one explains the genius of the wheel (page 215), and another connects $\vec{\omega}$ based kinematics to $\hat{e}_r$ and $\hat{e}_\theta$ based kinematics (page 874);
- **Sample problems** (marked with a gray border) at the end of each section show how to do homework-like calculations. These set an example by their consistent use of free-body diagrams, systematic application of basic principles, vector notation, units, and checks against both intuition and special cases;
- **Homework problems** at the end of each chapter give students a chance to practice mechanics calculations. The first problems for each section build a student's confidence with the basic ideas. The problems are ranked in approximate order of difficulty, with theoretical problems coming later. Problems marked with a * have an answer at the back of the book;
- **Reference tables** on the inside covers and end pages concisely summarize much of the content in the book. These tables can save students the time of hunting for formulas and definitions.

Notation

Clear **vector notation** helps students do problems. One common class of student errors comes from copying a textbook's printed bold vector $\mathbf{F}$ the same way as a plain-text scalar F. We help reduce this error by use a redundant vector notation, a bold and harpooned $\vec{F}$.

As for all authors and teachers concerned with motion in two and three dimensions (**kinematics**) we have struggled with the tradeoffs between a precise notation and a simple notation. Perfectly precise notations are complex and intimidating. Simple notations are ambiguous and hide key information. Our attempt at clarity without too-much clutter is summarized in the box on page 42.

Relation to other mechanics books

The bulk of the content of this book can be found in other places including freshman physics texts, other engineering texts, and hundreds of classics.

Freshman physics texts encompass much of this book's contents. However, this book is a bit deeper, more rigorous and more oriented to engineering. After freshman physics students often have only a vague notion of what mechanics is, and how it can be used. For example many students leave freshman physics with the sense that a free-body diagram (or 'force diagram') is a vague conceptual picture with arrows for various forces and motions drawn on it this way and that. Even the freshman-text illustrations sometimes do not make clear which force is acting on which object. Also, because freshman physics tends to avoid use of college math, many students leave freshman physics with little sense of how to use vectors or calculus to solve mechanics problems. This book aims to lead students who may start with these fuzzy freshman-physics notions into a world of precise, yet still intuitive, mechanics.

Various statics and dynamics textbooks cover much of the same material as this one. These textbooks have modern applications, ample samples, lots of pictures, and lots of homework problems. Many are excellent in some ways. Most of today's engineering professors learned from one of these books. Nonetheless we wrote this book hoping to do still better. This book is somewhat different in organization and approach. Some of our goals include

- showing the unity of the subject,
- presenting a complete description of the subject,
- clear notation in figures and equations,
- integration of the applicability of computers,
- consistent use of units throughout,
- introduction of various insights into how things work,
- a friendly writing style.

This book also uses some important but not well-enough known concepts①.

Between about 1689 and 1960 hundreds of books were written with titles like *Statics*, *Engineering mechanics*, *Dynamics*, *Machines*, *Mechanisms*, *Kinematics*, or *Elementary physics*. Many thoughtfully cover most of the material here and sometimes much more. But none are good modern textbooks; they lack an appropriate pace, style and organization; they are too reliant on geometry skills and not enough on vectors and numerics; and they don't have enough modern applications, samples calculations, illustrations, or homework problems. But much good mechanics can be found only in these older books②. If you love mechanics you will enjoy pondering ideas in some of these books.

What do you think?

We have tried to make it as easy as possible for you to learn basic mechanics from this book. We present truth as we know it and as we think it is effectively communicated. Nonetheless we have surely made some technical and strategic errors. Please let us know your thoughts so that we can improve future editions.

Rudra Pratap, pratap@mecheng.iisc.ernet.in
Andy Ruina, ruina@cornell.edu

①For example, we use angular momentum balance (appropriately expressed) with respect to any possibly-accelerating point, not just points selected from an arcane list.

②Here are three good and universally respected classics:

J.P. Den Hartog's *Mechanics* originally published in 1948 but still available as an inexpensive reprint (well written and insightful);

J.L. Synge and B.A. Griffith, *Principles of Mechanics* through page 408. Originally published in 1942, reprinted in 1959 (good pedagogy but dry); and

E.J. Routh's, *Dynamics of a System of rigid bodies*, Vol 1 (the "elementary" part through chapter 7. Originally published in 1905, but reprinted in 1960). Routh also has 5 other idea- packed statics and dynamics books. Routh shared college graduation honors with the now-more-famous physicist James Clerk Maxwell.

To the student

How to study. The use of computers.

Nature's rules are so strict that, to the extent that you know the rules, you can make reliable predictions about how Nature, the set of all things, behaves. In particular, most objects of concern to engineers obediently follow a subset of Nature's rules called the laws of Newtonian mechanics. So, if you learn the laws of mechanics, as this book should help you to do, you will be able to make quantitative predictions about how things stand, move, and fall. And you will gain intuition about the mechanics part of Nature's rules.

How to use this book

Here is some general guidance.

Check your own understanding

Most likely you want a decent grade by successfully getting through the homework assignements and exams. You will naturally get help by looking at examples and samples in the text or lecture notes, by looking up formulas in the front and back covers of this book, and by asking questions of friends, teaching assistants and professors. What good are books, notes, classmates or teachers if they don't help you do the homework? All the examples and sample problems in this book, for example, are just for this purpose.

But watch out. Too-much use of help from books, notes and people can lead to self deception③. After you have got through a problem using such help you should, at least sometimes, check that you have actually learned to solve the problem.

③**"Exams are harder than homework."** Some struggling students say "I can do the homework problems. I just can't do the exams. Exams are harder and trickier." These students may be fooling themselves. Most exams are not trickier than the homework. And when we have checked, many students who got through homework with help, can't do simplified versions of those same problems when they have no help.

> To see if you have learned to do a problem, do it again, justifying each step, without looking up *even one small ('oh, I almost knew that') thing*.

If you find that you can't do a problem totally alone, you gain two learning opportunities. First, you can learn the missing skill or idea. But more deeply, by getting stuck after you have been able to get through with help, you can learn things about your learning process. Often the real source of difficulty isn't a key formula or fact, but something more subtle. We hope you can learn some of these useful, and more subtle, ideas from the general text discussions here.

Read the parts that are at your level

You might be science and math school-smart, mechanically inclined, and already especially interested mechanics. Or you might be reluctantly taking this class to fulfil a requirement. In either case this book is meant for you. The sections start with generally accessible introductory material and include simple examples. The early sample problems in each section are also easy. But we also have discussions of the theory and other more advanced applications and asides to challenge more motivated students. If you are a nerd, please be patient with the slow introductions and the calculations that go line by line without skipping steps. On the other hand, if you are just trying to get through a course using this book, don't get hung up by every side discussion about history or theory.

Calculation strategies and skills

We try to demonstrate a systematic approach to solving problems. But its impossible to reduce all mechanics problem solutions to one clear recipe (despite the generally applicable recipe on the inside back cover). If a precise recipe existed to solve all statics and dynamics problems then someone could write a computer program that followed the recipe, and the course you are taking could be cancelled. Your mind could be freed from mechanics problem solutions like a calculator frees you from the tedium of long division④. There is an art to solving mechanics problems and understanding their solutions. This applies to homework problems and also engineering design problems. Art and insight, as opposed to application of a fixed precise algorithm, is what makes engineering require humans and not just computers. We hope you learn some of this art. For starters, here are some tips.

Understand the question

It is tempting to start writing equations and quoting principles when you first see a problem. However, it is usually worth a few minutes (and sometimes a few hours) to try to

> Get an intuitive sense of a problem before jumping to equations.

Before you draw any sketches or write equations, think: does the problem make sense? What information has been given? What are you trying to find? Is what you are trying to find determined by what is given? What physical laws make the problem solvable? What extra information do you think you need? What information have you been given that you don't need? You should first get a general sense of the problem to steer you through the technical details.

Some students find they can read every line of sample problems yet cannot do test problems, or, later on, cannot do applied design work effectively.

④**Actually, computers can do mechanics.** To be honest, this book presents some methods which computers *can* handle well. Once a problem has been reduced to a precise mechanical model a computer code could take over. Say a finite-element program or a 'rigid-body' dynamics program. But you will do better at mechanics, even with a computers help, if you can do simple mechanics problems without a computer.

Analogy with long-division. Since the mid 1970s, division by a 3 (or more) digit number is not done by pencil-and-paper long-division but with a calculator or computer. Nonetheless, understanding division (that, for example, it is inverse multiplication, or that division by zero is bad or which number to divide by which in real problems) is necessary. And such knowledge comes better by practice manipulating numbers in one's head and on paper.. Similarly it is useful to know mechanics solution methods well, even for problems can be solved by a computer package.

This failing may come from following details without spending time, thinking and gaining an overall sense of the problems.

Think through your solution strategy

For problem solutions you read, like those in this book, someone had to think about the order of work. You also have to think about the order of your work. You will find some tips in the text and samples. But it is your job to own the material, to learn how to think about it your own way, to become an expert in your own style, and to do the work in the way that makes things most clear to you.

The order of calculation is often backwards from the order of thinking

When working out how to solve a problem you often start 'backwards', with general principles, then look at terms you need to know. If these are not given, you think how to figure those from other terms, and so on. On the other hand, when you go to calculate an answer you have to start with the information given and work your way 'forwards' into the equation which has your answer from the information given⑤. To find the net worth of a corporation you add the value of the various divisions. To get the value of a division you add up the values of the factories. For each factory you add up the value of the pieces of machinery. But to get an actual corporate value you have to start by evaluating the pieces of machinery in each factory and working from the known towards the answer. Beware that

> A polished calculation, especially an algorithmic recipe or computer program, is often written in the inverse order of the thinking that went into making it.

⑤**A tree analogy.** Say you are interested in how energy gets stored in a tree. The energy flows to the roots from the trunk. The branches feed the trunk, the twigs feed the branches, and the leaves feed the twigs with energy from the sun. But the flow goes the opposite way, from the leaves on down to the roots. But if you try to invent a tree by starting at the leaves with no knowledge of the root you could easily get lost and connect leaves to electric wires or gas pipes — all nonsense. There's no point in connecting the leaves to anything until you have a sense of the whole tree from the roots up.

Real problem solving goes both ways. You think about what you need in order to calculate what you want. But you also think about what you can calculate easily from what is plainly given to you. You reach from the unknown towards the known details. And you work with known details towards answers of any kind, wanted or not. And you thus hunt out, building from details and simultaneously reaching back from the goal, a route leading all the way from the known details to the goal.

Look for equations containing unknowns. Don't look for formulas that evaluate unknowns.

In elementary science and math we often learn formulas like

$$V = LWH, \qquad d = \frac{1}{2}at^2, \quad \text{and} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to find V, d, or x. So it is common wishful thinking for newcomers to hope for a formula that generates the sought unknown in terms of given quantities. Rather, you should

> Find relations that contain variables of interest; don't worry about whether they are on the right or left side of an equation. Don't worry about whether the variables are alone or isolated.

Most often, you will not know a formula where the thing you want is on the left and everything given is on the right. You will have, say,

$$\begin{aligned} V &= LWH && \text{when you want to find } W \text{ from } V, L, \text{ and } H, \\ d &= \frac{1}{2}at^2 && \text{when you want to find } t \text{ from } a \text{ and } d, \text{ and} \\ ax^2 + bx + c &= 0 && \text{when you want to find } x \text{ from } a, b, \text{ and } c. \end{aligned}$$

Once you have got this far the only problem is math⑥. Here are two tricks of the mind

1) **You know a math and computer genius.** She is helpful but doesn't know any mechanics. Your first, and main, task is to write things down so she could finish up for you. She doesn't want to help? Then realize that finishing up without her is a separate job for you. You will do this later when you take of your mechanics hat and put on *your* math-genius hat.
2) **Be an egotist.** Pretend you are omniscient and know everything. Then write down true statements about those things; equations that contain terms that omniscient-you already know: "If I knew x, y and z the following equation would be true." Then relax your ego a bit. Count equations and unknowns to see if you, or at least your math genius friend, could solve for some of the things you previously pretended to know.

⑥For this and other courses, you should be good at solving math problems with your pencil and with a computer. But you should distinguish between the task of setting up a math problem and the task of solving it. Solving often takes most of the time and most of the space on your paper, but it's not where your thoughts should start. The important new material for you in this book is about setting up the math problems that arise in mechanics, not about solving them. Of course you should develop your math skills too, but that's not the main new content here.

Vectors and free-body diagrams

In the toolbox of someone who can solve lots of mechanics problems are two well-worn tools:

- A vector calculator that always keeps vectors and scalars distinct, and
- A reliable and clear free-body diagram drawing tool.

Because many of the terms in mechanics equations are vectors, the ability to do vector calculations is essential (here is some math you may have to learn better). Because the concept of an isolated system is at the core of mechanics, every mechanics practitioner needs the ability to draw a good free-body diagram. The second and third chapters will help you build your own set of these two most-important tools.

Guarantee: If you learn to do

- clear correct vector algebra and to draw
- good free-body diagrams

you will do well at mechanics.

(Assuming, of course, that you don't totally stop studying then and there.)

Thinking outside the books

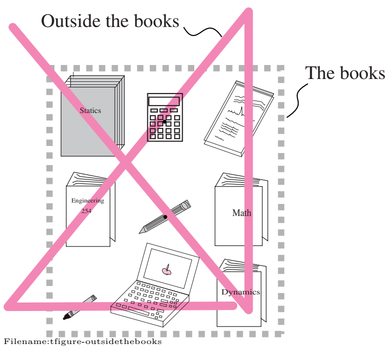


Figure 0.1: Thinking outside of the books. A famous puzzle asks: using 4 contiguous straightline segments connect all 9 dots that are in a square 3×3 array. The only solution has segments extending outside the "box" of 9 points. Hence the expression "thinking outside of the box".

It is fun to puzzle out how things work. Its satisfying to do calculations that make realistic predictions. Mechanics is interesting in its own right and, interesting or not, it feels good to take pride in new skills. We wrote this book because we want to help you learn the subject if you are interested, and get through it if you must. But we don't know the sure path through your resources (say a path with 4 straight segments, see fig. 0.1) that will get you to deeper understanding.

We do know that to learn deeply you need to

think outside of the confines of your usual study resources.

That is, think when you are relaxed, away from the pressures of books, notes, pencils or paper, say when you are walking, showering or lying down. These are the places where you naturally work out life problems, but they are good places to work out mechanics problems too.

Having an animated mechanics discussion with friends is also good. You should enjoy your inner nerd socially. Are your friends turned off by tech-talk? There are billions of people out there, you should be able to find one or two that would like to talk shop with you.

0.1 A note on computation

Mechanics is a physical subject. The concepts in mechanics do not depend on computers. But mechanics is also a quantitative subject; relevant amounts (of length, mass, force, moment, time, etc) are described with numbers, and relations are described using equations and formulas. Computers are very good with numbers and formulas. Thus the modern practice of engineering mechanics uses computers. The most-needed computer skills for mechanics are:

- solution of simultaneous linear algebraic equations,
- plotting, and
- numerical solution of ODEs (Ordinary Differential Equations).

More basically, an engineer also needs the ability to routinely evaluate standard functions (x^3, $\cos^{-1}\theta$, *etc.*), to enter and manipulate lists and arrays of numbers, and to write short programs.

Classical languages, applied packages, and simulators

Programming in standard languages such as Fortran, Basic, Pascal, C++, or Java probably take too much time to use in solving simple mechanics problems. Thus an engineer needs to learn to use one or another widely available computational package (*e.g.,* MATLAB, O-MATRIX, SCI-LAB, OCTAVE, MAPLE, MATHEMATICA, MATHCAD, TKSOLVER, LABVIEW, etc). We assume that students have learned, or are learning such a package. Although none of the homework here depends on such, we also encourage you to play with packaged mechanics simulators (*e.g.,* INVENTOR, WORKING MODEL, ADAMS, DADS, ODE, etc) for testing and building your intuition.

How we explain computation

Solving a mechanics problem involves

1. Reducing a physical problem to a well posed mathematical problem;
2. Solving the math problem using some combination of pencil and paper and numerical computation; and
3. Giving physical interpretation of the mathematical solution.

This book is primarily about setup (a) and interpretation (c), neither of which particularly depends on what method is used to solve the equations. If a problem requires computation, the exact computer commands vary from package to package. And we don't know which computer package you are using. So in this book we express our computer calculations using an informal pseudo computer language. For reference, typical commands are summarized on page 22.

Required computer skills

Here, in a little more detail, are the primary computer skills you need.

- **Linear algebraic equations.** Many mechanics problems are statics or 'instantaneous mechanics' problems. These problems involve trying to find some forces or accelerations at a given configuration of a system. These problems can generally be reduced to the **solution of linear algebraic equations** of this general type: solve

$$\begin{array}{rcrcl} 3\,x & + & 4\,y & = & 8 \\ -7\,x & + & \sqrt{2}\,y & = & 3.5 \end{array}$$

  for x and y. In practice the number of variables and equations can be quite large. Some computer packages will let you enter equations almost as written above. In our pseudo language we would write:

```
eqset = {   3*x +       4*y = 8
           -7*x + sqrt(2)*y = 3.5  }
solve eqset for x and y
```

Other packages may require you to set up your equations in matrix form

$$\underbrace{\begin{bmatrix} 3 & 4 \\ -7 & \sqrt{2} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{z} = \underbrace{\begin{bmatrix} 8 \\ 3.5 \end{bmatrix}}_{b} \quad \text{or} \quad Az = b$$

which in computer-speak might look something like this:

```
A =  [ 3      4
      -7   sqrt(2) ]
b =  [ 8 3.5 ]'
solve  A*z=b for z
```

where `A` is a 2×2 matrix, `b` is a column of 2 numbers (the `'` indicates that the row of numbers `b` should be transposed into a column), and the two elements of `z` are x and y. For systems of two equations, like above, a computer is hardly needed. But for systems of three equations pencil and paper work is sometimes error prone. Given the tedium, the propensity for error, and the availability of electronic alternatives, pencil and paper solution of four or more equations is an anachronism.

- **Plotting.** In order to see how a result depends on a parameter, or to see how a quantity varies with position or time, it is useful to see a **plot**. Any plot based on more than a few data points or a complex formula is far more easily drawn using a computer than by hand. Most often you can organize your data into a set of (x, y) pairs stored in an `x` list and a corresponding `y` list. A simple computer command will then plot x *vs* y. The pseudo-code below, for example, plots a circle using 100 points

```
npoints = [0 1 2 3 ... 100]
theta   = npoints * 2 * pi / 100
x       = cos(theta)
y       = sin(theta)
plot y vs x
```

where `npoints` is the list of numbers from 1 to 100, `theta` is a list of 100 numbers evenly spaced between 0 and 2π and `x` and `y` are lists of 100 corresponding x, y coordinate points on a circle.

- **ODEs** The result of using the laws of dynamics is often a set of **o**rdinary **d**ifferential **e**quations which need to be solved. A simple example would be:

Find x at $t = 5$ given that $\dfrac{dx}{dt} = x$ and that at $t = 0$, $x = 1$.

The solution to this problem can be found easily enough by hand to be $x(5) = e^5$. But often the differential equations are just too hard for pencil and paper solution. Fortunately the **numerical solution of Ordinary Differential Equations** (ODEs) is already programmed into scientific and engineering computer packages. The simple problem above is solved with computer code equivalent to these informal commands:

```
ODES =  { xdot  = x }
ICS  =  { xzero = 1 }
solve ODES with ICS until t=5
```

which will yield a list of values for paired values for t and x the last of which will be $t = 5$ and x close to $e^5 \approx 148.4$.

Examples of informal computer commands

We use informal computer commands that are not as strict as any real computer package. You will translate informal commands, like those below, into commands your package understands. This reference table uses mathematical ideas which you may or may not know before you read this book, but these will be introduced in the text when needed.

`x=7`	Set the variable x to 7.
`omega=13`	Set ω to 13.
`u=[1 0 -1 0]` `v=[2 3 4 pi]`	Define u and v to be the lists shown.
`t= [.1 .2 .3 ... 5]`	Set t to the list of 50 numbers implied by the expression.
`y=v(3)`	sets y to the third value of v (in this case 4).
`A=[1 2 3 6.9` `5 0 1 12 ]`	Set A to the array shown.
`z= A(2,3)`	Set z to the element of A in the second row and third column.

`w=[3`
`4`
`2`
`5]`
	Define w to be a column vector.

`w = [3 4 2 5]'`	Same as above. $'$ means transpose.
`u+v`	Vector addition. In this case the result is $[3\ 3\ 3\ \pi]$.
`u*v`	Element by element multiplication, in this case $[2\ 0\ -4\ 0]$.
`sum(w)`	Add the elements of w, in this case 14.
`cos(w)`	Make a new list, each element of which is the cosine of the corresponding element of $[w]$.
`mag(u)`	The square root of the sum of the squares of the elements in $[u]$, in this case 1.41421...
`u dot v`	The vector dot product of component lists $[u]$ and $[v]$, (we could also write `sum(A*B)`.
`C cross D`	The vector cross product of $\vec{C}$ and $\vec{D}$, assuming the three element component lists for $[C]$ and $[D]$ have been defined.
`A matmult w`	Use the rules of matrix multiplication to multiply $[A]$ and $[w]$.
`eqset = {3x + 2y = 6` `6x + 7y = 8}`	Define 'eqset' to stand for the set of 2 equations in braces.
`solve eqset` `for x and y`	Solve the equations in 'eqset' for x and y.
`solve Ax=b for x`	Solve the matrix equation $[A][x]=[b]$ for the list of numbers x. This assumes A and b have already been defined.

`for i = 1 to N`
`such and such`
`end`
	Execute the commands 'such and such' N times, the first time with $i=1$, the second with $i=2$, etc

`plot y vs x`	Assuming x and y are two lists of numbers of the same length, plot the y values *vs* the x values.

`solve ODEs`
`with ICs`
`until t=5`
	Assuming a set of ODEs and ICs have been defined, use numerical integration to solve them and evaluate the result at $t=5$.

With an informality consistent with what is written above, other commands are introduced as needed.

CHAPTER 1

What is mechanics?

Mechanics can predict forces and motions by using the three pillars of the subject: I. models of physical behavior, II. geometry, and III. the basic mechanics balance laws. The laws of mechanics are informally summarized in this introductory chapter. The extreme accuracy of Newtonian mechanics is emphasized, despite relativity and quantum mechanics supposedly having 'overthrown' seventeenth century physics. Various uses of the word 'model' are described.

Contents

Mechanics is the study of force, deformation and motion, and the relations between them. We care about forces because we want to know how hard to push something to make it move or whether it will break when we push. We care about deformation and motion because we want things to move or not move in certain ways. Towards these ends our goals are to solve special versions of this general mechanics problem:

The general mechanics problem: Given some (possibly idealized) information about the properties, forces, deformations, and motions of a mechanical system, make useful predictions about other aspects of its properties, forces, deformations, and motions.

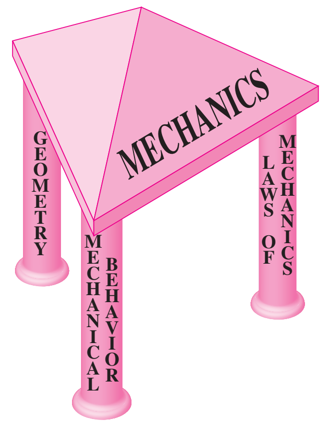
By *system*, we mean a tangible thing such as a wheel, a gear, a car, a bridge, a human finger, a butterfly, a skateboard and rider, a quartz-watch timing crystal, a building in an earthquake, a rocket, or the piston in an engine. Will a wheel slip? a gear tooth break? a car tip over? What is the biggest truck that can cross a given bridge? What muscles are used when you hit a key on your computer? How do people balance on skateboards? How does size effect the frequency of crystal vibration? Which buildings are more likely to fall in what kinds of earthquakes? What is the relation between gas-ejection rate and thrust in a rocket? What forces are on the connecting rod in an engine?

For each special case of the general mechanics problem we need to identify the system(s) of interest, idealize the system(s), use classical (high-school, Euclidean) geometry to describe the layout, deformation and motion, and finally use the laws of Newtonian mechanics. Those who want to know how machines, structures, plants, animals and planets hold together and move about need to know Newtonian mechanics. As best we can extrapolate, in another two or three hundred years people who want to design robots, buildings, airplanes, boats, prosthetic devices, and large or microscopic machines will likely still use the equations and principles we now call Newtonian mechanics①.

①The laws of classical mechanics, however expressed, are named for Isaac Newton because his theory of the world, the *Principia* published in 1689, contains much of the still-used theory. Newton used his theory to explain the motions of planets, the trajectory of a cannon ball, why there are tides, and many other things.

1.1 The three pillars

Any mechanics problem can be divided into 3 parts which we think of as the 3 pillars that hold up the subject:



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1. *Constitutive laws*: the mechanical behavior of objects and materials;
2. *Kinematics*: the geometry of motion and distortion; and
3. *Kinetics*: the laws of mechanics ($\vec{F} = m\vec{a}$, etc.).

Let's discuss each of these ideas a little more so you can get an overview before digging into the details in later chapters.

Pillar 1: Mechanical behavior, constitutive laws

The first pillar of mechanics is mechanical behavior. The *Mechanical behavior* of something is the description of how loads cause deformation (or vice versa). When something carries a force it stretches, shortens, shears, bends, or breaks. Your finger tip squishes when you poke something. Too large a force on a gear in an engine causes it to break. The force of air on an insect wing makes it bend. Various geologic forces bend, compress and break rock. This relation between force and deformation can be viewed in a few ways.

Definition of force. First, the relation between force and deformation gives us a definition of force. Force can be defined by the amount of spring stretch it causes. Thus most modern force measurement devices measure force indirectly by measuring the deformation it causes in a calibrated spring of some kind. That force can be defined in terms of deformation is one justification for calling 'mechanical behavior' the first pillar. It gives us a notion of force even before we introduce the laws of mechanics.

Steel vs chewing gum. Second, a piece of steel distorts under a given load differently than a same-sized piece of chewing gum. This observation, that different objects deform differently with the same loads, implies that an object's properties affect its mechanics. The relations of an object's deformations to the forces that are applied are called the *mechanical properties* of the object. Mechanical properties are sometimes called *constitutive laws* because the mechanical properties describe how an object is constituted (meaning 'what it is made from') at least from a mechanics point of view. The classic example of a constitutive law is that of a linear spring which you remember from your elementary physics classes:

$$\text{`}F = kx\text{'}$$

(spring tension is proportional to stretch). To do mechanics we have to make assumptions and idealizations about the constitutive laws applicable to the parts of a system. How stretchy (elastic) or gooey (viscous) or otherwise deformable is an object? The set of assumptions about the mechanical behavior of the system is sometimes called the *constitutive model*.

Deformation is often hard to see. Distortion in the presence of forces is easy to see or imagine in the flesh of squeezed fingertips, in chewing gum between teeth or when a piece of paper bends. But pieces of rock or metal have deformation that is essentially invisible and sometimes hard to imagine. With the exceptions of things like rubber, flesh, or objects that are very small in one or two directions (thin sheets and wires), solid objects that are not in the process of breaking typically change their sizes much less than 1% when loaded. Most structural materials deform less than one part per thousand with working loads. These small deformations, even though essentially invisible, are important because they are enough to break bones and collapse bridges.

Rigid-object mechanics. Part of good engineering is to idealize away things that are not important. Unimportant features unnecessarily clutter the mind and also make calculations harder. When deformations are not of much consequence engineers usually wish them away. Mechanics calculations in which deformation has been neglected are called *rigid-object* (or *rigid-body*①) mechanics because a rigid (infinitely stiff) solid would not deform at all. Rigidity, the assumption of infinite stiffness, is an extreme constitutive assumption. However, the assumption of rigidity greatly simplifies many calculations while still generating adequate predictions for many practical problems. The assumption of rigidity also simplifies the introduction of more general mechanics concepts. Thus for understanding the steering dynamics of a car we might treat the car as a rigid object, whereas for crash analysis where rigidity is clearly a poor approximation, we might treat a car as highly deformable.

①**'Rigid body' is another phrase meaning 'rigid object'.** Things idealized as 'Rigid-objects' are often called 'rigid-bodies', using the old-fashioned language that physical things were abstractly called 'bodies'. Think of a guy with a robe and beard squinting through a brass telescope and deeply pondering 'celestial bodies'. Now that mechanics is used widely to describe biological things like people, the word 'body' can be confusing. For example, 'rigid-body' biomechanics might be inferred to be the study of people with rigid *rigamortis* muscles, so called 'stiffs'. Or, often in biomechanics we think of the parts of the body as rigid, say the fore-arm or the shank of the leg. It is confusing to say that the human body is modeled as a collection of rigid bodies. Easier to say the body is modeled as a collection of rigid objects. Here we will often, although not religiously, adopt the ordinary English that things are objects and things whose deformation we neglect are rigid objects.

Contact behavior. Most constitutive models describe the material inside an object. But to solve a mechanics problem involving friction or collisions one also has to have a constitutive model for the contact interactions. The standard friction model (or idealization) '$F \leq \mu N$' is an example of a contact constitutive model, as is the elementary 'restitution' model for collisions '$v^+ = ev^-$'.

Summarizing, we need a model of a system's mechanical behavior before we can make useful predictions. Useful models can sound absurdly extreme, as in the assumption that a piece of a human body is rigid.

Pillar 2: The geometry of motion and deformation, kinematics

In mechanics we use classical Greek (Euclidean) geometry to describe the layout, deformation and large-motions of objects. Deformation is defined by changes of lengths and angles between various pairs and triplets of points. Motion is defined by the changes of the position of points in time. Length, angle, similar triangles, the curves that particles follow and so on can be studied and understood without Newton's laws and thus make up the second independent pillar: geometry and kinematics.

Large motions. Many machines and machine parts are designed to move something relatively far. Bicycles, planes, elevators, and hearses are designed to move people; a clockwork, to move clock hands; insect wings, to move insect bodies; and forks, to move potatoes. A connecting rod is designed to move a crankshaft; a crankshaft, to move a transmission; and a transmission, to move a wheel. And wheels are designed to move skateboards, bicycles and cars of various kinds.

The description of the motion of these things, of how the positions of the pieces change with time, of how the connections between pieces restrict the motions, of the curves traversed by the parts of a machine, and of the relations of these curves to each other is called *kinematics*. Kinematics is the study of the geometry of motion (or of geometry in motion).

Motion versus deformation. The idea behind the word deformation is correctly conveyed by the mis-spelling, 'deform-motion'. Deformations usually involve small changes of distance between points on one object, whereas net motion (see the paragraph above) involves large changes of distance between points on different objects. We often need to understand deformation of individual parts to predict when they will break. Sometimes the motion associated with deformation is important in itself, say you would like a building to not sway too much in the wind. And sometimes the larger net transport motion is of interest; for example we would like all points on a plane to travel about the same large distance from New York to Bangalore. Really, deformation and motion are not distinct topics, both involve keeping track of the positions of points. The distinction we have made is for simplicity. Trying to simultaneously describe deformations and large motions is just too complicated for beginners to understand and too complicated for most engineering practice. So the ideas are kept (somewhat artificially) separate in elementary mechanics courses such as this one. As separate topics, the geometry needed to understand small deformations (called 'strains') and the geometry needed to understand large motions of rigid objects (called 'particle and rigid-object kinematics') are both basic parts of mechanics. (This book, however, has little about deformation and strain.)

Pillar 3: Relation of force to motion, the laws of mechanics, kinetics

The same intuitive 'force' that causes deformation also causes motion, or more precisely, acceleration of mass. The relation between force and acceleration of mass makes up the third pillar holding up mechanics. We loosely call this *Newton's laws*; synonyms include *the laws of mechanics*, *momentum and energy balance* and *kinetics*.②Force is related to deformation by material properties (elasticity, viscosity, etc.) and force is related to motion by the laws of mechanics summarized in the front cover. In words and informally, these are: ③

0) The laws of mechanics apply to any system (rigid or not):
   a) Force and moment are *the* measure of mechanical interaction; and
   b) Action = minus reaction applies to all interactions, ('every action has an equal and opposite reaction');

I) The net force on a system causes a net linear acceleration (*linear momentum balance*),

II) The net turning effect of forces on system causes it to rotationally accelerate (*angular momentum balance*), and

III) The change of energy of a system is due to the energy flow into the system (*energy balance*).

A non-minimal set of assumptions. The principles of action and reaction, linear momentum balance, angular momentum balance, and energy balance, are actually redundant in various ways. Linear momentum balance can be derived from angular momentum balance and sometimes vice-versa (see page 1004). Energy balance equations can often be derived from the momentum balance equations. And the principle of action and reaction can be derived from the momentum balance equations. In engineering practice, however, we worry little about which idea could be derived from the others for the problem under consideration. The four assumptions in O-III above are not a mathematically minimal set, but they are all accepted truths by practitioners of mechanics.

A lot follows from the laws of Newtonian mechanics, including the contents of this book. When these ideas are supplemented with idealizations of the mechanical behavior of particular systems (e.g., of machines, buildings or human bodies), they lead to predictions about motions and forces. There

②**Kinetics and kinematics.** It is easy to confuse these similar looking and sounding words. *Kinematics* concerns geometry with no mention of force and *kinetics* concerns the relation of force to motion. The following (backwards) anti-pnemonic device might help you. Adding 'ma' to the middle of the word *kinetics* gives the word 'kinematics', whereas adding the concept $m\vec{a}$ (as in mass times acceleration) to the concept of kinematics gives the concept called kinetics.

③**Newton's laws vs the modern appraoch.** Isaac Newton's original three laws are:

1) an object in motion tends to stay in motion,
2) $\vec{F} = m\vec{a}$ for a particle, and
3) the principle of action and reaction.

These three Newton laws could be used as a starting point for the study of mechanics.

The more modern approach here leads to the same ends. Why not just do it Newton's way? One confusion in using Newton's original statements is trying to understand how the first law is not just a special case of the second law. One thought of modern historians of Science is that Newton's first law is implicitly, by describing what happens when there is no force, defining force. In this view Newton's first law is somewhat equivalent to what we call law (0a). Another advantage to the more modern approach is that we can think of angular momentum and energy as fundamental quantities with general import, not just quantities relevant to the particular models or systems for which we can make derivations based on Newton's particle mechanics.

is an endless stream of results about the mechanics of one or another special system. Some of these results are classified into entire fields of research such as 'fluid mechanics,' 'vibrations,' 'seismology,' 'granular flow,' 'biomechanics,' or 'celestial mechanics.'

The four basic ideas also lead to mathematically advanced formulations of mechanics with names like 'Lagrange's equations,' 'Hamilton's equations,' 'virtual work', and 'variational principles.' If you go on in mechanics, you may learn some of these things in more advanced courses.

Statics, dynamics, and strength of materials

Elementary mechanics is sometimes partitioned into three courses named 'statics', 'dynamics', and 'strength of materials'. These subjects vary in how much they emphasize material properties, geometry, and Newton's laws.

Statics is mechanics with the idealization that the acceleration of mass is negligible in Newton's laws. The first eight chapters of this book provide a thorough introduction to statics. Things need not be standing exactly still, nothing is, to be well idealized with statics. But, as the name implies, statics is generally about things that don't move much. The first pillar of mechanics, constitutive laws, is generally introduced without fanfare into statics problems by the (implicit) assumption of rigidity. Other constitutive assumptions used include inextensible ropes, linear springs, and frictional contact. The material properties used as examples in elementary statics are very simple. Also, because things don't move or deform much in statics, the geometry of deformation and motion are all but ignored. Despite the commonly applied vast simplifications, statics is useful for the analysis of natural and engineered structures, slow machines or the light parts of fast machines, and other things (say, the stability of boats).

Dynamics concerns the non-negligible acceleration of mass. Chapters 9 and on of this book introduce dynamics. As with statics, the first pillar of mechanics, constitutive laws, is given a relatively minor role in the elementary dynamics presented here. For the most part, the same library of elementary properties are used with little fanfare (rigidity, inextensibility, linear elasticity, and friction). Dynamics thus concerns kinematics and kinetics. Once one has mastered statics, the hard part of dynamics is the kinematics. Dynamics is useful for the analysis of, for example, fast machines, vibrations, and ballistics.

Strength of materials expands statics to include material properties and also pays more attention to distributed forces (*e.g.,* 'traction' and 'stress'). This book only occasionally touches lightly on strength-of-materials topics like stress (loosely, force per unit area), strain (a way to measure deformation), and linear elasticity (a commonly used constitutive idealization of solids that generalizes the concept of a spring). Strength of materials gives equal emphasis to all three pillars of mechanics. Strength of materials is useful for predicting the amount of deformation in a structure or machine, where it is most likely to break with a given load, and whether or not it is likely to break with that load.

1.2 Why study Newtonian mechanics when it has been overthrown by modern physics?

We are repeatedly reminded that Newtonian ideas have been replaced by relativity and quantum mechanics. So why, in the 21st century, should you read this book and learn ideas, remnants of the nineteenth century, which are known to be wrong?

First off, this criticism is maybe a bit off base: general relativity and quantum mechanics are inconsistent with each other, not yet united by a universally-accepted deeper theory of everything. So strict consistency with modern physics, as we know it, isn't possible. But how big are the errors we make when we do classical mechanics, neglecting various more modern physics discoveries?

Special relativity. The errors from neglecting the effects of special relativity are on the order of v^2/c^2 where v is a typical speed in your problem and c is the speed of light. The biggest errors are associated with the fastest objects. For, say, calculating space shuttle trajectories this leads to an error of about

$$\frac{v^2}{c^2} \approx \left(\frac{5\,\text{mi/s}}{3 \times 10^8\,\text{m/s}}\right)^2 \approx .000000001 \approx \text{one millionth of one percent}$$

General relativity errors having to do with the non-flatness of space are so small that the genius Einstein had trouble finding a place where the deviations from Newtonian mechanics could be observed at all. Finally he predicted a small, barely measurable effect on the predicted motion of the planet Mercury. Newtonion mechanics predicts a fixed eliptical orbit. Einstein's equations correctly predicted that the eliptical path itself rotates (precesses) once every 3 million years, that's about 45 arcsec (an 80th of a degree) per century. So the Newtonian 'error' is about one part in 10^8 (like a one cent error in a millionaire's bank balance). Global positioning sattelites (GPS) do actually take general relativity into account to prevent errors of about one part in a billion (a millimeter error over a thousand kilometers).

Uncertainty principle. In classical mechanics we assume we can know exactly where something is and how fast it is going. But according to quantum mechanics this is impossible. The product of the uncertainty δx in position of an object and the the uncertainty δp of its momentum must be greater than Planck's constant $\hbar$. Planck's constant is small; $\hbar \approx 1 \times 10^{-34}$joule$\cdot$s. The fractional error in position is biggest for small objects moving slowly. So if one measures the location of a computer chip with mass $m = 10^{-4}\,\text{kg}$ to within $\delta x = 10^{-6}\,\text{m} \approx$ a twenty fifth of a thousands of an inch, the uncertainty in its velocity $\delta v = \delta p/m$ is only

$$\delta x \delta p = \hbar \Rightarrow \delta v = m\hbar/\delta x \approx 10^{-24}\,\text{m/s} \approx 10^{-15}\text{inches per year.}$$

Brownian motion. In classical mechanics we usually (although not always) neglect fluctuations associated with the thermal vibrations of atoms. But any object in thermal equilibrium with its surroundings constantly undergoes changes in size, pressure, and energy, as it interacts with the environment. For example, the internal energy per particle of a sample at temperature T fluctuates with amplitude

$$\frac{\Delta E}{N} = \frac{1}{\sqrt{N}}\sqrt{k_B T^2 c_V},$$

where k_B is Boltzmann's constant, T is the absolute temperature, N is the number of particles in the sample, and c_V is the specific heat. Water has a specific heat of 1 cal/K, or around 4 Joule/K. At room temperature of 300 Kelvin, for 10^{23} molecules of water, these values lead to an uncertainty of only 7.2×10^{-21} Joule in the the internal energy of the water. Thermal fluctuations are big enough to visibly move pieces of dust in an optical microscope (Brownian motion), and to generate variations in electric currents that are easily measured, but for most engineering mechanics purposes they are negligible. But if thermal fluctuations are of interest, they can be modeled reasonably accurately using Newtonian mechanics at the atomic scale.

Physics errors vs modeling errors. As described above, classical Newtonian physics is an accurate approximation of Nature for engineers, with errors typically on the order of parts per billion. On the other hand, the errors within mechanics, due to imperfect modeling or inaccurate measurement, are, except in extreme situations (like GPS), *far* greater than the errors due to the imperfection of Newtonian mechanics theory. For example, mechanical force measurements are typically off by a percent or so, distance measurements by a part in a thousand, and material properties are rarely known to one part in a hundred and often not even one part in 10. That is, even in the most accurate of circumstances, your mechanics calculations will typically be off by at least 100,000 times more than the laws of mechanics themselves are off.

On the other hand, if your engineering mechanics calculations make inaccurate predictions this will surely be because of errors in modeling or measurement (lets assume no math mistakes), not inaccuracies in the laws of mechanics.

Only in special circumstances are classical mechanics predictions off because of neglect of relativity, quantum mechanics, or statistical mechanics.

You can trust Newtonian mechanics. In summary, Newtonian mechanics is accurate enough, and also much simpler to use than the theories which have 'overthrown' it. You have trusted your life many times to engineers who treated classical mechanics as 'truth' and in turn, your engineering mechanics work will justly be based on the laws of classical mechanics. Although perhaps philosophically objectionable, it is reasonable engineering practice to

Think of the laws of mechanics as absolute truth.

1.3 Models, modeling, and the heirarchy of models

A plastic toy car guided by a child's hand crashes into another toy car (fig. 1.1). In common English the toys are models of cars. But the word *model* has a broader meaning in Engineering and Science. In this broader sense, for example, the toy crash is a model of a real car crash. The model of the crash *event* is that two plastic things are guided together by human hands. Its as if there are two parallel universes, the 'real' one and the 'model' one. And the whole real process of car collision is 'modeled by' the crashing of toy cars. The word model then means that cars are replaced by plastic toys and the laws of mechanics replaced by the guiding of the child's hands. And the results of the collision are replaced by whatever damage occurs to the plastic toys.

The commuting diagram. A model, in this broader sense, is represented abstractly by a *commuting diagram*, as shown in Fig. 1.2. The top row is the system to be modeled, say the real cars. The real car collision is the workings of the system w as dictated by natures laws in their full subtlety and complexity, taking account all known and as-yet unknown physics. And the way the cars move and deform and end up damaged is the system behavior SB. Parallel to this in the bottom row of the figure is the model universe. A plastic car R represents a real car by having about the same shape. The laws of nature w are 'modeled by' the manipulation rules in the model m, in this case the guidance of the child's hands. And the result of the real crash SB is 'modeled by' the result of the play crash. The model is compared to reality by making an association between bent car metal with scratched toy plastic.

We will rate this as a ‘good model’ if the damage to the plastic mimics the damage to a real car. This is expressed by the success at ‘commuting’, in the mathematical sense of the word commuting. Is the result of making a model and then carrying out the model process (down then right) the same as the result of the process then modelded (right then down)? In the language of the commuting diagram the question is,

Does $S \xrightarrow{t} R \xrightarrow{m} RB$ give the same result as $S \xrightarrow{w} SB \xrightarrow{b} RB$?

For example, we compare the prediction of damaged plastic to what the real car damage would translate to as cracks and scratches on the plastic? If they agree well then the model ‘commutes’. That is, starting with the real system do you get the same answer by applying the real workings and then the translate to what you would expect to see in the model as you would by modeling the car in plastic and applying the model workings. Using the toy

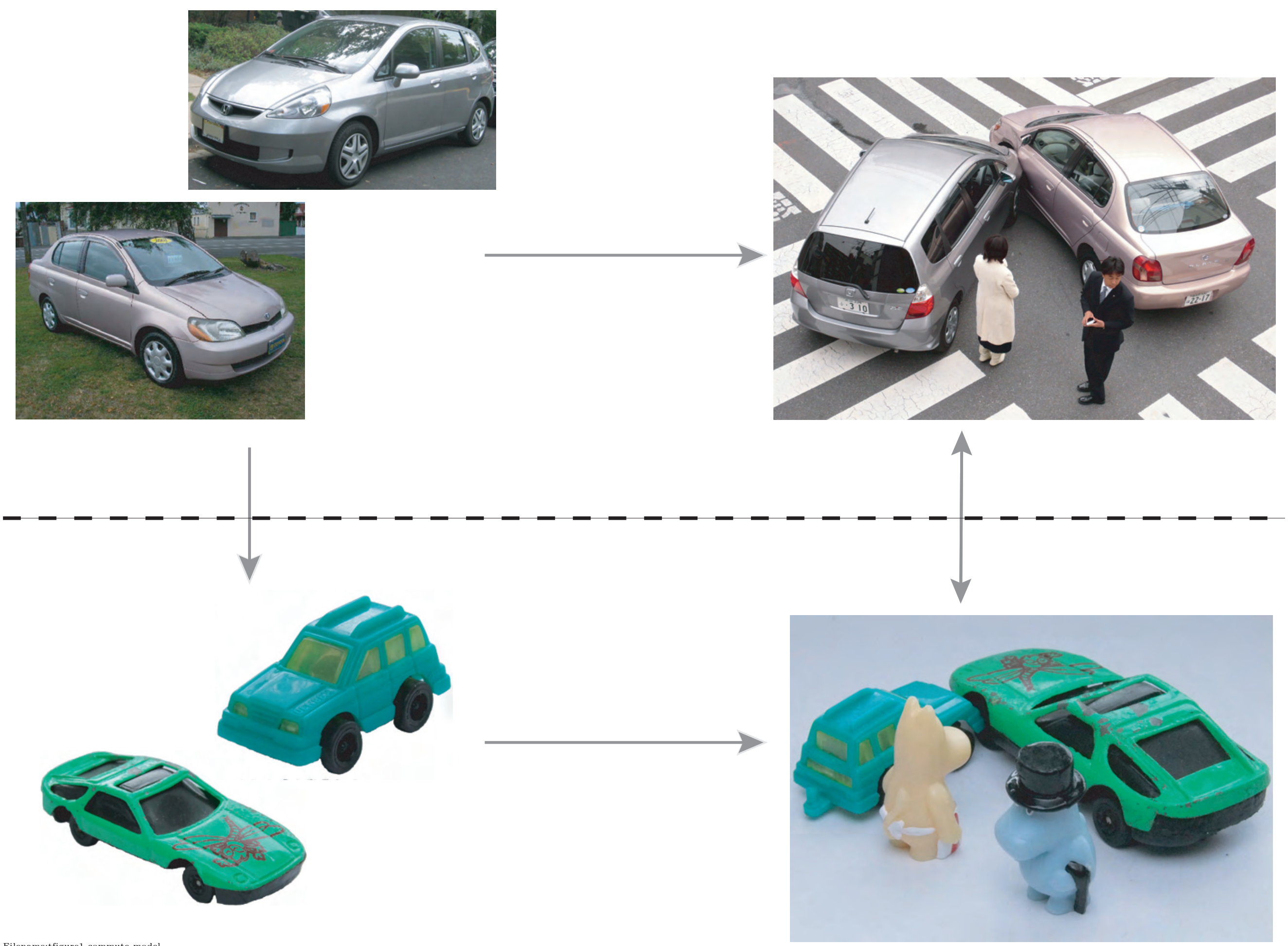

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Figure 1.1: **Toy cars and real cars.**

car example we can see aspects that commute and aspects that don't. That both the real cars and the toy cars end up with a crooked orientation is a sign of the model "commuting". Thats a good feature of the model. That the toy car passengers have no scratches and that the people in the real cars have aching necks is a lack of commuting, and a defect in 'the model'.

Mathematical vs physical models. In the toy crash example above the 'model' included a physical object, the toy car. More commonly in science and engineering the model is a constellation of ideas with no physical object involved. For example, if a solid 'is modeled as' a rigid object that means the motion of the object will be calculated by assuming that the solid does not deform. No piece of plastic representing the object is needed. ①

①**'Models' in biology.** In biology the word model is used sometimes used casually to mean'experimental subject'. But the intent of an 'animal model' is that, say, the growth of cancer in a monkey is meant to mimic ('model') the growth of similar cancer in a human.

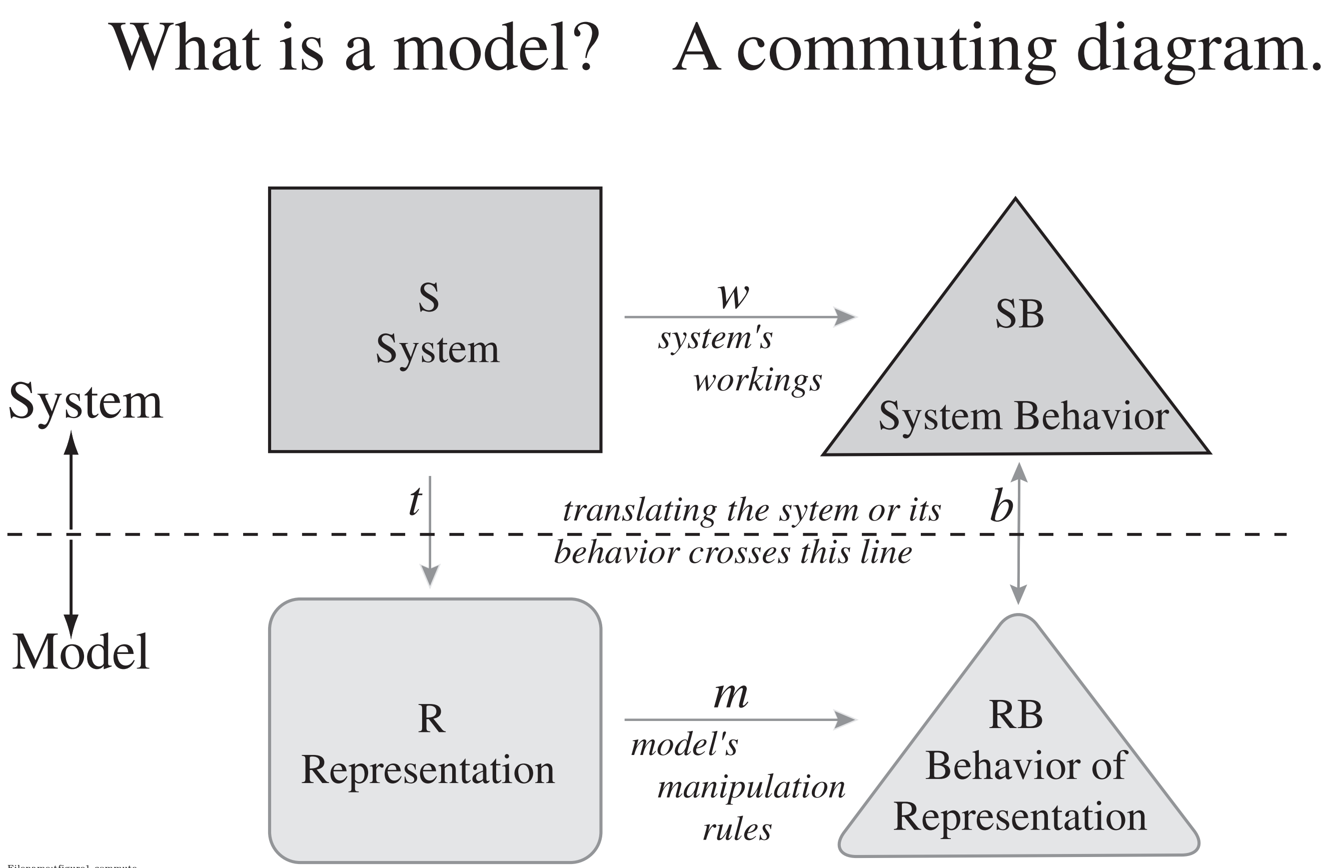

Figure 1.2: **The commuting diagram.** This 4-block diagram gives one definition of a model. **The system** S has behaviors SB that happen because of the systems workings w. **What is a model?** Overall the model includes a representation R of the system, the manipulation rules m which yield the behavior RB of the model. Translation rules t determine the Representation R from the System S. And once one has made a prediction of the Behavior RB one needs translation rules b to describe what the model behavior predicts about the system behavior. In science, engineering and math models the Representations are often a list of numbers for, say the masses and lengths of parts, *etc.*. The manipulation rules are often algebraic or differential equations. The predictions are numbers or graphs that come from the solution of the equations. The laws of Newtonian mechanics make up a model for the motions of objects which, in turn, depend on many sub-models, such as the concept of a force and of a rigid object.

Models in engineering. In engineering we use models to make predictions about reality. So the 'commuting' ability is usually expressed by comparing the model predictions to reality, wrapping three quarters of the way counter-clockwise around the diagram from the system (at the upper left) down to its model representation through the model manipulations to the model behavior and back up to the prediction for reality (at the upper right).

Models are pervasive. All this abstraction about modeling is confusing partly because we are surrounded by it all the time. Explaining modeling to you is like explaining water to a fish. For example, language and thought are themselves, in a sense, models of reality.

What makes a good model? Good features for a model include:

- Applicability to a broad range of systems,
- Prediction of a broad range of phenomena,
- Making accurate predictions (ie, 'commuting' so that the route S$\xrightarrow{t}$R$\xrightarrow{m}$RB gives the same result as S$\xrightarrow{w}$SB$\xrightarrow{b}$RB),
- Simplicity, and
- Lack of ambiguity in the rules t, m and b and thus clear model predictions.

Usually when setting up or choosing a model you need to make tradeoffs. For example, accurate (good) models are often complicated (bad) and simple (good) models are often ambiguous (bad), *etc.*.

Mechanics models

In a course like this we are concerned with a hierarchy of models.

Space and time. Most basically we model space, time and matter as having all the common-sense features that we are used to. For example we assume that the location of any point in space can be described by its x, y, and z coordinates relative to some origin.

The laws of mechanics. Second, we model all of nature's rules for motion with the basic laws of (classical, Newtonian) mechanics. As stated in the previous section with reference to modern physics concepts, Newtonian mechanics is a high-quality model whose errors (or lack of ability to commute) will likely be of no significance to you ever in your life.

Properties. Third, we have models of objects and forces. In this book, as opposed to a book about structural mechanics, we generally 'model' solid things as particles or as non-deforming rigid objects. This non-deformation model gives an error that typically ranges from a small fraction of a percent up to a few percent. Models of forces can be very accurate, for example you can know gravity forces, if you know where you are on the earth (see page A.3), to about one part in 10^6. Some other force models are also reasonably accurate like the description of linear springs (typically 1% accurate or so). And some force models are basically poor, like for friction and collisions (with typical errors of 20-50%), we just don't know good models for friction and collision forces, so we use and try to understand the bad models we have cooked up so far.

The modeling process. Given this hierarchical collection of mechanics models we next get to the engineer's task of 'modeling'. Given a real machine, how do we 'model' it as made up of various mechanics models from the paragraphs above? Which parts do we approximate as rigid objects, which as massless linear springs, etc? This modeling task is an important part of engineering practice.

However, before one can develop the art of engineering modeling one needs to know how to work with the range of common engineering models. In terms of the diagram in Fig. 1.2 you need to know how to do the manipulations m for a given candidate models before you can develop the art of determining what particular models should be used to represent your system of interest. Much more specifically for elementary mechanics you need to know how particles and rigid objects interact and move if governed by the common models for their interactions. Understanding how particles and rigid objects interact and move is the core of this book.

Models in homework problems. Most often the problem statement implicitly tells you what model to use, although sometimes in a mildly disguised language (in order to start training your modeling skills). Judging whether or not a given model is good (*i.e.,* commutes, corresponds well with reality) is an important part of engineering practice. So we will point out deficiencies in various models here and there. Further, because some of these models are pretty good, you can use your intuition (another model!) to guide your learning of mechanics models and, conversely, you can use your new understanding of mechanics models to improve your intuition about reality.

Utility of rigid-object-mechanics models. The bottom line is this. If you understand how particles and rigid bodies interact and move according to the 'rigid-object-mechanics' model, basically the contents of this book, you will understand a lot about how many real things hold together, fall apart, stay in place and move.

CHAPTER 2

Vector algebra for mechanics: position, force and moment

The key vectors for statics, namely relative position, force, and moment, are used to develop vector skills. Notational clarity is emphasized because good vector calculation demands distinguishing vectors from scalars. Vector addition is motivated by the need to add forces and relative positions. Dot products are motivated as the tool which reduces vector equations to scalar equations. And cross products are motivated as the formula which correctly calculates the heuristically motivated quantities of moment and moment about an axis.

Contents

In this book you will learn to use the laws of mechanics which were informally introduced in Chapter 1. The most fundamental quantities in mechanics are the two scalars,

- mass m and
- time t,

and the two vectors,

- relative position $\vec{\boldsymbol{r}}_{i/\mathrm{O}}$ (position of point i relative to point O), and
- force $\vec{\boldsymbol{F}}$ (acting on a system of interest) ①

Scalars are typed with an ordinary font (t and m) and vectors are typed in bold with a harpoon on top ($\vec{\boldsymbol{r}}_{i/\mathrm{O}}$, $\vec{\boldsymbol{F}}$). All the other scalars (see box 2.1 on page 41) and vectors (see box 2.2 on page 42) we use in mechanics are defined in terms of $m, t, \vec{\boldsymbol{r}}_{i/\mathrm{O}}$ and $\vec{\boldsymbol{F}}$. You are already good at scalar arithmetic and algebra (adding, subtracting, multiplying and dividing ordinary numbers and symbols representing numbers). For mechanics you also need facility with the vector arithmetic and algebra explained in this chapter.

①Actually, one can argue that $\vec{\boldsymbol{F}}$ isn't fundamental because it can be defined in terms of m and $\vec{\boldsymbol{a}}$ which is, in turn defined in terms of $\vec{\boldsymbol{r}}$ and t. But both historically and in most engineering practice, $\vec{\boldsymbol{F}}$ is a treated as a fundamental quantity.

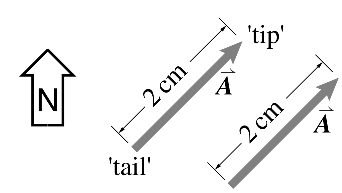

Figure 2.1: Vector $\vec{\boldsymbol{A}}$ is 2 cm long and points Northeast. Two copies of $\vec{\boldsymbol{A}}$ are drawn to show that where the drawing is placed doesn't effect the vector. Its the same vector drawn in two different places.

What is a vector?

Whereas a scalar is just a (possibly dimensional) single number, a thing with magnitude and a sign②,

> a vector is a (possibly dimensional) quantity that is fully described by both magnitude and direction.

②By 'dimensional' we mean 'with units' like meters, Newtons, or kg. We don't mean having an abstract vector-space dimension, as in one, two or three dimensional.

Example: NorthEast 2 cm
As a first vector example, consider a line segment with a length (magnitude) of 2 cm. The segment has a tail end and a head end and is pointed Northeast. Lets call this vector $\vec{A}$ (see fig. 2.1).

$$\vec{A} \stackrel{def}{=} 2\,\text{cm long line segment pointed Northeast}$$

In terms of our basic list at the top of the page, $\vec{A}$ is the relative position $\vec{r}_{h/t}$ of its head h relative to its tail t.

Every vector in mechanics is well visualized as an arrow. The direction of the arrow is the direction of the vector. The length of the arrow is proportional to the magnitude of the vector. The magnitude of $\vec{A}$ is a positive scalar indicated by $|\vec{A}|$. A vector does not lose its identity if it is picked up and moved around in space (so long as it is not rotated nor stretched). Thus both vectors drawn in fig. 2.1 are the same vector $\vec{A}$.

Vector arithmetic makes sense

We have oversimplified. We said that a vector is something with magnitude and direction. In fact, by common modern convention, that's not enough. A one way street sign, for example, is not considered a vector even though is has a magnitude (its mass is, say, half a kilogram) and a direction (the direction of most of the traffic). A thing is only called a vector if, additionally, elementary vector arithmetic, vector addition in particular, has a sensible meaning③.

③In abstract mathematics they don't bother to talk about magnitudes and directions. All they care about is vector arithmetic. So, to the mathematicians, anything which obeys simple vector arithmetic is a vector, arrow-like or not. In math talk lots of strange things are vectors, like arrays of numbers and functions. As special cases of the mathematicians' 'abstract vectors', the vectors in this book always have magnitude and direction.

The following sentence summarizes centuries of thought and also motivates this chapter:

The vectors in mechanics have magnitude and direction *and* elementary vector arithmetic operations have sensible physical meanings.

This chapter is about vector arithmetic. In this chapter you will learn how to add and subtract vectors, how to stretch them, how to find their components, and how to multiply them with each other two different ways. Each of these operations has use in mechanics.

2.1 Vector notation and vector addition

Facility with vectors has several aspects.

1. You must recognize which quantities are vectors (such as relative position) and which are scalars (such as length).
2. You have to use a notation that distinguishes between vectors and scalars using, for example, $\vec{a}$, or $\underline{a}$ for acceleration and a for a scalar with the same magnitude so that $|a| = |\vec{a}|$.
3. You need skills in vector arithmetic, perhaps more than you learned in your previous math and physics courses.

In this first section (2.1) we start with notation and go on to finding the relative position vector from a picture, multiplication of a vector by a scalar, vector addition and vector subtraction.

How to write vectors

A scalar is written as a single English or Greek letter. This book uses slanted type for scalars (*e.g., m* for mass) but ordinary printing is fine for hand work (*e.g.,* m for mass). A vector is also represented by a single letter of the alphabet, either English or Greek, but ornamented to indicate that it is a vector and not a scalar. The common ornamentations are described below.

*Use one of these vector notations in all of your work*①.

① **Be careful to distinguish vectors from scalars** in all of your written work. Clear notation helps clear thinking and will help you solve problems. If you notice that you are not using clear vector notation, *stop*, determine which quantities are vectors and which scalars, and fix your notation. Rare is the student who consistently gets correct answers to exam questions without clear vector notation. And almost as rare is the student who has clear vector usage and can't do problems. For some students, accepting this vector language and syntax is a bitter pill. Just swallow it. You'll feel much better.

$\vec{\boldsymbol{F}}$ Putting a harpoon (or arrow) over the letter $\boldsymbol{F}$ is the suggestive notation used in in this book for vectors.

$\boldsymbol{F}$ In most texts a bold $\boldsymbol{F}$ represents the vector $\vec{\boldsymbol{F}}$. But bold face is inconvenient for hand written work. The lack of bold face pens and pencils tempts students to transcribe a bold $\boldsymbol{F}$ as F. But F with no adornment represents a scalar and not a vector. Beware not to transcribe $\boldsymbol{F}$ as F.

$\underline{F}$ Underlining or undersquiggling ($\underset{\sim}{\boldsymbol{F}}$) is an easy and unambiguous notation for hand writing vectors. A recent survey found that 11 out of 17 mechanics professors use this notation. These professors would copy a $\vec{\boldsymbol{F}}$ from this book by writing $\underline{F}$. The origin of the notation seems to be from old-fashioned typesetting where an author would indicate that a letter should be printed in bold by underlining it.

2.1 The scalars in mechanics

Most of the scalars in this book are listed below. Dimensions (units) in given in brackets [ ], (M for mass, L for length, T for time, F for force, and E for energy).

- mass m, $[M]$;
- length or distance ℓ, w, x, r, ρ, d, or s, $[L]$;
- time t, $[T]$;
- pressure p, $[F/L^2] = [M/(L \cdot T^2)]$;
- angles θ 'theta', ϕ 'phi', γ 'gamma', and ψ 'psi', [dimensionless];
- energy E, kinetic energy E_{K}, potential energy E_{P}, $[E]$=$[F \cdot L]$=$[M \cdot L^2/T^2]$;
- work W, $[E]$=$[F \cdot L]$=$[M \cdot L^2/T^2]$;
- tension T, $[M \cdot L/T^2] = [F]$;
- power P, $[E/T]$=$[M \cdot L^2/T^3]$;
- the magnitudes of all the vector quantities are also scalars, for example
  - speed $|\vec{\boldsymbol{v}}|$, $[L/T]$;
  - magnitude of acceleration $|\vec{\boldsymbol{a}}|$, $[L/T^2]$;
  - magnitude of angular momentum $|\vec{\boldsymbol{H}}|$, $[M \cdot L^2/T]$;
- the components of vectors, for example
  - r_x (where $\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}}$), or
  - $L_{x'}$ (where $\vec{\boldsymbol{L}} = L_{x'}\hat{\boldsymbol{\imath}}' + L_{y'}\hat{\boldsymbol{\jmath}}'$);
- coefficient of friction μ 'mu', or friction angle ϕ 'phi';
- coefficient of restitution e;
- mass per unit length, area, or volume ρ;
- oscillation frequency β or λ.

$\bar{\boldsymbol{F}}$ It is a stroke simpler to put a bar rather than a harpoon over a symbol. But the saved effort causes ambiguity because an over-bar is often used to indicate average. There could be confusion, say, between the velocity $\bar{v}$ and the average speed $\bar{v}$.

$\hat{\boldsymbol{\imath}}$ Over-hat. Putting a hat on top is like an over-arrow or over-bar. In this book we reserve the hat for unit vectors. For example, we use $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$, or $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$, and $\hat{\boldsymbol{e}}_3$ for unit vectors parallel to the x, y, and z axes, respectively. The same poll of 17 mechanics professors found that 8 of them used no special notation for unit vectors and just wrote them like, *e.g.*, $\underline{\imath}$.

Drawing vectors

In fig. 2.1 on page 39, the magnitude of $\vec{\boldsymbol{A}}$ was used as the drawing length. But drawing a vector using its magnitude as length would be awkward if, say, we were interested in vector $\vec{\boldsymbol{B}}$ that points Northwest and has a magnitude of 2 meters. To well contain $\vec{\boldsymbol{B}}$ in a drawing would require a piece of paper about 2 meters square (each edge the length of a basketball player). This situation moves from difficult to ridiculous if the magnitude of the vector of interest is 2 km and it would take half an hour to stroll from tail to tip dragging a purple crayon. Thus in pictures we merely make scale drawings of vectors with, say, one centimeter of graph paper representing 1 kilometer of vector magnitude.

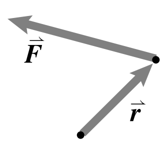

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Figure 2.2: Position and force vectors are drawn with different scales.

2.2 The Vectors in Mechanics

The vector quantities used in mechanics and the notations used in this book are shown below. The dimensional symbols of each are shown in brackets [ ].

- position $\vec{\boldsymbol{r}}$ or $\vec{\boldsymbol{x}}$, $[L]$;
- velocity $\vec{\boldsymbol{v}}$ or $\dot{\vec{\boldsymbol{x}}}$ or $\dot{\vec{\boldsymbol{r}}}$, $[L/t]$;
- acceleration $\vec{\boldsymbol{a}}$ or $\dot{\vec{\boldsymbol{v}}}$ or $\ddot{\vec{\boldsymbol{r}}}$, $[L/t^2]$;
- angular velocity $\vec{\boldsymbol{\omega}}$ 'omega' (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\dot{\theta}\hat{\boldsymbol{k}}$), $[1/t]$;
- rate of change of angular velocity $\vec{\boldsymbol{\alpha}}$ 'alpha' or $\dot{\vec{\boldsymbol{\omega}}}$ (or, if aligned with the $\hat{\boldsymbol{k}}$ axis, $\ddot{\theta}\hat{\boldsymbol{k}}$), $[1/t^2]$;
- force $\vec{\boldsymbol{F}}$ or $\vec{\boldsymbol{N}}$, $[m \cdot L/t^2]{=}[F]$;
- moment or torque $\vec{\boldsymbol{M}}$, $[m \cdot L^2/t^2]{=}[F \cdot L]$;
- linear momentum $\vec{\boldsymbol{L}}$, $[m \cdot L/t]$ and its rate of change $\dot{\vec{\boldsymbol{L}}}$, $[m \cdot L/t^2]$;
- angular momentum $\vec{\boldsymbol{H}}$, $[m \cdot L^2/t]$; and its rate of change $\dot{\vec{\boldsymbol{H}}}$, $[m \cdot L^2/t^2]$.
- unit vectors to help write other vectors [dimensionless]:
  - $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ for cartesian coordinates,
  - $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$, and $\hat{\boldsymbol{k}}'$ for crooked cartesian coordinates,
  - $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_\theta$ for polar coordinates,
  - $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$ for path coordinates, and
  - $\hat{\boldsymbol{\lambda}}$ 'lambda' and $\hat{\boldsymbol{n}}$ as miscellaneous unit vectors.

Ornamentation of vectors Subscripts and superscripts are often added to indicate the point, points, object, or objects the vectors are describing. Upper case letters (O, A, B, C,...) are used to denote points. Upper case calligraphic (or script if you are writing by hand) letters ($\mathcal{A}, \mathcal{B}, \mathcal{C} \ldots \mathcal{F} \ldots$) are for labeling rigid objects or reference frames. $\mathcal{F}$ is the fixed, Newtonian, or 'absolute' reference frame (think of $\mathcal{F}$ as the ground if you are a first time reader). For example, $\vec{\boldsymbol{r}}_{\mathrm{AB}}$ or $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ is the position of the point $\boldsymbol{B}$ relative to point A. $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is the absolute angular velocity of the object called $\mathcal{B}$ ($\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ is short hand for $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$). And $\vec{\boldsymbol{H}}_{\mathcal{A}/C}$ is the angular momentum of object $\mathcal{A}$ relative to point $\boldsymbol{C}$.

The notation is further complicated when we want to take derivatives with respect to moving frames, a topic which comes up later in the book. For completeness here: ${}^{\mathcal{B}}\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{D}/\mathcal{E}}$ is the time derivative with respect to reference frame $\mathcal{B}$ of the angular velocity of object $\mathcal{D}$ with respect to object (or frame) $\mathcal{E}$. (If this paragraph doesn't read like gibberish to you, you have already studied dynamics. Its here for the experts who are looking back.)

The necessity for using scale drawings to represent vectors is apparent for a vector whose magnitude is *not* length. Force is a vector since it has magnitude and direction. Say $\vec{F}_{\mathrm{gr}}$ is the 700 N force that the ground pushes up on your chair as you sit reading. We can't draw a line segment with length 700 N for $\vec{F}_{\mathrm{gr}}$ because a Newton is a unit of force not length. So a scale drawing is the only choice.

One often needs to draw vectors with different units on the same picture, as for showing the position $\vec{r}$ at which a force $\vec{F}$ is applied (see fig. 2.2). In this case different scale factors are used for the drawing of the vectors that have different units.

Drawing and measuring are tedious and also not very accurate. And drawing in 3 dimensions is particularly hard (given the short supply of 3D graph paper nowadays). So the magnitudes and directions of vectors are usually defined with numbers and units rather than scale drawings. Nonetheless, the drawing rules and geometric descriptions define all the vector concepts.

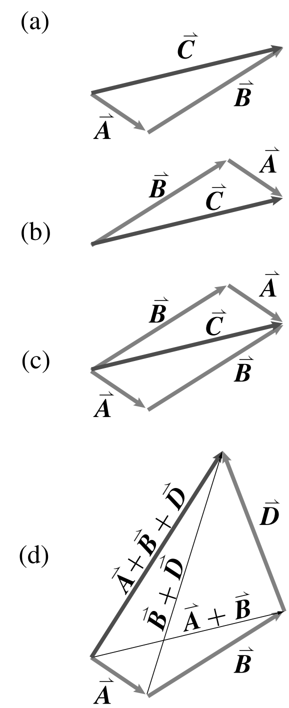

Figure 2.3: (a) tip to tail addition of $\vec{A} + \vec{B}$, (b) tip to tail addition of $\vec{B} + \vec{A}$, (c) the parallelogram interpretation of vector addition which shows the commutative law of vector addition: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$, and (d) The associative law of vector addition: $(\vec{A}+\vec{B})+\vec{D} = \vec{A}+(\vec{B}+\vec{D})$.

This figure also makes sense in 3D. Drawings (a), (b) and (c) are all on a tilted planes. And the 6 vectors drawn in (d) lie on the edges of a tetrahedron.

Adding vectors

Tip to tail rule. The sum of two vectors $\vec{A}$ and $\vec{B}$ is defined by the *tip to tail* rule of vector addition shown in fig. 2.3a for the sum $\vec{C} = \vec{A}+\vec{B}$. Vector $\vec{A}$ is drawn. Then vector $\vec{B}$ is drawn with its tail at the tip (or head) of $\vec{A}$. The sum $\vec{C}$ is the vector from the tail of $\vec{A}$ to the tip of $\vec{B}$.

Parallelogram rule. The same sum is achieved if $\vec{B}$ is drawn first, as shown in fig. 2.3b. Putting both ways of adding $\vec{A}$ and $\vec{B}$ on the same picture draws a parallelogram as shown in fig. 2.3c. Hence the tip to tail rule of vector addition is also called the *parallelogram* rule. The parallelogram construction shows the *commutative* property of vector addition, namely that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

3D. Note that you can view fig. 2.3a-c as 3D pictures. In 3D, the parallelogram will be on a plane but that may well be tilted relative to the x, y and z axes.

Adding many vectors. Three vectors are added by the same tip to tail rule. The construction shown in fig. 2.3d shows that $(\vec{A}+\vec{B})+\vec{D} = \vec{A}+(\vec{B}+\vec{D})$ so that the expression $\vec{A} + \vec{B} + \vec{D}$ is unambiguous. This is the *associative* property of vector addition.

With these two laws we see that the sum $\vec{A}+\vec{B}+\vec{D}+\ldots$ can be permuted to $\vec{D}+\vec{A}+\vec{B}+\ldots$ or any which way without changing the result. So vector addition shares the associativity and commutivity of scalar addition that you are used to *e.g.*, that $3 + (7 + \pi) = (\pi + 3) + 7$.

Concurrent forces. We can reconsider the statement 'force is a vector' and see that it hides one of the basic assumptions in mechanics, namely:

If forces $\vec{F}_1$ and $\vec{F}_2$ are applied to a point on a structure they can be replaced, for all mechanics considerations, with a single force $\vec{F} = \vec{F}_1 + \vec{F}_2$ applied to that point

as illustrated in fig. 2.4. The force $\vec{F}$ is said to be *equivalent* to the *concurrent* (acting at one point) force system consisting of $\vec{F}_1$ and $\vec{F}_2$ acting at the same point.

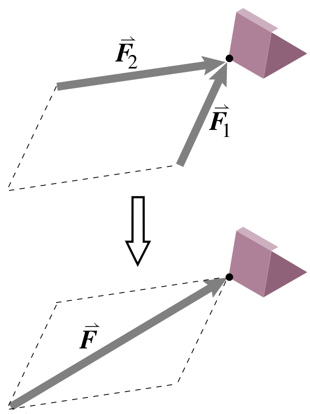

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Figure 2.4: Two forces acting at a point may be replaced by their sum for all mechanics purposes.

Apples and oranges. Note that two vectors with different dimensions cannot be added. Figure 2.2 on page 42 can no more sensibly be taken to represent meaningful vector addition than can the scalar sum of a length and a weight, "2 ft + 3 N", be taken as meaningful.

Subtraction, negation, and the zero vector

Subtraction is most simply defined by inverse addition. Find $\vec{C} - \vec{A}$ means find the vector which when added to $\vec{A}$ gives $\vec{C}$. We can draw $\vec{C}$, draw $\vec{A}$ and then find the vector which, when added tip to tail to $\vec{A}$ give $\vec{C}$. Figure 2.3a shows that $\vec{B}$ answers the question. Another interpretation comes from defining the negative of a vector $-\vec{A}$ as $\vec{A}$ with the head and tail switched. Again you can see from fig. 2.3b, by imagining that the head and tail on $\vec{A}$ were switched that $\vec{C} + (-\vec{A}) = \vec{B}$. The negative of a vector evidently has the expected property that $\vec{A} + (-\vec{A}) = \vec{0}$, where $\vec{0}$ is the vector with no magnitude so that $\vec{C} + \vec{0} = \vec{C}$ for all vectors $\vec{C}$.

Relative position vectors

The concept of relative position permeates most mechanics equations. The position of point B relative to point A is represented by the vector

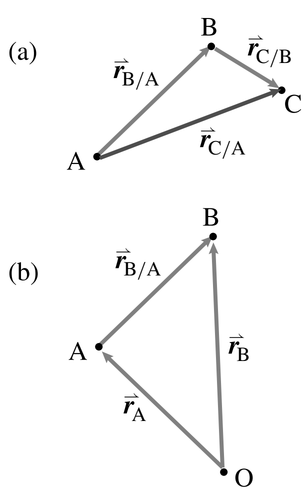
$$\vec{r}_{B/A} \text{ (pronounced 'r of B relative to A')}$$

drawn from A and to B (as shown in fig. 2.5). An alternate notation for the relative position vector $\vec{r}_{B/A}$ is

$$\vec{r}_{B/A} \equiv \vec{r}_{AB} \text{ (pronounced 'r A B' or 'r A to B').}$$

You can think of the position of B relative to A as being the position of B relative to you if you were standing on A. Similarly $\vec{r}_{C/B} = \vec{r}_{BC}$ is the position of C relative to B.

Figure 2.5a shows that relative positions add by the tip to tail rule. That is,



Filename:tfigure-relpos

Figure 2.5: a) Relative position of points A, B, and C; b) Relative position of points O, A, and B.

$$\vec{r}_{C/A} = \vec{r}_{B/A} + \vec{r}_{C/B} \qquad \text{or} \qquad \vec{r}_{AC} = \vec{r}_{AB} + \vec{r}_{BC}$$

so vector addition has a sensible meaning for relative position vectors.

Note that the position of B relative to A is the opposite (negative vector) of the position of A relative to B, so

$$\vec{\boldsymbol{r}}_{\mathrm{B/A}} = -\vec{\boldsymbol{r}}_{\mathrm{B/A}} \qquad \text{and} \qquad \vec{\boldsymbol{r}}_{\mathrm{AB}} = -\vec{\boldsymbol{r}}_{\mathrm{BA}}.$$

Position relative to the origin. Often when doing problems we pick a distinguished point in space, say a prominent point or corner of a machine or structure, and use it as the origin of a coordinate system O. The position of point A relative to O is $\vec{\boldsymbol{r}}_{\mathrm{A/0}}$ or $\vec{\boldsymbol{r}}_{\mathrm{OA}}$ but we often adopt the shorthand notation $\vec{\boldsymbol{r}}_{\mathrm{A}}$ (pronounced 'r A') leaving the reference point O as implied,

$$\vec{\boldsymbol{r}}_{\mathrm{A}} \quad \text{means} \quad \vec{\boldsymbol{r}}_{\mathrm{A/0}}.$$

Figure 2.5b shows that

$$\vec{\boldsymbol{r}}_{\mathrm{B/A}} = \vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}$$

which rolls off the tongue more easily than $\vec{\boldsymbol{r}}_{\mathrm{B/0}} - \vec{\boldsymbol{r}}_{\mathrm{A/0}}$ and makes the concept of relative position easier to remember.②

Multiplying by a scalar stretches a vector

Naturally enough $2\vec{\boldsymbol{F}}$ means $\vec{\boldsymbol{F}} + \vec{\boldsymbol{F}}$ (see fig. 2.6) and $127\vec{\boldsymbol{A}}$ means $\vec{\boldsymbol{A}}$ added to itself 127 times. Similarly $\vec{\boldsymbol{A}}/7$ or $\frac{1}{7}\vec{\boldsymbol{A}}$ means a vector in the direction of $\vec{\boldsymbol{A}}$ that when added to itself 7 times gives $\vec{\boldsymbol{A}}$. By combining these two ideas we can define any rational multiple of $\vec{\boldsymbol{A}}$. For example $\frac{29}{13}\vec{\boldsymbol{A}}$ means add 29 copies of the vector that when added 13 times to itself gives $\vec{\boldsymbol{A}}$. It is a mathematical fine point to extend the definition to $c\vec{\boldsymbol{A}}$ for c that are irrational.

Combining our abilities to negate a vector and multiply it by a positive scalar, we define $-17\vec{\boldsymbol{A}}$ as $17(-\vec{\boldsymbol{A}})$. In general, for any positive scalar c we define $c\vec{\boldsymbol{A}}$ as the vector that is in the same direction as $\vec{\boldsymbol{A}}$, or opposite if c is negative, but whose magnitude is multiplied by $|c|$. Five times a 5 N force pointed Northeast is a 25 N force pointed Northeast. Minus 5 times a 5 N force pointed Northeast is a 25 N force pointed SouthWest.

Distributive rule for scalar multiplication. If you imagine stretching a whole vector addition diagram (*e.g.,* fig. 2.3a on page 43) equally in all directions the distributive rule for scalar multiplication is apparent:

$$c(\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}) = c\vec{\boldsymbol{A}} + c\vec{\boldsymbol{B}}$$

②For the first several chapters of this book (until rotating reference frames) you can just translate 'relative to' to mean 'minus' as in english. 'How much money does Rudra have *relative to* Andy?' means what is Rudra's wealth *minus* Andy's wealth? What is the position of B *relative to* A? It is the position of B *minus* the position of A.

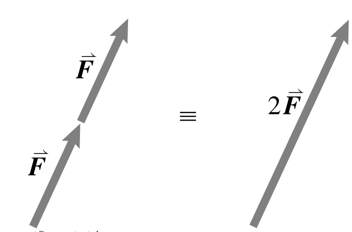


Figure 2.6: Multiplying a vector by a scalar stretches it.

Unit vectors have magnitude 1

Unit vectors are vectors with a magnitude of one. Unit vectors are useful for indicating direction. Key examples are the unit vectors pointed in the positive x, y and z directions $\hat{\boldsymbol{\imath}}$ (called 'i hat' or just 'i'), $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.

An easy way to find a unit vector in the direction of a vector $\vec{\boldsymbol{A}}$ is to divide $\vec{\boldsymbol{A}}$ by its magnitude. Thus

$$\hat{\boldsymbol{\lambda}}_A \equiv \frac{\vec{\boldsymbol{A}}}{|\vec{\boldsymbol{A}}|}$$

is a unit vector in the $\vec{\boldsymbol{A}}$ direction. We can check that this defines a unit vector by looking up at the rules for multiplication by a scalar: multiplying $\vec{\boldsymbol{A}}$ by the scalar $1/|\vec{\boldsymbol{A}}|$ gives a new vector with magnitude $|\vec{\boldsymbol{A}}|/|\vec{\boldsymbol{A}}| = 1$.

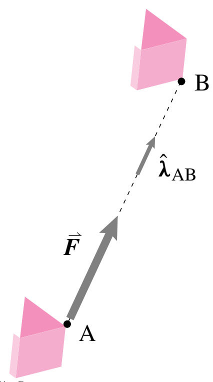


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Figure 2.7: The force $\vec{\boldsymbol{F}}$ that points from A to B can be represented as the product of a scalar F and a unit vector $\hat{\boldsymbol{\lambda}}_{\text{AB}}$: $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\lambda}}_{\text{AB}}$.

A vector as a scalar times a unit vector. Often we know that a force $\vec{\boldsymbol{F}}$ is a yet unknown scalar F multiplied by a unit vector pointing between known points A and B. (fig. 2.7). We can then write $\vec{\boldsymbol{F}}$ as

$$\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\lambda}}_{\text{AB}} = F\frac{\vec{\boldsymbol{r}}_{\text{AB}}}{|\vec{\boldsymbol{r}}_{\text{AB}}|} = F\frac{\vec{\boldsymbol{r}}_{\text{B}} - \vec{\boldsymbol{r}}_{\text{A}}}{|\vec{\boldsymbol{r}}_{\text{B}} - \vec{\boldsymbol{r}}_{\text{A}}|}$$

where we have used $\hat{\boldsymbol{\lambda}}_{\text{AB}}$ as the unit vector pointing from A to B. Note that in this usage, one we will use often, the scalar need not be positive. So the 'scalar part' might be plus or minus the magnitude of the vector.

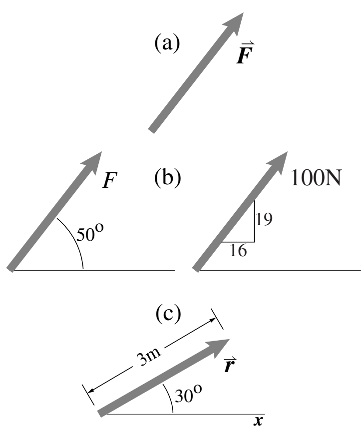

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Figure 2.8: Three different ways of drawing a vector (a) Symbolic. The magnitude and direction of the vector is given by the symbol $\vec{\boldsymbol{F}}$, the drawn arrow has no quantitative information; (b) "Scalar times arrow" shows an arrow with clearly indicated orientation next to the scalar F or the scalar 100N. The vector indicated is the scalar multiplied by a unit vector in the direction drawn; (c) Combined. The symbol $\vec{\boldsymbol{r}}$ is defined (set equal to) a vector with the magnitude and orientation shown.

Three notations for vectors in pictures and diagrams.

Some options for drawing vectors are shown in sample 2.1 on page 52. The three notations below are the most common.

Symbolic: labeling an arrow with a vector symbol. Indicate a vector, say a force $\vec{\boldsymbol{F}}$, by drawing an arrow and then labeling it with one of the symbolic notations above as in fig. 2.8a. *In this notation, the arrow is only schematic*, the magnitude and direction are determined by the algebraic symbol $\vec{\boldsymbol{F}}$. It is most clear if you draw the arrow roughly in the vector's direction and roughly to scale, but

> If the symbol and drawing disagree the symbol takes precedence (see sample 2.1j)

Graphical: "scalar times arrow", a scalar multiplies a unit vector in the direction of a drawn arrow (fig. 2.8b). Indicate a vector's direction by drawing an arrow. The direction should be made clear with a marked angle or slope. The length drawn is irrelevant. Write a letter of the alphabet, say F, or a (possibly dimensional number, say 100N) near the vector. The vector indicated is a scalar F (or the number) multiplying a unit vector in the direction of the arrow. Often you know that a force acts along a known line but you don't know which way. This is accomodated by allowing the scalar F to be positive or negative (See examples in sample 2.1.)

Combined: graphical representation used to define a symbolic vector. The symbolic notation can be used with the graphical notation to define the vector symbol. In fig. 2.8c $\vec{r}$ is being defined (being set equal) to the vector with magnitude 3m and direction 30° CCW from the $+x$ axis.

The cartesian components of a vector

A given vector, say $\vec{F}$, can be described as the sum of vectors each of which is parallel to a coordinate axis. Most often we use Cartesian axes, with the x, y, and z axes all orthogonal to each other. Thus $\vec{F} = \vec{F}_x + \vec{F}_y$ in 2D and $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$ in 3D (see fig. 2.9). Each of these vectors can in turn be written as the product of a scalar and a unit vector along the positive axes, *e.g.*, $\vec{F}_x = F_x\hat{\imath}$. So

$$\vec{F} = \vec{F}_x + \vec{F}_y = F_x\hat{\imath} + F_y\hat{\jmath} \qquad \text{(2D)}$$

or

$$\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}. \qquad \text{(3D)}$$

The scalars F_x, F_y, and F_z are called the components, or coordinates, of the vector with respect to the axes xyz. The components may also be thought of as the orthogonal projections (the shadows) of the vector onto the coordinate axes.

Because the list of components is such a handy way to describe a vector we have a special notation for it. The bracketed expression $[\vec{F}]_{xyz}$ stands for the list of components of $\vec{F}$ presented as a horizontal or vertical array (depending on context), as shown below.

$$[\vec{F}]_{xyz} = [F_x, \quad F_y, \quad F_z] \quad \text{or} \quad [\vec{F}]_{xyz} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

If we had an xy coordinate system with x pointing East and y pointing North we could write the components of a 5 N force pointed Northeast as $[\vec{F}]_{xy} = [(5/\sqrt{2})\,\text{N}, \quad (5/\sqrt{2})\,\text{N}]$③.

Rather than using new letters to repeat the same concept we sometimes label the coordinate axes x_1, x_2 and x_3 and the unit vectors along them $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ (thus freeing our minds from silently pronouncing the extra letters y,z,j, and k)④.

Study sample 2.1 on page 52 to master the various graphical and component representations.

③Note that the components of a vector in some tilted coordinate system $x'y'z'$ are different from its components in the coordinate system xyz because the projections are different.

Even though $\vec{F} = \vec{F}$ it is *not true* that $[\vec{F}]_{xyz} = [\vec{F}]_{x'y'z'}$ (see fig. 2.34 on page 59).

Understanding the relation between $[\vec{F}]_{xyz}$ and $[\vec{F}]_{x'y'z'}$ is especially important in dynamics (see sec. 14.1 on page 754). Because we often make use of multiple coordinate systems, when we define a vector by its components the coordinate system used must be specified

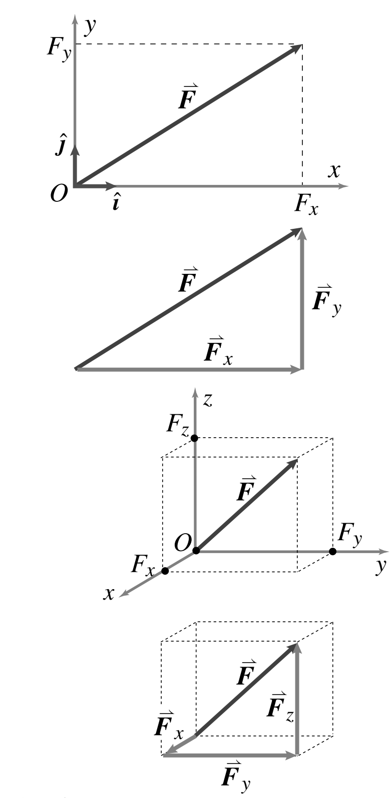


Figure 2.9: A vector can be broken into a sum of vectors, each parallel to the axis of a coordinate system. Each of these is a component multiplied by a unit vector along the coordinate axis, *e.g.*, $\vec{F}_x = F_x\hat{\imath}$.

④Note that non-Cartesian coordinates, most especially polar coordinates, are often useful in dynamics (*e.g.,* , see sec. 13.1).

Manipulating vectors by manipulating components

A vector can be represented by its components (once given a coordinate system), so we should be able to translate the rules for manipulating geometric vectors into rules about manipulating their components. This is important because in practice, when push comes to shove, most calculations with vectors are done with components.

Adding and subtracting vectors using components

Because a vector can be broken into a sum of orthogonal vectors, because addition is associative, and because each orthogonal vector can be written as a component times a unit vector we get the addition rule:

$$[\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}}]_{xyz} = [(A_x + B_x), \quad (A_y + B_y), \quad (A_z + B_z)]$$

which can be described by the tricky words 'the components of the sum of two vectors are given by the sums of the corresponding components.' Similarly,

$$[\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}}]_{xyz} = [(A_x - B_x), \quad (A_y - B_y), \quad (A_z - B_z)].$$

Multiplying a vector by a scalar using components

The vector $\vec{\boldsymbol{A}}$ can be decomposed into the sum of three orthogonal vectors. If $\vec{\boldsymbol{A}}$ is multiplied by 7 then so must be each of the component vectors. Thus

$$[c\vec{\boldsymbol{A}}]_{xyz} = [cA_x, \quad cA_y, \quad cA_z].$$

The cartesian components of a scaled vector are the corresponding scaled components. For example if $c = 3$ and $[\vec{\boldsymbol{A}}]_{xyz} = [2, 4, -5]$ then $[c\vec{\boldsymbol{A}}]_{xyz} = [6, 12, -15]$.

Often the components of vectors are written as columns rather than rows of numbers. Thus we would write

$$[\vec{\boldsymbol{A}}]_{xyz} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = [2, 4, -5]' = \begin{bmatrix} 2 \\ 4 \\ -5 \end{bmatrix}.$$

The $'$ means 'matrix transpose', turning the rows into columns and *vice versa*. We can add the components of vectors using this notation, so if $d = -0.5$ and $[\vec{\boldsymbol{B}}]_{xyz} = [100, 200, -300]'$ then

$$[c\vec{\boldsymbol{A}} + d\vec{\boldsymbol{B}}]_{xyz} = c[\vec{\boldsymbol{A}}]_{xyz} + d[\vec{\boldsymbol{B}}]_{xyz} = \begin{bmatrix} cA_x + dB_x \\ cA_y + dBy \\ cA_z + dBz \end{bmatrix} = \begin{bmatrix} -44 \\ -88 \\ 135 \end{bmatrix}$$

Finally we can use matrix notation and the definition of matrix multiplication to add multiples of vectors

$$\underbrace{\begin{bmatrix} A_x & B_x \\ A_y & By \\ A_z & Bz \end{bmatrix}}_{\text{A 3 by 2 matrix}} \begin{bmatrix} c \\ d \end{bmatrix} \underbrace{\equiv}_{\boxed{\text{Is defined to mean}}} c \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} + d \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} cA_x + dB_x \\ cA_y + dBy \\ cA_z + dBz \end{bmatrix}.$$

So, for example,

$$
\begin{aligned}
[c\vec{A} + d\vec{B}]_{xyz} &= \begin{bmatrix} 2 & 100 \\ 4 & 200 \\ -5 & -300 \end{bmatrix} \begin{bmatrix} 3 \\ -0.5 \end{bmatrix} \\
&= \begin{bmatrix} 3\cdot 2 & + & -0.5\cdot 100 \\ 3\cdot 4 & + & -0.5\cdot 200 \\ 3\cdot(-5) & + & -0.5\cdot(-300) \end{bmatrix} = \begin{bmatrix} -44 \\ -88 \\ 135 \end{bmatrix}.
\end{aligned}
$$

In the language of linear algebra (skip this sentence if you never took such a course), a matrix multiplied by a column vector is a linear combination of the matrix columns with weights (coefficients) given by the elements of the column vector.

Adding vectors on a computer

Computers deal well with lists of numbers but not generally with units. So only the numerical part of a calculation shows in the computer work. For example, when we write on the computer

```
F = [ 3 5 -7]
```

we take that to be computereze for $[\vec{F}]_{xyz} = [3\,\text{N}, \quad 5\,\text{N}, -7\,\text{N}]$. To do computer work we have to be clear about what units and what coordinate system we are using. In particular, at this point in the course, we advise you to only use one coordinate system and one consistent set of units in any one problem that uses computer calculations. We can add multiples of vectors on a computer with commands something like this:

```
A = [   2    4    -5]'
B = [ 100  200  -300]'
c = 3
d = -0.5
C = c*A + d*B
```

or using the matrix notation, like this.

```
A = [   2    4    -5]'
B = [ 100  200  -300]'
M = [A  B]   %M is column A next to column B
c =  3
d = -0.5
v =  [c d]'
C = M*v
```

Or, if you like to just put in the numbers and type as little as possible,

```
M = [  2    100
       4    200
      -5   -300]
C = M * [3  -0.5]'.
```

Although this last approach is compact, it makes deciphering your work later more difficult.

Magnitude of a vector using components

The Pythagorean Theorem for right triangles ('$A^2 + B^2 = C^2$') tells us that

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2}, \qquad \text{(2D)}$$

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}. \qquad \text{(3D)} \qquad (2.1)$$

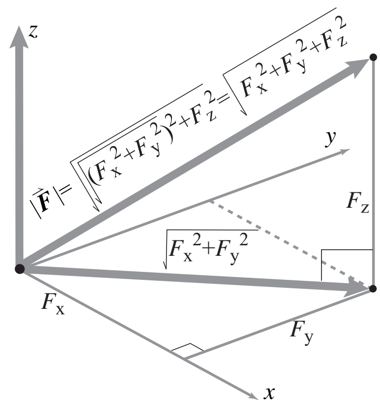


Figure 2.10: The magnitude of the diagonal line on the xy plane is $\sqrt{F_x^2 + F_y^2}$. Using the Pythagorean theorem again, gives that $|\vec{F}| = \sqrt{\sqrt{F_x^2 + F_y^2}^2 + F_z^2} = \sqrt{F_x^2 + F_y^2 + F_z^2}$.

To get the result in 3D the 2D Pythagorean Theorem needs to be applied twice successively, first to get the magnitude of the sum $\vec{F}_x + \vec{F}_y$ and once more to add in $\vec{F}_z$, which is orthogonal to the sum $\vec{F}_x + \vec{F}_y$ (see fig. 2.10).

On a computer one might write something like this

```
  F      = [10 -20 30]
answer =  sqrt( F(1)^2 + F(2)^2 + F(3)^2 )
```

However this formula is so commonly needed that many computer languages will have a command like `norm` or `mag` so computer code something like `answer = norm(F)` or `answer = mag(F)` might replace the second line in the calculation above.

A Given vector can be written as various sums and products

A vector $\vec{A}$ has many representations. The equivalence of different representations of a vector is partially analogous to the case of a dimensional scalar which has the same value no matter what units are used (*e.g.,* the mass $m = 4.41$ lbm is equal to $m = 2$ kg). Here are some common representations of vectors.

Scalar times a unit vector in the vector's direction. $\vec{F} = F\hat{\boldsymbol{\lambda}}$ means the scalar F multiplied by the unit vector $\hat{\boldsymbol{\lambda}}$.

Sum of orthogonal component vectors. $\vec{F} = \vec{F}_x + \vec{F}_y$ is a sum of two vectors parallel to the x and y axes, respectively. In three dimensions, $\vec{F} = \vec{F}_x + \vec{F}_y + \vec{F}_z$.

Components times unit base vectors. $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ or $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ in three dimensions. One way to think of this sum is to realize that $\vec{F}_x = F_x\hat{\imath}$, $\vec{F}_y = F_y\hat{\jmath}$ and $\vec{F}_z = F_z\hat{k}$.

Components times rotated unit base vectors. $\vec{F} = F'_x\mathbf{i}' + F'_y\mathbf{j}'$ or $\vec{F} = F'_x\mathbf{i}' + F'_y\mathbf{j}' + F'_z\mathbf{k}'$ in three dimensions. Here the base vectors marked with primes, $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$, are unit vectors parallel to some mutually orthogonal x', y', and z' axes. These x', y', and z' axes may be tilted in relation to the x, y, and z axes. That is, the x' axis need not be parallel to the x axis, the y' not parallel to the y axis, and the z' axis not parallel to the z axis.

Components times other unit base vectors. If you use polar or cylindrical coordinates the unit base vectors are $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_R$, so in 2-D , $\vec{\boldsymbol{F}} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta$ and in 3-D, $\vec{\boldsymbol{F}} = F_R\hat{\boldsymbol{e}}_R + F_\theta\hat{\boldsymbol{e}}_\theta + F_z\hat{\boldsymbol{k}}$. If you use 'path' coordinates, you will use the path-defined unit vectors $\hat{\boldsymbol{e}}_t$, $\hat{\boldsymbol{e}}_n$, and $\hat{\boldsymbol{e}}_b$ so in 2-D $\vec{\boldsymbol{F}} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n$. In 3-D $\vec{\boldsymbol{F}} = F_t\hat{\boldsymbol{e}}_t + F_n\hat{\boldsymbol{e}}_n + F_b\hat{\boldsymbol{e}}_b$.

A list of components. $[\vec{\boldsymbol{F}}]_{xy} = [F_x, F_y]$ or $[\vec{\boldsymbol{F}}]_{xyz} = [F_x, F_y, F_z]$ in three dimensions. This form coincides best with the way computers handle vectors. The row vector $[F_x, F_y]$ coincides with $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}}$ and the row vector $[F_x, F_y, F_z]$ coincides with $F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\hat{\boldsymbol{k}}$.

In summary:

$$
\begin{array}{rcll}
\vec{\boldsymbol{A}} & = & \vec{\boldsymbol{A}} & \\
 & = & |\vec{\boldsymbol{A}}|\hat{\boldsymbol{\lambda}}_A = A\hat{\boldsymbol{\lambda}}_A, & \text{where } \hat{\boldsymbol{\lambda}}_A \parallel \vec{\boldsymbol{A}},\ A = |\vec{\boldsymbol{A}}| \text{ and } |\hat{\boldsymbol{\lambda}}_A| = 1 \\
 & = & \vec{\boldsymbol{A}}_x + \vec{\boldsymbol{A}}_y + \vec{\boldsymbol{A}}_z & \text{where } \vec{\boldsymbol{A}}_x, \vec{\boldsymbol{A}}_y, \vec{\boldsymbol{A}}_z \text{ are parallel to the } x, y, z \text{ axes} \\
 & = & A_x\hat{\boldsymbol{\imath}} + A_y\hat{\boldsymbol{\jmath}} + A_z\hat{\boldsymbol{k}}, & \text{where } \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}} \text{ are parallel to the } x, y, z \text{ axes} \\
 & = & A_{x'}\hat{\boldsymbol{\imath}}' + A_{y'}\hat{\boldsymbol{\jmath}}' + A_{z'}\hat{\boldsymbol{k}}', & \text{where } \hat{\boldsymbol{\imath}}', \hat{\boldsymbol{\jmath}}', \hat{\boldsymbol{k}}' \text{ are } \parallel \text{ to skewed } x', y', z' \text{ axes} \\
 & = & A_R\hat{\boldsymbol{e}}_R + A_\theta\hat{\boldsymbol{e}}_\theta + A_z\hat{\boldsymbol{k}}, & \text{using cylindrical coordinate basis vectors.} \\
[\vec{\boldsymbol{A}}]_{xyz} & = & [A_x, A_y, A_z] & [\vec{\boldsymbol{A}}]_{xyz} \text{ stands for the component list in } xyz \\
[\vec{\boldsymbol{A}}]_{x'y'z'} & = & [A_{x'}, A_{y'}, A_{z'}] & [\vec{\boldsymbol{A}}]_{x'y'z'} \text{ stands for the component list in } x'y'z'
\end{array}
$$

SAMPLE 2.1 Various ways of representing a vector:

This sample should be mastered before proceeding to other samples.

A vector $\vec{F} = 3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$ is represented in various ways below, some incorrect. For each representation, determine whether it is correct or incorrect, and why. The base vectors used are shown first.

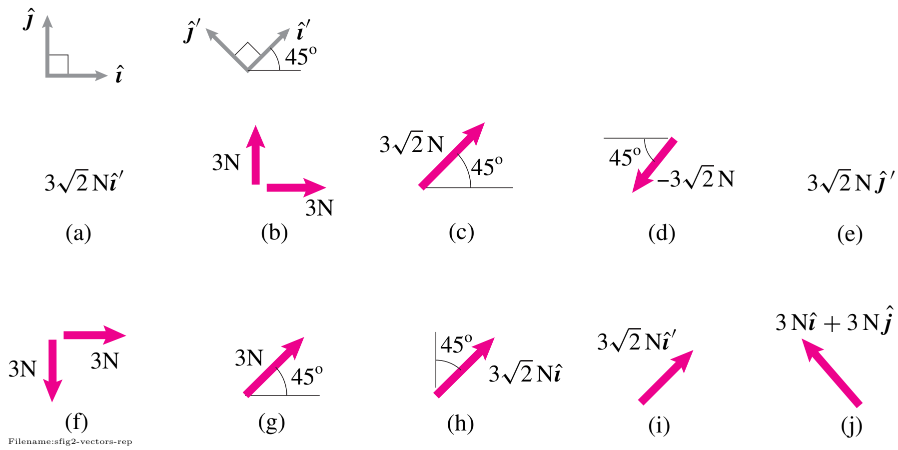


Figure 2.14:

Solution First note that the unit vectors $\hat{\imath}'$ and $\hat{\jmath}'$ can be expressed in terms of their components along $\hat{\imath}$ and $\hat{\jmath}$ as follows:

$$\begin{aligned} \hat{\imath}' &= |\hat{\imath}'|\cos 45^\circ \hat{\imath} + |\hat{\imath}'|\sin 45^\circ \hat{\jmath} \\ &= \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}). \end{aligned} \tag{2.2}$$

Similarly,

$$\hat{\jmath}' = -|\hat{\jmath}'|\cos 45^\circ \hat{\imath} + |\hat{\jmath}'|\sin 45^\circ \hat{\jmath} = \frac{1}{\sqrt{2}}(-\hat{\imath} + \hat{\jmath}).$$

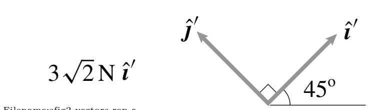


Figure 2.11: Case (a): Correct representation of $\vec{F}$.

a) Correct: $3\sqrt{2}\,\mathrm{N}\hat{\imath}'$. From the picture defining $\hat{\imath}'$, you can see that $\hat{\imath}'$ is a unit vector with equal components in the $\hat{\imath}$ and $\hat{\jmath}$ directions; *i.e.,* it is parallel to $\vec{F}$. So $\vec{F}$ is given by its magnitude $\sqrt{(3\,\mathrm{N})^2 + (3\,\mathrm{N})^2}$ times a unit vector in its direction, in this case $\hat{\imath}'$. It is the same vector. Algebraically,

$$3\sqrt{2}\,\mathrm{N}\hat{\imath}' = 3\sqrt{2}\,\mathrm{N}\cdot\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) = 3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath} = \vec{F}.$$

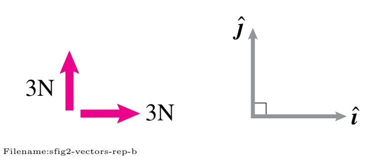


Figure 2.12: Case (b): Correct representation of $\vec{F}$.

b) Correct: Here two vectors are shown: one with magnitude 3 N in the direction of the horizontal arrow $\hat{\imath}$, and one with magnitude 3 N in the direction of the vertical arrow $\hat{\jmath}$. When two forces act on an object at a point, their effect is additive. So the net vector is the sum of the vectors shown. That is, $3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$. It is the same vector.

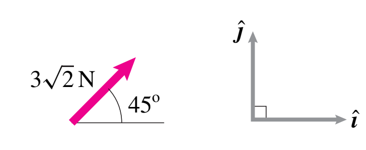
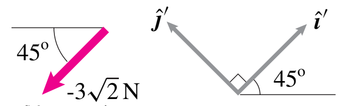


Figure 2.13: Case (c): Correct representation of $\vec{F}$.

c) Correct: Here we have a scalar $3\sqrt{2}\,\mathrm{N}$ next to an arrow. The vector described is the scalar multiplied by a unit vector in the direction of the arrow. Since the arrow's direction is marked as the same direction as $\hat{\imath}'$, which we already know is parallel to $\vec{F}$, this vector represents the same vector $\vec{F}$. Using the standard base vectors we can write,

$$3\sqrt{2}\,\mathrm{N}(\cos 45^\circ\hat{\imath} + \sin 45^\circ\hat{\jmath}) \quad = \quad = 3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath} = \vec{F}.$$

d) Correct: The scalar $-3\sqrt{2}\,\mathrm{N}$ is multiplied by a unit vector in the direction indicated, $-\hat{\imath}'$. So we get $(-3\sqrt{2}\,\mathrm{N})(-\hat{\imath}')$ which is $3\sqrt{2}\,\mathrm{N}\hat{\imath}'$ as before. It is the same vector.

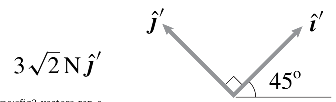


Filename:sfig2-vectors-rep-d

Figure 2.15: Case (d): Correct representation of $\vec{F}$.

e) Incorrect: $3\sqrt{2}\,\mathrm{N}\hat{\jmath}'$. The magnitude is right, but the direction is off by 90 degrees. It is a different vector. Algebraically,

$$\begin{aligned} 3\sqrt{2}\,\mathrm{N}\hat{\jmath}' &= 3\sqrt{2}\,\mathrm{N}\cdot\frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}) \\ &= -3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath} \\ &\neq \vec{F}. \end{aligned}$$



Filename:sfig2-vectors-rep-e

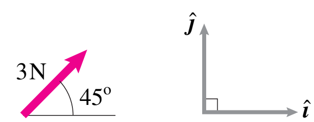
Figure 2.16: Case (e): Incorrect representation of $\vec{F}$.

f) Incorrect: $3\,\mathrm{N}\hat{\imath} - 3\,\mathrm{N}\hat{\jmath}$. The $\hat{\imath}$ component of the vector is correct but the $\hat{\jmath}$ component is in the opposite direction. The vector is in the wrong direction by 90 degrees. It is a different vector.

Filename:sfig2-vectors-rep-f

Figure 2.17: Case (f): Incorrect representation of $\vec{F}$.

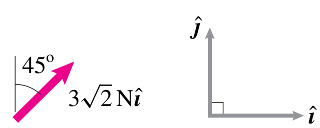
g) Incorrect: Right direction but the magnitude is off by a factor of $\sqrt{2}$.



Filename:sfig2-vectors-rep-g

Figure 2.18: Case (g): Incorrect representation of $\vec{F}$.

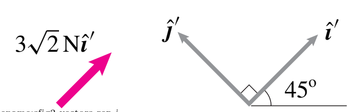
h) Incorrect: The magnitude is right. The direction indicated is right. But, the algebraic symbol $3\sqrt{2}\,\mathrm{N}\hat{\imath}$ takes precedence and it is in the wrong direction ($\hat{\imath}$ instead of $\hat{\imath}'$). It is a different vector.



Filename:sfig2-vectors-rep-h

Figure 2.19: Case (h): Incorrect representation of $\vec{F}$.

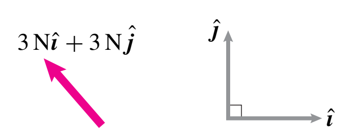
i) Correct: A labeled arrow. The arrow is only schematic. The algebraic symbol $3\sqrt{2}\,\mathrm{N}\hat{\imath}'$ defines the vector. We draw the arrow to remind us that there is a vector to represent. The tip or tail of the arrow would be drawn at the point of the force application. In this case, the arrow is drawn in the direction of $\vec{F}$, but strictly speaking, it need not.



Filename:sfig2-vectors-rep-i

Figure 2.20: Case (i): Correct representation of $\vec{F}$.

j) Correct: Like (i) above, the directional and magnitude information are embedded in the algebraic symbol $3\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$. The arrow is there to indicate a vector. In this case, it points in the wrong direction so is not ideally communicative. In fact, it is confusing and therefore, not recommended. But it still correctly represents the given vector because the algebraic symbol takes precedence over the graphical symbol.



Filename:sfig2-vectors-rep-j

Figure 2.21: Case (j): Correct but misleading representation of $\vec{F}$.

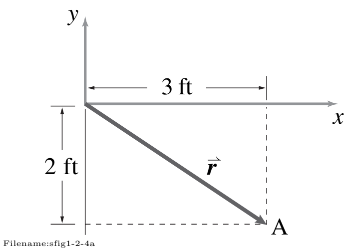


Figure 2.22: Drawing vector $\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath}$ using its components.

SAMPLE 2.2 **Drawing a vector from its components:** Draw the vector $\vec{r} = 3\,\text{ft}\hat{\imath} - 2\,\text{ft}\hat{\jmath}$ using its components.

Solution To draw $\vec{r}$ using its components, we first draw the axes and measure 3 units (any units that we choose on the ruler) along the x-axis and 2 units along the negative y-axis. We mark this point as A (say) on the paper and draw a line from the origin to the point A. We write the dimensions '3 ft' and '2 ft' on the figure. Finally, we put an arrowhead on this line pointing towards A.

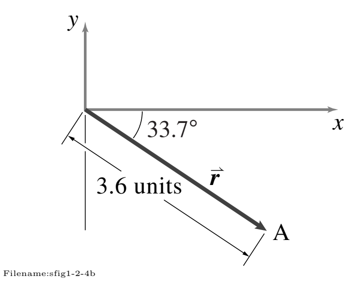


Figure 2.23: Drawing vector $\vec{r}$ given its length (3.6 ft) and direction (slope angle $\theta = -33.7^{\circ}$).

SAMPLE 2.3 **Drawing a vector from its length and direction:** A vector $\vec{r}$ is 3.6 ft long and is directed 33.7° clockwise (CW) from the positive x-axis. Draw $\vec{r}$.

Solution We first draw the x and y axes and then draw $\vec{r}$ as a line from the origin at an angle -33.7° from the x-axis (minus sign means measuring clockwise), measure 3.6 units (magnitude of $\vec{r}$) along this line and finally put an arrowhead pointing away from the origin.

Comments Note that this is about (at least to 2 digit accuracy) the same vector as in Sample 2.2. In fact, you can easily verify that $r_x = r\cos\theta = 3.6\,\text{ft}\cdot\cos(-33.7^{\circ}) = 3\,\text{ft}$, $r_y = r\sin\theta = 3.6\,\text{ft}\cdot\sin(-33.7^{\circ}) = -2\,\text{ft}$.

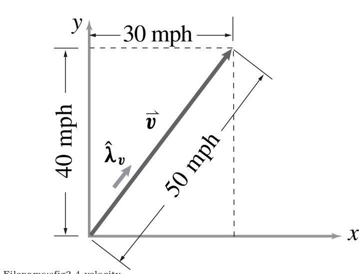


Figure 2.24: Speed and direction (indicated by the unit vector $\hat{\lambda}_v$) from the given velocity vector $\vec{v} = 30\,\text{mph}\hat{\imath} + 40\,\text{mph}\hat{\jmath}$.

SAMPLE 2.4 **Magnitude and direction of a vector:** The velocity of a car is given as $\vec{v} = (30\hat{\imath} + 40\hat{\jmath})$ mph. Find the speed (magnitude of $\vec{v}$) of the car, its direction as a unit vector, and write the velocity in terms of its magnitude and the unit vector.

Solution

1. **Speed of the car $v = |\vec{v}|$:**

$$\begin{aligned}
\vec{v} &= 30\,\text{mph}\hat{\imath} + 40\,\text{mph}\hat{\jmath}, \\
v = |\vec{v}| &= \sqrt{v_x^2 + v_y^2} = \sqrt{(30\,\text{mph})^2 + (40\,\text{mph})^2} \\
&= 50\,\text{mph}
\end{aligned}$$

2. **Direction of $\vec{v}$ as a unit vector along $\vec{v}$:** The unit vector along a given vector is found by dividing the given vector with its magnitude. Let $\hat{\lambda}_v$ be the unit vector along $\vec{v}$. Then,

$$\hat{\lambda}_v = \frac{\vec{v}}{|\vec{v}|} = \frac{30\,\text{mph}\hat{\imath} + 40\,\text{mph}\hat{\jmath}}{50\,\text{mph}} = 0.6\hat{\imath} + 0.8\hat{\jmath}.$$

3. **$\vec{v}$ as a product of its magnitude and the unit vector $\hat{\lambda}_v$:**

$$\vec{v} = |\vec{v}|\hat{\lambda}_v = 50\,\text{mph}(0.6\hat{\imath} + 0.8\hat{\jmath})$$

which, of course, is the same vector as given in the problem.

speed $v = 50$ mph, $\hat{\lambda}_v = 0.6\hat{\imath} + 0.8\hat{\jmath}$, $\vec{v} = 50(0.6\hat{\imath} + 0.8\hat{\jmath})$ mph

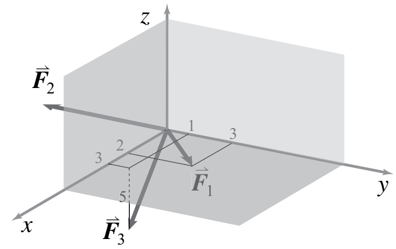
SAMPLE 2.5 Adding vectors: Three forces, $\vec{F}_1 = 2\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}$, $\vec{F}_2 = -10\,\mathrm{N}\hat{\jmath}$, and $\vec{F}_3 = 3\,\mathrm{N}\hat{\imath} + 1\,\mathrm{N}\hat{\jmath} - 5\,\mathrm{N}\hat{k}$, act on a particle. Find the net force on the particle.

Solution The net force on the particle is the vector sum of all the forces, *i.e.,*

$$
\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\
&= (2\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath}) + (-10\,\mathrm{N}\hat{\jmath}) + (3\,\mathrm{N}\hat{\imath} + 1\,\mathrm{N}\hat{\jmath} - 5\,\mathrm{N}\hat{k}) \\
&= \quad 2\,\mathrm{N}\hat{\imath} \;+\; 3\,\mathrm{N}\hat{\jmath} \;+\; 0\hat{k} \\
&\quad + \;\; 0\hat{\imath} \;-\; 10\,\mathrm{N}\hat{\jmath} \;+\; 0\hat{k} \\
&\quad + \; 3\,\mathrm{N}\hat{\imath} \;+\; 1\,\mathrm{N}\hat{\jmath} \;-\; 5\hat{k} \\
&= (2\,\mathrm{N} + 3\,\mathrm{N})\hat{\imath} + (3\,\mathrm{N} - 10\,\mathrm{N} + 1\,\mathrm{N})\hat{\jmath} + (-5\,\mathrm{N})\hat{k} \\
&= 5\,\mathrm{N}\hat{\imath} - 6\,\mathrm{N}\hat{\jmath} - 5\,\mathrm{N}\hat{k}.
\end{aligned}
$$

$$\boxed{\vec{F}_{\text{net}} = 5\,\mathrm{N}\hat{\imath} - 6\,\mathrm{N}\hat{\jmath} - 5\,\mathrm{N}\hat{k}}$$

Comments: In general, we do not need to write the summation so elaborately. Once you feel comfortable with the idea of summing only similar components in a vector sum, you can do the calculation in two lines.



Filename:sfig2-3Dvec-add

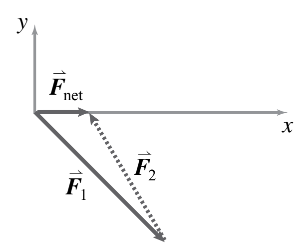
Figure 2.25: Force vectors in 3D space.

SAMPLE 2.6 Subtracting vectors: Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a body. The net force on the body is $\vec{F}_{\text{net}} = 2\,\mathrm{N}\hat{\imath}$. If $\vec{F}_1 = 10\,\mathrm{N}\hat{\imath} - 10\,\mathrm{N}\hat{\jmath}$, find the other force $\vec{F}_2$.

Solution

$$
\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 \\
\Rightarrow \quad \vec{F}_2 &= \vec{F}_{\text{net}} - \vec{F}_1 \\
&= 2\,\mathrm{N}\hat{\imath} - (10\,\mathrm{N}\hat{\imath} - 10\,\mathrm{N}\hat{\jmath}) \\
&= -8\,\mathrm{N}\hat{\imath} + 10\,\mathrm{N}\hat{\jmath}.
\end{aligned}
$$

$$\boxed{\vec{F}_2 = -8\,\mathrm{N}\hat{\imath} + 10\,\mathrm{N}\hat{\jmath}}$$



Filename:sfig2-vec-subtract

Figure 2.26: $\vec{F}_2 = \vec{F}_{\text{net}} - \vec{F}_1$.

SAMPLE 2.7 Scalar times a vector: Two forces acting on a particle are $\vec{F}_1 = -16\,\mathrm{N}\hat{\imath} + 8\,\mathrm{N}\hat{\jmath}$ and $\vec{F}_2 = 15\,\mathrm{N}\hat{\imath}$. If $\vec{F}_2$ is doubled, does the net force double?

Solution

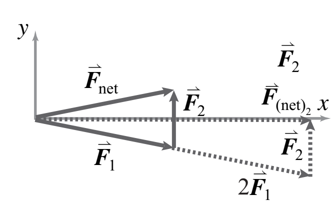
$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 = (-16\,\mathrm{N}\hat{\imath} + 8\,\mathrm{N}\hat{\jmath}) + (15\,\mathrm{N}\hat{\imath}) = -1\,\mathrm{N}\hat{\imath} + 8\,\mathrm{N}\hat{\jmath}.$$

After $\vec{F}_2$ is doubled, the new net force $\vec{F}_{(\text{net})_2}$ is

$$\vec{F}_{(\text{net})_2} = \vec{F}_1 + 2\vec{F}_2 = (-16\,\mathrm{N}\hat{\imath} + 8\,\mathrm{N}\hat{\jmath}) + 2(15\,\mathrm{N}\hat{\imath}) = 14\,\mathrm{N}\hat{\imath} + 8N\hat{\jmath} \neq 2\underbrace{(-1\,\mathrm{N}\hat{\imath} + 8\,\mathrm{N}\hat{\jmath})}_{\vec{F}_{\text{net}}}$$

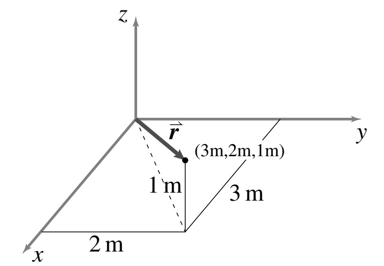
However, $|\vec{F}_{(\text{net})_2}| = \sqrt{14^2 + 8^2}\,\mathrm{N} = \sqrt{260}\,\mathrm{N} = 2(\sqrt{65})\,\mathrm{N} = 2|\vec{F}_{\text{net}}|$. Thus doubling the magnitude does not mean two times the vector, but doubling the vector will certainly double the magnitude.

$$\boxed{\text{No, the net force vector does not double.}}$$



Filename:sfig2-vec-6

Figure 2.27: $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2$ and $\vec{F}_{(\text{net})_2} = 2\vec{F}_1 + \vec{F}_2$. $|\vec{F}_{(\text{net})_2}| = 2|\vec{F}_{\text{net}}|$ but $\vec{F}_{(\text{net})_2} \neq 2\vec{F}_{\text{net}}$.



Filename:sfig2-vec1-6

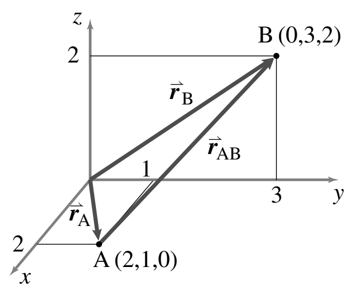
Figure 2.28: The position vector of the particle is a vector drawn from the origin of the coordinate system to the position of the particle.

SAMPLE 2.8 Position vector from the origin: In the xyz coordinate system, a particle is located at the coordinate (3m, 2m, 1m). Find the position vector of the particle.

Solution The position vector of the particle at P is a vector drawn from the origin of the coordinate system to the position P of the particle. See Fig. 2.28. We can write this vector as

$$\begin{aligned} \vec{r}_{\mathrm{P}} &= (3\,\mathrm{m})\hat{\imath} + (2\,\mathrm{m})\hat{\jmath} + (1\,\mathrm{m})\hat{k} \\ \text{or} \qquad \vec{r}_{\mathrm{P}} &= (3\hat{\imath} + 2\hat{\jmath} + \hat{k})\,\mathrm{m}. \end{aligned}$$

$$\boxed{\vec{r}_{\mathrm{P}} = 3\,\mathrm{m}\hat{\imath} + 2\,\mathrm{m}\hat{\jmath} + 1\,\mathrm{m}\hat{k}}$$



Filename:sfig2-vec1-7

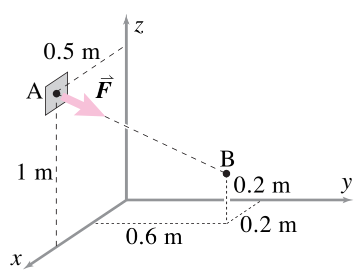
Figure 2.29: The position vector of B with respect to A is found from $\vec{r}_{\mathrm{AB}} = \vec{r}_{\mathrm{B}} - \vec{r}_{\mathrm{A}}$.

SAMPLE 2.9 Relative position vector: Let A (2m, 1m, 0) and B (0, 3m, 2m) be two points in the xyz coordinate system. Find the position vector of point B with respect to point A, *i.e.,* find $\vec{r}_{\mathrm{AB}}$ (or $\vec{r}_{\mathrm{B/A}}$).

Solution From the geometry of the position vectors shown in Fig. 2.29 and the rules of vector sums, we can write,

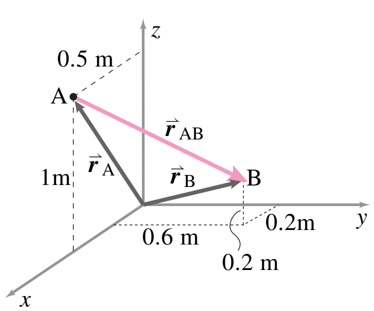
$$\begin{aligned} \vec{r}_{\mathrm{B}} &= \vec{r}_{\mathrm{A}} + \vec{r}_{\mathrm{AB}} \\ \Rightarrow \quad \vec{r}_{\mathrm{AB}} &= \vec{r}_{\mathrm{B}} - \vec{r}_{\mathrm{A}} = (0\hat{\imath} + 3\,\mathrm{m}\hat{\jmath} + 2\,\mathrm{m}\hat{k}) - (2\,\mathrm{m}\hat{\imath} + 1\,\mathrm{m}\hat{\jmath} + 0\hat{k}) \\ &= -2\,\mathrm{m}\hat{\imath} + 2\,\mathrm{m}\hat{\jmath} + 2\,\mathrm{m}\hat{k}. \end{aligned}$$

$$\boxed{\vec{r}_{\mathrm{AB}} \equiv \vec{r}_{\mathrm{B/A}} = -2\,\mathrm{m}\hat{\imath} + 2\,\mathrm{m}\hat{\jmath} + 2\,\mathrm{m}\hat{k}}$$



Filename:sfig1-2-2

Figure 2.30:



Filename:sfig1-2-2b

Figure 2.31: $\vec{r}_{AB} = \vec{r}_B - \vec{r}_A$.

SAMPLE 2.10 Finding a force vector given its magnitude and line of action: A string is pulled with a force $F = 100\,\mathrm{N}$ as shown in fig. 2.30. Write F as a vector.

Solution A vector can be written as the product of a scalar and a unit vector along its direction. Here, the magnitude of the force is given and we know it acts along AB. Therefore, we may write $\vec{F} = F\hat{\lambda}_{AB}$, where $\hat{\lambda}_{AB}$ is a unit vector along AB. So now we need to find $\hat{\lambda}_{AB}$. We can easily find $\hat{\lambda}_{AB}$ if we know vector AB. Let us denote vector AB by $\vec{r}_{AB}$ (same as $\vec{r}_{B/A}$). To find $\vec{r}_{AB}$, we note that (see Fig. 2.31)

$$\vec{r}_A + \vec{r}_{AB} = \vec{r}_B$$

where $\vec{r}_A$ and $\vec{r}_B$ are the position vectors of point A and point B respectively. Hence,

$$\begin{aligned} \vec{r}_{B/A} &\equiv \vec{r}_{AB} = \vec{r}_B - \vec{r}_A \\ &= (0.2\,\mathrm{m}\hat{\imath} + 0.6\,\mathrm{m}\hat{\jmath} + 0.2\,\mathrm{m}\hat{k}) - (0.5\,\mathrm{m}\hat{\imath} + 1.0\,\mathrm{m}\hat{k}) \\ &= -0.3\,\mathrm{m}\hat{\imath} + 0.6\,\mathrm{m}\hat{\jmath} - 0.8\,\mathrm{m}\hat{k}. \end{aligned}$$

Therefore,

$$\hat{\lambda}_{AB} = \frac{-0.3\,\mathrm{m}\hat{\imath} + 0.6\,\mathrm{m}\hat{\jmath} - 0.8\,\mathrm{m}\hat{k}}{\sqrt{(-0.3)^2 + (0.6)^2 + (-0.8)^2}\ \ \mathrm{m}} = -0.29\hat{\imath} + 0.57\hat{\jmath} - 0.77\hat{k},$$

and, finally,

$$\vec{F} = \overbrace{(100\,\mathrm{N})}^{F}\,\hat{\lambda}_{AB} = -29\,\mathrm{N}\hat{\imath} + 57\,\mathrm{N}\hat{\jmath} - 77\,\mathrm{N}\hat{k}.$$

$$\boxed{\vec{F} = -29\,\mathrm{N}\hat{\imath} + 57\,\mathrm{N}\hat{\jmath} - 77\,\mathrm{N}\hat{k}}$$

SAMPLE 2.11 Adding vectors on computers: The following six forces act at different points of a structure. $\vec{F}_1 = -3\,\text{N}\hat{\jmath}$, $\vec{F}_2 = 20\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath}$, $\vec{F}_3 = 1\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$, $\vec{F}_4 = 10\,\text{N}\hat{\imath}$, $\vec{F}_5 = 5\,\text{N}(\hat{\imath} + \hat{\jmath} + \hat{k})$, $\vec{F}_6 = -10\,\text{N}\hat{\imath} - 10\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$.

1. Write all the force vectors in column form.
2. Find the net force by hand calculation.
3. Write a computer program to sum n vectors, each with three components. Use your program to compute the net force.

Solution

1. The 3-D vector $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ is represented as a column (or a row) as follows:

$$[\vec{F}]_{xyz} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

Following this convention, we write the given forces as

$$[\vec{F}_1]_{xyz} = \begin{pmatrix} 0 \\ -3\,\text{N} \\ 0 \end{pmatrix}, [\vec{F}_2]_{xyz} = \begin{pmatrix} 20\,\text{N} \\ -10\,\text{N} \\ 0 \end{pmatrix}, \cdots, [\vec{F}_6]_{xyz} = \begin{pmatrix} -10\,\text{N} \\ -10\,\text{N} \\ 2\,\text{N} \end{pmatrix}$$

2. The net force $\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6$, or

$$\begin{aligned}
[\vec{F}_{\text{net}}]_{xyz} &= \begin{pmatrix} 0 & & 20 & & 1 & & 10 & & 5 & & -10 \\ -3 & + & -10 & + & 20 & + & 0 & + & 5 & + & -10 \\ 0 & & 0 & & -5 & & 0 & & 5 & & 2 \end{pmatrix} \text{N} \\
&= \begin{pmatrix} 26 \\ 2 \\ 2 \end{pmatrix} \text{N}
\end{aligned}$$

3. The steps to do this addition on computers are as follows.
   - Enter the vectors as rows or columns:

```
F1 = [0  -3  0]
F2 = [20  -10  0]
F3 = [1  20  -5]
F4 = [10  0  0]
F5 = [5  5  5]
F6 = [-10  -10  2]
```

   - Sum the vectors, using a summing operation that automatically does element by element addition of vectors:

```
Fnet = F1 + F2 + F3 + F4 + F5 + F6
```

   - The computer generated answer is:

```
Fnet = [26  2  2].
```

$$\boxed{\vec{F}_{\text{net}} = 26\,\text{N}\hat{\imath} + 2\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}}$$

2.2 The dot product of two vectors

The dot product is used to project a vector in a given direction, to reduce a vector to components, to reduce vector equations to scalar equations, to define work and power, and to help solve geometry problems.

The *dot product* of two vectors $\vec{A}$ and $\vec{B}$ is written $\vec{A}\cdot\vec{B}$ (pronounced 'A dot B'). The dot product of $\vec{A}$ and $\vec{B}$ is the product of the magnitudes of the two vectors times a number that expresses the degree to which $\vec{A}$ and $\vec{B}$ are parallel: $\cos\theta_{AB}$, where θ_{AB} is the angle between $\vec{A}$ and $\vec{B}$. That is,

$$\vec{A}\cdot\vec{B} \stackrel{def}{=} |\vec{A}|\,|\vec{B}|\cos\theta_{AB}$$

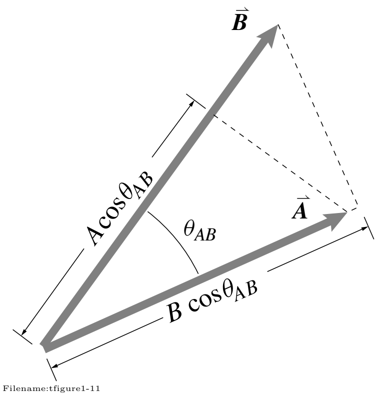


Figure 2.32: The dot product of $\vec{A}$ and $\vec{B}$ is a scalar and so is not easily drawn. It is given by $\vec{A}\cdot\vec{B} = AB\cos\theta_{AB}$ which is A times the projection of $\vec{B}$ in the A direction and also B times the projection of $\vec{A}$ in the B direction.

which is sometimes written more concisely as $\vec{A}\cdot\vec{B} = AB\cos\theta$. One special case occurs when $\cos\theta_{AB} = 1$, $\vec{A}$ and $\vec{B}$ are parallel, and $\vec{A}\cdot\vec{B} = AB$. Another is when $\cos\theta_{AB} = 0$, $\vec{A}$ and $\vec{B}$ are perpendicular, and $\vec{A}\cdot\vec{B} = 0$.①

The dot product of two vectors is a scalar. So the dot product is sometimes called the *scalar product*.

Using the geometric definition of dot product, and the rules for vector addition we have already discussed, you can convince yourself of (or believe) the features of the dot products in box 2.2. The identities in box 2.2 lead to the following equivalent ways of expressing the dot product of $\vec{A}$ and $\vec{B}$ (see box 2.4 on page 63 to see how the component formula follows from the geometric definition above).

①Make sure you know, almost without a thought, that $\cos 0 = 1, \cos\pi/2 = 0, \sin 0 = 0$, and $\sin\pi/2 = 1$. Draw as many triangles and unit circles as it takes to cement these 4 special cases into your head.

2.3 Basic features of the vector dot product.

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All of the following features of the dot product follow naturally from the definition

$$\vec{A}\cdot\vec{B} = AB\cos\theta.$$

Commutative law. Order doesn't matter with dot products. $AB\cos\theta = BA\cos\theta$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}.$$

A distributive law. Scalars slide through vector products like mercury through a chicken. $(aA)B\cos\theta = A(aB)\cos\theta$

$$\Rightarrow \quad (a\vec{A})\cdot\vec{B} = \vec{A}\cdot(a\vec{B}) = a(\vec{A}\cdot\vec{B}).$$

Another distributive law. Vector dot products distribute like regular multiplication. the projection of $\vec{B}+\vec{C}$ onto $\vec{A}$ is the sum of the two separate projections, so

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}.$$

Perpendicular vectors have zero for a dot product. If $\vec{A}\perp\vec{B}$ then the angle between them is $\pi/2$. Because $AB\cos\pi/2 = 0$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = 0 \quad \text{if} \quad \vec{A}\perp\vec{B}.$$

The dot product of parallel vectors is the product of their magnitudes. The angle between parallel vectors is zero and $AB\cos 0 = AB$. In particular, $\vec{A}\cdot\vec{A} = A^2$ or $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$

$$\Rightarrow \quad \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}| \quad \text{if} \quad \vec{A}\parallel\vec{B}.$$

The standard unit base vectors are orthonormal. They are unit vectors (in this case 'normal' means normalized, meaning taken down to size) and they are perpendicular (ortho) to each other.

$$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1 \quad \text{and} \quad \hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{k} = \hat{k}\cdot\hat{\imath} = 0$$

Also, the standard tilted base vectors are orthonormal.

$$\hat{\imath}'\cdot\hat{\imath}' = \hat{\jmath}'\cdot\hat{\jmath}' = \hat{k}'\cdot\hat{k}' = 1 \quad \text{and} \quad \hat{\imath}'\cdot\hat{\jmath}' = \hat{\jmath}'\cdot\hat{k}' = \hat{k}'\cdot\hat{\imath}' = 0$$

$$\begin{aligned}\vec{A}\cdot\vec{B} &= |\vec{A}||\vec{B}|\cos\theta_{AB}\\ &= A_x B_x + A_y B_y + A_z B_z \quad \text{(component form)} \quad (2.3)\\ &= A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'}\\ &= |\vec{A}|\cdot[\text{projection of } \vec{B} \text{ in the } \vec{A} \text{ direction}]\\ &= |\vec{B}|\cdot[\text{projection of } \vec{A} \text{ in the } \vec{B} \text{ direction}]\end{aligned}$$

Mechanics solutions use of all of these relations. The most famous, of course, is the second, sometimes written as

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3.$$

Using the dot product to find components

To find the x component of a vector (or vector expression) one can use the dot product (fig. 2.33),

$$v_x = \text{projection of } \vec{v} \text{ in the } \hat{\imath} \text{ direction} = \vec{v}\cdot\hat{\imath}. \qquad (2.4)$$

This idea can be used for finding components in any direction.

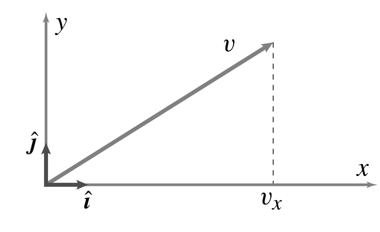


Figure 2.33: The dot product with unit vectors gives projection. For example, $v_x = \vec{v}\cdot\hat{\imath}$.

Tilted base vectors. If one knows the orientation of the tilted unit vectors $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ relative to the standard bases $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ then you can find the dot products between the standard base vectors and the tilted base vectors.

In 2D, assume that (fig. 2.34) $\hat{\imath}\cdot\hat{\jmath}' = -\sin\theta$ and $\hat{\jmath}\cdot\hat{\jmath}' = \cos\theta$. One can then use the dot product to find the $x'y'$ components $(A_{x'}, A_{y'})$ from the xy components (A_x, A_y). We start with the obvious (and ridiculously useful) equation

$$\vec{A} = \vec{A}$$

and dot both sides with $\hat{\jmath}'$ to get:

$$\begin{aligned}\vec{A}\cdot\hat{\jmath}' &= \vec{A}\cdot\hat{\jmath}'\\ \underbrace{(A_{x'}\hat{\imath}' + A_{y'}\hat{\jmath}')}_{\vec{A}}\cdot\hat{\jmath}' &= \underbrace{(A_x\hat{\imath} + A_y\hat{\jmath})}_{\vec{A}}\cdot\hat{\jmath}'\\ A_{x'}\underbrace{\hat{\imath}'\cdot\hat{\jmath}'}_{0} + A_{y'}\underbrace{\hat{\jmath}'\cdot\hat{\jmath}'}_{1} &= A_x\hat{\imath}\cdot\hat{\jmath}' + A_y\hat{\jmath}\cdot\hat{\jmath}'\\ \Rightarrow \quad A_{y'} &= A_x\underbrace{(\hat{\imath}\cdot\hat{\jmath}')}_{-\sin\theta} + A_y\underbrace{(\hat{\jmath}\cdot\hat{\jmath}')}_{\cos\theta}.\end{aligned}$$

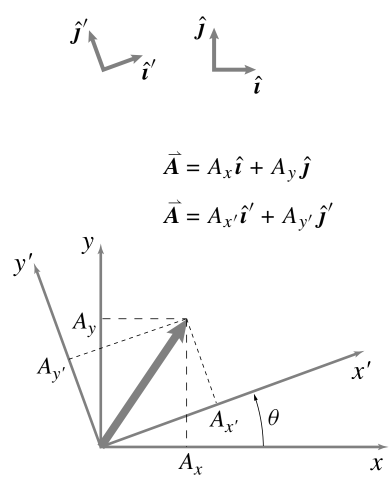


Figure 2.34: The dot product helps find components in terms of tilted unit vectors. For example, $A_{y'} = \vec{A}\cdot\hat{\jmath}' = A_x(\hat{\imath}\cdot\hat{\jmath}') + A_y(\hat{\imath}\cdot\hat{\jmath}') = A_x(-\sin\theta) + A_y(\cos\theta)$.

Similarly, one could find the component $A_{x'}$ using a dot product with $\hat{\boldsymbol{\imath}}'$ as

$$A_{x'} = A_x \underbrace{(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\imath}}')}_{\cos\theta} + A_y \underbrace{(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\imath}}')}_{\sin\theta}.$$

This way to find components is useful when a problem uses more than one base vector system.

Using dot products to get scalar equations

Dot products are the way to get scalar equations from vector equations.

For example the statics vector force balance equation

$$\vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2 + \vec{\boldsymbol{F}}_3 = \vec{\boldsymbol{0}} \tag{2.5}$$

can be reduced to two scalar equations by taking the dot product of both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$

$$\{(2.5)\} \cdot \hat{\boldsymbol{\imath}} \quad \Rightarrow \quad F_{1x} + F_{2x} + F_{3x} = 0 \tag{2.6}$$

$$\{(2.5)\} \cdot \hat{\boldsymbol{\jmath}} \quad \Rightarrow \quad F_{1y} + F_{2y} + F_{3y} = 0. \tag{2.7}$$

This approach is more general than the common one of 'taking x and y components'.

Dotting with vectors other than $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, or $\hat{\boldsymbol{k}}$ It is often useful to use dot products to get scalar equations using unit vectors other than $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$.

Example: Getting scalar equations without dotting with $\hat{\imath}$, $\hat{\jmath}$, or $\hat{k}$.
Given the vector equation

$$-mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}} = ma\hat{\boldsymbol{\lambda}}$$

where it is known that the unit vector $\hat{\boldsymbol{n}}$ is perpendicular to the unit vector $\hat{\boldsymbol{\lambda}}$, we can get a scalar equation and eliminate an unknown at the same time by dotting both sides with $\hat{\boldsymbol{\lambda}}$,

$$\begin{aligned}
\Big\{ \; (-mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}}) &= (ma\hat{\boldsymbol{\lambda}}) \Big\} \cdot \hat{\boldsymbol{\lambda}} \\
(-mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}}) \cdot \hat{\boldsymbol{\lambda}} &= (ma\hat{\boldsymbol{\lambda}}) \cdot \hat{\boldsymbol{\lambda}} \\
-mg\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\lambda}} + N \underbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}}}_{0} &= ma \underbrace{\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{\lambda}}}_{1} \\
-mg\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\lambda}} &= ma \quad \Rightarrow \quad a = -g/\cos\theta
\end{aligned}$$

with $\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\lambda}}$ being the cosine of the angle θ between $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{\lambda}}$.

Using dot products to solve geometry problems

Perpendicular and parallel parts. Given any vector $\vec{\boldsymbol{A}}$ and a unit vector $\hat{\boldsymbol{\lambda}}$, vector $\vec{\boldsymbol{A}}$ can be written as the sum of two parts,

$$\vec{\boldsymbol{A}} = \vec{\boldsymbol{A}}^{\parallel} + \vec{\boldsymbol{A}}^{\perp}$$

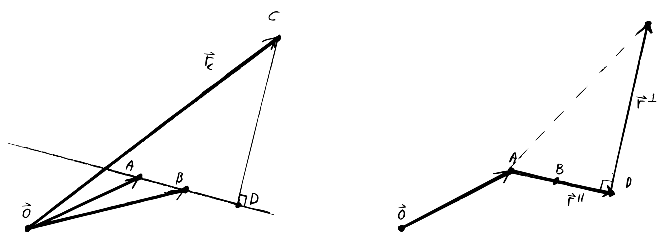
where $\vec{A}^{\parallel}$ ('A parallel') is parallel to $\hat{\boldsymbol{\lambda}}$ and $\vec{A}^{\perp}$ ('A perp') is perpendicular to $\hat{\boldsymbol{\lambda}}$ (see fig. 2.35). The part parallel to $\hat{\boldsymbol{\lambda}}$ is

$$\vec{A}^{\parallel} = (\text{projection of } \vec{A} \text{ in } \hat{\boldsymbol{\lambda}} \text{ direction})\hat{\boldsymbol{\lambda}} = (\vec{A}\cdot\hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}}.$$

The perpendicular part of $\vec{A}$ is just what you get when you subtract out the parallel part, namely,

$$\vec{A}^{\perp} = \vec{A} - \vec{A}^{\parallel} = \vec{A} - (\vec{A}\cdot\hat{\boldsymbol{\lambda}})\hat{\boldsymbol{\lambda}}$$

The claimed properties of the decomposition can now be checked, namely that $\vec{A} = \vec{A}^{\parallel} + \vec{A}^{\perp}$ (just add the 2 equations above and see), that $\vec{A}^{\parallel}$ is in the direction of $\hat{\boldsymbol{\lambda}}$ (its a scalar multiple), and that $\vec{A}^{\perp}$ is perpendicular to $\hat{\boldsymbol{\lambda}}$ ($\vec{A}^{\perp}\cdot\hat{\boldsymbol{\lambda}} = 0$).



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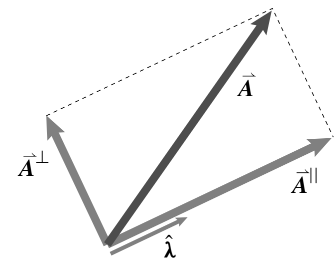

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Figure 2.35: For any $\vec{A}$ and $\hat{\boldsymbol{\lambda}}$, $\vec{A}$ can be decomposed into a part parallel to $\hat{\boldsymbol{\lambda}}$ and a part perpendicular to $\hat{\boldsymbol{\lambda}}$.

Example: Closest point.
What point D on line AB is closest to point C? That is, given the positions $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_C$ what is $\vec{r}_D$? The answer is,

$$\vec{r}_D = \vec{r}_A + \vec{r}_{C/A}{}^{\parallel}$$

where $\vec{r}_{C/A}{}^{\parallel}$ is the part of $\vec{r}_{C/A}$ that is parallel to the line segment AB. Thus,

$$\vec{r}_D = \vec{r}_A + (\vec{r}_C - \vec{r}_A)\cdot\frac{\vec{r}_B - \vec{r}_A}{|\vec{r}_B - \vec{r}_A|}.$$

Example: Graham-Schmidt orthogonalization
A plane is defined in 3D by two vectors $\vec{A}$ and $\vec{B}$ that lie in a plane. The goal of 'Graham-Schmidt orthogonalization' is to find a pair of orthonormal vectors that lie in the plane. First find a unit vector in the $\vec{A}$ direction:

$$\hat{\boldsymbol{\lambda}}_A = \vec{A}/|\vec{A}|.$$

Then find the part $\vec{B}^{\perp}$ of $\vec{B}$ that is orthogonal to $\hat{\boldsymbol{\lambda}}_A$:

$$\vec{B}^{\perp} = \vec{B} - \vec{B}^{\parallel} = \vec{B} - (\vec{B}\cdot\hat{\boldsymbol{\lambda}}_A)\hat{\boldsymbol{\lambda}}_A.$$

Finally, normalize:

$$\hat{\boldsymbol{\lambda}}_B = \vec{B}^{\perp}/|\vec{B}^{\perp}|$$

From two vectors in a plane, $\vec{A}$ and $\vec{B}$, we have found a pair of vectors ($\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$) in the same plane that are orthonormal (unit vectors that are orthogonal to each other).

Components perpendicular and parallel to a plane. The ideas above also apply to planes. What parts of a vector $\vec{C}$ are in ($\vec{C}^{\parallel}$) and orthogonal ($\vec{C}^{\perp}$) to a given plane?

Method 1. If the plane is defined by two vectors $\vec{A}$ and $\vec{B}$ (in the plane) that are not necessarily orthogonal we first use Graham-Schmidt orthogonalization (sample above) to find orthonormal vectors $\hat{\lambda}_A$ and $\hat{\lambda}_B$. Then we have that

$$\vec{C}^{\parallel} = (\vec{C} \cdot \hat{\lambda}_A)\hat{\lambda}_{\mathrm{A}} + (\vec{C} \cdot \hat{\lambda}_B)\hat{\lambda}_{\mathrm{B}} \qquad \text{and} \qquad \vec{C}^{\perp} = \vec{C} - \vec{C}^{\parallel}$$

Method 2. If the plane has known normal $\hat{n}$ then

$$\vec{C}^{\perp} = (\vec{C} \cdot \hat{n})\hat{n} \qquad \text{and} \qquad \vec{C}^{\parallel} = \vec{C} - \vec{C}^{\perp}.$$

Vector algebra

Vectors are algebraic quantities and manipulated algebraically in equations. The rules for vector algebra are similar to the rules for ordinary (scalar) algebra. For example, if vector $\vec{A}$ is the same as vector $\vec{B}$, $\vec{A} = \vec{B}$, for any scalar a and any vector $\vec{C}$, we then

$$\begin{aligned} \vec{A} + \vec{C} &= \vec{B} + \vec{C}, \\ a\vec{A} &= a\vec{B}, \text{ and} \\ \vec{A} \cdot \vec{C} &= \vec{B} \cdot \vec{C} \end{aligned}$$

because performing the same operation on equal quantities maintains the equality. The vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ might themselves be expressions involving other vectors.

The equations above show the allowable manipulations of vector equations: adding a common term to both sides, multiplying both sides by a common scalar, taking the dot product of both sides with a common vector. All the distributive, associative, and commutative laws of ordinary addition and multiplication hold but for when there is no sensible meaning to the expressions②.

② **Beware.** It does not make sense to add a vector and a scalar; $7 + \vec{A}$ is a *nonsense* expression. And you cannot divide a vector by a vector or a scalar by a vector; $7/\hat{\imath}$ and $\vec{A}/\vec{C}$ are *nonsense* expressions.

More about vector algebra and the use of vector algebra to 'solve triangles' is discussed in sec. 2.5 starting on page 92. Before that we will enrich our vector algebra with one more operation in the following section, the cross product.

Dot products on the computer

Computer use for vector addition was discussed on page 49. Most computer languages will submit to calculating a dot product in response to commands something like this:

```
A   = [  1  2  5 ]
B   = [ -2  4 19 ]
D   = A(1)*B(1) + A(2)*B(2) + A(3)*B(3).
```

In pseudo code we could write `D = A dot B`. Many computer languages have a shorter way to write the dot product like `dot(A,B)`. In a language built for linear algebra `D = A*B'`③ will work because the rules of matrix

③ Recall that `B'` means the transpose of `B`, in this case turning the row of numbers `B` into a column of numbers.

multiplication are then the same as the component formula for the dot product.

2.4 Using the geometric definition of the dot product to find the dot product in terms of components

Vectors are essentially a geometric concept and we have consequently defined the dot product geometrically as $\vec{A}\cdot\vec{B} = ABcos\theta$. Almost 400 years ago René Descartes discovered that you could do geometry by doing algebra on the coordinates of points.

So we should be able to figure out the dot product of two vectors by knowing their components. The central key to finding this component formula is the distributive law

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$$

which we derived geometrically. If we write $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$ then we just repeatedly use the distributive law to derive the component formulae, as follows.

$$\begin{aligned}
\vec{A}\cdot\vec{B} &= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\cdot(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}) \\
&= (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\cdot B_x\hat{\imath} \;+ \\
&\quad (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\cdot B_y\hat{\jmath} \;+ \\
&\quad (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\cdot B_z\hat{k} \\
&= A_xB_x\hat{\imath}\cdot\hat{\imath} + A_yB_x\hat{\jmath}\cdot\hat{\imath} + A_zB_x\hat{k}\cdot\hat{\imath} \;+ \\
&\quad A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_zB_y\hat{k}\cdot\hat{\jmath} \;+ \\
&\quad A_xB_z\hat{\imath}\cdot\hat{k} + A_yB_z\hat{\jmath}\cdot\hat{k} + A_zB_z\hat{k}\cdot\hat{k} \\
&= A_xB_x(1) + A_yB_x(0) + A_zB_x(0) \;+ \\
&\quad A_xB_y(0) + A_yB_y(1) + A_zB_y(0) \;+ \\
&\quad A_xB_z(0) + A_yB_z(0) + A_zB_z(1)
\end{aligned}$$

$$\Rightarrow \qquad \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z \quad \text{(3D).}$$
$$\Rightarrow \qquad \vec{A}\cdot\vec{B} = A_xB_x + A_yB_y \quad \text{(2D).}$$

If we call our coordinate x_1, x_2, and x_3; and our unit base vectors $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ we would have $\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$ and $\vec{B} = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3$ and the dot product has the familiar tidy form:

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_{i=1}^{3} A_iB_i\,.$$

Tilted coordinates: xyz *vs* $x'y'z'$.

The demonstration above could have been carried out using a different orthogonal coordinate system $x'y'z'$ that was tilted with respect to the xyz system. By identical reasoning we would find that $\vec{A}\cdot\vec{B} = A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'}$. Even though all of the numbers in the list $[A_x, A_y, A_z]$ might be different from the numbers in the list $[A_{x'}, A_{y'}, A_{z'}]$ and similarly all the list $[\vec{B}]_{xyz}$ might be different than the list $[\vec{B}]_{x'y'z'}$, so (remarkably, luckily and necesarily),

$$A_xB_x + A_yB_y + A_zB_z = A_{x'}B_{x'} + A_{y'}B_{y'} + A_{z'}B_{z'}\,.$$

The formula for the dot product is the same in the different coordinate systems. And the value of the dot product is the same in the different coordinate systems. Yet all the numbers on the two sides of the formula above are likely different from each other.

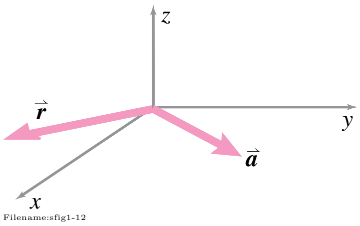

Figure 2.36:

SAMPLE 2.12 Calculating dot products: Find the dot product of the two vectors $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{r} = 5\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\hat{\jmath}$.

Solution The dot product of the two vectors is

$$\begin{aligned}
\vec{a}\cdot\vec{r} &= (2\hat{\imath} + 3\hat{\jmath} - 2\hat{k})\cdot(5\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\hat{\jmath}) \\
&= (2\cdot 5\,\mathrm{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (2\cdot 2\,\mathrm{m})\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} \\
&\quad + (3\cdot 5\,\mathrm{m})\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} - (3\cdot 2\,\mathrm{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} \\
&\quad - (2\cdot 5\,\mathrm{m})\underbrace{\hat{k}\cdot\hat{\imath}}_{0} + (2\cdot 2\,\mathrm{m})\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} \\
&= 10\,\mathrm{m} - 6\,\mathrm{m} = 4\,\mathrm{m}.
\end{aligned}$$

$$\boxed{\vec{a}\cdot\vec{r} = 4\,\mathrm{m}}$$

Comments: Note that with just a little bit of foresight, we could totally ignore the $\hat{k}$ component of $\vec{a}$ since $\vec{r}$ has no $\hat{k}$ component, *i.e.,* $\hat{k}\cdot\vec{r} = 0$. Also, if we keep in mind that $\hat{\imath}\cdot\hat{\jmath} = \hat{\jmath}\cdot\hat{\imath} = 0$, we could compute the above dot product in one line:

$$\vec{a}\cdot\vec{r} = (2\hat{\imath} + 3\hat{\jmath})\cdot(5\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\hat{\jmath}) = (2\cdot 5\,\mathrm{m})\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} - (3\cdot 2\,\mathrm{m})\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} = 4\,\mathrm{m}.$$

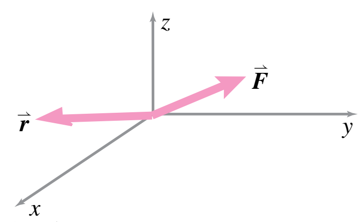

Figure 2.37:

SAMPLE 2.13 Component of a vector in a given direction: A force acting at some point is given as $\vec{F} = 5\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath} + 2\,\mathrm{N}\hat{k}$.

1. Find the y-component of $\vec{F}$.
2. Find the component of $\vec{F}$ along the vector $\vec{r} = 3\,\mathrm{m}\hat{\imath} - 4\,\mathrm{m}\hat{\jmath}$.

Solution

1. *Component along the y-direction:* The y-component of $\vec{F}$ is the scalar quantity multiplying the unit vector $\hat{\jmath}$, that is, 3 N. Although the y-component of $\vec{F}$ is obvious here (and hence the problem is trivial), we can find it in a formal way using the dot product between $\vec{F}$ and $\hat{\jmath}$.

$$\begin{aligned}
F_y &= \vec{F}\cdot(\text{a unit vector along } y\text{-axis}) \\
&= (5\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath} + 2\,\mathrm{N}\hat{k})\cdot\hat{\jmath} \\
&= 5\,\mathrm{N}\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0} + 3\,\mathrm{N}\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{1} + 2\,\mathrm{N}\underbrace{\hat{k}\cdot\hat{\jmath}}_{0} = 3\,\mathrm{N}.
\end{aligned}$$

$$\boxed{F_y = \vec{F}\cdot\hat{\jmath} = 3\,\mathrm{N}.}$$

2. *Component along the $\vec{r}$-direction:* The component of $\vec{F}$ along $\vec{r}$ is obtained from the dot product of $\vec{F}$ with a unit vector along $\vec{r}$. Therefore, we first need to find a unit vector $\hat{\boldsymbol{\lambda}}_r$ along $\vec{r}$ and then dot it with $\vec{F}$.

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_r &= \frac{\vec{r}}{|\vec{r}|} = \frac{3\,\mathrm{m}\hat{\imath} - 4\,\mathrm{m}\hat{\jmath}}{\sqrt{3^2 + 4^2}\,\mathrm{m}} = 0.6\hat{\imath} - 0.8\hat{\jmath} \\
F_r &= \vec{F}\cdot\hat{\boldsymbol{\lambda}}_r \\
&= (5\,\mathrm{N}\hat{\imath} + 3\,\mathrm{N}\hat{\jmath} + 2\,\mathrm{N}\hat{k})\cdot(0.6\hat{\imath} - 0.8\hat{\jmath}) \\
&= 3.0\,\mathrm{N} + 2.4\,\mathrm{N} = 5.4\,\mathrm{N}.
\end{aligned}$$

$$\boxed{F_r = \vec{F}\cdot\hat{\boldsymbol{\lambda}}_r = 5.4\,\mathrm{N}}$$

SAMPLE 2.14 Finding angle between two vectors using dot product: Find the angle between the vectors $\vec{r}_1 = 2\hat{\imath} + 3\hat{\jmath}$ and $\vec{r}_2 = 2\hat{\imath} - \hat{\jmath}$.

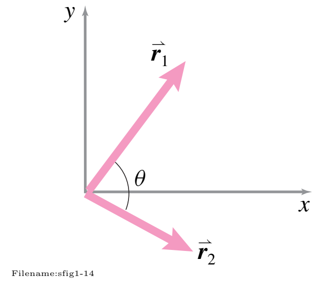

Figure 2.38:

Solution From the definition of dot product between two vectors

$$
\begin{aligned}
\vec{r}_1 \cdot \vec{r}_2 &= |\vec{r}_1||\vec{r}_2|\cos\theta \\
\text{or} \qquad \cos\theta &= \frac{\vec{r}_1 \cdot \vec{r}_2}{|\vec{r}_1||\vec{r}_2|} \\
&= \frac{(2\hat{\imath} + 3\hat{\jmath}) \cdot (2\hat{\imath} - \hat{\jmath})}{(\sqrt{2^2 + 3^2})(\sqrt{2^2 + 1^2})} \\
&= \frac{4 - 3}{\sqrt{13}\sqrt{5}} = 0.124 \\
\text{Therefore,} \qquad \theta &= \cos^{-1}(0.124) = 82.87^\circ.
\end{aligned}
$$

$$\boxed{\theta = 83^\circ}$$

SAMPLE 2.15 Finding direction cosines from unit vectors: Find the angles between $\vec{F} = 4\,\mathrm{N}\hat{\imath} + 6\,\mathrm{N}\hat{\jmath} + 7\,\mathrm{N}\hat{k}$ and each of the three axes.

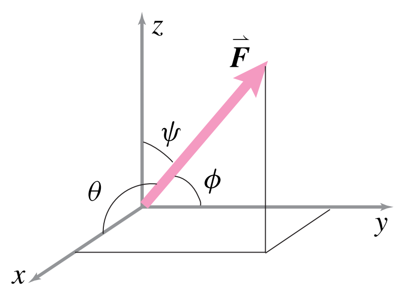

Figure 2.39:

Solution

$$
\begin{aligned}
\vec{F} &= F\hat{\lambda} \\
\hat{\lambda} &= \frac{\vec{F}}{F} \\
&= \frac{4\,\mathrm{N}\hat{\imath} + 6\,\mathrm{N}\hat{\jmath} + 7\,\mathrm{N}\hat{k}}{\sqrt{4^2 + 6^2 + 7^2}\,\mathrm{N}} \\
&= 0.4\hat{\imath} + 0.6\hat{\jmath} + 0.7\hat{k}.
\end{aligned}
$$

Let the angles between $\hat{\lambda}$ and the x, y, and z axes be θ, ϕ and ψ respectively. Then

$$
\begin{aligned}
\cos\theta &= \frac{\hat{\imath} \cdot \hat{\lambda}}{|\hat{\imath}||\hat{\lambda}|} = \frac{0.4}{|1||1|} = 0.4. \\
\Rightarrow \quad \theta &= \cos^{-1}(0.4) = 66.4^\circ.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\cos\phi &= 0.6 \quad \text{or} \quad \phi = 53.1^\circ \\
\cos\psi &= 0.7 \quad \text{or} \quad \psi = 45.6^\circ.
\end{aligned}
$$

$$\boxed{\theta = 66.4^\circ,\ \phi = 53.1^\circ,\ \psi = 45.6^\circ}$$

Comments: The components of a unit vector give the direction cosines with the respective axes. That is, if the angle between the unit vector and the x, y, and z axes are θ, ϕ and ψ, respectively (as above), then

$$\hat{\lambda} = \underbrace{\cos\theta}_{\lambda_x}\,\hat{\imath} + \underbrace{\cos\phi}_{\lambda_y}\,\hat{\jmath} + \underbrace{\cos\psi}_{\lambda_z}\,\hat{k}.$$

SAMPLE 2.16 Separating a vector equation into scalar equations: Assume that after writing the equation $\sum \vec{F} = m\vec{a}$ in a particular problem, a student finds $\sum \vec{F} = (20\,\mathrm{N} - P_1)\hat{\imath} + 7\,\mathrm{N}\hat{\jmath} - P_2\hat{k}$ and $\vec{a} = 2.4\,\mathrm{m/s^2}\hat{\imath} + a_3\hat{\jmath}$. Separate the scalar equations in the $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ directions.

Solution From a vector equation, separating the scalar equations is trivial as long as both sides of a vector equation are in the same basis — individual components on both sides must equal. That is

$$
\begin{aligned}
\overbrace{(20\,\mathrm{N} - P_1)\hat{\imath} + 7\,\mathrm{N}\hat{\jmath} - P_2\hat{k}}^{\sum \vec{F}} &= m\,\overbrace{(2.4\,\mathrm{m/s^2}\hat{\imath} + a_3\hat{\jmath})}^{\vec{a}} \\
\Rightarrow \quad 20\,\mathrm{N} - P_1 &= m(2.4\,\mathrm{m/s^2}) && (\hat{\imath} \text{ component}) \\
7\,\mathrm{N} &= ma_3 && (\hat{\jmath} \text{ component}) \\
-P_2 &= 0. && (\hat{k} \text{ component})
\end{aligned}
$$

These are the three independent scalar equations in the $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ directions.

$$\boxed{20\,\mathrm{N} - P_1 = m(2.4\,\mathrm{m/s^2}), \quad 7\,\mathrm{N} = ma_3,\ P_2 = 0}$$

Comments: The results obtained by equating individual components on both sides of the vector equation are based on the general technique of taking the dot product of both sides of an equation with a vector. It gives a scalar equation valid in any direction that one desires. For the example at hand, the long but easily readable and illustrative calculation is as follows. Taking the dot product of both sides of $\sum \vec{F} = m\vec{a}$ equation with $\hat{\imath}$, we write

$$
\begin{aligned}
\hat{\imath} \cdot \Big[(20\,\mathrm{N} - P_1)\hat{\imath} + 7\,\mathrm{N}\hat{\jmath} - P_2\hat{k} &= m(2.4\,\mathrm{m/s^2}\hat{\imath} + a_3\hat{\jmath})\Big] \\
\Rightarrow \quad \underbrace{(20\,\mathrm{N} - P_1)}_{F_x}\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + 7\,\mathrm{N}\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0} - P - 2\underbrace{\hat{k}\cdot\hat{\imath}}_{0} &= m(\underbrace{2.4\,\mathrm{m/s^2}}_{a_x}\underbrace{\hat{\imath}\cdot\hat{\imath}}_{1} + a_3\underbrace{\hat{\jmath}\cdot\hat{\imath}}_{0}) \\
\Rightarrow \quad 20\,\mathrm{N} - P_1 &= m(2.4\,\mathrm{m/s^2}) \\
\textit{i.e.,} \quad F_x &= ma_x
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\hat{\jmath} \cdot \Big[\sum \vec{F} = m\vec{a}\Big] \quad &\Rightarrow \quad 7\,\mathrm{N} = ma_3 \\
\hat{k} \cdot \Big[\sum \vec{F} = m\vec{a}\Big] \quad &\Rightarrow \quad -P_2 = 0.
\end{aligned}
$$

2.3 Vector cross product

The vector cross product① is a second way of multiplying vectors together (the first was the dot product).

①**Nomenclature.** The vector cross product is sometimes just called 'the vector product'. In this book we usually call it 'the cross product'.

The cross product of vectors $\vec{A}$ and $\vec{B} = \vec{A} \times \vec{B}$.

The vector cross product is used to define (and calculate) moment, to solve geometry problems and to calculate various quantities associated with rotations in dynamics. Most uses of the cross product used in this book are listed in box 2.5 on page 68.

In this first section we treat the cross product as a mathematical and geometrical calculation. Deeper understanding will come with applying the cross product to moments in sec. 2.4. Comfort with the cross product is a tremendous aid for solving three-dimensional statics problems and for doing all of dynamics. If this is a new topic for you, don't gloss over it.

The 2D cross product

Although the cross product is fundamentally a three-dimensional idea, we start with the two-dimensional version. The 2D cross product is defined as :

$$\underbrace{\vec{A} \times \vec{B}}_{\text{`A cross B'}} \quad \stackrel{def}{=} \quad |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{k}. \tag{2.8}$$

where θ is the amount that $\vec{A}$ would need to be rotated counterclockwise to point in the same direction as $\vec{B}$. An equivalent alternative approach is to define the cross product as

$$\vec{A} \times \vec{B} \quad \stackrel{def}{=} \quad |\vec{A}|\,|\vec{B}|\,\sin\theta\,\hat{n}. \tag{2.9}$$

with θ defined to be less than 180° and $\hat{n}$ defined as the unit vector pointing in the direction of the thumb when the fingers are curled from the direction of $\vec{A}$ towards the direction of $\vec{B}$. For the $\vec{r}$ and $\vec{F}$ on the right of the teeter totter this definition forces us to point our thumb into the plane (in the negative $\hat{k}$ direction). With this definition $\sin\theta$ is always positive and the negative moments come from $\hat{n}$ being in the $-\hat{k}$ direction.

Study of fig. 2.40 should you could convince you that the definition of cross product in eqn.2.8 obeys these standard algebra rules (for any 3 2D vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ and any scalar d):

$$\begin{aligned} d(\vec{A} \times \vec{B}) &= (d\vec{A}) \times \vec{B} = \vec{A} \times (d\vec{B}) \\ \vec{A} \times (\vec{B} + \vec{C}) &= \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \end{aligned}$$

A difference between the algebra rules for scalar multiplication and vector cross product multiplication is that for scalar multiplication $AB = BA$ whereas for the cross product $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ (because the definition of θ in eqn. 2.8 and $\hat{n}$ in 2.9 depends on order). In particular $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

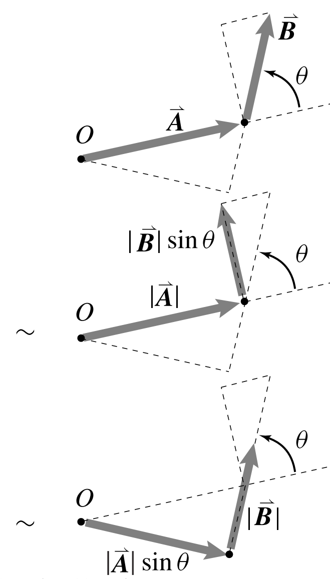

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Figure 2.40: The cross product of $\vec{A}$ and $\vec{B}$ is $|\vec{A}||\vec{B}|\sin\theta\hat{k}$. Grouping the $\sin\theta$ with $|\vec{A}|$ or with $|\vec{B}|$ gives two different geometric interpretations of the 2D cross product. First, you can think of the magnitude of the cross product as being the magnitude of $\vec{A}$ times the projection of $\vec{B}$ perpendicular to $\vec{A}$. That's $|\vec{A}|\left(|\vec{B}|\sin\theta\right)$. Or, you can think of it as the magnitude of $\vec{B}$ times the distance $|\vec{A}|\sin\theta$ marked in the tip-to-tail construction in the third picture above. $\left(|\vec{A}|\sin\theta\right)|\vec{B}|$.

2.5 Uses of the cross product

Here are the key uses of the cross product in this book. A first time reader is not supposed to understand all of these at the start, or even what the terms mean. These are here for reference and inspiration.

Geometry

1. **3D Normal to a plane.** Find the normal $\vec{N}$ to a plane containing points A,B and C as
$$\vec{N} = \vec{r}_{B/A} \times \vec{r}_{C/A}.$$

2. **3D Unit normal to a plane.** Use $\vec{N}$ above to find the unit normal to a plane containing points A,B and C as
$$\hat{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{|\vec{r}_{B/A} \times \vec{r}_{C/A}|}$$

3. **3D Distance between a point and a plane.** Use $\hat{n}$ above to find the distance between a point D and the plane containing points A,B and C as
$$\begin{aligned} d = \quad \vec{r}_{D/A} \cdot \hat{n} \quad &= \quad \vec{r}_{D/A} \cdot \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{|\vec{r}_{B/A} \times \vec{r}_{C/A}|} \\ = \quad \vec{r}_{D/B} \cdot \hat{n} \quad &= \quad \vec{r}_{D/B} \cdot \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{|\vec{r}_{B/A} \times \vec{r}_{C/A}|} \\ = \quad \vec{r}_{D/C} \cdot \hat{n} \quad &= \quad \vec{r}_{D/C} \cdot \frac{\vec{r}_{B/A} \times \vec{r}_{C/A}}{|\vec{r}_{B/A} \times \vec{r}_{C/A}|} \end{aligned}$$

4. **3D Distance between two lines.** Find the distance between the line containing the points A and B and the line containing the points C and D as
$$d = \left|\left(\vec{r}_{B/A} \times \vec{r}_{D/C}\right) \cdot \vec{r}_{C/A}\right| = \left|\left(\vec{r}_{B/A} \times \vec{r}_{D/C}\right) \cdot \vec{r}_{D/A}\right| \ldots$$

5. **2D Perpendicular in the xy plane.** Find the in-plane normal $\vec{r}^{\perp}$ to a vector $\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath}$ in the xy plane as
$$\vec{r}^{\perp} = \hat{k} \times \vec{r}.$$
(The vector $\vec{r}^{\perp}$ is also rotation of $\vec{r}$ counterclockwise by $90°$.)

6. **2D unit perpendicular in xy plane.** Use $\vec{r}^{\perp}$ to find a unit vector perpendicular to $\vec{r}$ as
$$\hat{n}^{\perp} = \frac{\vec{r}^{\perp}}{|\vec{r}^{\perp}|} = \frac{\hat{k} \times \vec{r}}{\left|\hat{k} \times \vec{r}\right|}$$

7. **3D distance between a point and a line.** Find the distance between point C and the line going through A and B as
$$d = \left|\vec{r}_{C/A} - \vec{r}_{C/A} \cdot \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}\right|$$

8. **2D Distance between a point and line in the plane.** Use $\hat{n}$ to find the distance between a line AB in the plane and a point C in the plane as
$$d = \left|\hat{n} \cdot \vec{r}_{C/A}\right| = \left|\hat{n} \cdot \vec{r}_{C/B}\right| = \left|\frac{\left(\hat{k} \times \vec{r}_{B/A}\right) \cdot \vec{r}_{C/A}}{\hat{k} \times \vec{r}_{B/A}}\right| = \ldots$$

9. **3D Volume V of a parallelopiped** with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$ is
$$V = \left(\vec{A} \times \vec{B}\right) \cdot \vec{C}$$

Statics

10. **Moment of a force.** Calculate moment $\vec{M}$ of a force $\vec{F}$ at position $\vec{r}$ as
$$\vec{M} = \vec{r} \times \vec{F}$$

11. **Moment about an axis.** Calculate moment M of a force $\vec{F}$ at position $\vec{r}$ relative to a point on the axis in the direction of a unit vector $\hat{\lambda}$ as
$$M = \left(\vec{r} \times \vec{F}\right) \cdot \hat{\lambda}.$$

12. **Moment about an axis.** Use the result above to calculate the moment of a force $\vec{F}$ at position C about a line through A and B as
$$M = \left(\vec{r}_{C/A} \times \vec{F}\right) \cdot \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|} = \left(\vec{r}_{C/B} \times \vec{F}\right) \cdot \frac{\vec{r}_{B/A}}{|\vec{r}_{B/A}|}$$

Kinematics and Dynamics

13. **Relative velocity of two points on a rigid object.** The velocity of point B relative to point A, where both points are on the same rigid object with angular velocity $\vec{\omega}$ is
$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}.$$

14. **Angular momentum of a particle** with velocity $\vec{v}$, mass m and at position $\vec{r}$ is
$$\vec{H} = m\vec{r} \times \vec{v}.$$

15. **Centripital acceleration** of point B relative to point A on the same rigid object with angular velocity $\vec{\omega}$ is
$$\text{Centripital part of } \vec{a}_{B/A} = \vec{\omega} \times \left(\vec{\omega} \times \vec{r}_{B/A}\right).$$

16. **Relative acceleration due to angular acceleration.** The relative acceleration of points A and B on the same rigid object with angular acceleration $\vec{\alpha}$ is
$$\text{Contribution of angular acceleration to } \vec{a}_{B/A} = \vec{\alpha} \times \vec{r}_{B/A}.$$

17. **Coriolis acceleration** of a particle moving at velocity $\vec{v}_{rel}$ relative to a rotating rigid object is
$$\vec{a}_{Coriolis} = 2\vec{\omega} \times \vec{v}_{rel}.$$

n. Variants and extensions of the kinematics and dynamics formulas are given in the tables at the back of the dynamics book.

Because the magnitude of the cross product of $\vec{A}$ and $\vec{B}$ is the magnitude of $\vec{A}$ times the magnitude of the projection of $\vec{B}$ in the direction perpendicular to $\vec{A}$ (as shown in the top two illustrations of fig. 2.53) you can think of the cross product as a measure of how much two vectors are perpendicular to each other. In particular

$$\begin{aligned}
\text{if } \vec{A} \perp \vec{B} \quad &\Rightarrow \quad |\vec{A}\times\vec{B}| &= \quad |\vec{A}|\,|\vec{B}|, \text{ and}\\
\text{if } \vec{A} \parallel \vec{B} \quad &\Rightarrow \quad |\vec{A}\times\vec{B}| &= \quad \vec{0}.
\end{aligned}$$

For example, $\hat{\imath}\times\hat{\jmath}=\hat{k}$, $\hat{\jmath}\times\hat{\imath}=-\hat{k}$, $\hat{\imath}\times\hat{\imath}=\vec{0}$, and $\hat{\jmath}\times\hat{\jmath}=\vec{0}$.

Component form for the 2D cross product

Just like the dot product, the cross product can be expressed using components. As can be verified by writing $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$, and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$ and using the distributive rules:

$$\vec{A}\times\vec{B} \quad = (A_xB_y - B_xA_y)\hat{k}. \tag{2.10}$$

Some people remember this formula by putting the components of $\vec{A}$ and $\vec{B}$ into a matrix and calculating the determinant $\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$. If you number the components of $\vec{A}$ and $\vec{B}$ (*e.g.*, $[\vec{A}]_{x_1x_2} = [A_1, A_2]$), the cross product is $\vec{A}\times\vec{B} \quad = (A_1B_2 - B_2A_1)\hat{e}_3$. This you might remember as "first times second minus second times first."

Example: Given that $\vec{A} = 1\hat{\imath} + 2\hat{\jmath}$ and $\vec{B} = 10\hat{\imath} + 20\hat{\jmath}$ then $\vec{A}\times\vec{B} = (1\cdot 20 - 2\cdot 10)\hat{k} = 0\hat{k} = \vec{0}$.

For vectors with just a few components it is often most convenient to use the distributive rule directly.

Example: Given that $\vec{A} = 7\hat{\imath}$ and $\vec{B} = 37.6\hat{\imath} + 10\hat{\jmath}$ then $\vec{A}\times\vec{B} = (7\hat{\imath})\times(37.6\hat{\imath} + 10\hat{\jmath}) = (7\hat{\imath})\times(37.6\hat{\imath}) + (7\hat{\imath})\times(10\hat{\jmath}) = \vec{0} + 70\hat{k} = 70\hat{k}$.

There are many ways of calculating a 2D cross product

You have several options for calculating the 2D cross product. Which you choose depends on taste and convenience. You can use the geometric definition directly, the first times the perpendicular part of the second (distance times perpendicular component of force), the second times the perpendicular part of the first (lever arm times the force), components, or break each of the vectors into a sum of vectors and use the distributive rule.

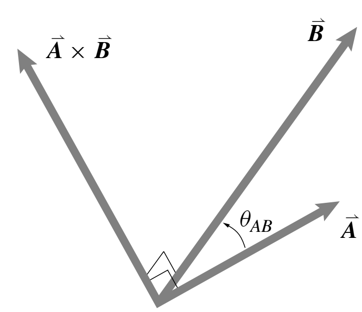

Figure 2.41: The cross product of $\vec{A}$ and $\vec{B}$ is perpendicular to $\vec{A}$ and $\vec{B}$ in the direction given by the right hand rule. The magnitude of $\vec{A}\times\vec{B}$ is $AB\sin\theta_{AB}$.

The 3D vector cross product

The cross product of two vectors $\vec{A}$ and $\vec{B}$ is written $\vec{A}\times\vec{B}$ and pronounced 'A cross B.' In contrast to the dot product, which gives a scalar and measures how much two vectors are parallel, the cross product is a vector and measures

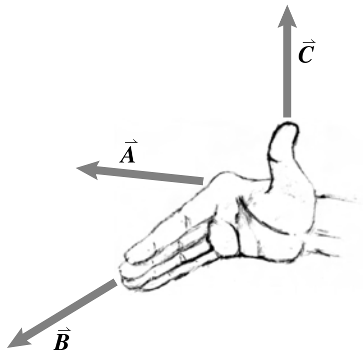

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Figure 2.42: The right hand rule for determining the direction of the cross product of two vectors. $\vec{C} = \vec{A} \times \vec{B}$.

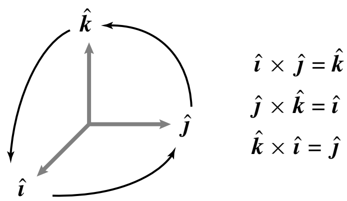

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Figure 2.43: Mnemonic device to remember the cross product of the standard base unit vectors.

how much they are perpendicular. The cross product is also called *the vector product*.

The cross product is defined by:

$$\vec{A} \times \vec{B} \stackrel{def}{=} |\vec{A}||\vec{B}| \sin\theta_{AB}\, \hat{n} \tag{2.11}$$

where $|\hat{n}| = 1$,
$\hat{n} \perp \vec{A}$,
$\hat{n} \perp \vec{B}$,
$0 \leq \theta_{AB} \leq \pi$, and
$\hat{n}$ is in the direction given by the right hand rule, that is, in the direction of the right thumb when the fingers of the right hand are pointed in the direction of $\vec{A}$ and then wrapped towards the direction of $\vec{B}$.

If $\vec{A}$ and $\vec{B}$ are perpendicular then θ_{AB} is $\pi/2$, $\sin\theta_{AB} = 1$, and the magnitude of the cross product is AB. If $\vec{A}$ and $\vec{B}$ are parallel then θ_{AB} is 0, $\sin\theta_{AB} = 0$ and the cross product is $\vec{0}$ (the zero vector). This is why we say the cross product is a measure of the degree of orthogonalityof two vectors.

Using the definition above you should be able to verify to your own satisfaction that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$. Applying the definition to the standard base unit vectors you can see that $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, and $\hat{k} \times \hat{\imath} = \hat{\jmath}$ (fig. 2.43).

The geometric definition above and the geometric (tip to tale) definition of vector addition imply that the cross product follows the distributive rule.

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

Applying the distributive rule to the cross products of $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$ leads to the algebraic formula for the Cartesian components of the cross product.

$$\begin{aligned} \vec{A} \times \vec{B} \quad = \quad & [A_yB_z - A_zB_y]\hat{\imath} & (2.12)\\ & +[A_zB_x - A_xB_z]\hat{\jmath} & (2.13)\\ & +[A_xB_y - A_yB_x]\hat{k} & (2.14)\\ & & (2.15) \end{aligned}$$

The distributive rule, and how it gives the component formula, is described in box 2.7 on page 74. There are various mnemonics for remembering the component formula for cross products. The most common is to calculate a 'determinant' of the 3×3 matrix with one row given by $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the other two rows the components of $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

The following identities and special cases of cross products are worth knowing well:

- $(a\vec{A}) \times \vec{B} = \vec{A} \times (a\vec{B}) = a(\vec{A} \times \vec{B})$ (a distributive law)
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (the cross product is not commutative!)
- $\vec{A} \times \vec{B} = \vec{0}$ if $\vec{A} \parallel \vec{B}$ (parallel vectors have zero cross product)
- $|\vec{A} \times \vec{B}| = AB$ if $\vec{A} \perp \vec{B}$
- $\hat{\imath} \times \hat{\jmath} = \hat{k}$, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$ (assuming the x, y, z coordinate system is right handed — if you use your right hand and point your fingers along the positive x axis, then curl them towards the positive y axis, your thumb will point in the same direction as the positive z axis.)
- $\hat{\imath}' \times \hat{\jmath}' = \hat{k}'$, $\hat{\jmath}' \times \hat{k}' = \hat{\imath}'$, $\hat{k}' \times \hat{\imath}' = \hat{\jmath}'$ (assuming the $x'y'z'$ coordinate system is right handed.)
- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$, $\hat{\imath}' \times \hat{\imath}' = \hat{\jmath}' \times \hat{\jmath}' = \hat{k}' \times \hat{k}' = \vec{0}$

The mixed triple product

The 'mixed triple product' of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is so called because it mixes both the dot product and cross product in a single expression. The mixed triple product is also sometimes called the scalar triple product because its value is a scalar. The mixed triple product is useful for calculating the moment of a force about an axis and for related dynamics quantities.The *mixed triple product* of $\vec{A}$, $\vec{B}$, and $\vec{C}$ is defined by and written as

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right)$$

and pronounced 'A dot B cross C.' The parentheses () are sometimes omitted, *i.e.,*

$$\vec{A} \cdot \vec{B} \times \vec{C},$$

because the wrong grouping $(\vec{A} \cdot \vec{B}) \times \vec{C}$ is nonsense (you can't take the cross product of a scalar with a vector) . It is apparent that one way of calculating the mixed triple product is to calculate the cross product of $\vec{B}$ and $\vec{C}$ and then to take the dot product of that result with $\vec{A}$. Some people use the notation $\left(\vec{A}, \vec{B}, \vec{C}\right)$ for the mixed triple product but it will not occur again in this book.

The mixed triple product has the same value if one takes the cross product of $\vec{A}$ and $\vec{B}$ and then the dot product of the result with $\vec{C}$. That is $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$. This identity can be verified using the geometric description below, or by looking at the (complicated) expression for the mixed triple product of three general vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in terms of their components as calculated the two different ways. One thus obtains the string of results

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C} = -\vec{B} \times \vec{A} \cdot \vec{C} = -\vec{B} \cdot \vec{A} \times \vec{C} = \ldots$$

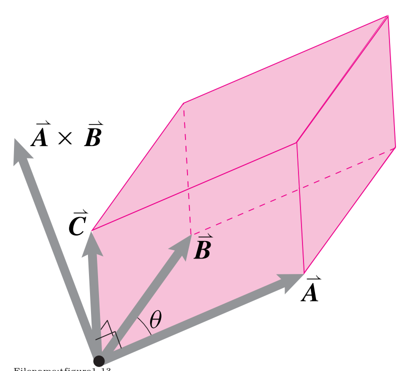


Figure 2.44: One interpretation of the mixed triple product of $\vec{A} \times \vec{B} \cdot \vec{C}$ is as the volume (a scalar) of a parallelepiped with $\vec{A}$, $\vec{B}$, and $\vec{C}$ as the three edges emanating from one corner. This interpretation only works if $\vec{A}$, $\vec{B}$, and $\vec{C}$ are taken in the appropriate order, otherwise $\vec{A} \times \vec{B} \cdot \vec{C}$ is minus the volume which is calculated.

The minus signs in the above expressions follow from the cross product identity that $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.

The mixed triple product has various geometric interpretations, one of them is that $\vec{A} \cdot \vec{B} \times \vec{C}$ is (plus or minus) the volume of the parallelepiped, the crooked shoe box, edged by $\vec{A}$, $\vec{B}$ and $\vec{C}$ as shown in fig. 2.44.

Another way of calculating the value of the mixed triple product is with the determinant of a matrix whose rows are the components of the vectors.

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{array}{r} A_x(B_yC_Z - B_zC_y) \\ +A_y(B_zC_x - B_xC_z) \\ +A_z(B_xC_y - B_yC_x) \end{array}$$

②In the language of linear algebra, the mixed triple product of three vectors is zero if the vectors are linearly dependent.

The mixed triple product of three vectors is zero if②

- any two of them are parallel, or
- all three of the vectors have one common plane.

A different triple product, sometimes called the vector triple product, $\vec{A} \times (\vec{B} \times \vec{C})$, is discussed later (see box 14.4 on page 779).

Cross products and computers

The components of the cross product can be calculated with computer code that may look something like this.

```
A = [  1   2   5 ]
B = [ -2   4  19 ]
C = [  ( A(2)*B(3)  - A(3)*B(2)  ) ...
       ( A(3)*B(1)  - A(1)*B(3)  ) ...
       ( A(1)*B(2)  - A(2)*B(1)  ) ]
```

giving the result `C=[18  -29  8]`. Many computer languages have a shorter way to write the cross product like `cross(A,B)`. The mixed triple product might be calculated by assembling a 3×3 matrix of rows and then taking a determinant like this:

```
        A    = [  1   2   5 ]
        B    = [ -2   4  19 ]
        C    = [ 32   4   5 ]
   matrix    = [A ;      % Each row under the one above
               B ;
               C   ]
 mixedprod = det(matrix)
```

giving the result `mixedprod = 500`. A versatile computer language might allow a command like `dot( A, cross(B,C) )` to calculate the mixed triple product.

2.6 The cross product of vectors as matrix multiplication

For hand calculations in statics, this box is probably not useful. This box is for the theoretically inclined, and for people interested in doing complex dynamics calculations on a computer. We assume here that you know some linear algebra.

The cross product between two vectors $\vec{a}$ and $\vec{b}$ gives a new vector

$$\vec{c} = \vec{a} \times \vec{b}$$

All three of these vectors have components:

$$[\vec{a}]_{xy} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad [\vec{b}]_{xy} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

and

$$[\vec{c}]_{xy} = \begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix}.$$

The components of $\vec{c}$ can be calculated from those of $\vec{a}$ and $\vec{b}$ using the component rules from this section as

$$\begin{bmatrix} c_x \\ c_y \\ c_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}.$$

Because the cross product $\vec{a} \times \vec{b}$ is linear in $\vec{b}$ (meaning $\vec{a} \times (\vec{b}_1 + \vec{b}_2) = \vec{a} \times \vec{b}_1 + \vec{b}_2$, *etc.*) we can represent the cross product as some matrix times $\vec{b}$.

We now define a matrix $[A]$, that does the job. We associate the the vector $\vec{a}$ with the matrix $[A]$ as follows:

$$[A] \equiv \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}. \tag{2.16}$$

$[A]$ is an anti-symmetric matrix (also called a skew symmetric matrix), with

$$[A]' = -[A] \qquad \text{(A transpose is minus A)}.$$

The terms of $[A]$ on the diagonal are zero and those off the diagonal are negative of their corresponding (transposed) terms. The conversion rule (Equation 2.16) that takes a vector and puts its components in a matrix in the right place we can call $[\mathcal{S}]$ thus

$$[A] = [\mathcal{S}(\vec{a})].$$

$[A]$ is the anti-symetric matrix defined in terms of the components of $\vec{a}$ by $[\mathcal{S}(\vec{a})]$.

Just multiplying out the terms we see that

$$\mathcal{S}(\vec{a}) \times \vec{b} = \underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{[\mathcal{S}(\vec{a})]} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} \tag{2.17}$$

$$= \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix}. \tag{2.18}$$

So we get our main result:

$$[\vec{a} \times \vec{b}]_{xyz} = [\mathcal{S}(\vec{a})][\vec{b}]_{xyz} = [A][\vec{b}]_{xyz}.$$

That is, the components of the cross product of $\vec{a}$ and $\vec{b}$ can be found by multiplying the matrix $[A]$ by the components of $\vec{b}$.

Writing the cross product as a matrix multiplication is sometimes useful for dynamics when the first vector $\vec{a}$ is $\vec{\omega}$ (see box 13.7 on page 682). The advantage of the matrix representation over the cross product is that matrix multiplication satisfies the associative rule

$$[A]([B][C]) = ([A][B])[C]$$

whereas the vector cross product does not:

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}.$$

Computer calculation

One could write a short program (computer function) that does the conversion from vector $\vec{a}$ to matrix $[A] = \mathcal{S}(\vec{a})$ and call it `skew`. That is, *skew* would calculate eqn. (2.16). Using *skew* we could carry out a cross product like this (see box 0.1 on page 22)

`a = [1 2 3]'`	Define vector $\vec{a}$ with components
`b = [4 5 6]'`	Define vector $\vec{b}$ with components
`A = skew(a)`	Use the function `skew`' to find A
`c = A*b`	Calculate the cross product $\vec{c} = \vec{a} \times \vec{b}$ using ordinary matrix multiplication.

For just one cross product this would be silly. But for a long calculation involving various vectors and matrices it often makes things simpler.

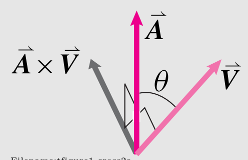
2.7 The cross product: from geometry to components

Why is the 3D vector given by the component formula for the cross product ((2.12) **on page 70**)

$$[\vec{A} \times \vec{B}]_{xyz} = \begin{bmatrix} (A_y B_z - A_z B_y) \\ (A_z B_x - A_x B_z) \\ (A_x B_y - A_y B_x) \end{bmatrix} \tag{2.19}$$

the same vector as that given by the geometric definition ((2.11) **on page 70**),

$$\vec{A} \times \vec{B} \equiv |\vec{A}||\vec{B}| \sin\theta_{AV}\, \hat{n}_{\perp AV}\,?$$



Filename:tfigure1-cross2a

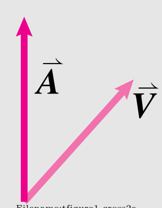
That these different-looking formulas should give the same answer is not obvious. Here we show why. Although the result is often used, the reasoning is not. But for logical completeness, and to entertain the curious, we present it here. We assume we know the geometric definition and want to find the component formula (2.19).

The reasoning has two big steps:

- First we will show that the geometric definitions of the vector cross product 'distributes' over vector addition.
- Then we apply the distributive rule to get our component formula.

A new definition of cross product

Start with $\vec{A}$ and $\vec{V}$:



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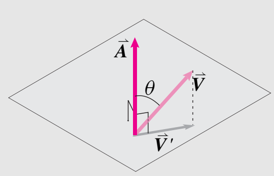
Now we will show that the geometric formula

$$\vec{A} \times \vec{B} \equiv |\vec{A}||\vec{B}| \sin\theta_{AV} * \hat{n}_{\perp AV}$$

is equivalent to a sequence of three operations: Project, rotate and stretch.

1. **Project $\vec{V}$ on to the plane P (normal to $\vec{A}$).** The projection of $\vec{V}$ onto the plane orthogonal to $\vec{A}$ is $\vec{V}'$.



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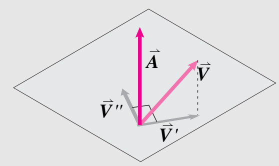
The magnitude of $\vec{V}'$ is

$$|\vec{V}'| = |\vec{V}| \sin\theta_{AV}\,.$$

$\vec{V}'$ is in the plane defined by $\vec{A}$ and $\vec{V}$ and also in the plane orthogonal to $\vec{A}$.

2. **Then rotate that projection by 90° about $\vec{A}$.**



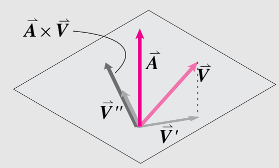
Filename:tfigure1-cross2c

Call the result of this rotation $\vec{V}''$. The magnitude is unchanged by rotation so we still have

$$|\vec{V}''| = |\vec{V}| \sin\theta_{AV}\,.$$

Note that $\vec{V}''$ is in the $\hat{n}$ direction that is perpendicular to both $\vec{A}$ (it's in the plane $\perp$ to $\vec{A}$) and to $\vec{V}$ (it's rotated 90° from $\vec{V}'$. So $\vec{V}''$ is in the direction of $\vec{A} \times \vec{V}$.

3. **Finally, stretch $\vec{V}''$ by $|\vec{A}|$.**

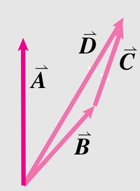


Filename:tfigure1-cross2b

This result has magnitude $|\vec{A}||\vec{V}| \sin\theta_{AV}$ which is the magnitude of $\vec{A} \times \vec{V}$. This vector is in the direction normal to both $\vec{A}$ and $\vec{V}$ given by the right-hand rule, that's the direction of $\vec{A} \times \vec{V}$. Having the same magnitude and direction as $\vec{A} \times \vec{V}$, it *is* $\vec{A} \times \vec{V}$. As infamously reasoned by Joseph McCarthy, "If it looks like a duck, walks like a duck and quacks like a duck, its a duck."

Apply the new definition to $\vec{B} + \vec{C}$

Consider $\vec{D} = \vec{B} + \vec{C}$. We are interested in all three cross products: $\vec{A} \times \vec{D}$, $\vec{A} \times \vec{B}$, and $\vec{A} \times \vec{C}$.



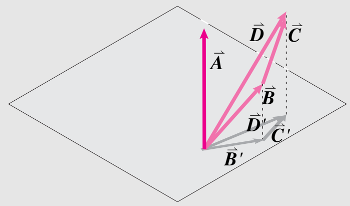
Filename:tfigure1-crossc

First we will check that each of the operations above (project, rotate, stretch) is distributive.

1. **Project.** The projection of a sum is the sum of the projections ($\vec{D}' = \vec{B}' + \vec{C}'$);

(continued...)

2.7 The cross product: from geometry to components (continued)



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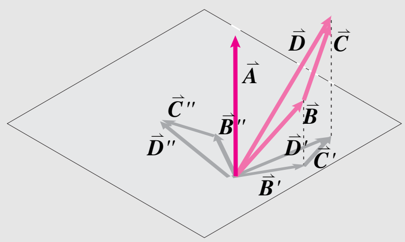
2. **Rotate.** The sum of two 90° rotated vectors is the rotation of the sum ($\vec{D}'' = \vec{B}'' + \vec{C}''$);



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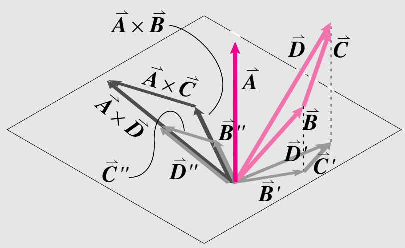
3. **Stretch.** (stretched $\vec{D}''$) = (stretched $\vec{B}''$) + (stretched $\vec{C}''$), scalar multiplication is distributive.



Filename:tfigure1-cross

Thus the act of taking the cross product of $\vec{A}$ with $\vec{B}$ and adding that to the cross product of $\vec{A}$ with $\vec{C}$ gives the same result as taking the cross product of $\vec{A}$ with $\vec{D} \equiv \vec{B} + \vec{C}$. That's the distributive law for vector cross products over vector addition:

$$\vec{A} \times \left(\vec{B} + \vec{C}\right) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}.$$

The distributive rule also works for a sum of 3 or more vectors (all of the reasoning above would be the same). And that the distributive rule works when the sum is on the left (*i.e.,* that $(\vec{B} + \vec{C}) \times \vec{A} = \vec{B} \times \vec{A} + \vec{C} \times \vec{A})$ also follows by similar very similar reasoning, so we skip those details. In sum, we know now that $(\vec{A} + \vec{B} + \vec{C}) \times (\vec{D} + \vec{E} + \vec{F}) = \vec{A} \times \vec{D} + \vec{A} \times \vec{E} + \ldots$ (9 terms in all).

Applying the distributive rule to the vectors' components

A vector is a some of component parts. First note that $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$

From the distributive rule (demonstrated with the 8 pictures above) we know that

$$\vec{A} \times \vec{B} = [A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}] \times [B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}].$$

Now we just apply the distributive law, using what we know about the cross products of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ with each other (*e.g.,* that $\hat{\imath} \times \hat{\imath} = \vec{0}$ and that $\hat{\imath} \times \hat{\jmath} = \hat{k}$).

First in 2D, to better show the patterns in the algebra:

$$\begin{aligned}
\vec{A} \times \vec{B} &= [A_x\hat{\imath} + A_y\hat{\jmath}] \times [B_x\hat{\imath} + B_y\hat{\jmath}] \\
&= [A_x\hat{\imath} + A_y\hat{\jmath}] \times B_x\hat{\imath} + [A_x\hat{\imath} + A_y\hat{\jmath}] \times B_y\hat{\jmath} \\
&= \quad A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} \\
&\quad + A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} \\
&= \quad A_xB_x\vec{0} + A_xB_y\hat{k} - A_yB_x\hat{k} + A_yB_y\vec{0}
\end{aligned}$$

$$\text{So } \vec{A} \times \vec{B} = [A_xB_y - A_yB_x]\hat{k} \qquad \text{(2D)}$$

Now in 3D, applying the distributive rule multiple times,

$$\begin{aligned}
\vec{A} \times \vec{B} &= [A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}] \times [B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}] \\
&= \quad A_xB_x\hat{\imath} \times \hat{\imath} + A_xB_y\hat{\imath} \times \hat{\jmath} + A_xB_z\hat{\imath} \times \hat{k} \\
&\quad + A_yB_x\hat{\jmath} \times \hat{\imath} + A_yB_y\hat{\jmath} \times \hat{\jmath} + A_yB_z\hat{\jmath} \times \hat{k} \\
&\quad + A_zB_x\hat{k} \times \hat{\imath} + A_zB_y\hat{k} \times \hat{\jmath} + A_zB_z\hat{k} \times \hat{k} \\
&= \quad A_xB_x\vec{0} + A_xB_y\hat{k} - A_xB_z\hat{\jmath} \\
&\quad - A_yB_x\hat{k} + A_yB_y\vec{0} + A_yB_z\hat{\imath} \\
&\quad + A_zB_x\hat{\jmath} - A_zB_y\hat{\imath} + A_zB_z\vec{0}
\end{aligned}$$

$$\begin{aligned}
\text{So} \qquad \vec{A} \times \vec{B} = \; & [A_yB_z - A_zB_y]\hat{\imath} && (2.20) \\
& + [A_zB_x - A_xB_z]\hat{\jmath} && (2.21) \\
& + [A_xB_y - A_yB_x]\hat{k} && (2.22)
\end{aligned}$$

from which you can pick out the familiar xyz components of the cross product. As hoped, we have derived the component formula for the cross product from it's geometric definition.

[The other way around. On the other hand, if we are *given* the component formula we can (almost) verify that it corresponds to the geometric definition: Use the component formulas for the magnitude, dot product and cross product (2.1,2.3& 2.19) and tediously evaluate $|\vec{A} \times \vec{B}|^2$ and note that it is equal to $|\vec{A}|^2|\vec{B}|^2 - |\vec{A} \cdot \vec{B}|^2$.. Because we already know that $|\vec{A} \cdot \vec{B}| = |\vec{A}||\vec{B}|\cos\theta$ and that $\sin^2\theta = 1 - \cos^2\theta$ this gives $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}||\sin\theta|$. To show that $\vec{A} \times \vec{B}$ is orthogonal to $\vec{A}$ and $\vec{B}$ use the component formulas to show that $(\vec{A} \times \vec{B}) \cdot \vec{A} = 0$ and $(\vec{A} \times \vec{B}) \cdot \vec{B} = 0$. That the right hand rule is satisfied is a final tricky point.]

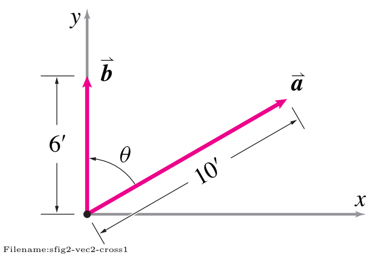


Figure 2.45:

SAMPLE 2.17 Cross product in 2-D: Two vectors $\vec{a}$ and $\vec{b}$ of length 10 ft and 6 ft, respectively, are shown in the figure. The angle between the two vectors is $\theta = 60^\circ$. Find the cross product of the two vectors.

Solution Both vectors $\vec{a}$ and $\vec{b}$ are in the xy plane. Therefore, their cross product is,

$$\begin{aligned}\vec{a}\times\vec{b} &= |\vec{a}||\vec{b}|\sin\theta\,\hat{n}\\ &= (10\,\text{ft})\cdot(6\,\text{ft})\cdot\sin 60^\circ\hat{k}\\ &= 60\,\text{ft}^2\cdot\frac{\sqrt{3}}{2}\hat{k}\\ &= 30\sqrt{3}\,\text{ft}^2\hat{k}.\end{aligned}$$

$$\boxed{\vec{a}\times\vec{b} = 30\sqrt{3}\,\text{ft}^2\hat{k}}$$

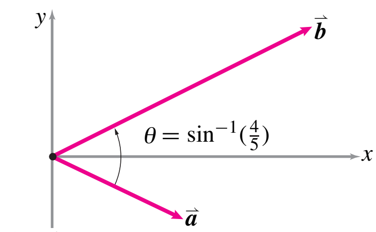


Figure 2.46:

SAMPLE 2.18 Computing 2-D cross product in different ways: The two vectors shown in the figure are $\vec{a} = 2\hat{\imath} - \hat{\jmath}$ and $\vec{b} = 4\hat{\imath} + 2\hat{\jmath}$. The angle between the two vectors turns out to be $\theta = \sin^{-1}(4/5)$. Find the cross product of the two vectors

1. using the angle θ, and
2. using the components of the vectors.

Solution

1. Cross product using the angle θ:

$$\begin{aligned}\vec{a}\times\vec{b} &= |\vec{a}||\vec{b}|\sin\theta\,\hat{n}\\ &= |2\hat{\imath}-\hat{\jmath}||4\hat{\imath}+2\hat{\jmath}|\cdot\sin(\sin^{-1}\frac{4}{5})\hat{k}\\ &= (\sqrt{2^2+1^2})(\sqrt{4^2+2^2})\cdot\frac{4}{5}\hat{k}\\ &= \sqrt{5}\cdot\sqrt{20}\cdot\frac{4}{5}\hat{k} = 10\cdot\frac{4}{5}\hat{k}\\ &= 8\hat{k}.\end{aligned}$$

2. Cross product using components:

$$\begin{aligned}\vec{a}\times\vec{b} &= (2\hat{\imath}-\hat{\jmath})\times(4\hat{\imath}+2\hat{\jmath})\\ &= 2\hat{\imath}\times(4\hat{\imath}+2\hat{\jmath}) - \hat{\jmath}\times(4\hat{\imath}+2\hat{\jmath})\\ &= 8\underbrace{\hat{\imath}\times\hat{\imath}}_{\vec{0}} + 4\underbrace{\hat{\imath}\times\hat{\jmath}}_{\hat{k}} - 4\underbrace{\hat{\jmath}\times\hat{\imath}}_{-\hat{k}} - 2\underbrace{\hat{\jmath}\times\hat{\jmath}}_{\vec{0}}\\ &= 4\hat{k}+4\hat{k}\\ &= 8\hat{k}.\end{aligned}$$

The answers obtained from the two methods are, of course, the same as they must be.

$$\boxed{\vec{a}\times\vec{b} = 8\hat{k}}$$

SAMPLE 2.19 Finding the minimum distance from a point to a line: A straight line passes through two points, A (-1,4) and B (2,2), in the xy plane. Find the shortest distance from the origin to the line.

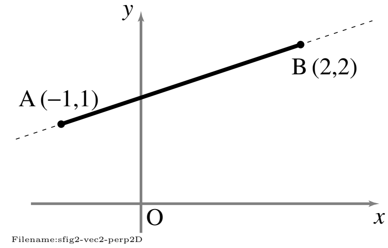


Figure 2.47:

Solution Let $\hat{\boldsymbol{\lambda}}_{AB}$ be a unit vector along line AB. Then,

$$\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B = \underbrace{|\hat{\boldsymbol{\lambda}}_{AB}|}_{1} |\vec{\boldsymbol{r}}_B| \sin\theta \hat{\boldsymbol{n}} = |\vec{\boldsymbol{r}}_B| \sin\theta \hat{\boldsymbol{k}}.$$

Now $|\vec{\boldsymbol{r}}_B| \sin\theta$ is the component of $\vec{\boldsymbol{r}}_B$ that is perpendicular to $\hat{\boldsymbol{\lambda}}_{AB}$ or line AB, *i.e.,* it is the perpendicular, and hence the shortest, distance from the origin (the root of vector $\vec{\boldsymbol{r}}_B$) to the line AB. Thus, the shortest distance d from the origin to the line AB is computed from,

$$\begin{aligned} d &= |\hat{\boldsymbol{\lambda}}_{AB} \times \vec{\boldsymbol{r}}_B| \\ &= \left| (\frac{3\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{\sqrt{3^2+1^2}}) \times (2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}) \right| \\ &= \left| \frac{6}{\sqrt{10}}\hat{\boldsymbol{k}} - \frac{2}{\sqrt{10}}\hat{\boldsymbol{k}} \right| \\ &= \left| \frac{4}{\sqrt{10}}\hat{\boldsymbol{k}} \right| = \frac{4}{\sqrt{10}}. \end{aligned}$$

$$\boxed{d = 4/\sqrt{10}}$$

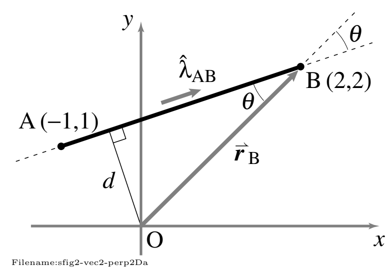


Figure 2.48:

Comments: In this calculation, $\vec{\boldsymbol{r}}_B$ is a vector from the origin to an arbitrary point on line AB. You can take any convenient vector. Since the shortest distance is unique, any such vector will give you the same answer. In fact, you can check your answer by selecting another vector (*e.g.,* $\vec{\boldsymbol{r}}_A$) and repeating the calculations.

SAMPLE 2.20 **Computing cross product in 3-D:** Compute $\vec{a} \times \vec{b}$, where $\vec{a} = \hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\vec{b} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}$.

Solution The calculation of a cross product between two 3-D vectors can be carried out by either using a determinant or the distributive rule. Usually, if the vectors involved have just one or two components, it is easier to use the distributive rule. We show you both methods here and encourage you to learn both. We are given two vectors:

$$\begin{aligned} \vec{a} &= a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k} = \hat{\imath} + \hat{\jmath} - 2\hat{k}, \\ \vec{b} &= b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k} = 3\hat{\imath} + -4\hat{\jmath} + \hat{k}. \end{aligned}$$

- **Calculation using the determinant formula:** In this method, we first write a 3×3 matrix whose first row has the basis vectors as its elements, the second row has the components of the first vector as its elements, and the third row has the components of the second vector as its elements. Thus,

$$\begin{aligned} \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{\imath}(1-8) + \hat{\jmath}(-6-1) + \hat{k}(-4-3) \\ &= -7(\hat{\imath} + \hat{\jmath} + \hat{k}). \end{aligned}$$

- **Calculation using the distributive rule:** In this method, we carry out the cross product by distributing the cross product properly over the three basis vectors. The steps involved are shown below.

$$\begin{aligned} \vec{a} \times \vec{b} &= (a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}) \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \\ &= a_1\hat{\imath} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) + \\ &\quad\; a_2\hat{\jmath} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) + \\ &\quad\; a_3\hat{k} \times (b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}) \\ &= a_1b_1(\overbrace{\hat{\imath} \times \hat{\imath}}^{\mathbf{0}}) + a_1b_2(\overbrace{\hat{\imath} \times \hat{\jmath}}^{\hat{k}}) + a_1b_3(\overbrace{\hat{\imath} \times \hat{k}}^{-\hat{\jmath}}) + \\ &\quad\; a_2b_1(\overbrace{\hat{\jmath} \times \hat{\imath}}^{-\hat{k}}) + a_2b_2(\overbrace{\hat{\jmath} \times \hat{\jmath}}^{\mathbf{0}}) + a_2b_3(\overbrace{\hat{\jmath} \times \hat{k}}^{\hat{\imath}}) + \\ &\quad\; a_3b_1(\overbrace{\hat{k} \times \hat{\imath}}^{\hat{\jmath}}) + a_3b_2(\overbrace{\hat{k} \times \hat{\jmath}}^{-\hat{\imath}}) + a_3b_3(\overbrace{\hat{k} \times \hat{k}}^{\mathbf{0}}) \\ &= \hat{\imath}(a_2b_3 - a_3b_2) + \hat{\jmath}(a_3b_1 - a_1b_3) + \hat{k}(a_1b_2 - b_1a_2) \\ &= \hat{\imath}(1-8) + \hat{\jmath}(-6-1) + \hat{k}(-4-3) \\ &= -7(\hat{\imath} + \hat{\jmath} + \hat{k}) \end{aligned}$$

which, of course, is the same result as obtained above using the determinant. Making a sketch such as Fig. 2.49 is helpful while calculating cross products this way. The product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, *e.g.,* $\hat{\imath} \times \hat{\jmath} = \hat{k}$ but $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.

$$\boxed{\vec{a} \times \vec{b} = -7(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

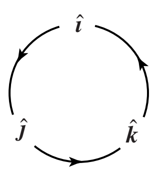


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Figure 2.49: The cross product of any two basis vectors is positive in the direction of the arrow and negative if carried out backwards, e.g. $\hat{\imath} \times \hat{\jmath} = \hat{k}$ but $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.

SAMPLE 2.21 Finding a vector normal to two given vectors: Find a unit vector perpendicular to the vectors $\vec{r}_A = \hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{r}_b = 3\hat{\jmath} + 2\hat{k}$.

Solution The cross product between two vectors gives a vector perpendicular to the plane formed by the two vectors. The sense of direction is determined by the right hand rule.

Let $\vec{N} = N\hat{\lambda}$ be the perpendicular vector.

$$\begin{aligned} \vec{N} &= \vec{r}_A \times \vec{r}_B \\ &= (\hat{\imath} - 2\hat{\jmath} + \hat{k}) \times (3\hat{\jmath} + 2\hat{k}) \\ &= -7\hat{\imath} - 2\hat{\jmath} + 3\hat{k}. \end{aligned}$$

This calculation can be done in either of the two ways shown in the previous sample.

Therefore,

$$\begin{aligned} \hat{\lambda} &= \frac{\vec{N}}{N} \\ &= \frac{-7\hat{\imath} - 2\hat{\jmath} + 3\hat{k}}{\sqrt{7^2 + 2^2 + 3^2}} \\ &= -0.89\hat{\imath} - 0.25\hat{\jmath} + 0.38\hat{k} \end{aligned}$$

$$\boxed{\hat{\lambda} = -0.89\hat{\imath} - 0.25\hat{\jmath} + 0.38\hat{k}}$$

Check:

- $|\hat{\lambda}| = (0.89)^2 + (0.25)^2 + (0.38)^2 \stackrel{\checkmark}{=} 1.$ (it is a unit vector)
- $\hat{\lambda} \cdot \vec{r}_A = 1(-0.89) - 2(-0.25) + 1(0.38) \stackrel{\checkmark}{=} 0.$ $(\hat{\lambda} \perp \vec{r}_A)$.
- $\hat{\lambda} \cdot \vec{r}_B = 3(-0.25) + 2(0.38) \stackrel{\checkmark}{=} 0.$ $(\hat{\lambda} \perp \vec{r}_B)$.

Comments: If $\hat{\lambda}$ is perpendicular to $\vec{r}_{\mathrm{A}}$ and $\vec{r}_{\mathrm{B}}$, then so is $-\hat{\lambda}$. The perpendicularity does not change by changing the *sense* of direction (from positive to negative) of the vector. In fact, if $\hat{\lambda}$ is perpendicular to a vector $\vec{r}$ then any scalar multiple of $\hat{\lambda}$, *i.e.,* $\alpha\hat{\lambda}$, is also perpendicular to $\vec{r}$. This follows because

$$\alpha\hat{\lambda} \cdot \vec{r} = \alpha(\hat{\lambda} \cdot \vec{r}) = \alpha(0) = 0.$$

The case of $-\hat{\lambda}$ is just a particular instance of this rule with $\alpha = -1$.

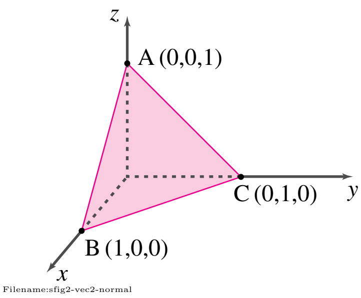


Figure 2.50:

SAMPLE 2.22 Finding a vector normal to a plane: Find a unit vector normal to the plane ABC shown in the figure.

Solution A vector normal to the plane ABC would be normal to any vector in that plane. In particular, if we take any two vectors, say $\vec{r}_{AB}$ and $\vec{r}_{AC}$, the normal to the plane would be perpendicular to both $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Since the cross product of two vectors gives a vector perpendicular to both vectors, we can find the desired normal vector by taking the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Thus,

$$\begin{aligned}
\vec{N} &= \vec{r}_{AB} \times \vec{r}_{AC} \\
&= (\hat{\imath} - \hat{k}) \times (\hat{\jmath} - \hat{k}) \\
&= \underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{k}} - \underbrace{\hat{\imath} \times \hat{k}}_{-\hat{\jmath}} - \underbrace{\hat{k} \times \hat{\jmath}}_{-\hat{\imath}} + \underbrace{\hat{k} \times \hat{k}}_{\vec{0}} \\
&= \hat{\imath} + \hat{\jmath} + \hat{k} \\
\Rightarrow \quad \hat{n} &= \frac{\vec{N}}{|\vec{N}|} \\
&= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}).
\end{aligned}$$

$$\boxed{\hat{n} = \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})}$$

Check: Now let us check if $\hat{n}$ is normal to any vector in the plane ABC. It is fairly easy to show that $\hat{n} \cdot \vec{r}_{AB} = \hat{n} \cdot \vec{r}_{AC} = 0$. This is not a surprise because we found $\hat{n}$ from the cross product of $\vec{r}_{AB}$ and $\vec{r}_{AC}$. Let us check if $\hat{n}$ is normal to $\vec{r}_{BC}$:

$$\begin{aligned}
\hat{n} \cdot \vec{r}_{BC} &= \frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (-\hat{\imath} + \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-\hat{\imath} \cdot \hat{\imath} + \hat{\jmath} \cdot \hat{\jmath}) \\
&= \frac{1}{\sqrt{3}}(-1 + 1) \stackrel{\checkmark}{=} 0.
\end{aligned}$$

SAMPLE 2.23 The shortest distance between two lines: Two lines, AB and CD, in 3-D space are defined by four specified points, A(0,2 m,1 m), B(2 m,1 m,3 m), C(-1 m,0,0), and D(2 m,2 m,2 m) as shown in the figure. Find the shortest distance between the two lines.

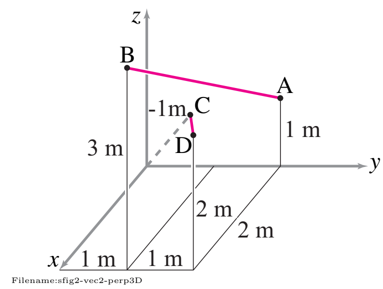


Figure 2.51:

Solution The shortest distance between any pair of lines is the length of the line that is perpendicular to both the lines. We can find the shortest distance in three steps:

1. First find a vector that is perpendicular to both the lines. This is easy. Take two vectors $\vec{r}_1$ and $\vec{r}_2$, one along each of the two given lines. Take the cross product of the two unit vectors and make the resulting vector a unit vector $\hat{\boldsymbol{n}}$.
2. Find a vector parallel to $\hat{\boldsymbol{n}}$ that connects the two lines. This is a little tricky. We don't know where to start on any of the two lines. However, we can take any vector from one line to the other and then, take its component along $\hat{\boldsymbol{n}}$.
3. Find the length (magnitude) of the vector just found (in the direction of $\hat{\boldsymbol{n}}$). This is simply the component we find in step (b) devoid of its sign.

Now let us carry out these steps on the given problem.

1. Step-1: Find a unit vector $\hat{\boldsymbol{n}}$ that is perpendicular to both the lines.

$$\begin{aligned}
\vec{r}_{AB} &= 2\,\text{m}\hat{\imath} - 1\,\text{m}\hat{\jmath} + 2\,\text{m}\hat{\boldsymbol{k}} \\
\vec{r}_{CD} &= 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath} + 2\,\text{m}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \vec{r}_{AB} \times \vec{r}_{CD} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ 2 & -1 & 2 \\ 3 & 2 & 2 \end{vmatrix} \text{m}^2 \\
&= \hat{\imath}(-2-4)\,\text{m}^2 + \hat{\jmath}(6-4)\,\text{m}^2 + \hat{\boldsymbol{k}}(4+3)\,\text{m}^2 \\
&= (-6\hat{\imath} + 2\hat{\jmath} + 7\hat{\boldsymbol{k}})\,\text{m}^2
\end{aligned}$$

Therefore,

$$\begin{aligned}
\hat{\boldsymbol{n}} &= \frac{\vec{r}_{AB} \times \vec{r}_{CD}}{|\vec{r}_{AB} \times \vec{r}_{CD}|} \\
&= \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{\boldsymbol{k}}).
\end{aligned}$$

2. Step-2: Find any vector from one line to the other line and find its component along $\hat{\boldsymbol{n}}$.

$$\begin{aligned}
\vec{r}_{AC} &= -1\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath} - 1\,\text{m}\hat{\boldsymbol{k}} \\
\vec{r}_{AC} \cdot \hat{\boldsymbol{n}} &= -(\hat{\imath} + 2\hat{\jmath} + \hat{\boldsymbol{k}})\,\text{m} \cdot \frac{1}{\sqrt{89}}(-6\hat{\imath} + 2\hat{\jmath} + 7\hat{\boldsymbol{k}}) \\
&= \frac{1}{\sqrt{89}}(6-4-7)\,\text{m} = -\frac{5}{\sqrt{89}}\,\text{m}.
\end{aligned}$$

3. Step-3: Find the required distance d by taking the magnitude of the component along $\hat{\boldsymbol{n}}$.

$$d = |\vec{r}_{AC} \cdot \hat{\boldsymbol{n}}| = \left| -\frac{5}{\sqrt{89}}\,\text{m} \right| = 0.53\,\text{m}$$

$$\boxed{d = 0.53\,\text{m}}$$

SAMPLE 2.24 The mixed triple product: Calculate the mixed triple product $\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}})$ for $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$, $\vec{\boldsymbol{a}} = 3\hat{\imath}$, and $\vec{\boldsymbol{b}} = \hat{\imath} + \hat{\jmath} + 3\hat{\boldsymbol{k}}$.

Solution We compute the given mixed triple product in two ways here:

- Method-1: Straight calculation using cross product and dot product.

$$\begin{aligned}
\text{Let} \quad \vec{\boldsymbol{c}} &= \vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}} \\
&= (3\hat{\imath}) \times (\hat{\imath} + \hat{\jmath} + 3\hat{\boldsymbol{k}}) \\
&= 3\underbrace{\hat{\imath} \times \hat{\imath}}_{\vec{\mathbf{0}}} + 3\underbrace{\hat{\imath} \times \hat{\jmath}}_{\hat{\boldsymbol{k}}} + 9\underbrace{\hat{\imath} \times \hat{\boldsymbol{k}}}_{-\hat{\jmath}} = -9\hat{\jmath} + 3\hat{\boldsymbol{k}} \\
\text{So,} \quad \hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) &= \hat{\boldsymbol{\lambda}} \cdot \vec{\boldsymbol{c}} \\
&= \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) \cdot (-9\hat{\jmath} + 3\hat{\boldsymbol{k}}) = -\frac{9}{\sqrt{2}}.
\end{aligned}$$

- Method-2: Using the determinant formula for mixed product.

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) &= \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 3 & 0 & 0 \\ 1 & 1 & 3 \end{vmatrix} \\
&= \frac{1}{\sqrt{2}}(0-0) + \frac{1}{\sqrt{2}}(0-9) + 0 = -\frac{9}{\sqrt{2}}.
\end{aligned}$$

$$\hat{\boldsymbol{\lambda}} \cdot (\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) = -\frac{9}{\sqrt{2}}$$

2.4 Moment and moment about an axis

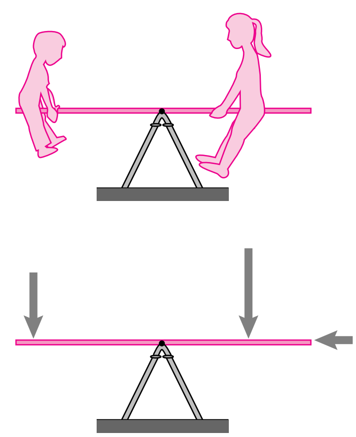
When you try to move something you can push it and you can turn it. In mechanics, the measure of your pushing is the net force you apply. The measure of your turning is the net *moment*, also sometimes called the net *torque* or net *couple*.

Although concepts involving moment (and rotation) are often harder for beginners than force (and translation), they were understood first. The ancient principle of the lever, which can be viewed as the root of all mechanics, is the basic idea incorporated by moments.

Ultimately you can take on faith the vector definition of moment (given opposite the inside cover) and its role in Moment and angular momentum balance (eqs. II). In this section we will define the moment of a force intuitively, geometrically, and finally using vector algebra. We will do this first in 2 dimensions and then in 3. The main mathematical tool here is the vector cross product from sec. 2.3. To start with, however, we can more or less deduce the concept of moment by generalizing from common experience.

Teeter totter mechanics

The two people weighing down on the teeter totter in fig. 2.52 tend to rotate it about its hinge, the right one clockwise and the left one counterclockwise. We will now cook up a measure of the tendency of each force to cause rotation about the hinge and call it *the moment of the force about the hinge*.



(not a free body diagram)

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Figure 2.52: On a balanced teeter totter the bigger person gets the short end of the stick. A sideways force directed towards the hinge has no effect on the balance.

The poet Robert Burns suggested using a Teeter Totter to weigh hogs like this: 1) Get a perfetly symmetric plank and balance it accross a saw horse, 2) Put the hog on one end of the plank, 3) pile rocks on the other end of the plank until the plank is again perfectly balanced, 4) Carefully guess the weight of the rocks.

As is verified a million times a year by young future engineering students,

- To balance a teeter-totter the smaller person needs to be further from the hinge.
- If two people are on one side then the teeter totter is balanced by two similar people an equal distance from the hinge on the other side.
- Two people can balance one similar person by scooting twice as close to the hinge.

These proportionalities generalize to this:

> The tendency of a force to cause teeter totter rotation is proportional to the size of the force and to its distance from the hinge (for forces perpendicular to the teeter totter).

Further, if someone standing nearby adds a force that is directed towards the hinge it causes no tendency to rotate. We can deduce more. Because

- any force can be decomposed into a sum of forces, one perpendicular to the teeter totter and the other towards the hinge, and because
- we assume that the affect of the sum of these forces is the sum of the affects of each separately, and because
- the force towards the hinge has no tendency to rotate,

we can conclude that:

> The moment of a force about a hinge is the product of its distance from the hinge and the component of the force perpendicular to the line from the hinge to the force.

Here then is the formula for 2D *moment about C* or *moment with respect to C*.①

$$M_{/C} = |\vec{r}|\,(|\vec{F}|\sin\theta) = (|\vec{r}|\sin\theta)\,|\vec{F}|. \tag{2.23}$$

①The '/' in the subscript of $\vec{M}$ reads as 'relative to' or 'about'. For simplicity we often leave the / out and just write $\vec{M}_{\mathrm{C}}$.

Here, θ is the angle between $\vec{r}$ (the position of the point of force application relative to the hinge) and $\vec{F}$ (see fig. 2.53). This formula for moment has all the teeter totter deduced properties. Moment is proportional to r, and to the part of $\vec{F}$ that is perpendicular to $\vec{r}$. The re-grouping as $(|\vec{r}|\sin\theta)$ shows that a force $\vec{F}$ has the same effect if it is applied at a new location that is displaced in the direction of $\vec{F}$. That is, the force $\vec{F}$ can slide along its length without changing its $M_{/C}$ and is *equivalent* in its effect on the teeter totter. The quantity $|\vec{r}|\sin\theta$ is sometimes called the *lever arm* of the force.

By common convention we define as positive a moment that causes a counterclockwise rotation. A moment that causes a clockwise rotation is negative. If we define θ appropriately then eqn. (2.23) obeys this sign convention. We define θ as the angle from the positive vector $\vec{r}$ to the positive vector $\vec{F}$ measured counterclockwise. Point the thumb of your right hand towards yourself. Point the fingers of your right hand along $\vec{r}$ and curl them towards the direction of $\vec{F}$ and see how far you have to rotate them. The force caused by the person on the left of the teeter totter has $\theta = 90^\circ$ so $\sin\theta = 1$ and the formula 2.23 gives a positive counterclockwise M. The force of the person on the right has $\theta = 270^\circ$ (3/4 of a revolution) so $\sin\theta = -1$ and the formula 2.23 gives a negative M.

In two dimensions moment is really a scalar concept, it is either positive or negative. In three dimensions moment is a vector. But even in 2D we find it easier to keep track of signs if we treat moment as a vector. In the xy plane, the 2D moment is a vector in the $\hat{k}$ direction (straight out of the plane). So eqn. 2.23 becomes

$$\vec{M}_{/\mathrm{C}} = |\vec{r}|\,|\vec{F}|\sin\theta\,\hat{k}. \tag{2.24}$$

If you curl the fingers of your *right hand* in the direction of rotation caused by a force your thumb points in the direction of the moment vector.

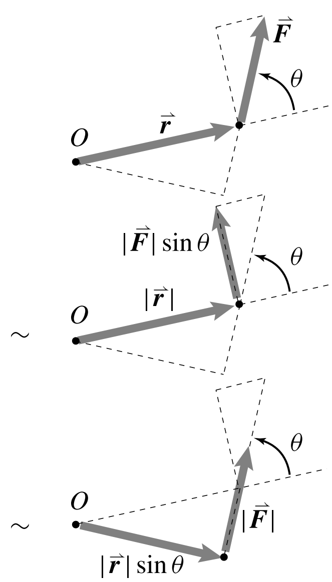

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Figure 2.53: The moment of a force $|\vec{r}||\vec{F}|\sin\theta$ is either the product of its distance $|\vec{r}|$ with its perpendicular component $|\vec{F}|\sin\theta$ or of its lever arm (perpendicular distance) $|\vec{r}|\sin\theta$ and the full force $|\vec{F}|$. The $\sim$ indicates that the lower two forces and positions have the same moment as the force at top.

2D moment by components

We can use the component form of the 2D cross product to find a component form for the moment $\vec{M}_{/\mathrm{C}}$ of eqn. 2.24. Given $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ acting at P, where $\vec{r}_{\mathrm{P/C}} = r_x\hat{\imath} + r_y\hat{\jmath}$, the moment of the force about C is

$$\vec{M}_{/\mathrm{C}} = (r_xF_y - r_yF_x))\hat{k}$$

or the moment of $\vec{F}$ about the axis at C is

$$M_C = r_x F_y - r_y F_x. \tag{2.25}$$

We can derive this component formula with the sequence of vector manipulations shown graphically in fig. 2.54.

3D moment about an axis

The concept of moment about an axis is historically, theoretically, and practically important. Moment about an axis describes the principle of the lever, which far precedes Newton's laws. The net moment of a force system about enough different axes determines everything needed in mechanics about a force system. And one can sometimes quickly solve a statics or dynamics problem by considering moment about a judiciously chosen axis.

Lets start by thinking about a teeter totter again. Looking from the side we thought of a teeter totter as a 2D system. But the teeter totter really lives in the 3D world (see fig. 2.55). We now re-interpret the 2D moment M as the moment of the 2D forces about the $\hat{\boldsymbol{k}}$ axis of rotation at the hinge. It is plain that a force $\vec{\boldsymbol{F}}^{\parallel}$ pushing a teeter totter parallel to the axle causes no tendency to rotate. And we already agreed that a radial force $\vec{\boldsymbol{F}}^{r}$ causes no rotation. So we see that the moment a force causes about an axis is the distance of the force from the axis times the part of the force that is neither parallel to the axis nor directed towards the axis.

Now look at this in the more 3-dimensional context of fig. 2.56. Here an imagined axis of rotation is defined as the line through C that is in the $\hat{\boldsymbol{\lambda}}$ direction. A force $\vec{\boldsymbol{F}}$ is applied at P. We can break $\vec{\boldsymbol{F}}$ into a sum of three vectors

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}^{\parallel} + \vec{\boldsymbol{F}}^{r} + \vec{\boldsymbol{F}}^{\perp}$$

where $\vec{\boldsymbol{F}}^{\parallel}$ is parallel to the axis, $\vec{\boldsymbol{F}}^{r}$ is directed along the shortest connection between the axis and P (and is thus perpendicular to the axis) and $\vec{\boldsymbol{F}}^{\perp}$ is out of the plane defined by C, P and $\hat{\boldsymbol{\lambda}}$. By analogy with the teeter totter we see that $\vec{\boldsymbol{F}}^{r}$ and $\vec{\boldsymbol{F}}^{\parallel}$ cause no tendency to rotate about the axis. So only the $\vec{\boldsymbol{F}}^{\perp}$ contributes.

Example: Try this. Stand facing a partially open door with the front of your body parallel to the plane of the door (a door with no springs is best). Hold the outer edge of the door with one hand. Press down (F_1) and note that the door is not opened or closed. Push towards the hinge (F_2) and note that the door is not opened or closed. Push and pull away and towards your body (F_3) and note how easily you cause the door to rotate. Thus the only force component that tends to rotate the door is perpendicular to the plane of the door (which is the plane of the hinge and line from the hinge to your hand). Now move your hand to the middle of the door, half the distance from the hinge. Note that it takes more force (F_4) to rotate the door with the same authority (push with your pinky if you have trouble feeling the difference).

Thus the only potent force for rotation is perpendicular to the plane defined by the hinge and point of force application. The potence of the force is increased with distance from the hinge.

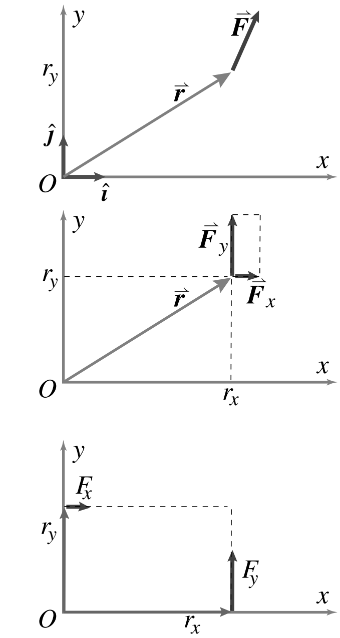

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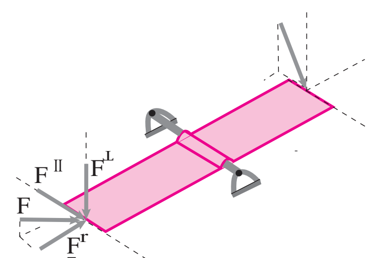
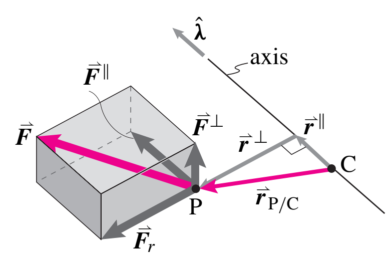
Figure 2.54: The component form of the 2D moment can be found by sequentially breaking the force into components, sliding each component along its line of action to the x and y axis, and adding the moments of the two components.



Filename:tfigure-3Dteeter

Figure 2.55: Teeter totter with applied forces broken into components parallel to the axis $\vec{\boldsymbol{F}}^{\parallel}$, radial $\vec{\boldsymbol{F}}^{r}$, and perpendicular to the plane containing the axis and the point of force application $\vec{\boldsymbol{F}}^{\perp}$.



Filename:tfigure2-mom-axis

Figure 2.56: Moment about an axis

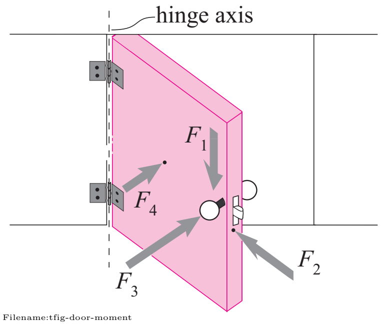


Figure 2.57: Force F_3 has moment about the door's hinge axis. But not forces F_1 and F_2 do not. Force F_4, with less moment arm than F_3, rotates the door with less authority.

We can also decompose $\vec{r} = \vec{r}_{P/C}$ into two parts, one parallel to the hinge and one radial, as

$$\vec{r} = \vec{r}^{\parallel} + \vec{r}^{r}.$$

Clearly $\vec{r}^{\parallel}$ has no affect on how much rotation $\vec{F}$ causes about the axis. If for example the point of force application was moved parallel to the axis a few centimeters, the tendency to rotate would not be changed. Altogether, we have that the moment of the force $\vec{F}$ about the axis $\hat{\lambda}$ through C is given by

$$M_{\lambda C} = r^r \ F^{\perp}.$$

The perpendicular distance from the axis to the point of force application is $|\vec{r}^{r}|$ and $\vec{F}^{\perp}$ is the part of the force that causes right-handed rotation about the axis. A moment about an axis is defined as positive if curling the fingers of your right hand gives the sense of rotation when your outstretched thumb is pointing along the axis (as in fig. 2.56). The force of the left person on the teeter totter causes a positive moment about the $\hat{k}$ axis through the hinge.

So long as you interpret the quantities correctly, the freshman physics line

"Moment is distance ($|\vec{r}^{r}|$) times force ($|\vec{F}^{\perp}|$)"

perfectly defines moment about an axis.

Three dimensional geometry is difficult, so a formula for moment about an axis in terms of components would be most useful. The needed formula depends on the 3D moment vector defined by the 3D cross product which we introduce now.

The moment vector

We now define the moment of a force $\vec{F}$ applied at P, relative to point C as

$$\vec{M}_C = \vec{r}_{P/C} \times \vec{F} \tag{2.26}$$

which we read as 'M is r cross F.' The moment vector is hard to intuit. A look at its components is helpful.

$$\vec{M}_C = (r_y F_z - r_z F_y)\hat{\imath} + (r_z F_x - r_x F_z)\hat{\jmath} + (r_x F_y - r_y F_x)\hat{k}$$

You can recognize the z component of the moment vector as the moment of the force about the $\hat{k}$ axis through C (eqn. 2.25). Similarly the x and y components of $\vec{M}_C$ are the moments about the $\hat{\imath}$ and $\hat{\jmath}$ axis through C. So at least the components of $\vec{M}_C$ have intuitive meaning. They are the moments around the positive x, y, and z axes respectively.

Starting with this moment-about-the-coordinate-axes interpretation of the moment vector, each of the three components can be deduced graphically by the moves shown in fig. 2.58. The force is first broken into components. The components are then moved along their lines of action to the coordinate planes. From the resulting picture you can see, say, that the moment about the positive y axis gets a positive contribution from F_x with lever arm r_z and a negative contribution from F_z with lever arm r_x. Thus the y component of $\vec{M}$ is $r_z F_x - r_x F_z$.

Maximum property. Finally, the moment of a force about C is a vector whose magnitude is the product of the distance of the force from C and the magnitude of the force. The direction is the orientation of the axis about which the force has the greatest moment.

More on moment about an axis

We defined moment about an axis geometrically using fig. 2.56 on page 85 as $M_{\hat{\boldsymbol{\lambda}}} = r^r F^{\perp}$. We can now verify that the mixed triple product gives the desired result by guessing the formula and seeing that it agrees with the geometric definition.

$$M_{\lambda \mathrm{C}} = \hat{\boldsymbol{\lambda}} \cdot \vec{M}_{/\mathrm{C}} \quad \text{(An inspired guess...)} \tag{2.27}$$

We break both $\vec{r}$ and $\vec{F}$ into sums indicated in the figure, use the distributive law, and note that the mixed triple product gives zero if any two of the vectors are parallel. Thus,

$$\begin{aligned}
\hat{\boldsymbol{\lambda}} \cdot \vec{M}_{/\mathrm{C}} &= \hat{\boldsymbol{\lambda}} \cdot \vec{r}_{\mathrm{P/C}} \times \vec{F} \\
&= \hat{\boldsymbol{\lambda}} \cdot (\vec{r}^{\,r} + \vec{r}^{\,\parallel}) \times (\vec{F}^{\perp} + \vec{F}^{\parallel} + \vec{F}^{r}) \\
&= \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,r} \times \vec{F}^{\perp} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,r} \times \vec{F}^{\parallel} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,r} \times \vec{F}^{r} \ldots \\
&\quad + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,\parallel} \times \vec{F}^{\perp} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,\parallel} \times \vec{F}^{\parallel} + \hat{\boldsymbol{\lambda}} \cdot \vec{r}^{\,\parallel} \times \vec{F}^{r} \\
&= r^r F^{\perp} + 0 + 0 + 0 + 0 + 0 \\
&= r^r F^{\perp}. \quad \text{(... and a good guess too.)}
\end{aligned}$$

We can calculate the cross and dot product in any convenient way, say using vector components.

Example: Moment about an axis
Given a force, $\vec{F}_1 = (5\hat{\imath} - 3\hat{\jmath} + 4\hat{k})\,\mathrm{N}$ acting at a point P whose position is given by $\vec{r}_{P/O} = (3\hat{\imath} + 2\hat{\jmath} - 2\hat{k})\,\mathrm{m}$, what is the moment about an axis through the origin O with direction $\hat{\boldsymbol{\lambda}} = \frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}$?

$$\begin{aligned}
M_{\hat{\boldsymbol{\lambda}}} &= (\vec{r}_{P/O} \times \vec{F}_1) \cdot \hat{\boldsymbol{\lambda}} \\
&= [(3\hat{\imath} + 2\hat{\jmath} - 2\hat{k})\,\mathrm{m} \times (5\hat{\imath} - 3\hat{\jmath} + 4\hat{k})\,\mathrm{N}] \cdot (\frac{1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}) \\
&= -\frac{17}{\sqrt{2}}\,\mathrm{m\,N}.
\end{aligned}$$

The power of our abstract reasoning is apparent when we consider calculating the moment of a force about an axis with two different coordinate systems. Each of the vectors in eqn. 2.4 will have different components in the different

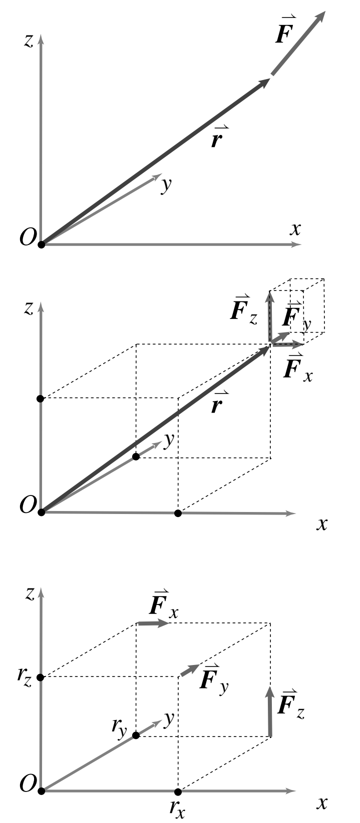


Figure 2.58: The three components of the 3D moment vector are the moments about the three axis. These can be found by sequentially breaking the force into components, sliding each component along its line of action to the coordinate planes, and noting the contribution of each component to moment about each axis. See fig. 2.54 on page 85 for the 2D version of this construction.

systems. Yet the resulting scalar, after all the arithmetic, will be the same no matter what the coordinate system.

Finally, the moment about an axis gives us an interpretation of the moment vector. The direction of the moment vector $\vec{M}_C$ is the direction of the axis through C about which $\vec{F}$ has the greatest moment. The magnitude of $\vec{M}_C$ is the moment of $\vec{F}$ about that axis.

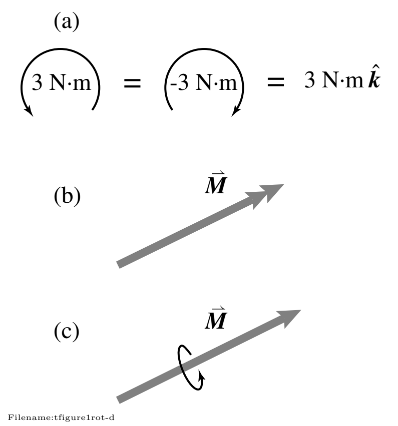


Figure 2.59: Optional drawing methods for moment vectors. (a) shows an arced arrow to represent vectors having to do with rotation in 2 dimensions. Such vectors point directly out of, or into, the page so are indicated with an arc in the direction of the rotation. (b) shows a double-headed arrow for torque or rotation quantity in three dimensions. (c) is another notation for a 3D vector quantity that uses rotations. But a vector is a vector and no special notation is usually used for vectors that concern turning or rotation.

Special optional ways to draw moment vectors

None of the special rotation notations below is needed because moment is a vector like any other. The same is true for the angular velocity vector and the angular momentum vector in dynamics. None-the-less, sometimes people like to use a notation that suggests the rotational nature of these quantities.

Arced arrow for 2-D moment and angular velocity. In 2D problems in the xy plane, the relevant moment, angular velocity, and angular momentum point straight out or into the plane in the z ($\hat{k}$) direction. A way of drawing this is to use an arced arrow. Wrap the fingers of your right hand in the direction of the arc and your thumb points in the direction of the unit vector that the scalar multiplies. The three representations in fig. 2.59a indicate the same moment vector.

Double headed arrow for 3-D rotations and moments. Two other ways of indicating rotation are to use double-headed arrows or to use an arrow with an arced arrow around it as shown in fig. 2.59b.

SAMPLE 2.25 **Moment of a force:** A force $\vec{F} = 1\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath}$ acts at point A of an object pinned at O as shown in the figure. The distance OA = 2 m. Find the moment of the force about the pin at point O.

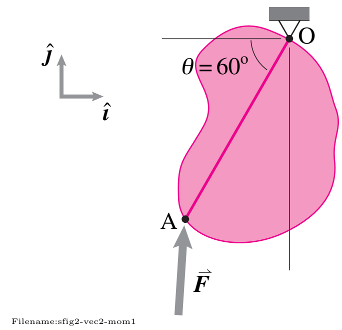


Figure 2.60:

Solution The given force acts through point A on the body. Therefore, we can compute its moment about O as follows.

$$\begin{aligned}
\vec{M}_O &= \vec{r}_{OA} \times \vec{F} \\
&= \underbrace{(-2\,\text{m} \cdot \cos 60^\circ \hat{\imath} - 2\,\text{m} \cdot \sin 60^\circ \hat{\jmath})}_{\vec{r}_{OA}} \times \underbrace{(1\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath})}_{\vec{F}} \\
&= (-1\,\text{m}\hat{\imath} - \sqrt{3}\,\text{m}\hat{\jmath}) \times (1\,\text{N}\hat{\imath} + 20\,\text{N}\hat{\jmath}) \\
&= -20\,\text{N·m}\hat{k} + 1.73\,\text{N·m}\hat{k} \\
&= -18.27\,\text{N·m}\hat{k}.
\end{aligned}$$

$$\boxed{\vec{M}_O = -18.27\,\text{N·m}\hat{k}}$$

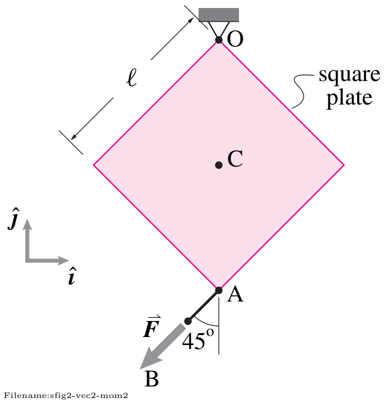


Figure 2.61:

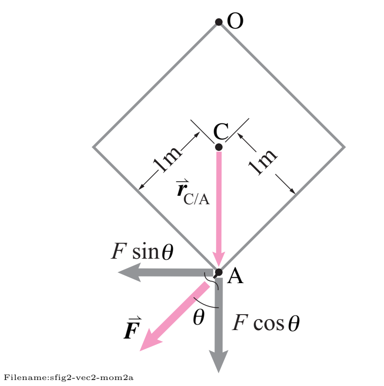


Figure 2.62: The moment $M_{/C} = F\sin\theta \cdot |\vec{r}_{A/C}|$ and acts clockwise.

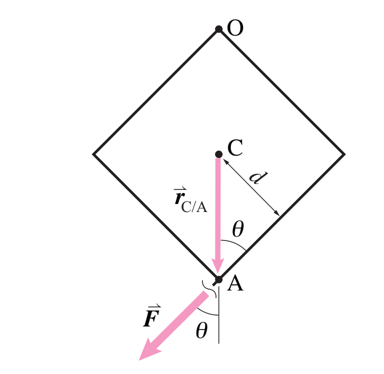


Figure 2.63: The moment $M_{/C} = Fd$ and acts clockwise.

SAMPLE 2.26 A $2\,\text{m}\times 2\,\text{m}$ square plate hangs from one of its corners as shown in the figure. At the diagonally opposite end, a force of 50 N is applied by pulling on the string AB. Find the moment of the applied force about the center C of the plate using

1. The component of the force perpendicular to $\vec{r}_{A/C}$,
2. The lever arm (the perpendicular distance from C to the force vector), and
3. The vectors $\vec{F}$ and $\vec{r}_{A/C}$.

Solution

1. To find the moment point C, we need to find the component of $\vec{F}$ perpendicular to CA. From the figure, we see that the desired component is $F\sin\theta$ where $\theta = 45°$. Therefore,

$$M_{/C} = |\vec{F}|\sin\theta|\vec{r}_{A/C}| = 50\,\text{N}\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{2}\,\text{m} = 50\,\text{N·m}.$$

The direction of this moment is obtained by curling the fingers of the the right hand from $\vec{r}_{A/C}$ towards $\vec{F}$ which points into the page, *i.e.,* $-\hat{k}$ direction. Thus, $\vec{M}_C = -50\,\text{N·m}\hat{k}$.

$$\boxed{\vec{M}_{/C} = -50\,\text{N·m}\hat{k}}$$

2. The lever arm is the perpendicular distance from point C to the line of action of $\vec{F}$. This perpendicular distance is $d = |\vec{r}_{A/C}|\sin\theta = \sqrt{2}\,\text{m}\cdot(1/\sqrt{2}) = 1\,\text{m}$ (see fig. 2.62). Therefore, the moment of $\vec{F}$ about point C is

$$\begin{aligned}\vec{M}_{/C} &= Fd(-\hat{k})\\ &= -(50\,\text{N})\cdot(1\,\text{m})\hat{k} = -50\,\text{N·m}\hat{k}.\end{aligned}$$

where the direction of the moment is evident from the right hand rule as pointed out above.

$$\boxed{\vec{M}_{/C} = -50\,\text{N·m}\hat{k}}$$

3. The moment $\vec{M}_C$ is calculated from $\vec{F}$ and $\vec{r}_{A/C}$ by carrying out the cross product $\vec{r}_{A/C}\times\vec{F}$ in a straightforward manner. For this calculation, we first need to find the vectors $\vec{r}_{A/C}$ and $\vec{F}$:

$$\begin{aligned}\vec{r}_{A/C} &= -CA\,\hat{\jmath} = -\frac{\ell}{\sqrt{2}}\hat{\jmath} \quad (\text{since OA} = 2\,\text{CA} = \sqrt{2}\ell)\\ \vec{F} &= F(-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) = -F(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}).\end{aligned}$$

Hence,

$$\begin{aligned}\vec{M}_{/C} &= \vec{r}_{A/C}\times\vec{F}\\ &= -\frac{\ell}{\sqrt{2}}\hat{\jmath}\times[-F(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})]\\ &= -\frac{\ell}{\sqrt{2}}F\cos\theta\hat{k}\\ &= -\frac{2\,\text{m}}{\sqrt{2}}\cdot 50\,\text{N}\cdot\cos 45°\hat{k} = -50\,\text{N·m}\hat{k}.\end{aligned}$$

$$\vec{M}_{/C} = -50\,\text{N·m}\hat{k}$$

SAMPLE 2.27 Moment about an axis: A vertical force of unknown magnitude F acts at point B of a triangular plate ABC shown in the figure. Find the moment of the force about the edge CA of the plate.

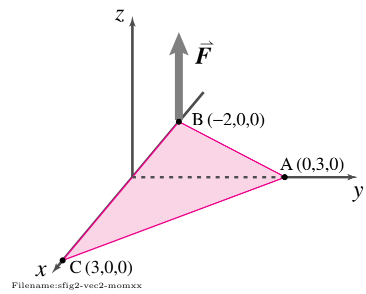


Figure 2.64:

Solution The moment of a force $\vec{F}$ about an axis x-x is given by

$$M_{xx} = \hat{\lambda}_{xx} \cdot (\vec{r} \times \vec{F})$$

where $\hat{\lambda}_{xx}$ is a unit vector along the axis x-x, $\vec{r}$ is a position vector from any point on the axis to the applied force. In this problem, the given axis is CA. Therefore, we can take $\vec{r}$ to be $\vec{r}_{AB}$ or $\vec{r}_{CB}$. Here,

$$\hat{\lambda}_{CA} = \frac{\vec{r}_{CA}}{|\vec{r}_{CA}|} = \frac{3(-\hat{\imath} + \hat{\jmath})}{\sqrt{9+9}} = -\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}.$$

Now, moment about point A is

$$\begin{aligned} \vec{M}_A &= \vec{r}_{AB} \times \vec{F} \\ &= (-2\hat{\imath} - 3\hat{\jmath}) \times F\hat{k} = 2F\hat{\jmath} - 3F\hat{\imath}. \end{aligned}$$

Therefore, the moment about CA is

$$\begin{aligned} M_{CA} &= \hat{\lambda}_{CA} \cdot (\vec{r}_{AB} \times \vec{F}) = \hat{\lambda}_{CA} \cdot \vec{M}_A \\ &= (-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}) \cdot (-3F\hat{\imath} + 2F\hat{\jmath}) \\ &= (\frac{3}{\sqrt{2}} + \frac{2}{\sqrt{2}})F = \frac{5}{\sqrt{2}}F. \end{aligned}$$

$$M_{CA} = \frac{5}{\sqrt{2}}F$$

2.5 Solving vector equations

Often you know less than you like. So you try to figure out more. One way to do this is with reasoning. In mechanics you reason with the laws of mechanics (including geometry and kinematics) to find some things you want to know from others that you already do know. Because many of the laws of mechanics (and geometry) are vector equations,

> Engineering analysis often involves solving vector equations.

How you calculate with vectors is the same whether the problems are in geometry, kinematics, statics, dynamics or a combination of these. In this section we will show a few methods for solving some of the more common vector equations. In a sense there are no new concepts here; its just a matter of guiding the rules of vector algebra that you already know.

Vector algebra

We want to manipulate equations that involve vectors (like $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{0}$) and scalars (like a, b, c, and 0). Without knowing anything about mechanics, or even geometry, you can learn to do correct vector algebra by just following the manipulation rules in box 2.8. These are elaborations of elementary scalar algebra to accommodate vectors and the three new kinds of multiplication (scalar times vector, dot product, and cross product). Just looking out for the exceptions is enough to make your manipulations correct. That's enough to keep the car on the road. And if you just follow the traffic rules you might by chance get to your goal. But to get to your goal efficiently you need to steer correctly. For vector algebra this means using the simplification rules to your advantage.

Example. Say you know $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ and you know that

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

but you don't know a, b, and c. How could you find a? First dot both sides with $\vec{B} \times \vec{C}$ and then blindly follow the rules:

$$\begin{aligned}
\left\{a\vec{A} + b\vec{B} + c\vec{C}\right. &= \left.\vec{D}\right\}\cdot\left(\vec{B}\times\vec{C}\right)\\
a\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) + b\underbrace{\vec{B}\cdot\left(\vec{B}\times\vec{C}\right)}_{0} + c\underbrace{\vec{C}\cdot\left(\vec{B}\times\vec{C}\right)}_{0} &= \vec{D}\cdot\left(\vec{B}\times\vec{C}\right)\\
\Rightarrow \quad a = \frac{\vec{D}\cdot\left(\vec{B}\times\vec{C}\right)}{\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)}. & \qquad (2.28)
\end{aligned}$$

The two zeros followed from the general rules that $\vec{D}\cdot(\vec{V}\times\vec{W}) = (\vec{D}\times\vec{V})\cdot\vec{W}$) and $\vec{D}\times\vec{D} = \vec{0}$. ①.

① The linear-algebra savvy reader may recognize the manipulation leading to eqn. (2.28) as a derivation of Cramer's rule for a 3×3 matrix whose columns are the components of the vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$, respectively.
Note, if the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ are co-planar then the last line of the calculation would have $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = 0$ in the denominator. Bad. If $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar the original problem is either nonsense (if $\vec{D}$ is off that plane) or has non-unique solutions (if $\vec{D}$ is on that plane). So the failure of the derivation in that case is sensible. See box 2.10 on page 107 for more discussion of when equations do and do not have solutions.)

The point of the example above was to show the vector algebra rules at work. However, to get to the end took the first 'move' of dotting the equation with the appropriate vector. That move could be motivated this way. We are trying to find a and not b or c. We can get rid of the terms in the equation that contain b and c if we can dot $\vec{B}$ and $\vec{C}$ with a vector perpendicular to both of them. $\vec{B} \times \vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ so can be used to kill them off with a dot product. The 0s in the example calculation were thus expected for geometric reasons.

2.8 The rules of vector algebra.

Without knowing any geometry or mechanics or even anything about vectors, you can do good vector calculations by just following the rules in this box. On the other hand, even if you know vector geometry and mechanics you are stuck with these rules.

Vector algebra

In the expressions below a, b and c are any scalars. And $\vec{A}, \vec{B}$ and $\vec{C}$ are any vectors.

You can add vectors

$$\vec{A} + \vec{B}$$

And you can multiply them three ways

1. **Scalar multiplication:** $a\vec{A}$.
2. **Dot product, inner product or scalar product:** $\vec{A} \cdot \vec{B}$.
3. **Cross product or vector product:** $\vec{A} \times \vec{B}$.

And you can combine expressions using usual rules you are used to in scalar arithmetic (with some exceptions) and the extra vector simplification rules.

The usual rules.

Vector addition and all three kinds of multiplication (scalar multiplication, dot product, cross product) all follow many of the usual commutative, associative, and distributive laws (usual, meaning that which you know from regular scalar algebra).

Associative rules. These have to do with how you group terms.

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$
$$a(b\vec{C}) = (ab)\vec{C}$$
$$(a\vec{B}) \times \vec{C} = a(\vec{B} \times \vec{C})$$
$$(a\vec{B}) \times \vec{C} = a(\vec{B} \times \vec{C})$$

Cummutative rules. These have to do with the order of terms in a calculation.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$
$$ab\vec{C} = ba\vec{C} = \vec{C}ab = \vec{C}ba$$
$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

Distributive rules. These have to do with multiplying individual terms that are added, rather than their sum.

$$a(\vec{B} + \vec{C}) = a\vec{B} + a\vec{C}$$
$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$$
$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

Extra simplification rules

As you proceed with using the rules above you can simplify using the following extra vector simplification rules.

- $a\vec{A}$ is a vector,
- $\vec{A} \cdot \vec{B}$ is a scalar,
- $\vec{A} \times \vec{B}$ is a vector,
- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ (so $\vec{A} \times \vec{A} = \vec{0}$),
- $\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} \quad \Rightarrow \quad \vec{B} \cdot (\vec{B} \times \vec{C}) = 0$,
- $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$, 'BAC -CAB' (see box 14.4 on page 779),

Exceptions

In the list of usual rules, above, we did not include all of the rules from scalar algebra, they don't all work. So look out for these exceptions.

- $a + \vec{A}$ is nonsense,
- $a/\vec{A}$ is nonsense,
- $\vec{A}/\vec{B}$ is nonsense (a common beginner's error),
- $a \cdot \vec{A}$ is nonsense (unless you mean by it $a\vec{A}$),
- $a \times \vec{A}$ is nonsense,
- $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$,
- $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$.

Substitutions

In all of the expressions in this box the scalars and vectors can be the result of other calculations. For example, all of the expressions above would be equally valid if every place you see a you substituted $\vec{A} \cdot \vec{B}$ and everyplace you see $\vec{A}$ you substituted $\vec{B} + \vec{C}$ or $\vec{B} \times \vec{C}$.

A simpler two-dimensional example using a judiciously chosen dot product, in the same spirit as the example above, is on page 98.

Count equations and unknowns.

One cannot (usually) find more unknowns than one has scalar equations②. Before you do lots of algebra, you should check that you have as many equations as unknowns. If not, you probably can't find all the unknowns. How do you count vector equations and vector unknowns? A two- dimensional vector is fully described by two numbers. For example, a $2D$ vector is described by its x and y components or its magnitude and the angle it makes with the positive x axis. A three-dimensional vector is described by three numbers. So a vector equation counts as 2 or 3 equations in 2 or 3 dimensional problems, respectively. And an unknown vector counts as 2 or 3 unknowns in 2 or 3 dimensions, respectively. If the direction of a vector is known but its magnitude is not, then the magnitude is the only unknown. Magnitude is a scalar, so it counts as one unknown.

②There are famous counter-examples where you can solve for more variables than you have equations. The simplest example, $x^2 + y^2 = 0$, is one equation which can be solved for both x and y to get $x = 0$ and $y = 0$. Although such examples seem to be mathematical trickery they do show up sometimes, but they are always nonlinear. The simultaneous equations in mechanics are most often linear equations (so you can safely ignore this margin comment).

Example: Counting equations
Say you are doing a 2-D problem where you already know the vector $\hat{\boldsymbol{\lambda}} = \sqrt{2}\hat{\boldsymbol{\imath}} + \sqrt{2}\hat{\boldsymbol{\jmath}}$ and you are given the vector equation

$$C\hat{\boldsymbol{\lambda}} = \vec{\boldsymbol{a}}.$$

You then have two equations (a vector equation in 2-D) and three unknowns (the scalar C and the vector $\vec{\boldsymbol{a}}$). There are more unknowns than equations so this vector equation is not sufficient for finding C and $\vec{\boldsymbol{a}}$.

Most often when you have as many equations as unknowns the equations have a unique solution. When you have more equations than unknowns there is most often no solution to the equations. When you have more unknowns than equations most often you have a whole family of solutions.

However these are only guidelines, no matter how many equations and unknowns you have, you could have no solutions, many solutions or a unique solution. The geometric significance of some cases that satisfy and that don't satisfy these guidelines is given in box 2.10 on page 107.

Vector triangles

In 2D one often wants to know all three vectors in a vector triangle, the diagram for expressions like

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{C}} \quad \text{or} \quad \vec{\boldsymbol{A}} - \vec{\boldsymbol{C}} = \vec{\boldsymbol{B}} \quad \text{or} \quad \vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} + \vec{\boldsymbol{C}} = \vec{\boldsymbol{0}} \quad \textit{etc.}$$

Usually at least one vector is given and some information is given about the others. The situation is much like the geometry problem of drawing a triangle given various bits of information about the lengths of its sides and its interior angles. If enough information is given to prove triangle congruence, then enough information is given to determine all angles and sides. A difference between vector triangles and proofs of triangle congruence is that triangle congruence does not depend on the overall orientation, whereas vector triangles need to have the correct orientation. Nonetheless, the tools used to solve triangles are useful for solving vector equations.

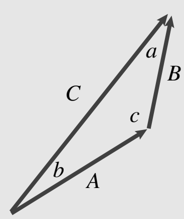
Vector addition

We start with a problem that is in some sense solved at the start. Say $\vec{A}$ and $\vec{B}$ are known and you want to find $\vec{C}$ given that

$$\vec{C} = \vec{A} + \vec{B}.$$

2.9 Vector triangles and the laws of sines and cosines

The tip to tail rule of vector addition defines a triangle. Knowing something about the vectors in this triangle how can we find more? One approach is to use the laws of sines and cosines.



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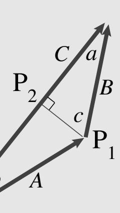
Consider the vector sum $\vec{A} + \vec{B} = \vec{C}$ represented by the triangle shown with traditionally labeled sides A, B, and C and internal angles a, b, and c. The sides and angles are related by

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C} \quad \text{the law of sines, and} \tag{2.29}$$

$$C^2 = A^2 + B^2 - 2AB\cos c \quad \text{the law of cosines.} \tag{2.30}$$

Proof of the law of sines The first equality in the law of sines can be proved by calculating the altitude from c two ways.



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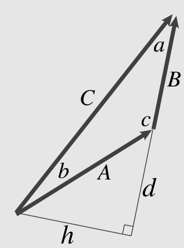
On the one hand length P_1P_2 is given by $P_1P_2 = B\sin a$ and on the other hand by $P_1P_2 = A\sin b$

so $B\sin a = A\sin b \quad \Rightarrow \quad \frac{\sin a}{A} = \frac{\sin b}{B}$.

We can do likewise with all three altitudes thus proving the triple equality.

Proof of the law of cosines. Look at altitude h of the triangle.



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This is the base of two different right triangles. So by the pythagorean theorem we have on the one hand that

$h^2 = A^2 - d^2$ and on the other that $h^2 = C^2 - (B + d)^2$.

Equating these expressions and expanding the square we get

$$\begin{aligned} A^2 - d^2 &= C^2 - (B^2 + d^2 - 2dB) \\ \Rightarrow \quad A^2 + B^2 + 2dB &= C^2 \end{aligned} \tag{2.31}$$

But $d = -A\cos c$ so

$$C^2 = A^2 + B^2 - 2AB\cos c.$$

Sometimes the angle we call c is called θ.

Applications. These laws are useful when you want to figure out the shape and size of a triangle when, of the six triangle quantities (thee sides and three angles), only 3 are given. At least one of these three has to be a length.

As noted, it is possible to give problems of this type that have no solutions. And it is possible to give problems that have either 1 or 2 solutions.

In this era where vector algebra is popular as is the representation of vectors in terms of their components, the laws of sines and cosines are used little. But sometimes they are the easiest approach.

The obvious and correct answer is that you find $\vec{C}$ by vector addition. You could do this addition graphically by drawing a scale picture, or by adding corresponding vector components. Suppose now that $\vec{A}$ and $\vec{B}$ are given to you in terms of magnitude and direction and that you are interested in the direction of $\vec{C}$.

Example: adding vectors defined by magnitude and direction
Say direction is indicated by angle measured counterclockwise form the positive x axis and that $A = 5\sqrt{2}$, $\theta_A = \pi/4$, $B = 4$, and $\theta_B = 2\pi/3$. So

$$\begin{aligned}
\vec{A} &= A\,(\cos\theta_A \hat{\imath} + \sin\theta_A \hat{\jmath}) \\
&= 5\sqrt{2}\,(\cos(\pi/4)\hat{\imath} + \sin(\pi/4)\hat{\jmath}) = 5\hat{\imath} + 5\hat{\jmath} \\
\vec{B} &= B\,(\cos\theta_B \hat{\imath} + \sin\theta_B \hat{\jmath}) \\
&= 4\,(\cos(2\pi/3)\hat{\imath} + \sin(2\pi/3)\hat{\jmath}) = -2\hat{\imath} + 2\sqrt{3}\hat{\jmath} \\
\vec{C} &= \vec{A} + \vec{B} = (5\hat{\imath} + 5\hat{\jmath}) + \left(-2\hat{\imath} + 2\sqrt{3}\hat{\jmath}\right) \\
&= 3\hat{\imath} + (5 + 2\sqrt{3})\hat{\jmath} \\
\Rightarrow \quad \theta_C &= \tan^{-1}(C_y/C_x) = \tan^{-1}\left(\left(5 + 2\sqrt{3}\right)/3\right) \approx 1.23 \approx 70.5^\circ \\
\text{and} \quad C &= \sqrt{3^2 + (5 + 2\sqrt{3})^2} \approx 8.98
\end{aligned}$$

To find θ_C we used the arctan (or $\tan^{-1}$) function which can be off by π ③.To find the angle of $\vec{C}$ we had to convert $\vec{A}$ and $\vec{B}$ to coordinate form, add components, and then convert back to find the angle of $\vec{C}$. That is, even though the desired answer is given by a sum, carrying out the sum takes a bit of effort. An alternative approach avoids some work.

③The problem is that, measuring angles between 0 and 2π (or equivalently between $-\pi$ and π) there are always two different angles that have the same tangent. The inverse tangent function picks one. Some computers or calculators always pick an angle between 0 and π and some always pick a value between $-\pi/2$ and $\pi/2$. Both of these could be the wrong answer. So you need to check and possibly add π to your answer, or, alternatively use one of these two commands: 1) the two-argument inverse tangent ($\arctan(x, y)$) or 2) rectangular-to-polar coordinate conversion, using the angle as the desired arctangent.

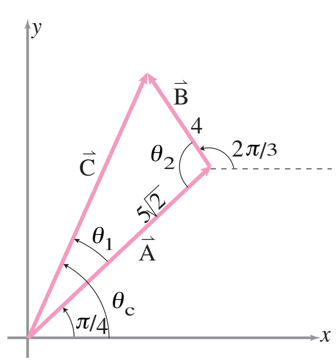

Figure 2.65: Using trig to solve vector triangles

Example: Same as above, different method
Start with picture of the situation, fig. 2.65. By adding angles,

$$\theta_2 = \pi/4 + \pi/3 = 7\pi/12.$$

From the law of cosines (see box 2.9 on page 95),

$$\begin{aligned}
C^2 &= A^2 + B^2 - 2AB\cos\theta_2 \\
\Rightarrow \quad C &= \sqrt{(5\sqrt{2})^2 + 4^2 - 2(5\sqrt{2})\cdot 4\cdot\cos(7\pi/12)} \\
&\approx 8.98 \quad \text{(as before)}
\end{aligned}$$

And from the law of sines (see box 2.9),

$$\begin{aligned}
\frac{\sin\theta_1}{B} &= \frac{\sin\theta_2}{C} \\
\Rightarrow \quad \theta_1 &= \sin^{-1}\left(\frac{B\sin\theta_2}{C}\right) \approx \sin^{-1}\left(\frac{4\sin(7\pi/12)}{8.98}\right) \\
&\approx .445 \\
\Rightarrow \quad \theta_C &= \theta_A + \theta_2 \approx \pi/4 + .445 \approx 1.23 \quad \text{(as before).}
\end{aligned}$$

This second approach is somewhat more direct in some situations.

The determination of a third vector by vector addition is analogous to the determination of a triangle in geometry by “side-angle-side”.

Vector subtraction

Say you want to find $\vec{C}$ given $\vec{A}$ and $\vec{B}$ and that $\vec{A}$, $\vec{B}$ and $\vec{C}$ add to zero. So, subtracting $\vec{C}$ from both sides and multiplying through by -1 we get

$$\begin{aligned} \vec{A}+\vec{B}+\vec{C} &= \vec{0} \\ \Rightarrow \qquad \vec{C} &= -\vec{A}-\vec{B}. \end{aligned}$$

The problem has now been reduced to one of addition which can be done by drawing, components, or trig as shown above.

Find the magnitude of two vectors given their directions and their sum (2D)

Often one knows that 2 vectors $\vec{A}$ and $\vec{B}$ add to a given third vector $\vec{C}$. The directions of $\vec{A}$ and $\vec{B}$ are known but not their magnitudes. That is, given $\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$ and $\vec{C}$ and that

$$\begin{aligned} \vec{A} \quad + \quad \vec{B} \quad &= \vec{C} \\ A\hat{\boldsymbol{\lambda}}_A \quad + \quad B\hat{\boldsymbol{\lambda}}_B \quad &= \vec{C} \end{aligned} \tag{2.32}$$

you would like to find $\vec{A}$ and $\vec{B}$ (which you will know if you find A and B).

Example: A walk
You walked SE (half way between South and East) for a while and NNW (half way between North and NorthWest, 22.5° West of North) for a while and ended up going a net distance of 200 m East. $\vec{A}$ and $\vec{B}$ are your displacements on the first and second parts of your walk.

So, taking xy axes aligned with East and North, the directions are

$$\hat{\boldsymbol{\lambda}}_A = \frac{\sqrt{2}}{2}\hat{\imath} - \frac{\sqrt{2}}{2}\hat{\jmath} \text{ and } \hat{\boldsymbol{\lambda}}_B = -\sin(\frac{\pi}{8})\hat{\imath} + \cos(\frac{\pi}{8})\hat{\jmath}$$

and the given sum is $\vec{C} = 200\,\mathrm{m}\hat{\imath}$. Still unknown are the distances A and B.

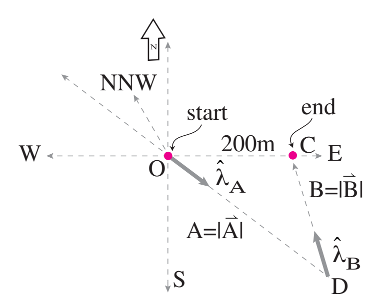

Figure 2.66: An indirect walk from O to D via C.

In statics problems of this type or frequent with A and B representing the unknown magnitudes of forces $\vec{A}$ and $\vec{B}$ and $\hat{\boldsymbol{\lambda}}_A$ and $B\hat{\boldsymbol{\lambda}}_B$ their known directions. Here are four ways to solve eqn. (2.32) which will be illustrated with "**a walk**".

Method I: Use dot products with $\hat{\imath}$ and $\hat{\jmath}$

If we take the dot product of both sides of eqn. (2.32) with $\hat{\imath}$ and then again with $\hat{\jmath}$ we get:

$$\begin{aligned} \hat{\imath}\cdot\{\text{eqn. (2.32)}\} \quad &\Rightarrow \quad A\lambda_{Ax} + B\lambda_{Bx} = C_x, \text{ and} \\ \hat{\jmath}\cdot\{\text{eqn. (2.32)}\} \quad &\Rightarrow \quad A\lambda_{Ay} + B\lambda_{By} = C_y \end{aligned} \tag{2.33}$$

where the components of the vectors $\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$, and $\vec{C}$ are known, or easily determined, because the vectors are known (however they are represented). Eqns. 2.33 are two scalar equations in the unknowns A and B. You can solve these any way that pleases you. One method would be to write the equations in matrix form

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} \\ \lambda_{Ay} & \lambda_{By} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} C_x \\ C_y \end{bmatrix} \tag{2.34}$$

Example: Solving "A walk": method I, simultaneous equations
For the walk example above we would have

$$\begin{bmatrix} \sqrt{2}/2 & -\sin(\frac{\pi}{8}) \\ -\sqrt{2}/2 & \cos(\frac{\pi}{8}) \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 200\,\mathrm{m} \\ 0 \end{bmatrix}$$

which solves (on a computer or calculator) to $A \approx 483\,\mathrm{m}$ and $B \approx 370$ (with the total walked distance being about $852\,\mathrm{m}$).

Taking dot products of a vector equation with $\hat{\imath}$ and $\hat{\jmath}$ is equivalent to extracting the x and y components of the equation. But we use the dot product notation to highlight that you could dot both sides of the vector equation with any vector that pleases you and you would get a legitimate scalar equation. Use any other vector that pleases you (not parallel with the first) and you will get a second independent equation. And the two resulting equations will have the same solution for A and B as the x and y (or "$\hat{\imath}$" and "$\hat{\jmath}$") equations above.

Method II: pick a vector for a dot product that gets rid of terms you don't know.

Pretend for a paragraph that you only want to find A in eqn. (2.32), for example that you only wanted to know the distance walked on the first leg of the indirect walk in the example above. It would be nice to reduce eqn. (2.32) to a single scalar equation in the single unknown A. We'd like to get rid of the term with B, a quantity that we do not know. Suppose we knew a vector $\hat{\boldsymbol{n}}_B$ that was perpendicular to $\hat{\boldsymbol{\lambda}}_B$. If we dotted both sides of eqn. (2.32) we'd get:

$$\begin{array}{rclcl} \hat{\boldsymbol{n}} \cdot \{\text{eqn. (2.32)}\} & \Rightarrow & \hat{\boldsymbol{n}}_B \cdot \left(A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B\right) & = & \hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}} \\ & \Rightarrow & \hat{\boldsymbol{n}}_B \cdot \left(A\hat{\boldsymbol{\lambda}}_A\right) + \hat{\boldsymbol{n}}_B \cdot \left(B\hat{\boldsymbol{\lambda}}_B\right) & = & \hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}} \\ \hat{\boldsymbol{n}}_B \cdot \hat{\boldsymbol{\lambda}}_B = 0 & \Rightarrow & \left(\hat{\boldsymbol{n}}_B \cdot \hat{\boldsymbol{\lambda}}_A\right) A & = & \hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}} \\ & \Rightarrow & A & = & \frac{\hat{\boldsymbol{n}}_B \cdot \vec{\boldsymbol{C}}}{\hat{\boldsymbol{n}}_B \cdot \hat{\boldsymbol{\lambda}}_A}. \end{array}$$

To make use of this method we have to cook up a vector $\hat{\boldsymbol{n}}_B$ that is perpendicular to $\hat{\boldsymbol{\lambda}}_B$ ④.Crossing $\hat{\boldsymbol{\lambda}}_B$ with $\hat{\boldsymbol{k}}$ serves the purpose:

$$\hat{\boldsymbol{n}}_B = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B = \hat{\boldsymbol{k}} \times \left(\lambda_{Bx}\hat{\imath} + \lambda_{By}\hat{\jmath}\right) = -\lambda_{By}\hat{\imath} + \lambda_{Bx}\hat{\jmath}.$$

Without doing the cross product explicitly you can remember that a vector orthogonal to a 2D vector $\hat{\boldsymbol{\lambda}}_B$ has the x and y components switched and the sign of first component then changed. So we get

$$A = \frac{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \hat{\boldsymbol{\lambda}}_A} = \frac{\lambda_{By}C_x - \lambda_{Bx}C_y}{\lambda_{By}\lambda_{Ax} - \lambda_{Bx}\lambda_{Ay}}$$

④The vector $\hat{\boldsymbol{k}}$ (the unit vector out of the page) is perpendicular to $\hat{\boldsymbol{\lambda}}_B$ but is unfortunately not suitable because it is also perpendicular to $\hat{\boldsymbol{\lambda}}_A$ and $\vec{\boldsymbol{C}}$ so only yields the equation $0 + 0 = 0$ or the nonsense that $A = 0/0$.

which is a direct formula for the desired answer ⑤. You could use this formula by substituting in numbers, but that requires memorization or look up. Rather, if you like this short cut, you should remember the idea and reproduce the steps with the symbols or numbers in your problem. Summarizing,

⑤This solution is identical to the Cramer's rule solution of eqn. (2.40) on page 102. That is, we have used dot products to derive Cramer's rule for 2×2 matrices.

> To reduce eqn. (2.32) to one scalar equation in the one unknown A, use a judiciously chosen dot product. For example, get rid of the $\hat{\boldsymbol{\lambda}}_B$ or $\vec{\boldsymbol{B}}$ term by dotting both sides of with $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B$ (or, to save the trouble of finding the unit vector $\hat{\boldsymbol{\lambda}}_B$ just dot with $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{B}}$). ⑥.

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Altogether you can think of this method as something like the "component" method. But we are taking components of the vectors in the direction perpendicular to $\vec{\boldsymbol{B}}$. Alternatively you can think of this method as taking the projection of the vector equation onto a line perpendicular to $\vec{\boldsymbol{B}}$.

Similarly dotting both sides of eqn. (2.32) with $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_A$ gives

$$B = \frac{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_A) \cdot \vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_A) \cdot \hat{\boldsymbol{\lambda}}_B}.$$

⑥**Judicious choice of vectors for dot products.** The basic idea is this: *Get rid of things you don't know and don't care about.*
Say a and b are unknown and appear in your equations in terms like $a\vec{\boldsymbol{A}}$ and $b\vec{\boldsymbol{B}}$. First take an interest in a. You don't know anything about b, and for a moment you don't care about it. So get rid of it. The two ways to get rid of a vector term you don't know and don't care about are 1) dotting the force-balance equation with a vector orthogonal to the vector of disinterest, and 2) using moment balance about a point or axis about which the force of disinterest has no moment. So, for example, dotting an equation containing $a\vec{\boldsymbol{A}}$ and $b\vec{\boldsymbol{B}}$ with a vector orthogonal to $\vec{\boldsymbol{B}}$ eliminates the unknown b from your equation.

Example: Solving "A walk": method II, judicious dot products
You should be able to derive the formulas above as needed. Dotting, for example, both sides of eqn. (2.32) with $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B$ and plugging in the known components yields

$$\begin{aligned} A = \frac{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \vec{\boldsymbol{C}}}{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_B) \cdot \hat{\boldsymbol{\lambda}}_A} &= \frac{\lambda_{By} C_x - \lambda_{Bx} C_y}{\lambda_{By}\lambda_{Ax} - \lambda_{Bx}\lambda_{Ay}} \\ &= \frac{\cos(\pi/8) \cdot 200\,\mathrm{m} - (-\sin(\pi/8)) \cdot 0}{\cos(\pi/8) \cdot (\sqrt{2}/2) - (-\sin(\pi/8)) \cdot (-\sqrt{2}/2)} \\ &\approx 483\,\mathrm{m} \qquad \text{(as before)} \end{aligned}$$

Method III, graphical solution

On the vector triangle defined by $\vec{\boldsymbol{A}} + \vec{\boldsymbol{B}} = \vec{\boldsymbol{C}}$ we call O the tail end of $\vec{\boldsymbol{A}}$. The location of the tip of $\vec{\boldsymbol{C}}$ at G can be drawn to scale. Then the point H can be located as at the intersection of two lines: one emanating from O and in the direction of $\hat{\boldsymbol{\lambda}}_A$ and one emanating from H and in the direction of $\hat{\boldsymbol{\lambda}}_B$. Once the point H is located, the lengths A and B can be measured.

Example: Solving "A walk": method III, graphing
Taking 100 m as drawn to scale as, say 1 cm, point G is drawn 2 cm to the right of O. The location of the point H is found as the intersection of two lines: one emanating from O and pointing 45° counterclockwise from the $-\hat{\boldsymbol{\jmath}}$ axis, and the other emanating from G and pointing 22.5° counterclockwise from the $-\hat{\boldsymbol{\jmath}}$ axis. The distance from O to H can be measured as about 4.8 cm yielding $A \approx 480$ m.

This construction can be done with pencil and paper or with a computer drawing program.

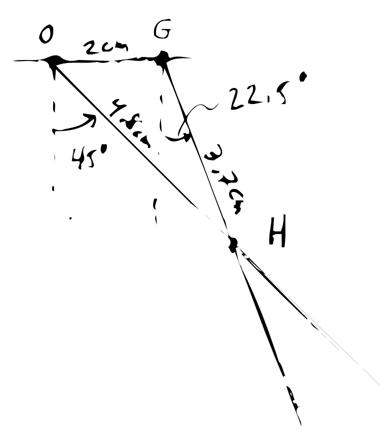

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Figure 2.67: Method III: Find point H as the intersection of two known lines.

Method IV, trigonometry

The final method, the classical method used predominantly before vector notation was well accepted, is to treat the vector triangle as a triangle with some known sides and some known angles, and to use the *law of sines* (discussed in box 2.9).

Because $\vec{C}$ and the directions of $\vec{A}$ and $\vec{B}$ are assumed known, the angles a (opposite side A) and b (opposite side B) are known. Because the sum of interior angles in a triangle is π we know the angle $c = \pi - a - b$. The law of sines tells us that

$$\frac{\sin a}{A} = \frac{\sin c}{C} \quad \text{and} \quad \frac{\sin b}{B} = \frac{\sin c}{C}$$

which we can rewrite as

$$A = \frac{C \sin a}{\sin c} \quad \text{and} \quad B = \frac{C \sin b}{\sin c}.$$

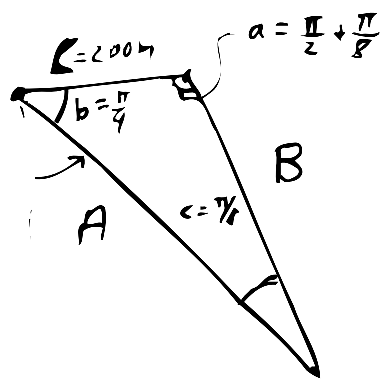

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Figure 2.68: Solving "A walk" using the law of sines.

Example: Solving "A walk": method IV, the law of sines
Referring to fig. 2.68 we get

$$\begin{aligned} A &= \frac{C \sin a}{\sin c} = \frac{200\,\mathrm{m} \cdot \sin(5\pi/8)}{\sin(\pi/8)} \approx 483\,\mathrm{m} \\ \text{and} \qquad B &= \frac{C \sin b}{\sin c} = \frac{200\,\mathrm{m} \cdot \sin(\pi/4)}{\sin(\pi/8)} \approx 370\,\mathrm{m} \end{aligned}$$

as we have found three times already.

The determination of two vectors by knowing their directions and their sum is analogous to determination of a triangle by "angle-side-angle".

The magnitudes and sum of two vectors are known (2D)

Two vectors $\vec{A}$ and $\vec{B}$ in the plane have known magnitudes A and B but unknown directions $\hat{\boldsymbol{\lambda}}_A$ and $\hat{\boldsymbol{\lambda}}_B$. Their sum $\vec{C}$ is known. So, measuring angles counterclockwise relative to the positive x axis, we have:

$$\begin{aligned} \vec{A} + \vec{B} &= \vec{C} \\ A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B &= \vec{C} \\ A\,(\cos\theta_A \hat{\imath} + \sin\theta_A \hat{\jmath}) + B\,(\cos\theta_B \hat{\imath} + \sin\theta_B \hat{\jmath}) &= \vec{C} \end{aligned} \tag{2.35}$$

where eqn. (2.35) is one 2D vector equation in 2 unknowns: θ_A and θ_B.

Method 1: using an appropriate dot product

This problem is really best solved with trig (see below) and getting it right with component method is a matter of hindsight. Eqn. 2.35 can be rewritten as

$$C\,(\cos\theta_C \hat{\imath} + \sin\theta_C \hat{\jmath}) - A\,(\cos\theta_A \hat{\imath} + \sin\theta_A \hat{\jmath}) = B\,(\cos\theta_B \hat{\imath} + \sin\theta_B \hat{\jmath})$$

Taking the dot product of each side with itself gives

$$C^2 + A^2 - 2AC\underbrace{(\cos\theta_C\cos\theta_A + \sin\theta_C\sin\theta_A)}_{\cos(\theta_C-\theta_A)} = B^2$$

so

$$\theta_A = \theta_C - \arccos\left(\frac{C^2+A^2-B^2}{2AC}\right).$$

Now $\vec{A}$ is fully determined and $\vec{B}$ can be found by vector subtraction. Note that the arccos function is always double valued (the negative of any arccos is also a legitimate arccos), so that the solution of this problem is not unique. Also, if the argument of the arccos function is greater than 1 in magnitude, there is no solution; this happens if any two of A, B, and C is greater than the third (that is, if the so-called "triangle inequality" is violated) and there is no way of making a triangle with the given lengths.)

Method II: The law of cosines

Referring to fig. 2.69, we can apply the law of cosines directly to get

$$B^2 = A^2 + C^2 - 2AB\cos\theta_B \quad (2.36)$$

which we can solve to get

$$\theta_1 = \arccos\left(\frac{C^2+A^2-B^2}{2AC}\right) \quad (2.37)$$

Thus the orientation of $\vec{A}$ is determined in relation to $\vec{C}$. This method is a bit quicker than the component method above because it skips the steps where, in effect, the component method derives the law of cosines.

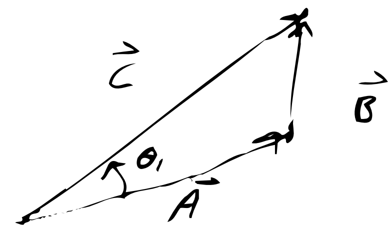

Figure 2.69: For use in using the law of cosines to solve a vector triangle.

Method III: graphical construction

From the tail of $\vec{C}$ draw a circle with radius A (see fig. 2.70). From the tip of $\vec{C}$ draw a circle with radius B. For each of the two points of intersection, P_1 and P_2, a solution has been found. Vector $\vec{A}$ goes from the tail of $\vec{C}$ to, say, P_1, and $\vec{B}$ goes from P_1 to the tip of $\vec{C}$. An $\vec{A}$ and $\vec{B}$ based on P_2 is also a legitimate solution. Each pair is a legitimate solution to the problem. To get a unique solution set other information would have to be provided.

Determining a vector triangle when one vector is known and only the magnitudes of the other two are known is analogous to determining a triangle from "side-side-side" in geometry. It is interesting that this, the most elementary of all geometric constructions does not have an equally simple analytic representation.

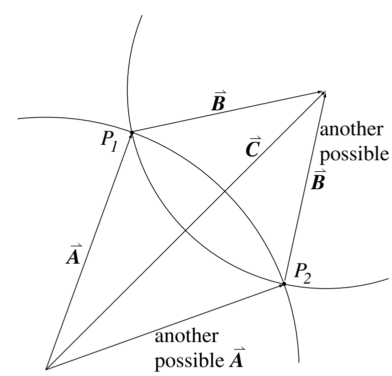

Figure 2.70: Solving a vector triangle where vectors $\vec{A}$ and $\vec{B}$ have known magnitude but unknown direction.

Find the magnitude of three vectors given their directions and their sum (3D)

This problem is close in approach to its junior 2D cousin on page 97 and to the example on page 92. It is the most common of the 3D vector equation problems. Assume that you know the directions of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$

(given, say, as the unit vectors $\hat{\boldsymbol{\lambda}}_A$, $\hat{\boldsymbol{\lambda}}_B$, and $\hat{\boldsymbol{\lambda}}_C$), as well as their sum $\vec{\boldsymbol{D}}$. So we have

$$\begin{aligned} \vec{\boldsymbol{A}} \quad + \quad \vec{\boldsymbol{B}} \quad + \quad \vec{\boldsymbol{C}} \quad &= \vec{\boldsymbol{D}} \\ A\hat{\boldsymbol{\lambda}}_A \quad + \quad B\hat{\boldsymbol{\lambda}}_B \quad + \quad C\hat{\boldsymbol{\lambda}}_C \quad &= \vec{\boldsymbol{D}} \end{aligned} \tag{2.38}$$

and we want to find A, B, and C from which we can find $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, and $\vec{\boldsymbol{C}}$ (*e.g.*, $\vec{\boldsymbol{A}} = A\hat{\boldsymbol{\lambda}}_A$). We can think of the last of eqn. (2.38) as one 3D vector equation in three unknowns.

In three dimensions the graphical approach is essentially impossible. And the trigonometric approach is awkward to say the least, and probably only generally practical for people with British accents who are long dead. The general ideas in the first two methods still stand, however. Thus the use of vector concepts is basically unavoidable in 3D problems.

Method I: dotting with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

We can dot the left and right sides of eqn. (2.38) with $\hat{\boldsymbol{\imath}}$ or $\hat{\boldsymbol{\jmath}}$ or $\hat{\boldsymbol{k}}$. This is equivalent to taking the x, y and z components of the equation. We get then

$$\begin{aligned} \hat{\boldsymbol{\imath}} \cdot \{\text{eqn. (2.38)}\} \quad &\Rightarrow \quad A\lambda_{Ax} + B\lambda_{Bx} + C\lambda_{Cx} = D_x, \\ \hat{\boldsymbol{\jmath}} \cdot \{\text{eqn. (2.38)}\} \quad &\Rightarrow \quad A\lambda_{Ay} + B\lambda_{By} + C\lambda_{Cy} = D_y, \text{ and} \\ \hat{\boldsymbol{k}} \cdot \{\text{eqn. (2.38)}\} \quad &\Rightarrow \quad A\lambda_{Az} + B\lambda_{Bz} + C\lambda_{Cz} = D_z \end{aligned} \tag{2.39}$$

which can be written in matrix form as

$$\begin{bmatrix} \lambda_{Ax} & \lambda_{Bx} & \lambda_{Cx} \\ \lambda_{Ay} & \lambda_{By} & \lambda_{Cy} \\ \lambda_{Az} & \lambda_{Bz} & \lambda_{Cz} \end{bmatrix} \cdot \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}. \tag{2.40}$$

Unless the matrix is sparse (has a lot of zeros as entries) it is probably best to solve such a set of equations for A, B and C on a computer or calculator.

Method II: pick a vector for dot product that kills terms you don't know.

The philosophy here is the same as for method II in 2D (page 98). Pretend for a paragraph that you only want to find A in eqn. (2.38). We can kill the terms involving the unknowns B and C by dotting both sides of the equation with a vector perpendicular to $\hat{\boldsymbol{\lambda}}_B$ and $\hat{\boldsymbol{\lambda}}_C$. Such a vector is $\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C$. Thus

$$\begin{aligned} &(\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \{\text{eqn. (2.32)}\} \\ \Rightarrow \quad & (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \left(A\hat{\boldsymbol{\lambda}}_A + B\hat{\boldsymbol{\lambda}}_B + C\hat{\boldsymbol{\lambda}}_C\right) = (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \vec{\boldsymbol{D}} \\ \Rightarrow \quad & (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \left(A\hat{\boldsymbol{\lambda}}_A\right) + \vec{\boldsymbol{0}} + \vec{\boldsymbol{0}} = (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C) \cdot \vec{\boldsymbol{D}} \\ \Rightarrow \quad & A = \frac{\vec{\boldsymbol{D}} \cdot (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)}{\hat{\boldsymbol{\lambda}}_A \cdot (\hat{\boldsymbol{\lambda}}_B \times \hat{\boldsymbol{\lambda}}_C)}. \end{aligned}$$

If you use a matrix determinant to evaluate the mixed triple product you can recognize this formula (like the formula solving the example on 92) as

Cramer's rule. By a judicious dot product we have reduced the vector equation to a scalar equation in one unknown. Similarly we could get one equation in one unknown for B or for C by doting eqn. (2.38) with $\hat{\boldsymbol{\lambda}}_A \times \hat{\boldsymbol{\lambda}}_C$ and $\hat{\boldsymbol{\lambda}}_A \times \hat{\boldsymbol{\lambda}}_B$, respectively⑦.

⑦Note that the key to the method was dotting with a vector in an appropriate direction, the magnitude of the vector did not matter. So if, for example, you knew any vector $\vec{\boldsymbol{v}}_B$ in the direction of $\hat{\boldsymbol{\lambda}}_B$ and any vector $\vec{\boldsymbol{v}}_C$ in the direction of $\hat{\boldsymbol{\lambda}}_C$ you could dot both sides of eqn. (2.38) with $\vec{\boldsymbol{v}}_B \times \vec{\boldsymbol{v}}_C$ to get one scalar equation for A. This can simplify calculations by avoiding the square roots (which cancel in the end) that you calculate to find unit vectors.

Parametric equations for lines and planes

A line in 2D

In geometry a line on a plane is often describe as the set of x and y points that satisfy an equation like

$$Ax + By = D \qquad \text{or} \qquad y = mx + b$$

for given A, B, and D or m and b. However a line is a "one dimensional" object and it is nice to describe it that way. The parametric form that is often useful is:

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_\text{A} + s\vec{\boldsymbol{v}} \tag{2.41}$$

where $\vec{\boldsymbol{r}}$ are the position vectors of set of points on the line, one point for each value of the scalar parameter s. $\vec{\boldsymbol{r}}_\text{A}$ is the position vector of one given reference point on the line and $\vec{\boldsymbol{v}}$ is a vector parallel to the line. In the special case that $\vec{\boldsymbol{v}}$ is a unit vector, s is the distance from the point at $\vec{\boldsymbol{r}}_\text{A}$ to the point at $\vec{\boldsymbol{r}}$. If the vector $\vec{\boldsymbol{v}}$ was the velocity of a point moving on the line then $s|\vec{\boldsymbol{v}}|$ would be the distance of the point from the point at $\vec{\boldsymbol{r}}_\text{A}$.

Example: Parametric equation of a line

A parametric equation for the line going through the points with position vectors $\vec{\boldsymbol{r}}_\text{A}$ and $\vec{\boldsymbol{r}}_\text{B}$ is

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_\text{A} + s\left(\underbrace{\vec{\boldsymbol{r}}_\text{B} - \vec{\boldsymbol{r}}_\text{A}}_{\vec{\boldsymbol{v}}}\right) \qquad \text{or better} \qquad \vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_\text{A} + s\hat{\boldsymbol{\lambda}}_A$$

where $\hat{\boldsymbol{\lambda}}_A = (\vec{\boldsymbol{r}}_\text{B} - \vec{\boldsymbol{r}}_\text{A}) \,/\, |\vec{\boldsymbol{r}}_\text{B} - \vec{\boldsymbol{r}}_\text{A}|$

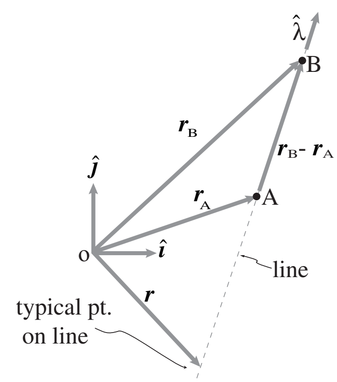

Figure 2.71: Parametric description of a line using vectors.

A line in 3D

In three dimensions a line is often described geometrically as the intersection of two planes. But a line in three dimensions is still a one dimensional object so the parametric form eqn. (2.41), applicable in three dimensions as well as two, is nice.

A plane

A plane in three dimensions can be described as the set of points x,y, and z that satisfy an equation like:

$$Ax + By + Cz = D$$

for a given A,B,C, and D. The parametric description of a plane uses two parameters s_1 and s_2 and is

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_\text{O} + s_1\vec{\boldsymbol{v}}_1 + s_2\vec{\boldsymbol{v}}_2 \tag{2.42}$$

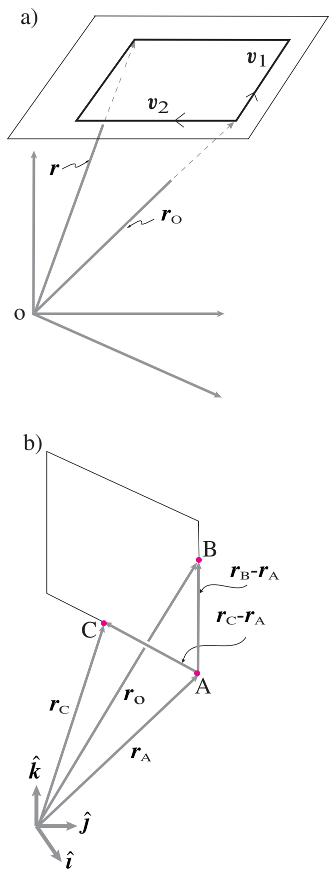

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Figure 2.72: a) Parametric equation of a plane. b) the plane through the points A,B, and C

where $\vec{r}$ is a typical point on the plane, $\vec{v}_1$ and $\vec{v}_2$ are any two non-parallel vectors that lie in the plane and s_1 and s_2 are any two real numbers. Each pair (s_1, s_2) corresponds to one point in the plane and vice versa. The numbers s_1 and s_2 can be thought of as in-plane distance coordinates if the vectors $\vec{v}_1$ and $\vec{v}_2$ are mutually orthogonal unit vectors.

Example: A plane
A parametric equation for the plane going through the three points with position vectors $\vec{r}_A$, $\vec{r}_B$, and $\vec{r}_C$ is

$$\vec{r} = \underbrace{\vec{r}_A}_{\vec{r}_0} + s_1 \underbrace{(\vec{r}_B - \vec{r}_A)}_{\vec{v}_1} + s_2 \underbrace{(\vec{r}_C - \vec{r}_A)}_{\vec{v}_2}$$

You can check that when $s_1 = s_2 = 0$ the point on the plane $\vec{r}_A$ is given. And when one of the s values is one and the other zero the points $\vec{r}_B$ and $\vec{r}_C$ are given.

Vectors, matrices, and linear algebraic equations

Once one has drawn a free body diagram and written the force and moment balance equations one is left with vector equations to solve for various unknowns. The vector equations of mechanics can be reduced to scalar equations by using dot products. The simplest dot product to use is with the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$. This use of dot products is equivalent to taking the x, y, and z components of the vector equation. The two vector equations

$$\begin{aligned} a\hat{\imath} + b\hat{\jmath} &= (c-5)\hat{\imath} + (d+7)\hat{\jmath} \\ (a-c)\hat{\imath} + (a+b)\hat{\jmath} &= (c+b)\hat{\imath} + (2a+c)\hat{\jmath} \end{aligned}$$

with four scalar unknowns a, b, c, and d, can be rewritten as four scalar equations, two from each two-dimensional vector equation. Taking the dot product of the first equation with $\hat{\imath}$ gives $a = c - 5$. Similarly dotting with $\hat{\jmath}$ gives $b = d + 7$. Repeating the procedure with the second equation gives 4 scalar equations:

$$\begin{aligned} a &= c-5 \\ b &= d+7 \\ a-c &= c+b \\ a+b &= 2a+c. \end{aligned}$$

These equations can be re-arranged putting unknowns on the left side and knowns on the right side:

$$\begin{array}{rcrcrcrcr} 1a &+& 0b &+& -1c &+& 0d &=& -5 \\ 0a &+& 1b &+& 0c &+& -1d &=& 7 \\ 1a &+& -1b &+& -2c &+& 0d &=& 0 \\ -1a &+& 1b &+& -1c &+& 0d &=& 0 \end{array}$$

These equations can in turn be written in standard matrix form. The standard matrix form is a short hand notation for writing (linear) equations, such as the equations above:

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix}}_{[\boldsymbol{A}]} \cdot \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}_{[\boldsymbol{x}]} = \underbrace{\begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}}_{[\boldsymbol{y}]}$$

$$\Rightarrow \quad [\boldsymbol{A}]\cdot[\boldsymbol{x}] \;=\; [\boldsymbol{y}].$$

The matrix equation $[\boldsymbol{A}]\cdot[\boldsymbol{x}] = [\boldsymbol{y}]$ is in a form that is easy to input to any of several programs that solve linear equations. The computer (or a do-able but probably untrustworthy hand calculation) should return the following solution for $[\boldsymbol{x}]$ (a, b, c, and d).

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix}.$$

That is, $a = -5$, $b = -5$, $c = 0$, and $d = -12$. If you doubt the solution, check it. To check the answer, plug it back into the original matrix equation and note the equality (or lack thereof!). In this case, we have done our calculations correctly and

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & -1 & -2 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ -5 \\ 0 \\ -12 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 0 \\ 0 \end{bmatrix}.$$

Going back to the original vector equations we can also check that

$$\begin{aligned} -5\hat{\imath} + -5\hat{\jmath} &= (0-5)\hat{\imath} + (-12+7)\hat{\jmath} \\ (-5-0)\hat{\imath} + (-5+-5)\hat{\jmath} &= (0+-5)\hat{\imath} + (2\cdot -5+0)\hat{\jmath}. \end{aligned}$$

Computer solution of simultaneous equations

Depending on your computer package you might solve the equations above like this

```
eqset  =  {  a      -  c       =  -5
                b        - d   =   7
             a - b - 2c        =   0
            -a + b -  c        =   0}
Solve eqset for a,b,c,d.
```

Or, if your computer package is set up especially for linear algebra then you could write something analogous to this:

```
M =  [  1  0 -1  0
        0  1  0 -1
```

```
          1 -1 -2  0
         -1  1 -1  0]
   w = [-5 7 0 0]'
  Solve M*z = w for z
             % the elements of z are a,b,c,d
```

'Physical' vectors and row or column vectors

The word 'vector' has two related but subtly different meanings. One is a physical vector like $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$, a quantity with magnitude and direction. The other meaning is a list of numbers like the row vector

$$[\boldsymbol{x}] = [x_1, x_2, x_3]$$

or the column vector

$$[\boldsymbol{y}] = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

Once you have picked a basis, like $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, you can represent a physical vector $\vec{F}$ as a row vector $\left[F_x, F_y, F_z\right]$ or a column vector $\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$. But the components of a given vector depend on the base coordinate system (or base vectors) that are used. For clarity it is best to distinguish a physical vector from a list of components using a notation like the following:

$$[\vec{F}]_{XYZ} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

The square brackets around $\vec{F}$ indicate that we are looking at its components. The subscript XYZ identifies what coordinate system or base vectors are being used. The right side is a list of three numbers (in this case arranged as a column, the default arrangement in linear algebra).

2.10 Existence, uniqueness, and geometry

Sometimes there is a *unique* solution set to a set of simultaneous solutions. Sometimes its impossible to solve a set of vector equations; no solutions *exist*. And sometimes there are lots of solutions; solutions *exist* but are not *unique*. These cases sometimes have simple geometric interpretations.

Example 1. Consider a very simple equation

$$a\vec{v}_1 = \vec{w}$$

where $\vec{v}_1$ and $\vec{w}$ are given and you are to find a. The left hand side is a parametric expression for points on a line through the origin in the direction $\vec{v}_1$.

- if $\vec{w}$ is parallel to $\vec{v}$ then the equation has exactly one solution for a;
- if $\vec{w}$ is not parallel to $\vec{v}$ then there is no possible a that could make the equation true. The equation has no solutions.

This vector equation is equivalent to 2 scalar equations (3 in 3D) with one scalar unknown and we expect generally to find no solution. That is, two random vectors $\vec{v}_1$ and $\vec{w}$ are unlikely to be parallel either in 2D or 3D.

Example 2. Now consider this 2D vector equation in two unknown scalars a and b:

$$a\vec{v}_1 + b\vec{v}_2 = \vec{w}.$$

- If $\vec{v}_1$ and $\vec{v}_2$ are not parallel $a\vec{v}_1 + b\vec{v}_2$ could be, with appropriate choice of a and b, any 2D vector. There would be a unique solution for every possible $\vec{w}$.
- But if $\vec{v}_1$ and $\vec{v}_2$ are parallel then the expression $a\vec{v}_1 + b\vec{v}_2$ describes a line.
  - If $\vec{w}$ is on this line there are many solutions for a and b because the two vectors $a\vec{v}_1$ and $\vec{v}_2$ can be added various ways that partially cancel.
  - If $\vec{w}$ is off the line then there are no combinations of a and b that get vectors off the line, there are no solutions.

In 2D a test to see if two vectors are parallel is to take their cross product. So, if

$$(\vec{v}_1 \times \vec{v}_2) \cdot \hat{k} = v_{1x}v_{2y} - v_{1y}v_{2x} = \det \begin{bmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{bmatrix} = 0$$

then $\vec{v}_1$ and $\vec{v}_2$ are parallel and there are either many solutions or no solutions depending on whether or not $\vec{w}$ is also parallel to $\vec{v}_1$ and $\vec{v}_2$.

Example 3. Consider the same example as above but now in 3D.

$$a\vec{v}_1 + b\vec{v}_2 = \vec{w}.$$

Now the question is whether the vector $\vec{w}$ is in the plane described parametrically by $a\vec{v}_1 + b\vec{v}_2$. We have more equations than unknowns, $3 > 2$ so solution should be unlikely. Given 3 random vectors in 3D $\vec{v}_1$, $\vec{v}_2$ and $\vec{w}$, it is unlikely that $\vec{w}$ would be in the plane determined by $\vec{v}_1$ and $\vec{v}_2$. If $\vec{w}$ is in that plane, we get again the three possibilities from the previous example.

Example 4. Finally consider this common equation in 3D.

$$a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{w}. \tag{2.43}$$

where $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, and $\vec{w}$ are given vectors and a, b and c are unknowns.

- If $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are not co-planar, then by imagining flying in through space in each of three directions, you can see that you can get to any point in space $\vec{w}$ by using one and only one set of multiples a, b and c of the three vectors.
- On the other hand, if $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are co-planar, they are redundant, and
  - there can only be a solution if $\vec{w}$ is on the plane and, assuming the three vectors are not also colinear, there are many solutions. There are various ways for combinations of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ to cancel each other out.
  - if $\vec{w}$ is off this plane there are no solutions.

If the vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are coplanar then there are either no solutions for a,b and c or many solutions. We can test for coplanarity of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ with geometric reasoning and cross products. The vector $\vec{v}_1 \times \vec{v}_2$ is orthogonal to the plane of $\vec{v}_1$ and $\vec{v}_2$. So, if $\vec{v}_3$ is in the plane defined by $\vec{v}_1$ and $\vec{v}_2$ it will be orthogonal to $\vec{v}_1 \times \vec{v}_2$. Thus if

$$(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = 0$$

the three vectors are co-planar and . This test can also be written as

$$\det \begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix} = 0$$

which is what we would expect from considering the matrix form of eqn. (2.43)

$$\begin{bmatrix} v_{1x} & v_{2x} & v_{3x} \\ v_{1y} & v_{2y} & v_{3y} \\ v_{1z} & v_{2z} & v_{3z} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

and checking to see if the 3×3 matrix is "singular" (a linear algebra word meaning that the determinant is zero).

Relation to more general linear algebra. For systems of equations in 4 or more dimensions we can't use our geometric intuition quite so directly. But the cases above are analagous to what one always finds. The geometric interpretations are helpful for gaining an intuition, even in higher than 3 dimensions when they don't strictly hold. Consider the matrix equation

$$M * v = b$$

with the square matrix M and the column vector b given.

- If the columns of M are not redundant (e.e., they are linearly independent) then there exists a unique v for any b. This is like having $\vec{v}_1, \vec{v}_2, \vec{v}_3$ not coplanar in 3D.
- If the columns of M are redundant (e.g., they are linearly dependent) this is like having coplanar $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and
  - if b is in the span of the columns of M, like $\vec{w}$ being in the plane, there are many solutions, and
  - if b is not in the span of the columns of M, like $\vec{W}$ being off the plane, there are no solutions.

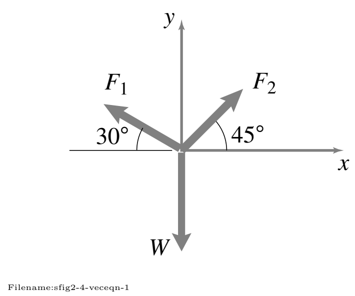

Figure 2.73:

SAMPLE 2.28 Plain vanilla vector equation in 2-D: Three forces act on a particle as shown in the figure. The equilibrium condition of the particle requires that $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$. It is given that $\vec{W} = -20\,\text{N}\,\hat{\boldsymbol{j}}$. Find the magnitudes of forces $\vec{F}_1$ and $\vec{F}_2$.

Solution We are given a vector equation, $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$, in which one vector $\vec{W}$ is completely known and the directions of the other two vectors are given. We need to find their magnitudes. Let us write the vectors as

$$\vec{F}_1 = F_1\hat{\boldsymbol{\lambda}}_1, \quad \vec{F}_2 = F_2\hat{\boldsymbol{\lambda}}_2, \quad \vec{W} = -W\hat{\boldsymbol{j}},$$

where $\hat{\boldsymbol{\lambda}}_1$ and $\hat{\boldsymbol{\lambda}}_2$ are unit vectors along $\vec{F}_1$ and $\vec{F}_2$, respectively (their directions are specified by the given angles in the figure), and $W = 20\,\text{N}$ as given. We can write the unit vectors in component form as

$$\hat{\boldsymbol{\lambda}}_1 = \lambda_{1_x}\hat{\boldsymbol{\imath}} + \lambda_{1_y}\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \hat{\boldsymbol{\lambda}}_2 = \lambda_{2_x}\hat{\boldsymbol{\imath}} + \lambda_{2_y}\hat{\boldsymbol{\jmath}}.$$

Now we can write the given vector equation as

$$F_1(\lambda_{1_x}\hat{\boldsymbol{\imath}} + \lambda_{1_y}\hat{\boldsymbol{\jmath}}) + F_2(\lambda_{2_x}\hat{\boldsymbol{\imath}} + \lambda_{2_y}\hat{\boldsymbol{\jmath}}) = W\hat{\boldsymbol{\jmath}}. \tag{2.44}$$

Dotting both sides of eqn. (2.44) with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ respectively, we get

$$\lambda_{1_x}F_1 + \lambda_{2_x}F_2 = 0 \tag{2.45}$$
$$\lambda_{1_y}F_1 + \lambda_{2_y}F_2 = W. \tag{2.46}$$

Here, we have two equations in two unknowns (F_1 and F_2). We can solve these equations for the unknowns. Let us solve these two linear equations by first putting them into a matrix form and then solving the matrix equation. The matrix equation is

$$\begin{bmatrix} \lambda_{1_x} & \lambda_{2_x} \\ \lambda_{1_y} & \lambda_{2_y} \end{bmatrix} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Using Cramer's rule for matrix inversion, we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \frac{1}{\lambda_{1_x}\lambda_{2_y} - \lambda_{2_x}\lambda_{1_y}} \begin{bmatrix} \lambda_{2_y} & -\lambda_{2_x} \\ -\lambda_{1_y} & \lambda_{1_x} \end{bmatrix} \begin{pmatrix} 0 \\ W \end{pmatrix}.$$

Substituting the numerical values of $\lambda_{1_x} = -\cos 30^\circ = -\sqrt{3}/2$, $\lambda_{1_y} = \sin 30^\circ = 1/2$ and similarly, $\lambda_{2_x} = 1/\sqrt{2}$, $\lambda_{2_y} = 1/\sqrt{2}$, and $W = 20\,\text{N}$, we get

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 14.64 \\ 17.93 \end{pmatrix} \text{N}.$$

$$\boxed{F_1 = 14.64\,\text{N}, \quad F_2 = 17.93\,\text{N}}$$

Check: We can easily check if the values we have got are correct. For example, substituting the numerical values in eqn. (2.45), we get

$$14.64\,\text{N}\cdot\left(-\frac{\sqrt{3}}{2}\right) + 17.93\,\text{N}\cdot\frac{1}{\sqrt{2}} \stackrel{\checkmark}{=} 0.$$

SAMPLE 2.29 Solving for a single unknown from a 2-D vector equation: Consider the same problem as in Sample 2.28. That is, you are given that $\vec{F}_1 + \vec{F}_2 + \vec{W} = \vec{0}$ where $\vec{W} = -20\,\text{N}\,\hat{\jmath}$ and $\vec{F}_1$ and $\vec{F}_2$ act along the directions shown in the figure. Find the magnitude of $\vec{F}_2$.

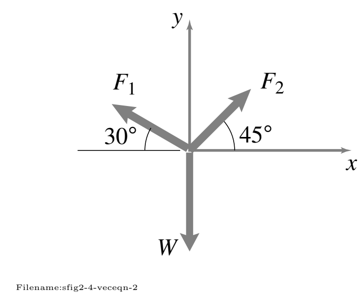

Figure 2.74:

Solution Once again, we write the given vector equation as

$$F_1\hat{\boldsymbol{\lambda}}_1 + F_2\hat{\boldsymbol{\lambda}}_2 = W\hat{\jmath},$$

where

$$\begin{aligned}
W &= 20\,\text{N}, \\
\hat{\boldsymbol{\lambda}}_1 &= \lambda_{1_x}\hat{\imath} + \lambda_{1_y}\hat{\jmath} \\
&= -\sqrt{3}/2\hat{\imath} + 1/2\hat{\jmath}, \quad \text{and} \\
\hat{\boldsymbol{\lambda}}_2 &= \lambda_{2_x}\hat{\imath} + \lambda_{2_y}\hat{\jmath} \\
&= 1/\sqrt{2}(\hat{\imath} + \hat{\jmath}).
\end{aligned}$$

We are interested in finding F_2 only. So, let us take a dot product of this equation with a vector that gets rid of the F_1 term. Any such vector would have to be perpendicular to $\hat{\boldsymbol{\lambda}}_1$. One such vector is $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_1$. Let us call this vector $\hat{\boldsymbol{n}}_1$, that is,

$$\hat{\boldsymbol{n}}_1 = \hat{\boldsymbol{k}} \times (\lambda_{1_x}\hat{\imath} + \lambda_{1_y}\hat{\jmath}) = \lambda_{1_x}\hat{\jmath} - \lambda_{1_y}\hat{\imath}.$$

Now, dotting the given vector equation with $\hat{\boldsymbol{n}}_1$, we get

$$\begin{aligned}
F_1\overbrace{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_1)}^{0} + F_2(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2) &= W(\hat{\boldsymbol{n}}_1 \cdot \hat{\jmath}) \\
\Rightarrow \quad F_2 &= W\,\frac{\hat{\boldsymbol{n}}_1 \cdot \hat{\jmath}}{\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2} \\
&= W\,\frac{(\lambda_{1_x}\hat{\jmath} - \lambda_{1_y}\hat{\imath}) \cdot \hat{\jmath}}{(\lambda_{1_x}\hat{\jmath} - \lambda_{1_y}\hat{\imath}) \cdot (\lambda_{2_x}\hat{\imath} + \lambda_{2_y}\hat{\jmath})} \\
&= W\,\frac{\lambda_{1_x}}{\lambda_{1_x}\lambda_{2_y} - \lambda_{1_y}\lambda_{2_x}} \\
&= 20\,\text{N}\,\frac{-\sqrt{3}/2}{-\sqrt{3}/2 \cdot 1/\sqrt{2} - 1/2 \cdot 1/\sqrt{2}} \\
&= 20\,\text{N}\,\frac{\sqrt{6}}{\sqrt{3}+1} = 17.93\,\text{N}
\end{aligned}$$

which, of course, is the same value we got in Sample 2.28. Note that here we obtained one scalar equation in one unknown by dotting the 2-D vector equation with an appropriate vector to get rid of the other unknown F_1.

$$F_2 = 17.93\,\text{N}$$

SAMPLE 2.30 Solving a 3-D vector equation on a computer: Four forces, $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ and $\vec{N}$ are in equilibrium, that is, $\vec{F}_1 + \vec{F}_2 + +\vec{F}_3 + \vec{N} = \vec{\mathbf{0}}$ where $\vec{N} = -100\,\mathrm{kN}\hat{\boldsymbol{k}}$ is known and the directions of the other three forces are known. $\vec{F}_1$ is directed from (0,0,0) to (1,-1,1), $\vec{F}_2$ from (0,0,0) to (-1,-1,1), and $\vec{F}_3$ from (0,0,0) to (0,1,1). Find the magnitudes of $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$.

Solution Let $\vec{F}_1 = F_1\hat{\boldsymbol{\lambda}}_1$, $\vec{F}_2 = F_2\hat{\boldsymbol{\lambda}}_2$, and $\vec{F}_1 = F_3\hat{\boldsymbol{\lambda}}_3$, where $\hat{\boldsymbol{\lambda}}_1$, $\hat{\boldsymbol{\lambda}}_2$ and $\hat{\boldsymbol{\lambda}}_3$ are unit vectors in the directions of $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$, respectively. Then the given vector equation can be written as

$$F_1\hat{\boldsymbol{\lambda}}_1 + F_2\hat{\boldsymbol{\lambda}}_2 + F_3\hat{\boldsymbol{\lambda}}_3 = -\vec{N} = -N\hat{\boldsymbol{k}}$$

where $N = -100\,\mathrm{kN}$. Dotting this equation with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{\boldsymbol{k}}$ respectively, and realizing that $\hat{\imath}\cdot\hat{\boldsymbol{\lambda}}_1 = \lambda_{1_x}$, $\hat{\jmath}\cdot\hat{\boldsymbol{\lambda}}_1 = \lambda_{1_y}$, etc., we get the following three scalar equations.

$$\begin{aligned}
\lambda_{1_x}F_1 + \lambda_{2_x}F_2 + \lambda_{3_x}F_3 &= 0\\
\lambda_{1_y}F_1 + \lambda_{2_y}F_2 + \lambda_{3_x}F_3 &= 0\\
\lambda_{1_z}F_1 + \lambda_{2_z}F_2 + \lambda_{3_z}F_3 &= -N.
\end{aligned}$$

Thus we get a system of three linear equations in three unknowns. To solve for the unknowns, we set up these equations as a matrix equation and then use a computer to solve it. In matrix form these equations are

$$\begin{bmatrix} \lambda_{1_x} & \lambda_{2_x} & \lambda_{3_x}\\ \lambda_{1_y} & \lambda_{2_y} & \lambda_{3_y}\\ \lambda_{1_z} & \lambda_{2_z} & \lambda_{3_z}\end{bmatrix}\begin{pmatrix}F_1\\F_2\\F_3\end{pmatrix} = \begin{pmatrix}0\\0\\-N\end{pmatrix}.$$

To solve this equation on a computer, we need to input the matrix of unit vector components and the known vector on the right hand side. From the given coordinates for the directions of forces, we have $\hat{\boldsymbol{\lambda}}_1 = (\hat{\imath} - \hat{\jmath} + \hat{\boldsymbol{k}})/\sqrt{3}$, $\hat{\boldsymbol{\lambda}}_2 = (-\hat{\imath} - \hat{\jmath} + \hat{\boldsymbol{k}})/\sqrt{3}$, and $\hat{\boldsymbol{\lambda}}_3 = (\hat{\jmath} + \hat{\boldsymbol{k}})/\sqrt{2}$. ⑧ We are also given that $N = -100\,\mathrm{kN}$. Now, we use the following pseudo-code to find the solution on a computer.

⑧ These unit vectors are computed by taking a vector from one end point to the other end point (as given) and then dividing by its magnitude. For example, we find $\hat{\boldsymbol{\lambda}}_1$ by first finding $\vec{r}_1 = (1)\hat{\imath} + (-1)\hat{\jmath} + (1)\hat{\boldsymbol{k}}$, a vector from (0,0,0) to (1,-1,1), and then $\hat{\boldsymbol{\lambda}}_1 = \frac{\vec{r}_1}{|\vec{r}_1|}$.

```
Let s2 = sqrt(2),  s3 = sqrt(3)
    A = [ 1/s3  -1/s3  0
         -1/s3  -1/s3  1/s2
          1/s3   1/s3  1/s2 ]
    b = [ 0   0   100]'
    solve A*F = b for F
```

Using this pseudo-code we find the solution to be

```
    F = [ 43.3013
          43.3013
          70.7107 ]
```

That is, $F_1 = F_2 = 43.3\,\mathrm{kN}$ and $F_3 = 70.7\,\mathrm{kN}$.

$$F_1 = 43.3\,\mathrm{kN},\ F_2 = 43.3\,\mathrm{kN},\ F_3 = 70.7\,\mathrm{kN}$$

SAMPLE 2.31 **Vector operations on a computer:** Consider the problem of Sample 2.30 again. That is, you are given the vector equation $\vec{F_1} + \vec{F_2} + +\vec{F_3} + \vec{N} = \vec{0}$ where $\vec{N} = -100\,\mathrm{kN}\hat{\boldsymbol{k}}$ and the directions of $\vec{F_1}$, $\vec{F_2}$ and $\vec{F_3}$ are given by the unit vectors $\hat{\boldsymbol{\lambda}}_1 = (\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})/\sqrt{3}$, $\hat{\boldsymbol{\lambda}}_2 = (-\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})/\sqrt{3}$, and $\hat{\boldsymbol{\lambda}}_3 = (\hat{\boldsymbol{\jmath}} + \hat{\boldsymbol{k}})/\sqrt{2}$, respectively. Find F_1.

Solution We can, of course, solve the problem as we did in Sample 2.30 and we get the answer as a part of the unknown forces we solved for. However, we would like to show here that we can extract one scalar equation in just one unknown (F_3) from the given 3-D vector equation and solve for the unknown without solving a matrix equation. Although we can carry out all required calculations by hand, we will show how we can use a computer to do these operations.

We can write the given vector equation as

$$F_1\hat{\boldsymbol{\lambda}}_1 + F_2\hat{\boldsymbol{\lambda}}_2 + F_3\hat{\boldsymbol{\lambda}}_3 = -\vec{N}. \tag{2.47}$$

We want to find F_1. Therefore, we should dot this equation with a vector that gets rid of both F_2 and F_3, i.e., with a vector which is perpendicular to both $\vec{F_2}$ and $\vec{F_3}$. One such vector is $\vec{F_2} \times \vec{F_3}$ or $\hat{\boldsymbol{\lambda}}_2 \times \hat{\boldsymbol{\lambda}}_3$. Let $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\lambda}}_2 \times \hat{\boldsymbol{\lambda}}_3$. Now, dotting both sides of eqn. (2.47) with $\hat{\boldsymbol{n}}$, we get

$$F_1(\hat{\boldsymbol{\lambda}}_1 \cdot \hat{\boldsymbol{n}}) + F_2(\hat{\boldsymbol{\lambda}}_2 \cdot \hat{\boldsymbol{n}}) + F_3(\hat{\boldsymbol{\lambda}}_3 \cdot \hat{\boldsymbol{n}}) = -\vec{N} \cdot \hat{\boldsymbol{n}}$$

Since $\hat{\boldsymbol{\lambda}}_2 \cdot \hat{\boldsymbol{n}} = 0$ and $\hat{\boldsymbol{\lambda}}_3 \cdot \hat{\boldsymbol{n}} = 0$ ($\hat{\boldsymbol{n}}$ is normal to both $\hat{\boldsymbol{\lambda}}_2$ and $\hat{\boldsymbol{\lambda}}_3$), we get

$$\begin{aligned} F_1(\hat{\boldsymbol{\lambda}}_1 \cdot \hat{\boldsymbol{n}}) &= -\vec{N} \cdot \hat{\boldsymbol{n}} \\ \Rightarrow \quad F_1 &= \frac{-\vec{N} \cdot \hat{\boldsymbol{n}}}{\hat{\boldsymbol{\lambda}}_1 \cdot \hat{\boldsymbol{n}}}. \end{aligned}$$

Thus we have found the solution. To compute the expression on the right hand side of the above equation we use the following pseudo-code which assumes that you have written (or have access to) two functions, `dot` and `cross`, that compute the dot and cross product of two given vectors.

```
lambda_1 = 1/sqrt(3)*[1  -1  1]';
lambda_2 = 1/sqrt(3)*[-1  -1  1]';
lambda_3 = 1/sqrt(2)*[0  1  1]';
       N = [0  0  -100]';
       n = cross(lambda_2, lambda_3);
      F1 = - dot(N, n)/dot(lambda_1, n)
```

By following these steps on a computer, we get the output `F1 = 43.3013`, that is, $F_1 = 43.3\,\mathrm{kN}$, which, of course, is the same answer we obtained in Sample 2.30.

$F_1 = 43.3\,\mathrm{kN}$

2.6 Equivalent force systems

Most often one does not want to know the complete details of all the forces acting on a system. When you think of the force of the ground on your bare foot you do not think of the thousands of little forces at each micro-asperity or the billions and billions of molecular interactions between the wood (say) and your skin. Instead you think of some kind of equivalent force. In what way equivalent? Well, because all that the equations of mechanics know about forces is their net force and net moment, you have a criterion. You replace the actual force system with a simpler force system, possibly just a single well-placed force, that has the same total force and same total moment with respect to a reference point C.

The replacement of one system with an equivalent system is often used to help simplify or solve mechanics problems. Further, the concept of equivalent force systems allows us to define a *couple*, a concept we will use throughout the book. Here is the definition of the word *equivalent*① when applied to force systems in mechanics.

①Other phrases used to describe the same concept include: *statically equivalent*, *mechanically equivalent*, and *equipollent*.

> Two force systems are said to be *equivalent* if they have the same sum (the same *resultant*) and the same net moment about any one point C.

We have already discussed two important cases of equivalent force systems. On page 44 we stated the mechanics assumption that a set of forces applied at one point is equivalent to a single resultant force, their sum, applied at that point. Thus when doing a mechanics analysis you can replace a collection of forces at a point with their sum. If you think of your whole foot as a 'point' this justifies the replacement of the billions of little atomic ground contact forces with a single force.

On page 84 we discovered that a force applied at a different point is equivalent to the same force applied at a point displaced in the direction of the force. You can thus harmlessly slide the point of force application along the line of the force.

More generally, we can compare two sets of forces. The first set consists of $\vec{F}_1^{(1)}, \vec{F}_2^{(1)}, \vec{F}_3^{(1)}$, *etc.* applied at positions $\vec{r}_{1/C}^{(1)}, \vec{r}_{2/C}^{(1)}, \vec{r}_{3/C}^{(1)}$, *etc.* In short hand, these forces are $\vec{F}_i^{(1)}$ applied at positions $\vec{r}_{i/C}^{(1)}$, where each value of i describes a different force ($i = 7$ refers to the seventh force in the set). The second set of forces consists of $\vec{F}_j^{(2)}$ applied at positions $\vec{r}_{j/C}^{(2)}$ where each value of j describes a different force in the second set.

Now we compare the net (resultant) force and net moment of the two sets. If

$$\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)} \qquad \text{and} \qquad \vec{M}_{\text{C}}^{(1)} = \vec{M}_{\text{C}}^{(2)} \tag{2.48}$$

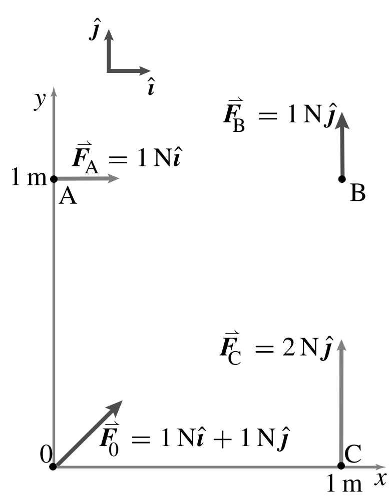

Figure 2.75: The force system $\vec{F}_A$, $\vec{F}_C$ is equivalent to the force system $\vec{F}_0$, $\vec{F}_B$.

then the two sets are *equivalent*. Here we have defined the net forces and net moments by

$$\vec{F}^{(1)}_{\text{tot}} = \sum_{\text{all forces } i} \vec{F}^{(1)}_i, \qquad \vec{M}^{(1)}_{\text{C}} = \sum_{\text{all forces } i} \vec{r}^{(1)}_{i/C} \times \vec{F}^{(1)}_i,$$
$$\vec{F}^{(2)}_{\text{tot}} = \sum_{\text{all forces } j} \vec{F}^{(2)}_j, \quad \text{and} \quad \vec{M}^{(2)}_{\text{C}} = \sum_{\text{all forces } j} \vec{r}^{(2)}_{j/C} \times \vec{F}^{(2)}_j.$$

If you find the $\sum$ (sum) symbol intimidating see box 2.11 on page 114.

Example:
Consider force system (1) with forces $\vec{F}_{\text{A}}$ and $\vec{F}_{\text{C}}$ and force system (2) with forces $\vec{F}_0$ and $\vec{F}_{\text{B}}$ as shown in fig. 2.75. Are the systems equivalent? First check the sum of forces.

$$\begin{aligned}
\vec{F}^{(1)}_{\text{tot}} &\stackrel{?}{=} \vec{F}^{(2)}_{\text{tot}} \\
\sum \vec{F}^{(1)}_i &\stackrel{?}{=} \sum \vec{F}^{(2)}_j \\
\vec{F}_{\text{A}} + \vec{F}_{\text{C}} &\stackrel{?}{=} \vec{F}_0 + \vec{F}_{\text{B}} \\
1\,\text{N}\hat{\imath} + 2\,\text{N}\hat{\jmath} &\stackrel{\checkmark}{=} (1\,\text{N}\hat{\imath} + 1\,\text{N}\hat{\jmath}) + 1\,\text{N}\hat{\jmath}
\end{aligned}$$

Then check the sum of moments about C.

$$\begin{aligned}
\vec{M}^{(1)}_{\text{C}} &\stackrel{?}{=} \vec{M}^{(2)}_{\text{C}} \\
\sum \vec{r}^{(1)}_{i/C} \times \vec{F}^{(1)}_i &\stackrel{?}{=} \sum \vec{r}^{(2)}_{j/C} \times \vec{F}^{(2)}_j \\
\vec{r}_{\text{A/C}} \times \vec{F}_{\text{A}} + \vec{r}_{\text{C/C}} \times \vec{F}_{\text{C}} &\stackrel{?}{=} \vec{r}_{0/\text{C}} \times \vec{F}_0 + \vec{r}_{\text{B/C}} \times \vec{F}_{\text{B}} \\
(-1\,\text{m}\hat{\imath} + 1\,\text{m}\hat{\jmath}) \times 1\,\text{N}\hat{\imath} + \vec{\mathbf{0}} \times 2\,\text{N}\hat{\jmath} &\stackrel{?}{=} (-1\,\text{m}\hat{\imath}) \times (1\,\text{N}\hat{\imath} + 1\,\text{N}\hat{\jmath}) + 1\,\text{m}\hat{\jmath} \times 1\,\text{N}\hat{\jmath} \\
-1\,\text{m}\,\text{N}\hat{k} &\stackrel{\checkmark}{=} -1\,\text{m}\,\text{N}\hat{k}
\end{aligned}$$

So the two force systems are indeed equivalent.

What is so special about the point C in the example above? Nothing.

> If two force systems are equivalent with respect to some point C, they are equivalent with respect to *any* point.

For example, both of the force systems in the example above have the same moment of $2\,\text{N}\,\text{m}\hat{k}$ about the point A. See box 2.12 for the proof of the general case.

Example: Frictionless wheel bearing
If the contact of an axle with a bearing housing is perfectly frictionless then each of the contact forces has no moment about the center of the wheel (fig. 2.76). Thus the whole force system is equivalent to a single force at the center of the wheel.

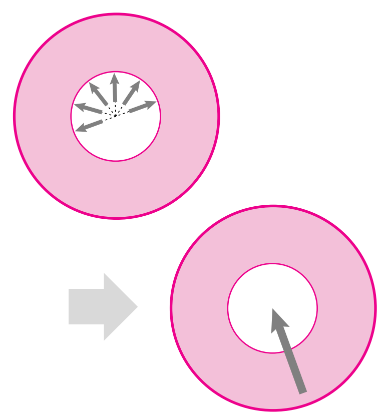

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Figure 2.76: Frictionless wheel bearing. All the bearing forces are *equivalent* to a single force acting at the center of the wheel.

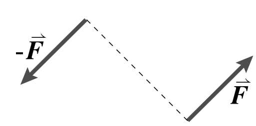

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Figure 2.77: One couple. The forces add to zero. The net moment they cause does not.

Couples

Consider a pair of equal and opposite forces that are not colinear. Such a pair is called a *couple*.The net moment caused by a couple is the size of the force times the perpendicular distance between the two lines of action and doesn't depend on the reference point. In fact, any force system that has $\vec{F}_{\text{tot}} = \vec{0}$ causes the same moment about all different reference points (as shown at the end of box 2.12).So, in modern usage, any force system with any number of forces and with $\vec{F}_{\text{tot}} = \vec{0}$ is called a couple. A couple is described by its net moment.②

②People who have been in difficult long term relationships don't need a mechanics text to know that a couple is a pair of equal and opposite forces that push each other around.

A *couple* is any force system that has a total force of $\vec{0}$. It is described by the net moment $\vec{M}$ that it causes.

We then think of $\vec{M}$ as representing an equivalent force system that contributes $\vec{0}$ to the net force and $\vec{M}$ to the net moment with respect to every reference point.

The concept of a couple (also called an applied moment or an applied torque) is especially useful for representing the net effect of a complicated collection of forces that causes some turning. The complicated set of electromagnetic forces turning a motor shaft can be replaced by a couple.

2.11 $\sum$ means add

In mechanics we often need to add up lots of things: all the forces on a body, all the moments they cause, all the mass of a system, *etc.* One notation for adding up all 14 forces on some body is

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 + \vec{F}_5 + \vec{F}_6 + \vec{F}_7 \\ &\quad + \vec{F}_8 + \vec{F}_9 + \vec{F}_{10} + \vec{F}_{11} + \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}.\end{aligned}$$

which is a bit long, so we might abbreviate it as

$$\vec{F}_{\text{net}} = \vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_{14}.$$

But this is definition by pattern recognition. A more explicit statement would be

$$\vec{F}_{\text{net}} = \text{The sum of all 14 forces} \vec{F}_i \text{ where } i = 1 \ldots 14$$

which is too space consuming. This kind of summing is so important that mathematicians use up a whole letter of the greek alphabet as a short hand for 'the sum of all'. They use the capital greek 'S' (for Sum) called sigma which looks like this:

$$\sum .$$

When you read $\sum$ aloud you don't say 'S' or'sigma' but rather 'the sum of.' The $\sum$ (sum) notation may remind you of infinite series, and convergence thereof. We will rarely be concerned with infinite sums in this book and never with convergence issues. So panic on those grounds is unjustified. We just want to easily write about adding things. For example we use the $\sum$ (sum) to write the sum of 14 forces $\vec{F}_i$ explicitly and concisely as

$$\sum_{i=1}^{14} \vec{F}_i$$

and say 'the sum of F sub i where i goes from one to fourteen'. Sometimes we don't know, say, how many forces are being added. We just want to add all of them so we write (a little informally)

$$\sum \vec{F}_i \quad \text{meaning} \quad \vec{F}_1 + \vec{F}_2 + etc.,$$

where the subscript i lets us know that the forces are numbered.

Rather than panic when you see something like $\sum_{i=1}^{14}$, just relax and think: oh, we want to add up a bunch of things all of which look like the next thing written. In general,

$$\sum (\text{thing})_i \quad \text{translates to} \quad (\text{thing})_1 + (\text{thing})_2 + (\text{thing})_3 + etc.$$

no matter how intimidating the 'thing' is. In time you can skip writing out the translation and will enjoy the concise notation.

Every system of forces is equivalent to a force and a couple

Given any point C, we can calculate the net moment of a system of forces relative to C. We then can replace the sum of forces with a single force at C and the net moment with a couple at C and we have an equivalent force system.

A force system is equivalent to a force $\vec{F} = \vec{F}_{\text{tot}}$ acting at C and a couple M equal to the net moment of the forces about C, *i.e.,* $\vec{M} = \vec{M}_{\text{C}}$.

If instead we want a force system at D we could recalculate the net moment about D or just use the translation formula (see box 2.12 on page 115).

$$\begin{aligned} \vec{F}_{\text{tot}} &= \vec{F}_{\text{tot}}, \quad \text{and} \\ \vec{M}_{\text{D}} &= \vec{M}_{\text{C}} + \vec{r}_{\text{C/D}} \times \vec{F}_{\text{tot}}. \end{aligned}$$

2.12 Two force systems that are equivalent for one reference point are equivalent for all reference points.

Consider two sets of forces $\vec{F}_i^{(1)}$ and $\vec{F}_j^{(2)}$ with corresponding points of application $\text{P}_i^{(1)}$ and $\text{P}_j^{(2)}$ at positions relative to the origin of $\vec{r}_i^{(1)}$ and $\vec{r}_j^{(2)}$. To simplify the discussion let's define the net forces of the two systems as

$$\vec{F}_{\text{tot}}^{(1)} \equiv \sum \vec{F}_i^{(1)} \quad \text{and} \quad \vec{F}_{\text{tot}}^{(2)} \equiv \sum \vec{F}_j^{(2)},$$

and the net moments about the origin as

$$\vec{M}_0^{(1)} \equiv \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} \quad \text{and} \quad \vec{M}_0^{(2)} \equiv \sum \vec{r}_j^{(2)} \times \vec{F}_j^{(2)}.$$

Using point 0 as a reference, the statement that the two systems are equivalent is then $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$ and $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$. Now consider point C with position $\vec{r}_{\text{C}} = \vec{r}_{\text{C}/0} = -\vec{r}_{0/\text{C}}$. What is the net moment of force system (1) about point C?

$$\begin{aligned} \vec{M}_{\text{C}}^{(1)} &\equiv \sum \vec{r}_{i/\text{C}}^{(1)} \times \vec{F}_i^{(1)} \\ &= \sum \left(\vec{r}_i^{(1)} - \vec{r}_{\text{C}}\right) \times \vec{F}_i^{(1)} \\ &= \sum \left(\vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_{\text{C}} \times \vec{F}_i^{(1)}\right) \\ &= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \sum \vec{r}_{\text{C}} \times \vec{F}_i^{(1)} \\ &= \sum \vec{r}_i^{(1)} \times \vec{F}_i^{(1)} - \vec{r}_{\text{C}} \times \left(\sum \vec{F}_i^{(1)}\right) \\ &= \vec{M}_0^{(1)} - \vec{r}_{\text{C}} \times \vec{F}_{\text{tot}}^{(1)}. \\ &= \vec{M}_0^{(1)} + \vec{r}_{0/\text{C}} \times \vec{F}_{\text{tot}}^{(1)}. \end{aligned}$$

[**Note.** The calculation above uses the 'move' of factoring a constant vector out of a sum. This mathematical move will be used again and again in the development of the theory of mechanics.] c

Similarly, for force system (2)

$$\vec{M}_{\text{C}}^{(2)} = \vec{M}_0^{(2)} + \vec{r}_{0/\text{C}} \times \vec{F}_{\text{tot}}^{(2)}.$$

If the two force systems are equivalent for reference point 0 then $\vec{F}_{\text{tot}}^{(1)} = \vec{F}_{\text{tot}}^{(2)}$ and $\vec{M}_0^{(1)} = \vec{M}_0^{(2)}$ and the expressions above imply that $\vec{M}_{\text{C}}^{(1)} = \vec{M}_{\text{C}}^{(2)}$. Because we specified nothing special about the point C, the systems are equivalent for *any* reference point. Thus, to demonstrate equivalence we need to use a reference point, but once equivalence is demonstrated we need not name the point since the equivalence holds for all points.

By the same reasoning we find that once we know the net force and net moment of a force system ($\vec{F}_{\text{tot}}$) relative to some point C (call it $\vec{M}_{\text{C}}$), we know the net moment relative to point D as

$$\vec{M}_{\text{D}} = \vec{M}_{\text{C}} + \vec{r}_{\text{C/D}} \times \vec{F}_{\text{tot}}.$$

Note that if the net force is $\vec{\mathbf{0}}$ (and the force system is then called a couple) that $\vec{M}_{\text{D}} = \vec{M}_{\text{C}}$ so the net moment is the same for all reference points.

The total force $\vec{F}_{\text{net}}$ stays the same and the moment at D is the moment at C plus the moment caused by $\vec{F}_{\text{net}}$ acting at position C relative to D. The net effect of the forces of the ground on a tree, for example, is of a force and a couple acting on the base of the tree.

Equivalent does not mean equivalent for all purposes

We have perhaps oversimplified.

Imagine you stayed up late studying and overslept. Your roommate was not so diligent; woke up on time and went to wake you by gently shaking you. Having read this chapter so far and no further, and being rather literal, your roommate gets down on the floor and presses on the linoleum underneath your bed applying a force that is *equivalent* to pressing on you. Obviously this is not equivalent in the ordinary sense of the word. It isn't even equivalent in all of its mechanics effects. One force moves you even if you don't wake up, and the other doesn't.

Any two force systems that are 'equivalent' but different *do* have different mechanical effects. In what sense are two force systems that have the same net force and the same net moment really equivalent?

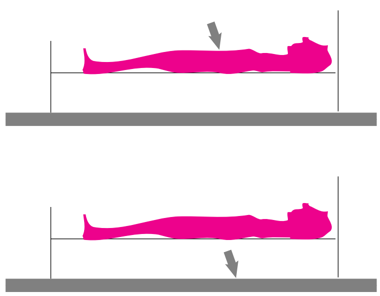

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Figure 2.78: It feels different if someone presses on you or presses on the floor underneath you with an 'equivalent' force. The equivalence of 'equivalent' force systems depends on them both being applied to the same system.

> 'Equivalent' force systems are equivalent in their contributions to the equations of mechanics (equations 0-II on the inside cover) for any system to which they are both applied.

But full mechanical analysis of a situation requires looking at the mechanics equations of many subsystems. In the mechanics equations for each subsystem, two 'equivalent' force systems are equivalent if they are both applied to that subsystem.

For the analysis of the subsystem that is you sleeping, the force of your roommate's hand on the floor isn't applied to you, so it doesn't show up in the mechanics equations for you, and doesn't have the same effect as a force on you.

2.13 The tidiest representation of a force system: A "wrench"

Any force system can be represented by an equivalent force and a couple at any point. But force systems can be reduced to simpler forms. That this is so is of more theoretical than practical import. We state the results here without proof.

In 2D one of these two things is true:

- The system is equivalent to a couple, or most often
- There is a line parallel to the force which the system can be described by an equivalent force with no couple.

In 3D one of these three things is true:

- The system is equivalent to a couple (applied anywhere), or
- The system is equivalent to a force (applied on a given line parallel to the force), or most often
- There is a line for which the system can be reduced to a force and a couple where the force, couple, and line are all parallel. The representation of the system of forces as a force and a parallel moment is called a *wrench*.

SAMPLE 2.32 Equivalent force on a particle: Four forces $\vec{F}_1 = 2\,\mathrm{N}\hat{\imath} - 1\,\mathrm{N}\hat{\jmath}$, $\vec{F}_2 = -5\,\mathrm{N}\hat{\jmath}$, $\vec{F}_3 = 3\,\mathrm{N}\hat{\imath} + 12\,\mathrm{N}\hat{\jmath}$, and $\vec{F}_4 = 1\,\mathrm{N}\hat{\imath}$ act on a particle. Find the equivalent force on the particle.

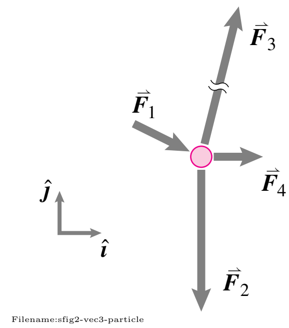


Figure 2.80:

Solution The equivalent force on the particle is the net force, *i.e.,* the vector sum of all forces acting on the particle. Thus,

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 \\
&= (2\,\mathrm{N}\hat{\imath} - 1\,\mathrm{N}\hat{\jmath}) + (-5\,\mathrm{N}\hat{\jmath}) + (3\,\mathrm{N}\hat{\imath} + 12\,\mathrm{N}\hat{\jmath}) + (1\,\mathrm{N}\hat{\imath}) \\
&= 6\,\mathrm{N}\hat{\imath} + 6\,\mathrm{N}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 6\,\mathrm{N}(\hat{\imath} + \hat{\jmath})}$$

Note that there is no net couple since all the four forces act at the same point. This is always true for particles. Thus, the equivalent force-couple system for particles consists of only the net force.

SAMPLE 2.33 Equivalent force with no net moment: In the figure shown, $F_1 = 50\,\mathrm{N}$, $F_2 = 10\,\mathrm{N}$, $F_3 = 30\,\mathrm{N}$, and $\theta = 60°$. Find the equivalent force-couple system about point D of the structure.

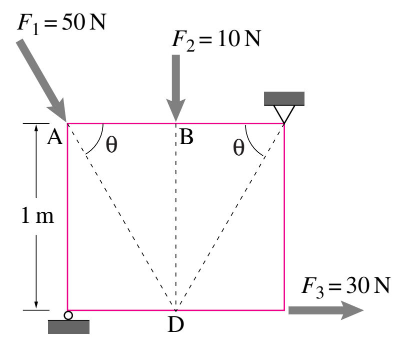


Figure 2.81:

Solution From the given geometry, we see that the three forces $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ pass through point D. Thus they are *concurrent* forces. Since point D is on the line of action of these forces, we can simply slide the three forces to point D without altering their mechanical effect on the structure. Then the equivalent force-couple system at point D consists of only the net force, $\vec{F}_{\text{net}}$, with no couple (the three forces passing through point D produce no moment about D). This is true for all concurrent forces. Thus,

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\
&= F_1(\cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath}) - F_2\hat{\jmath} + F_3\hat{\imath} \\
&= (F_1\cos\theta + F_3)\hat{\imath} - (F_1\sin\theta + F_2)\hat{\jmath} \\
&= (50\,\mathrm{N}\cdot\frac{1}{2} + 30\,\mathrm{N})\hat{\imath} - (50\,\mathrm{N}\cdot\frac{\sqrt{3}}{2} + 10\,\mathrm{N})\hat{\jmath} \\
&= 50\,\mathrm{N}\hat{\imath} - 53.3\,\mathrm{N}\hat{\jmath}, \\
\text{and}\quad \vec{M}_{\text{D}} &= \vec{0}.
\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = 50\,\mathrm{N}\hat{\imath} - 53.3\,\mathrm{N}\hat{\jmath},\ \vec{M}_{\text{D}} = \vec{0}}$$

Graphically, the solution is shown in Fig. 2.79

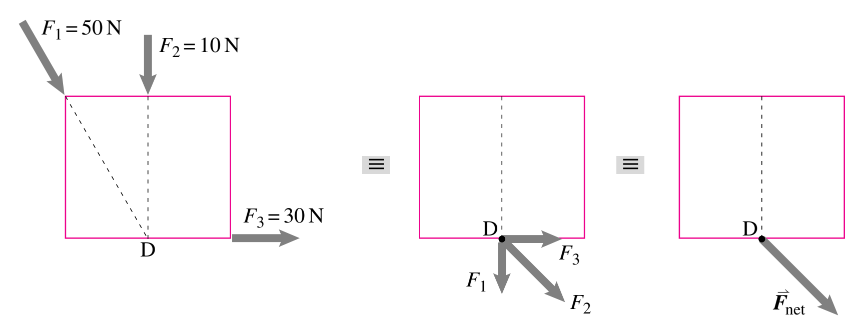


Figure 2.79:

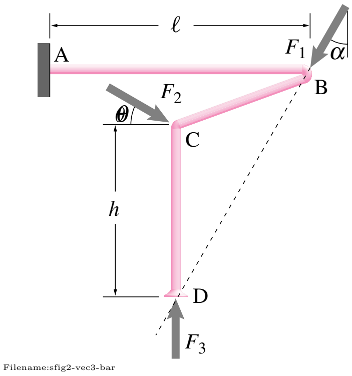


Figure 2.82:

SAMPLE 2.34 An equivalent force-couple system: Three forces $F_1 = 100\,\mathrm{N}$, $F_2 = 50\,\mathrm{N}$, and $F_3 = 30\,\mathrm{N}$ act on a structure as shown in the figure where $\alpha = 30°$, $\theta = 60°$, $\ell = 1\,\mathrm{m}$ and $h = 0.5\,\mathrm{m}$. Find the equivalent force-couple system about point D.

Solution The net force is the sum of all applied forces, *i.e.*,

$$\begin{aligned}
\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\
&= F_1(-\sin\alpha\hat{\imath} - \cos\alpha\hat{\jmath}) + F_2(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) + F_3\hat{\jmath} \\
&= (-F_1\sin\alpha + F_2\cos\theta)\hat{\imath} + (-F_1\cos\alpha - F_2\sin\theta + F_3)\hat{\jmath} \\
&= (-100\,\mathrm{N}\cdot\frac{1}{2} + 50\,\mathrm{N}\cdot\frac{1}{2})\hat{\imath} + (-100\,\mathrm{N}\cdot\frac{\sqrt{3}}{2} - 50\,\mathrm{N}\cdot\frac{\sqrt{3}}{2} + 30\,\mathrm{N})\hat{\jmath} \\
&= -25\,\mathrm{N}\hat{\imath} - 99.9\,\mathrm{N}\hat{\jmath}.
\end{aligned}$$

Forces $\vec{F}_1$ and $\vec{F}_3$ pass through point D. Therefore, they do not produce any moment about D. So, the net moment about D is the moment caused by force $\vec{F}_2$:

$$\begin{aligned}
\vec{M}_{\mathrm{D}} &= \vec{r}_{\mathrm{C/D}} \times \vec{F}_2 \\
&= h\hat{\jmath} \times F_2(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= -F_2 h\cos\theta\hat{k} \\
&= -50\,\mathrm{N}\cdot 0.5\,\mathrm{m}\frac{1}{2}\hat{k} = -12.5\,\mathrm{N{\cdot}m}\hat{k}.
\end{aligned}$$

The equivalent force-couple system is shown in Fig. 2.83

$$\boxed{\vec{F}_{\text{net}} = -25\,\mathrm{N}\hat{\imath} - 99.9\,\mathrm{N}\hat{\jmath} \quad \text{and} \quad \vec{M}_{\mathrm{D}} = -12.5\,\mathrm{N{\cdot}m}\hat{k}}$$

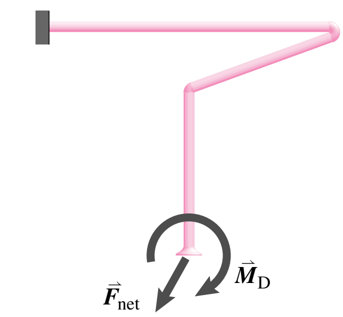


Figure 2.83:

SAMPLE 2.35 Translating a force-couple system: The net force and couple acting about point O on the 'L' shaped bar shown in the figure are 100 N and 20 N·m, respectively. Find the net force and moment about point G.

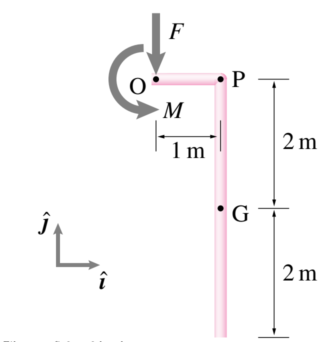


Figure 2.84:

Solution The net force on a structure is the same about any point since it is just the vector sum of all the forces acting on the structure and is independent of their point of application. Therefore,

$$\vec{F}_{\text{net}} = \vec{F} = -100\,\mathrm{N}\hat{\jmath}.$$

The net moment about a point, however, depends on the location of points of application of the forces with respect to that point. Thus,

$$\begin{aligned}
\vec{M}_{\mathrm{G}} &= \vec{M}_{\mathrm{O}} + \vec{r}_{\mathrm{O/G}} \times \vec{F} \\
&= M\hat{k} + (-\ell\hat{\imath} + h\hat{\jmath}) \times (-F\hat{\jmath}) \\
&= (M + F\ell)\hat{k} \\
&= (20\,\mathrm{N{\cdot}m} + 100\,\mathrm{N}\cdot 1\,\mathrm{m})\hat{k} = 120\,\mathrm{N{\cdot}m}\hat{k}.
\end{aligned}$$

$$\boxed{\vec{F}_{\text{net}} = -100\,\mathrm{N}\hat{\jmath}, \quad \text{and} \quad \vec{M}_{\mathrm{G}} = 120\,\mathrm{N{\cdot}m}\hat{k}}$$

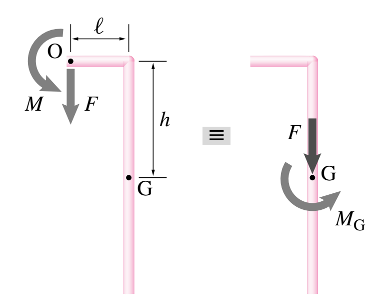


Figure 2.85:

SAMPLE 2.36 Checking equivalence of force-couple systems: In the figure shown below, which of the force-couple systems shown in (b), (c), and (d) are equivalent to the force system shown in (a)?

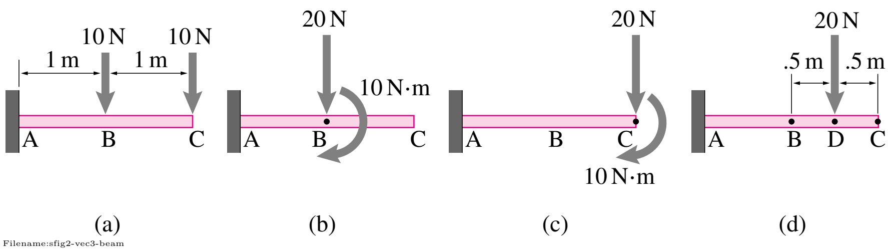


Figure 2.86:

Solution The equivalence of force-couple systems require that (i) the net force be the same, and (ii) the net moment about any reference point be the same. For the given systems, let us choose point B as our reference point for comparing their equivalence. For the force system shown in Fig. 2.86(a), we have,

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_1 + \vec{F}_2 = -10\,\text{N}\hat{\jmath} - 10\,\text{N}\hat{\jmath} = -20\,\text{N}\hat{\jmath}\\ \vec{M}_{\text{B}_{\text{net}}} &= \vec{r}_{\text{C/B}} \times \vec{F}_2 = 1\,\text{m}\hat{\imath} \times (-10\,\text{N}\hat{\jmath}) = -10\,\text{N}{\cdot}\text{m}\hat{k}.\end{aligned}$$

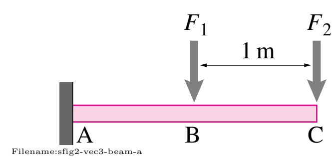


Figure 2.87:

Now, we can compare the systems shown in (b), (c), and (d) against the computed equivalent force-couple system, $\vec{F}_{\text{net}}$ and $\vec{M}_{\text{D}}$.

- Figure (b) shows exactly the system we calculated. Therefore, it represents an equivalent force-couple system.
- Figure (c): Let us calculate the net force and moment about point B for this system.

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_{\text{C}} \stackrel{\checkmark}{=} -20\,\text{N}\hat{\jmath}\\ \vec{M}_{\text{B}} &= \vec{M}_{\text{C}} + \vec{r}_{\text{C/B}} \times \vec{F}_{\text{C}}\\ &= -10\,\text{N}{\cdot}\text{m}\hat{k} + 1\,\text{m}\hat{\imath} \times (-20\,\text{N}\hat{\jmath}) = -30\,\text{N}{\cdot}\text{m}\hat{k} \neq \vec{M}_{\text{B}_{\text{net}}}.\end{aligned}$$

  Thus, the given force-couple system in this case is not equivalent to the force system in (a).

- Figure (d): Again, we compute the net force and the net couple about point B:

$$\begin{aligned}\vec{F}_{\text{net}} &= \vec{F}_{\text{D}} \stackrel{\checkmark}{=} -20\,\text{N}\hat{\jmath}\\ \vec{M}_{\text{B}} &= \vec{r}_{\text{D/B}} \times \vec{F}_{\text{D}}\\ &= 0.5\,\text{m}\hat{\imath} \times (-20\,\text{N}\hat{\jmath}) = -10\,\text{N}{\cdot}\text{m}\hat{k} \stackrel{\checkmark}{=} \vec{M}_{\text{B}_{\text{net}}}.\end{aligned}$$

  Thus, the given force-couple system (with zero couple) at D is equivalent to the force system in (a).

(b) and (d) are equivalent to (a); (c) is not.

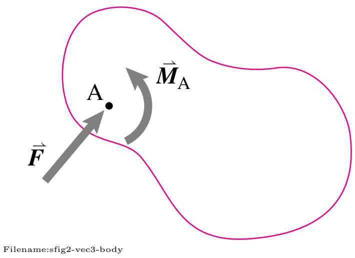


Figure 2.88:

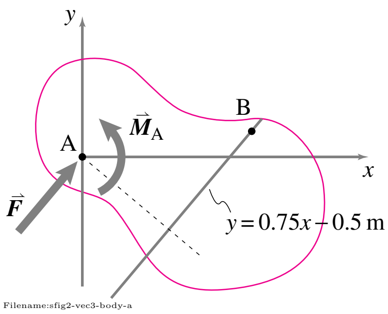


Figure 2.89:

SAMPLE 2.37 Equivalent force with no couple: For a body, an equivalent force-couple system at point A consists of a force $\vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$ and a couple $\vec{M}_A = 10\,\text{N·m}\hat{k}$. Find a point on the body such that the equivalent force-couple system at that point consists of only a force (zero couple).

Solution The net force in the two equivalent force-couple systems has to be the same. Therefore, for the new system, $\vec{F}_{\text{net}} = \vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$. Let B be the point at which the equivalent force-couple system consists of only the net force, with zero couple. We need to find the location of point B. Let A be the origin of a xy coordinate system in which the coordinates of B are (x, y). Then, the moment about point B is,

$$\begin{aligned}
\vec{M}_B &= \vec{M}_A + \vec{r}_{A/B} \times \vec{F} \\
&= M_A\hat{k} + (-x\hat{\imath} - y\hat{\jmath}) \times (F_x\hat{\imath} + F_y\hat{\jmath}) \\
&= M_A\hat{k} + (-F_y x + F_x y)\hat{k}.
\end{aligned}$$

Since we require that $\vec{M}_B$ be zero, we must have

$$\begin{aligned}
F_y x - F_x y &= M_A \\
\Rightarrow \quad y &= \frac{F_y}{F_x}x - \frac{M_A}{F_x} \\
&= \frac{15\,\text{N}}{20\,\text{N}}x - \frac{10\,\text{N·m}}{20\,\text{N}} \\
&= 0.75x - 0.5\,\text{m}.
\end{aligned}$$

This is the equation of a line. Thus, we can select any point on this line and apply the force $\vec{F} = 20\,\text{N}\hat{\imath} + 15\,\text{N}\hat{\jmath}$ with zero couple as an equivalent force-couple system.

Any point on the line $y = 0.75x - 0.5\,\text{m}$.

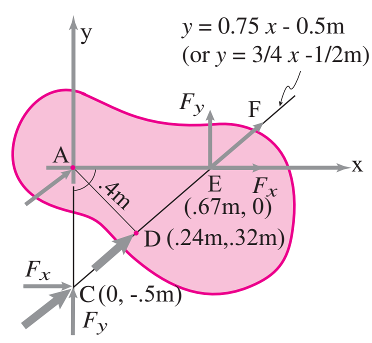


Figure 2.90:

So, how or why does it work? The line we obtained is shown in gray in Fig. 2.89. Note that this line has the same slope as that of the given force vector (slope $= 0.75 = F_y/F_x$) and the offset is such that shifting the force $\vec{F}$ to this line counter balances the given couple at A. To see this clearly, let us select three points C, D, and E on the line as shown in Fig. 2.90. From the equation of the line, we find the coordinates of C(0,-.5m), D(.24m,.32m) and E(.67m,0). Now imagine moving the force $\vec{F}$ to C, D, or E. In each case, it must produce the same moment $\vec{M}_A$ about point A. Let us do a quick check.

$\vec{F}$

- at point C: The moment about point A is due to the horizontal component $F_x = 20\,\text{N}$, since F_y passes through point A. The moment is $F_x \cdot AC = 20\,\text{N} \cdot 0.5\,\text{m} = 10\,\text{N·m}$, same as M_A. The direction is counterclockwise as required.

$\vec{F}$

- at point D: The moment about point A is $|\vec{F}| \cdot AD = 25\,\text{N} \cdot 0.4\,\text{m} = 10\,\text{N·m}$, same as M_A. The direction is counterclockwise as required.

$\vec{F}$

- at point E: The moment about point A is due to the vertical component F_y, since F_x passes through point A. The moment is $F_y \cdot AE = 15\,\text{N} \cdot 0.67\,\text{m} = 10\,\text{N·m}$, same as M_A. The direction here too is counterclockwise as required.

Once we check the calculation for one point on the line, we should not have to do any more checks since we know that sliding the force along its line of action (line CB) produces no couple and thus preserves the equivalence.

2.7 Center of mass and gravity

For every system and at every instant in time, there is a unique location in space that is the average position of the system's mass. This place is called the *center of mass*, commonly designated by cm, c.o.m., COM, G, c.g., or .

One of the routine but important tasks of many real engineers is to find the center-of-mass of a complex machine. Just knowing the location of the center-of-mass of a car, for example, is enough to estimate whether it can be tipped over by maneuvers on level ground. The center-of-mass of a boat must be low enough for the boat to be stable. Any propulsive force on a space craft must be directed towards the center-of-mass in order to not induce rotations. Tracking the trajectory of the center-of-mass of an exploding plane can determine whether or not it was hit by a massive object. Any rotating piece of machinery must have its center-of-mass on the axis of rotation if it is not to cause much vibration.

Also, many calculations in mechanics are greatly simplified by making use of a system's center-of-mass. In particular, the whole complicated distribution of near-earth gravity forces on a body is equivalent to a single force at the body's center-of-mass. Many of the important quantities in dynamics are similarly simplified using the center-of-mass.

The center-of-mass of a system is the point at the position $\vec{\boldsymbol{r}}_{\text{cm}}$ defined by

$$\begin{aligned} \vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\sum \vec{\boldsymbol{r}}_{\text{i}} m_i}{m_{\text{tot}}} \quad \text{for discrete systems} \\ &= \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} \quad \text{for continuous systems} \end{aligned} \tag{2.49}$$

where $m_{\text{tot}} = \sum m_i$ for discrete systems and $m_{\text{tot}} = \int dm$ for continuous systems (see boxes 2.11 and 2.14 on pages 114 and 122 for a discussion of the $\sum$ and $\int$ sum notations).

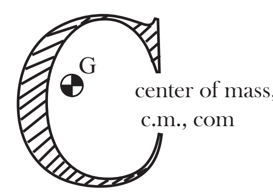
Often it is convenient to remember the rearranged definition of center of mass as the position that, when multiplied by the total mass gives the same result as all the sum of all the mass bits each multiplied by their positions:

$$m_{\text{tot}} \vec{\boldsymbol{r}}_{\text{cm}} = \sum m_i \vec{\boldsymbol{r}}_{\text{i}} \quad \text{or} \quad m_{\text{tot}} \vec{\boldsymbol{r}}_{\text{cm}} = \int \vec{\boldsymbol{r}}\, dm.$$

For theoretical purposes we rarely need to evaluate these sums and integrals, and for simple problems there are sometimes shortcuts that reduce the calculation to a matter of observation. For complex machines one or both of the formulas 2.49 must be evaluated in detail [1].

Example: System of two point masses

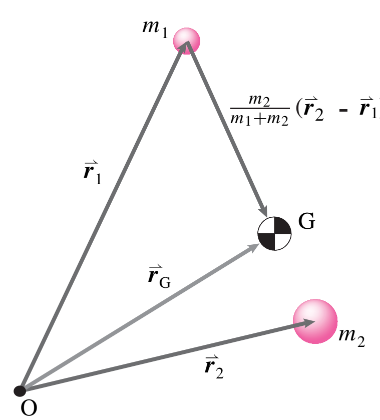
Intuitively, the center-of-mass of the two masses shown in fig. 2.92 is between the two masses and closer to the larger one. Referring to equation 2.49,



Filename:tfig-C-centerofmass

Figure 2.91: The center of mass of the ink making up the letter C is approximately at the location shown. The various notations for center of mass include: G, COM and cm.

[1] Nowadays this routine work is often done with CAD (computer aided design) software. But an engineer still needs to know the basic calculation skills in order to make make sense of, and roughly check, the computer calculations and in order to think of appropriate design changes.



Filename:tfigure3-com-twomass

Figure 2.92: Center of mass of a system consisting of two points.

$$
\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{\sum \vec{r}_i m_i}{m_{tot}} \\
&= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2} \\
&= \frac{\vec{r}_1 (m_1 + m_2) - \vec{r}_1 m_2 + \vec{r}_2 m_2}{m_1 + m_2} \\
&= \vec{r}_1 + \underbrace{\left(\frac{m_2}{m_1 + m_2}\right)}_{\text{the fraction of the distance that the cm is from } \vec{r}_1 \text{ to } \vec{r}_2} \underbrace{(\vec{r}_2 - \vec{r}_1)}_{\text{the vector from } \vec{r}_1 \text{ to } \vec{r}_2.} .
\end{aligned}
$$

so that the math agrees with common sense — the center-of-mass is on the line connecting the masses. If $m_2 \gg m_1$, then the center-of-mass is near m_2. If $m_1 \gg m_2$, then the center-of-mass is near m_1. If $m_1 = m_2$ the center-of-mass is right in the middle at $(\vec{r}_1 + \vec{r}_2)/2$.

2.14 Like $\sum$, the symbol $\int$ also means add

We often add things up in mechanics. For example, the total mass of some particles is

$$m_{\text{tot}} = m_1 + m_2 + m_3 + \cdots = \sum m_i$$

or more specifically the mass of 137 particles is, say, $m_{\text{tot}} = \sum_{i=1}^{137} m_i$. And the total mass of a bicycle is:

$$m_{\text{bike}} = \sum_{i=1}^{100{,}000{,}000{,}000{,}000{,}000{,}000{,}000} m_i$$

where m_i are the masses of each of the 10^{23} (or so) atoms of metal, rubber, plastic, cotton, and paint. But atoms are so small and there are so many of them. Instead we often think of a bike as built of macroscopic parts. The total mass of the bike is then the sum of the masses of the tires, the tubes, the wheel rims, the spokes and nipples, the ball bearings, the chain pins, and so on. And we would write:

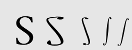
$$m_{\text{bike}} = \sum_{i=1}^{2{,}000} m_i$$

where now the m_i are the masses of the 2,000 or so bike parts. This sum is more manageable but still too detailed in concept for some purposes.

An approach that avoids attending to atoms or ball bearings, is to think of sending the bike to a big shredding machine that cuts it up into very small bits. Now we write

$$m_{\text{bike}} = \sum m_i$$

where the m_i are the masses of the very small bits. We don't fuss over whether one bit is a piece of ball bearing or fragment of cotton from the tire walls. We just chop the bike into bits and add up the contribution of each bit. If you take the letter S, as in SUM, and distort it and you get a big old fashioned German 'S' used in calculus as the integral sign



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So we write

$$m_{\text{bike}} = \int dm$$

to mean the $\int$um of all the teeny bits of mass. More formally we mean the value of that sum in the limit that all the bits are infinitesimal (not minding the technical fine point that its hard to chop atoms into infinitesimal pieces).

The mass is one of many things we would like to add up, though many of the others also involve mass. In center-of-mass calculations, for example, we add up the positions 'weighted' by mass.

$$\int \vec{r}\, dm \quad \text{which means} \quad \sum_{\lim m_i \to 0} \vec{r}_i m_i .$$

That is, you take your object of interest and chop it into a billion pieces and then re-assemble it. For each piece you make the vector which is the position vector of the piece multiplied by ('weighted by') its mass and then add up the billion vectors. Well really you chop the thing into a trillion trillion ... pieces, but a billion gives the idea.

Continuous systems

How do we evaluate integrals like $\int$(something) dm? In center-of-mass calculations, (something) is position, but we will evaluate similar integrals where (something) is some other scalar or vector function of position. Most often we label the material by its spatial position, and evaluate dm in terms of increments of position. For 3D solids $dm = \rho dV$ where ρ is density (mass per unit volume). So $\int$(something)dm turns into a standard volume integral $\int_V$(something)$\rho\, dV$②.For thin flat things like metal sheets we often take ρ to mean mass per unit area A so then $dm = \rho dA$ and $\int$(something)$dm = \int_A$(something)$\rho\, dA$. For mass distributed along a line or curve we take ρ to be the mass per unit length or arc length s and so $dm = \rho ds$ and $\int$(something)$dm = \int_{curve}$(something)$\rho\, ds$.

②Note: writing $\int_{m_1}^{m_2}$(something) dm is nonsense because m is not a scalar parameter which labels points in a material (there is no point at $m = 3\,\text{kg}$).

Example. The center-of-mass of a uniform rod is naturally in the middle, as the calculations here show (see fig. 2.93a). Assume the rod has length $L = 3\,\text{m}$ and mass $m = 7\,\text{kg}$.

$$\vec{r}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} = \frac{\int_0^L x\hat{\boldsymbol{\imath}} \overbrace{\rho dx}^{dm}}{\int_0^L \rho\, dx} = \frac{\rho(x^2/2)|_0^L}{\rho(1)|_1^L}\hat{\boldsymbol{\imath}} = \frac{\rho(L^2/2)}{\rho L}\hat{\boldsymbol{\imath}} = (L/2)\hat{\boldsymbol{\imath}}$$

So $\vec{r}_{\text{cm}} = (L/2)\hat{\boldsymbol{\imath}}$, or by dotting with $\hat{\boldsymbol{\imath}}$ (taking the x component) we get that the center-of-mass is on the rod a distance $d = L/2 = 1.5\,\text{m}$ from the end.

The center-of-mass calculation is *objective*. It describes something about the object that does not depend on the coordinate system. In different coordinate systems the center-of-mass for the rod above will have different coordinates, but it will always be at the middle of the rod.

Example. Find the center-of-mass using the coordinate system with s & $\hat{\boldsymbol{\lambda}}$ in fig. 2.93b:

$$\vec{r}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} = \frac{\int_0^L s\hat{\boldsymbol{\lambda}}\, \rho ds}{\int_0^L \rho\, ds}\hat{\boldsymbol{\lambda}} = \frac{\rho(s^2/2)|_0^L}{\rho(1)|_0^L}\hat{\boldsymbol{\lambda}} = \frac{\rho(L^2/2)}{\rho L}\hat{\boldsymbol{\lambda}} = (L/2)\hat{\boldsymbol{\lambda}},$$

again showing that the center-of-mass is in the middle.

Note, one can treat the center-of-mass vector calculations as separate scalar equations, one for each component. For example:

$$\hat{\boldsymbol{\imath}} \cdot \left\{ \vec{r}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} \right\} \quad \Rightarrow \quad r_{x\text{cm}} = x_{\text{cm}} = \frac{\int x\, dm}{m_{\text{tot}}}.$$

Finally, there is no law that says you have to use the best coordinate system. One is free to make trouble for oneself and use an inconvenient coordinate system.

Example. Use the xy coordinates of fig. 2.93c to find the center-of-mass of the

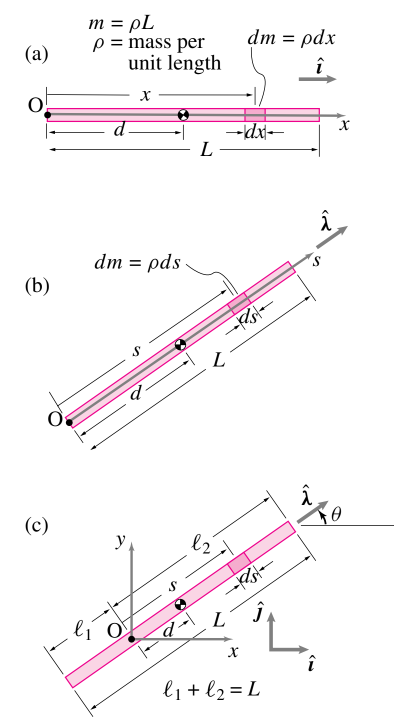


Figure 2.93: Where is the center-of-mass of a uniform rod? In the middle, as you can find calculating a few ways or by symmetry.

rod.

$$x_{cm} = \frac{\int x\,dm}{m_{tot}} = \frac{\int_{-\ell_1}^{\ell_2} \overbrace{s\cos\theta}^{x}\ \rho\,ds}{\int_0^L \rho\,ds} = \frac{\rho\cos\theta\,\frac{s^2}{2}\big|_{-\ell_1}^{\ell_2}}{\rho(1)\big|_{-\ell_1}^{\ell_2}} = \frac{\rho\cos\theta\,\frac{(\ell_2^2-\ell_1^2)}{2}}{\rho(\ell_1+\ell_2)}$$
$$= \frac{\cos\theta(\ell_2-\ell_1)}{2}$$

Similarly $y_{cm} = \sin\theta(\ell_2-\ell_1)/2$ so

$$\vec{r}_{cm} = \frac{\ell_2-\ell_1}{2}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

which is at the middle of the rod.

The most commonly needed center-of-mass that can be found analytically but not directly from symmetry is that of a triangle (see box 2.16 on page 130). In your calculus text you will find more examples of finding the center-of-mass using integration.

Center of mass and centroid

For objects with uniform material density we have

$$\vec{r}_{cm} = \frac{\int \vec{r}\,dm}{m_{tot}} = \frac{\int_V \vec{r}\rho\,dV}{\int_V \rho\,dV} = \frac{\rho\int_V \vec{r}\,dV}{\rho\int_V dV} = \frac{\int_V \vec{r}\,dV}{V}$$

where the last expression is just the formula for geometric centroid. Analogous calculations hold for 2D and 1D geometric objects.

For objects with density that does not vary from point to point, the geometric centroid and the center-of-mass coincide.

Center of mass and symmetry

The center-of-mass respects any symmetry in the mass distribution of a system. If the word ‘middle’ has unambiguous meaning in English then that is the location of the center-of-mass, as for the rod of fig. 2.93 and the other examples in fig. 2.95.

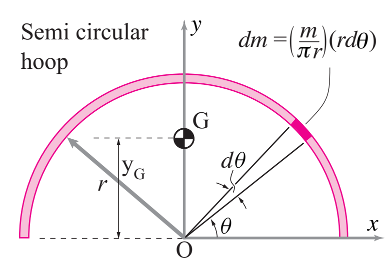


Figure 2.94: The center of mass of a simicircular arc is about $0.6\,r$ from the center of the circle.

Example: Center of mass of a semicircle.

(see fig. 2.94) The center of mass of a semicircular arc of mass m and radius r is on the y axis at $x = 0$, by symmetry. The location of y_G is found by

$$y_G = \frac{\int y\,dm}{m_{tot}} = \frac{1}{m}\int_0^\pi \overbrace{r\sin\theta}^{y}\ \overbrace{\rho\,ds}^{dm} = \frac{1}{m}\int_0^\pi r\sin\theta\ \overbrace{\frac{m}{\pi r}}^{\rho}\ \overbrace{r\,d\theta}^{ds}$$
$$= \frac{r}{\pi}\int_0^\pi \sin\theta\,d\theta = \frac{2}{\pi}r \approx 0.64\,r$$

The center of mass of a semicircle is almost 2/3 of the way towards the perimeter from the center of the circle. You can see that G has to be above the halfway point by noticing how much more mass is near to $y = r$ (where the circular arc is nearly horizontal) than to $y = 0$ (where the circular arc is running away from the x axis).

Systems of systems and composite objects

Another way of interpreting the formula

$$\vec{r}_{\text{cm}} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots}{m_1 + m_2 + \cdots}$$

is that the m's are the masses of subsystems, not just points, and that the $\vec{r}_i$ are the positions of the centers of mass of these systems. This subdivision is justified in box 2.15 on page 126. The center-of-mass of a single complex shaped object can be found by treating it as an assembly of simpler objects.

Example: Two rods

The center-of-mass of two rods shown in fig. 2.96 can be found as

$$\vec{r}_{\text{cm}} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the centers of mass of each rod and m_1 and m_2 are the masses.

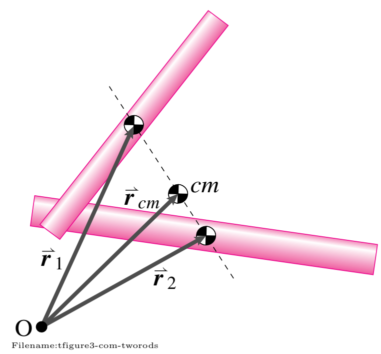


Figure 2.96: Center of mass of two rods

Example: 'L' shaped plate

Consider the plate with uniform mass per unit area ρ.

$$\begin{aligned}
\vec{r}_G &= \frac{\vec{r}_I m_I + \vec{r}_{II} m_{II}}{m_I + m_{II}} \\
&= \frac{(\frac{a}{2}\hat{\imath} + a\hat{\jmath})(2\rho a^2) + (\frac{3}{2}a\hat{\imath} + \frac{a}{2}\hat{\jmath})(\rho a^2)}{(2\rho a^2) + (\rho a^2)} \\
&= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).
\end{aligned}$$

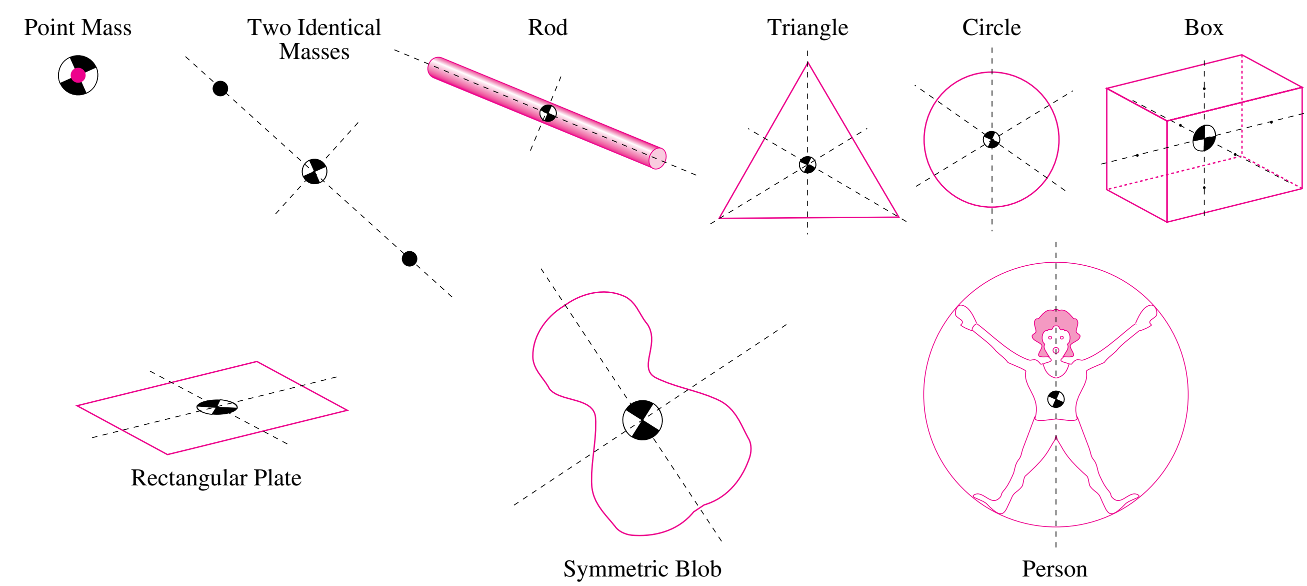


Figure 2.95: The center of mass and the geometric centroid share the symmetries of the object.

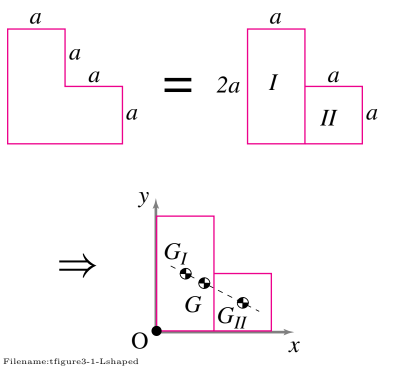


Figure 2.97: The center-of-mass of the 'L' shaped object can be found by thinking of it as a rectangle plus a square.

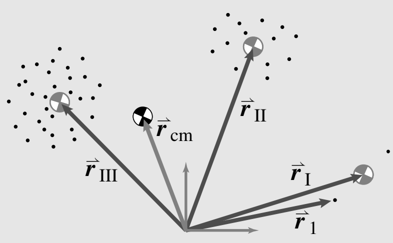
Composite objects using subtraction

It is sometimes useful to think of an object as composed of pieces, some of which have negative mass.

Example: 'L' shaped plate, again
Reconsider the plate from the previous example.

$$
\begin{aligned}
\vec{r}_{\mathrm{G}} &= \frac{\vec{r}_{\mathrm{I}} m_I + \vec{r}_{\mathrm{II}} m_{II}}{m_I + m_{II}} \\
&= \frac{(a\hat{\imath} + a\hat{\jmath})(\rho(2a)^2) + (\frac{3}{2}a\hat{\imath} + \frac{3}{2}a\hat{\jmath})\overbrace{(-\rho a^2)}^{m_{II}}}{(\rho(2a)^2) + \underbrace{(-\rho a^2)}_{m_{II}}} \\
&= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).
\end{aligned}
$$

2.15 Why can subsystems be treated like particles when finding the center-of-mass?



Look at the collection of 47 particles above and then think of it as a set of three subsystems: I, II, and III with 2, 14, and 31 particles, respectively. Masses 1 and 2 as subsystem I with center-of-mass $\vec{r}_{\mathrm{I}}$ and total mass m_I. Similarly, we call subsystem II masses m_3 to m_{16}, and subsystem III, masses m_{17} to m_{47}. We can calculate the center-of-mass of the system by treating it as 47 particles, or we can re-arrange the sum as follows:

$$
\begin{aligned}
\vec{r}_{\mathrm{cm}} &= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots + \vec{r}_{46} m_{46} + \vec{r}_{47} m_{47}}{m_1 + m_2 + \cdots + m_{47}} \\
&= \frac{\frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}(m_1 + m_2)}{m_1 + m_2 + \cdots + m_{47}} \\
&\quad + \frac{\frac{\vec{r}_3 m_3 + \cdots + \vec{r}_{16} m_{16}}{m_3 + \cdots + m_{16}}(m_3 + \cdots + m_{16})}{m_1 + m_2 + \cdots + m_{47}} \\
&\quad + \frac{\frac{\vec{r}_{17} m_{17} + \cdots \vec{r}_{47} m_{47}}{m_{17} + \cdots + m_{47}}(m_{17} + \cdots + m_{47})}{m_1 + m_2 + \cdots + m_{47}} \\
&= \frac{\vec{r}_{\mathrm{I}} m_I + \vec{r}_{\mathrm{II}} m_{II} + \vec{r}_{\mathrm{III}} m_{III}}{m_I + m_{II} + m_{III}}, \text{ where}
\end{aligned}
$$

$$
\begin{aligned}
\text{where } \vec{r}_{\mathrm{I}} &= \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}, \\
m_I &= m_1 + m_2 \\
\vec{r}_{\mathrm{II}} &= etc.
\end{aligned}
$$

That is, the center of mass of the 47 particles is the same as the center of mass of three particles, where each of the three particles has the total mass of its subsystem located at its subsystem's center of mass.

The reduction of subsystem of particles to one particle is easily generalized to the integral formulae as well like this.

$$
\begin{aligned}
\vec{r}_{\mathrm{cm}} &= \frac{\int \vec{r}\, dm}{\int dm} \\
&= \frac{\int_{\text{region 1}} \vec{r}\, dm + \int_{\text{region 2}} \vec{r}\, dm + \int_{\text{region 3}} \vec{r}\, dm + \cdots}{\int_{\text{region 1}} dm + \int_{\text{region 2}} dm + \int_{\text{region 1}} dm + \cdots} \\
&= \frac{\vec{r}_{\mathrm{I}} m_I + \vec{r}_{\mathrm{II}} m_{II} + \vec{r}_{\mathrm{III}} m_{III} + \cdots}{m_I + m_{II} + m_{III} + \cdots}.
\end{aligned}
$$

The general idea of the calculations above is that center-of-mass calculations are basically big sums (addition), and addition is 'associative.'

Center of gravity

The force of gravity on each little bit of an object is gm_i where g is the local gravitational 'constant' and m_i is the mass of the bit. For objects that are small compared to the radius of the earth (a reasonable assumption for all but a few special engineering calculations) the gravity constant is indeed constant from one point on the object to another (see box A.3 on page A.3 for a discussion of the meaning and history of g.)

Not only that, all the gravity forces point in the same direction, down. For engineering purposes, the two intersecting lines that go from your two hands to the center of the earth are parallel.) Lets call this the $-\hat{\boldsymbol{k}}$ direction. So the net force of gravity on an object is:

$$\begin{aligned}\vec{\boldsymbol{F}}_{\text{net}} &= \sum \vec{\boldsymbol{F}}_i = \sum m_i g(-\hat{\boldsymbol{k}}) = -mg\hat{\boldsymbol{k}} \quad \text{(discrete systems)}\\ &= \int d\vec{\boldsymbol{F}} = \int \underbrace{-g\hat{\boldsymbol{k}}\,dm}_{d\vec{\boldsymbol{F}}} = -mg\hat{\boldsymbol{k}} \quad \text{(continuous systems)}\end{aligned}$$

That's easy, the billions of gravity forces on an objects microscopic constituents add up to mg pointed down. What about the net moment of the gravity forces? The answer turns out to be simple. The top line of the calculation below poses the question, the last line gives the lucky answer.③

$$\begin{aligned}\vec{\boldsymbol{M}}_{\text{C}} &= \int \vec{\boldsymbol{r}} \times d\vec{\boldsymbol{F}} && \text{The net moment with respect to C.}\\ &= \int \vec{\boldsymbol{r}}_{/\text{C}} \times \left(-g\hat{\boldsymbol{k}}\,dm\right) && \text{A force bit is gravity acting on a mass bit.}\\ &= \left(\int \vec{\boldsymbol{r}}_{/\text{C}}dm\right) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Distributive law (g \& $\hat{\boldsymbol{k}}$ are constants).}\\ &= (\vec{\boldsymbol{r}}_{\text{cm}/\text{C}}m) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Definition of center-of-mass.}\\ &= \vec{\boldsymbol{r}}_{\text{cm}/\text{C}} \times \left(-mg\hat{\boldsymbol{k}}\right) && \text{Re-arranging terms.}\\ &= \vec{\boldsymbol{r}}_{\text{cm}/\text{C}} \times \vec{\boldsymbol{F}}_{\text{net}} && \text{Express in terms of net gravity force.}\end{aligned}$$

Thus the net moment is the same as for the total gravity force acting at the center-of-mass.

> The near-earth gravity forces acting on a system are *equivalent* to a single force, mg, acting at the system's center-of-mass.

For the purposes of calculating the net force and moment from near-earth (constant g) gravity forces, a system can be replaced by a point mass at the center of gravity. The words 'center-of-mass' and 'center of gravity' both describe the same point in space.

Although the result we have just found seems plain enough, here are two things to ponder about gravity when viewed as an inverse square law (and thus not constant like we have assumed) that may make the result above seem less obvious.

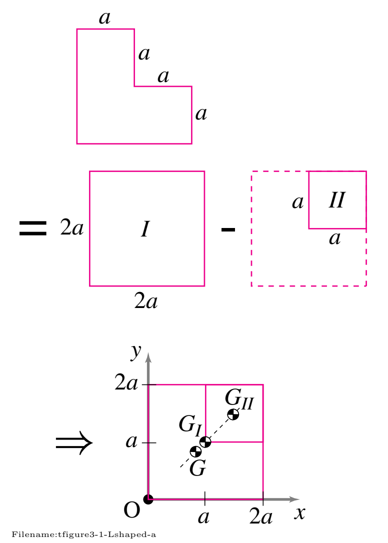


Figure 2.98: Another way of looking at the 'L' shaped object is as a square minus a smaller square in its upper right-hand corner.

③We do the calculation here using the $\int$ notation for sums. But it could be done just as well using $\sum$.

- The net gravity force on a sphere is indeed equivalent to the force of a point mass at the center of the sphere. It took the genius Isaac Newton 3 years to deduce this result and the reasoning involved is too advanced for this book.
- The net gravity force on systems that are not spheres is generally *not* equivalent to a force acting at the center-of-mass (this is important for the understanding of tides as well as the orientational stability of satellites).

How to find the center-of-mass of a complex system

You find the center-of-mass of a complex system by knowing the masses and mass centers of its components. You find each of these centers of mass by

- Treating it as a point mass, or
- Treating it as a symmetric body and locating the center-of-mass in the middle, or
- Using integration, or
- Using the result of an experiment (which we will discuss in statics), or
- Treating the component as a complex system itself and applying this very recipe.

The recipe is just an application of the basic definition of center-of-mass (eqn. 2.49) but with our accumulated wisdom that the locations and masses in that sum can be the centers of mass and total masses of complex subsystems.

One way to arrange one's data is in a table or spreadsheet, like below.

Center of mass spreadsheet

Subsys#	1	2	3	4	5	6	7
Subsys 1	x_1	y_1	z_1	m_1	m_1x_1	m_1y_1	m_1z_1
Subsys 2	x_2	y_2	z_2	m_2	m_2x_2	m_2y_2	m_2z_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Subsys N	x_N	y_N	z_N	m_N	m_Nx_N	m_Ny_N	m_Nz_N
Row $N+1$ =sums				$m_{\text{tot}} = \sum m_i$	$\sum m_i x_i$	$\sum m_i y_i$	$\sum m_i z_i$

divide row $N+1$ by m_{tot} $\Rightarrow$ Result					$x_{\text{cm}} = \dfrac{\sum m_i x_i}{m_{\text{tot}}}$	$y_{\text{cm}} = \dfrac{\sum m_i y_i}{m_{\text{tot}}}$	$z_{\text{cm}} = \dfrac{\sum m_i z_i}{m_{\text{tot}}}$

1. The first four columns are the basic data. They are the x, y, and z coordinates of the subsystem center-of-mass locations (relative to some clear reference point), and the masses of the subsystems, one row for each of the N subsystems.

2. One next calculates three new columns (5,6, and 7) which come from each coordinate multiplied by its mass. For example the entry in the 6th row and 7th column is the z component of the 6th subsystem's center-of-mass multiplied by the mass of the 6th subsystem.
3. Then one sums columns 4 through 7. The sum of column 4 is the total mass, the sums of columns 5 through 7 are the total mass-weighted positions.
4. Finally the result, the system center of mass coordinates, are found by dividing columns 5-7 of row N+1 by column 4 of row N+1.

Of course, there are multiple ways of systematically representing the data. The spreadsheet-like calculation above is just one oragnization scheme.

Summary of center-of-mass

All discussions in mechanics make frequent reference to the concept of center of mass

$$\begin{aligned} m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} &= \sum \vec{\boldsymbol{r}}_{\text{i}} m_i && \text{for discrete systems or systems of systems} \\ &= \int \vec{\boldsymbol{r}}\, dm && \text{for continuous systems} \end{aligned}$$

where

$$\begin{aligned} m_{\text{tot}} &= \sum \mathrm{m}_i && \text{for discrete systems or systems of systems} \\ &= \int dm && \text{for continuous systems.} \end{aligned}$$

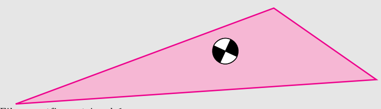
Who cares about the center of mass? We have demonstrated that the gravity moment is calculated correctly by applying the net gravity force at the center-of-mass. These other useful facts about center-of-mass will come later in the book.

For non-point-mass systems, the expressions for gravitational moment, linear momentum, angular momentum, and energy are all simplified by using the center-of-mass.

Simple center-of-mass calculations also can serve as a check of a more complicated analysis. For example, after a computer simulation of a system with many moving parts is complete, one way of checking the calculation is to see if the whole system's center of mass moves as would be expected by applying the net external force to the system.

2.16 The COM of a uniform triangle is $h/3$ up from the base

The center-of-mass of a 2D uniform triangular region is the centroid of the area.



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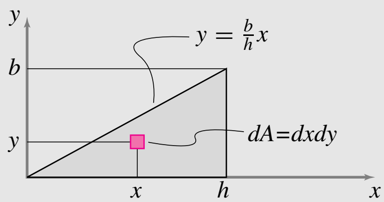
First we consider a right triangle with perpendicular sides b and h



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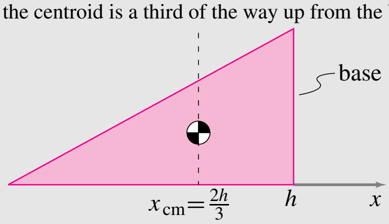
and find the x coordinate of the centroid as

$$\begin{aligned} x_{\rm cm} A &= \int x\, dA \\ &= \int_0^h \left[\int_0^{\frac{b}{h}x} x\, dy \right] dx = \int_0^h [xy] \Big|_{y=0}^{y=\frac{b}{h}x} dx \\ x_{\rm cm} \left(\frac{bh}{2} \right) &= \int_0^h x \left(\frac{b}{h} x \right) dx = \frac{b}{h} \frac{x^3}{3} \Big|_0^h = \frac{bh^2}{3} \end{aligned}$$

$\Rightarrow \quad x_{\rm cm} = \frac{2h}{3}$, a third of the way to the left of the vertical base on the right. By similar reasoning, but in the y direction, the centroid is a third of the way up from the base.



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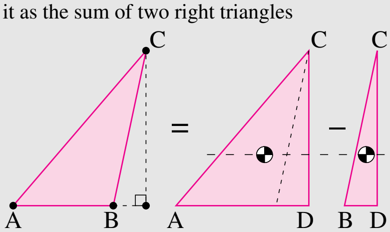
The center-of-mass of an arbitrary triangle can be found by treating it as the sum of two right triangles



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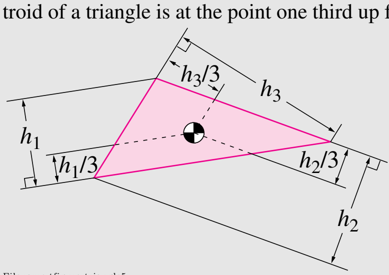
so the centroid is a third of the way up from the base of any triangle. Finally, the result holds for all three bases. Summarizing, the centroid of a triangle is at the point one third up from each of the bases.



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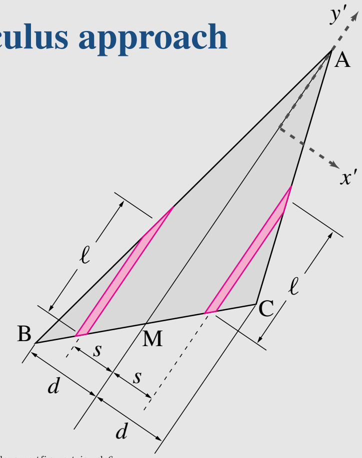
Non-calculus approach



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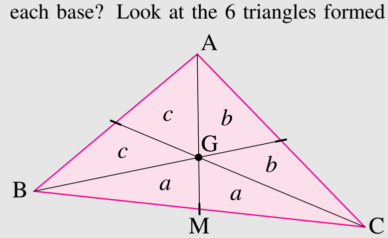
M is the midpoint of the line segment BC. Divide triangle ABC into equal width strips that are parallel to AM. Group these strips into pairs, each a distance s from AM. Because M is the midpoint of BC, by proportions each of these strips has the same length ℓ. What is the distance of the center-of-mass from the line AM? Because the strips are of equal area and equal distance from AM but on opposite sides, contributions to the sum come in canceling pairs. So the centroid is on AM. Likewise for all three sides. So the triangle's centroid is at the intersection of the three side bisectors.

Why do the three side bisectors intersect a third of the way up each base? Look at the 6 triangles formed by the side bisectors.



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The two triangles marked a and a have the same area (lcall it a) because they have the same height and bases of equal length (BM and CM). Likewise for the other side bisectors, so that the pairs marked b have equal area as do the pairs c. Triangle ABM has the same base and height and thus the same area as the triangle ACM. So $a + b + b = a + c + c$. Thus $b = c$ and similarly $a = b$: all six little triangles have equal area. Thus the area of ABC is 3 times the area of GBC. Because ABC and GBC share the base BC, ABC must have 3 times the height as GBC, and point G is thus a third of the way up from the base.

Where is the middle of a triangle?

We just showed that the centroid of a triangle is at the point that is at the intersection of: the three side bisectors; the three area bisectors (which are the side bisectors); and the three lines one third of the way up from the three bases.

And if the triangle only had three equal point masses on its vertices the center of mass lands on that same place. Thus the 'middle' of a triangle seems pretty well defined. *Yet*, there is ambiguity. If the triangle were made of bars along each edge, each with equal cross sections, the center-of-mass would be in a different location for all but equilateral triangles. Also, the three angle bisectors of a triangle do not intersect at the centroid. Unless we *define* middle to mean centroid, the "middle" of a triangle is not well defined.

SAMPLE 2.38 Center of mass in 1-D: Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center-of-mass of the assembly.

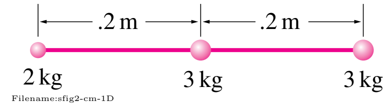


Figure 2.99:

Solution Let us select the first mass, $m_1 = 2\,\text{kg}$, to be at the origin of our co-ordinate system with the x-axis along the rod. Since all the three masses lie on the x-axis, the center-of-mass will also lie on this axis. Let the center-of-mass be located at x_{cm} on the x-axis. Then,

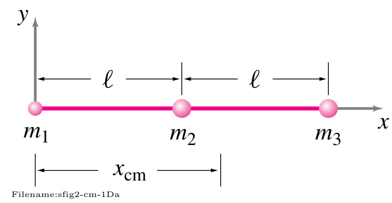


Figure 2.100:

$$\begin{aligned}
m_{\text{tot}} x_{cm} &= \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3 \\
&= m_1(0) + m_2(\ell) + m_3(2\ell) \\
\Rightarrow \quad x_{cm} &= \frac{m_2 \ell + m_3 2\ell}{m_1 + m_2 + m_3} \\
&= \frac{3\,\cancel{\text{k}}\text{g} \cdot 0.2\,\text{m} + 3\,\cancel{\text{k}}\text{g} \cdot 0.4\,\text{m}}{(2+3+3)\,\cancel{\text{k}}\text{g}} \\
&= \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}.
\end{aligned}$$

$$\boxed{x_{cm} = 0.225\,\text{m}}$$

Alternatively, we could find the center-of-mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center-of-mass of the two equal masses) and then calculate the value of x_{cm} for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. 2.101):

$$x_{cm} = \frac{6\,\text{kg} \cdot 0.3\,\text{m}}{8\,\text{kg}} = \frac{1.8\,\text{m}}{8} = 0.225\,\text{m}.$$

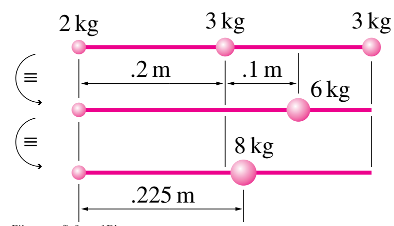


Figure 2.101:

SAMPLE 2.39 Center of mass in 2-D: Two particles of mass $m_1 = 1\,\text{kg}$ and $m_2 = 2\,\text{kg}$ are located at coordinates (1m, 2m) and (-2m, 5m), respectively, in the xy-plane. Find the location of their center-of-mass.

Solution Let $\vec{r}_{\text{cm}}$ be the position vector of the center-of-mass. Then,

$$\begin{aligned}
m_{\text{tot}} \vec{r}_{\text{cm}} &= m_1 \vec{r}_1 + m_2 \vec{r}_2 \\
\Rightarrow \quad \vec{r}_{\text{cm}} &= \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_{\text{tot}}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \\
&= \frac{1\,\cancel{\text{k}}\text{g}(1\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}) + 2\,\cancel{\text{k}}\text{g}(-2\,\text{m}\hat{\imath} + 5\,\text{m}\hat{\jmath})}{3\,\cancel{\text{k}}\text{g}} \\
&= \frac{(1\,\text{m} - 4\,\text{m})\hat{\imath} + (2\,\text{m} + 10\,\text{m})\hat{\jmath}}{3} = -1\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath}.
\end{aligned}$$

Thus the center-of-mass is located at the coordinates(-1m, 4m).

$$\boxed{(x_{cm}, y_{cm}) = (-1\,\text{m}, 4\,\text{m})}$$

Geometrically, this is just a 1-D problem like the previous sample. The center-of-mass has to be located on the straight line joining the two masses. Since the center-of-mass is a point about which the distribution of mass is *balanced*, it is easy to see (see Fig. 2.102) that the center-of-mass must lie one-third way from m_2 on the line joining the two masses so that $2\,\text{kg} \cdot (d/3) = 1\,\text{kg} \cdot (2d/3)$.

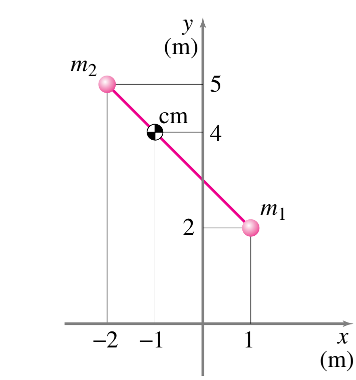


Figure 2.102:

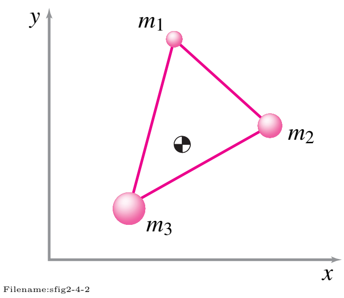


Figure 2.103:

SAMPLE 2.40 **Location of the center-of-mass.** A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$, connected rigidly by massless rods. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\boldsymbol{\imath}}$, $2\,\text{m/s}(\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}})$ and $1\,\text{m/s}\hat{\boldsymbol{\jmath}}$, respectively. Find the coordinates of the center-of-mass of the structure.

Solution Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center-of-mass, and second, using vector equations for the position of the center-of-mass.

1. **Scalar calculations:** Let (x_{cm}, y_{cm}) be the coordinates of the mass-center. Then from the definition of mass-center,

$$\begin{aligned} x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{7.5\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.25\,\text{m}. \end{aligned}$$

Similarly,

$$\begin{aligned} y_{cm} &= \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{8.5\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.42\,\text{m}. \end{aligned}$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

2. **Vector calculations:** Let $\vec{r}_{\text{cm}}$ be the position vector of the mass-center. Then,

$$\begin{aligned} m_{\text{tot}}\vec{r}_{\text{cm}} &= \sum_{i=1}^{3} m_i\vec{r}_i = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \\ \Rightarrow \quad \vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \end{aligned}$$

Substituting the values of m_1, m_2, and m_3, and $\vec{r}_1 = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$, $\vec{r}_2 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}$, and $\vec{r}_3 = 0.75\,\text{m}\hat{\boldsymbol{\imath}} + 0.5\,\text{m}\hat{\boldsymbol{\jmath}}$, we get,

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{1\,\text{kg}\cdot(1.25\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})\,\text{m} + 2\,\text{kg}\cdot(2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{m} + 3\,\text{kg}\cdot(0.75\hat{\boldsymbol{\imath}} + 0.5\hat{\boldsymbol{\jmath}})\,\text{m}}{(1+2+3)\,\text{kg}} \\ &= \frac{(7.5\hat{\boldsymbol{\imath}} + 8.5\hat{\boldsymbol{\jmath}})\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} \\ &= 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}} \end{aligned}$$

which, of course, gives the same location of the mass-center as above.

$$\boxed{\vec{r}_{\text{cm}} = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}}}$$

SAMPLE 2.41 Center of mass of a bent bar: A uniform bar of mass 4 kg is bent in the shape of an asymmetric 'Z' as shown in the figure. Locate the center-of-mass of the bar.

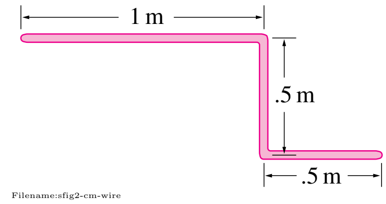


Figure 2.105:

Solution Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center-of-mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let $m_2 = m_3 = m$, then $m_1 = 2m$; and $m_1+m_2+m_3 = 4m = 4\,\text{kg}$ which gives $m = 1\,\text{kg}$. Now, from Fig. 2.106,

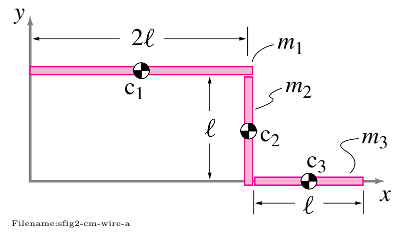


Figure 2.106:

$$
\begin{aligned}
\vec{r}_1 &= \ell\hat{\imath} + \ell\hat{\jmath} \\
\vec{r}_2 &= 2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath} \\
\vec{r}_3 &= (2\ell + \frac{\ell}{2})\hat{\imath} = \frac{5\ell}{2}\hat{\imath}
\end{aligned}
$$

So,

$$
\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_{\text{tot}}} \\
&= \frac{2m(\ell\hat{\imath} + \ell\hat{\jmath}) + m(2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath}) + m(\frac{5\ell}{2}\hat{\imath})}{4m} \\
&= \frac{\cancel{m}\ell(2\hat{\imath} + 2\hat{\jmath} + 2\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{5}{2}\hat{\imath})}{4\cancel{m}} \\
&= \frac{\ell}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= \frac{0.5\,\text{m}}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}.
\end{aligned}
$$

$$\boxed{\vec{r}_{\text{cm}} = 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}}$$

Geometrically, we could find the center-of-mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. 2.104.

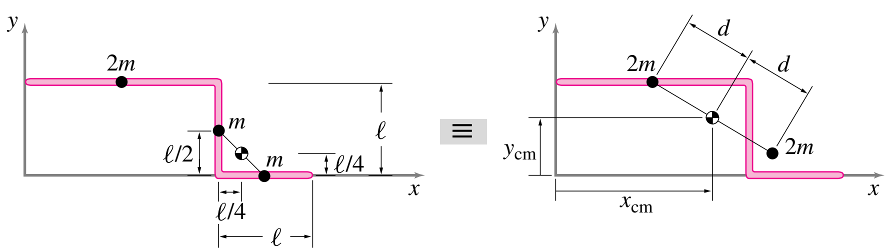


Figure 2.104:

The center-of-mass of m_2 and m_3 (each of mass m) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass $2m$ at their mass-center. Next, we connect this mass-center and m_1 with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center-of-mass of the entire bar.

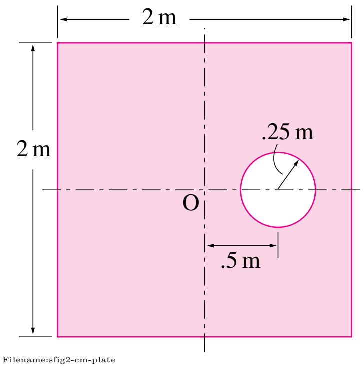


Figure 2.107:

SAMPLE 2.42 Shift of mass-center due to cut-outs: A $2\,\mathrm{m} \times 2\,\mathrm{m}$ uniform square plate has mass $m = 4\,\mathrm{kg}$. A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center-of-mass of the plate.

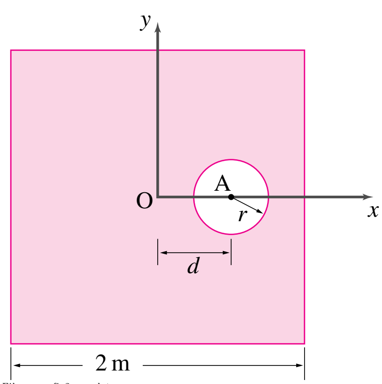


Figure 2.108:

Solution Let us use an xy-coordinate system with its origin at the geometric center of the plate and the x-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the x-axis, the new center-of-mass must lie somewhere on the x-axis. Thus, we only need to find x_{cm} (since $y_{cm} = 0$). Let m_1 be the mass of the plate with the hole, and m_2 be the mass of the circular cut-out. Clearly, $m_1 + m_2 = m = 4\,\mathrm{kg}$. The center-of-mass of the circular cut-out is at A, the center of the circle. The center-of-mass of the intact square plate (without the cut-out) must be at O, the middle of the square. Then,

$$\begin{aligned} m_1 x_{cm} + m_2 x_A &= m x_O = 0 \\ \Rightarrow \quad x_{cm} &= -\frac{m_2}{m_1} x_A. \end{aligned}$$

Now, since the plate is uniform, the masses m_1 and m_2 are proportional to the surface areas of the geometric objects they represent, *i.e.*,

$$\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi}.$$

Therefore,

$$\begin{aligned} x_{cm} &= -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d \qquad (2.50) \\ &= -\frac{\pi}{\left(\frac{2\,\mathrm{m}}{.25\,\mathrm{m}}\right)^2 - \pi} \cdot 0.5\,\mathrm{m} \\ &= -25.81 \times 10^{-3}\,\mathrm{m} = -25.81\,\mathrm{mm} \end{aligned}$$

Thus the center-of-mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

$$\boxed{x_{cm} = -25.81\,\mathrm{mm}}$$

Comments: The advantage of finding the expression for x_{cm} in terms of r and ℓ as in eqn. (2.50) is that you can easily find the center-of-mass of any size circular cut-out located at any distance d on the x-axis. This is useful in design where you like to select the size or location of the cut-out to have the center-of-mass at a particular location.

SAMPLE 2.43 **Center of mass of two objects:** A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center-of-mass of the combined system.

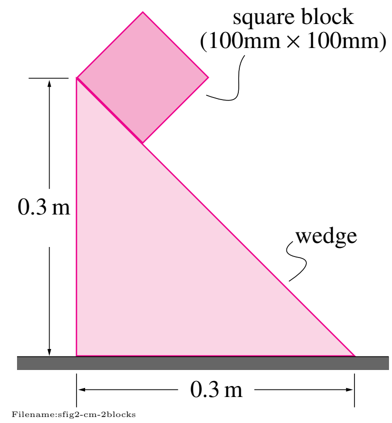


Figure 2.109:

Solution The center-of-mass of the triangular wedge is located at $h/3$ above the base and $\ell/3$ to the right of the vertical side. Let m_1 be the mass of the wedge and $\vec{r}_1$ be the position vector of its mass-center. Then, referring to Fig. 2.110,

$$\vec{r}_1 = \frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}.$$

The center-of-mass of the square block is located at its geometric center C_2. From geometry, we can see that the line AE that passes through C_2 is horizontal since $\angle OAB = 45°$ ($h = \ell = 0.3\,\text{m}$) and $\angle DAE = 45°$. Therefore, the coordinates of C_2 are ($d/\sqrt{2}, h$). Let m_2 and $\vec{r}_2$ be the mass and the position vector of the mass-center of the block, respectively. Then,

$$\vec{r}_2 = \frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath}.$$

Now, noting that $m_1 = 3m_2$ or $m_1 = 3m$, and $m_2 = m$ where $m = 2\,\text{kg}$, we find the center-of-mass of the combined system:

$$\begin{aligned}
\vec{r}_{cm} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{(m_1 + m_2)} \\
&= \frac{3m(\frac{\ell}{3}\hat{\imath} + \frac{h}{3}\hat{\jmath}) + m(\frac{d}{\sqrt{2}}\hat{\imath} + h\hat{\jmath})}{3m + m} \\
&= \frac{\cancel{m}[(\ell + \frac{d}{\sqrt{2}})\hat{\imath} + 2h\hat{\jmath}]}{4\cancel{m}} \\
&= \frac{1}{4}(\frac{d}{\sqrt{2}} + \ell)\hat{\imath} + \frac{h}{2}\hat{\jmath} \\
&= \frac{1}{4}(\frac{0.1\,\text{m}}{\sqrt{2}} + 0.3\,\text{m})\hat{\imath} + \frac{0.3\,\text{m}}{2}\hat{\jmath} \\
&= 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\vec{r}_{cm} = 0.093\,\text{m}\hat{\imath} + 0.150\,\text{m}\hat{\jmath}$$

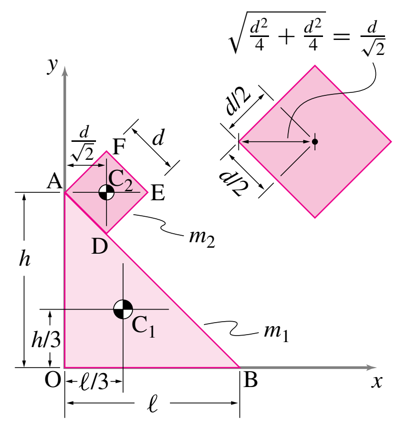


Figure 2.110:

Thus, the center-of-mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than $C_1(0.1\,\text{m}, 0.1\,\text{m})$. This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass $3m$ at C_1 and m at C_2, connected the two points by a straight line, and located the center-of-mass C on that line such that $CC_2 = 3CC_1$. You can verify that the distance from $C_1(0.1\,\text{m}, 0.1\,\text{m})$ to $C(.093\,\text{m}, 0.15\,\text{m})$ is one third the distance from C to $C_2(.071\,\text{m}, 0.3\,\text{m})$.

Problems for Chapter 2

Vector skills for mechanics

2.1 Vector notation and vector addition

2.1.1 Draw the vector $\vec{r} = (5\,\mathrm{m})\hat{\imath} + (5\,\mathrm{m})\hat{\jmath}$.

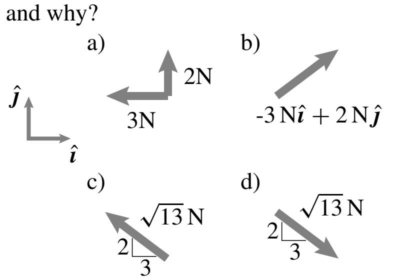
2.1.2 A vector $\vec{a}$ is 2 m long and points northwest at an angle 60° from the north. Draw the vector.

2.1.3 The position vector of a point B measures 3 m and is directed at 40° CCW from the negative x-axis. Show the position vector vector.

2.1.4 Draw a force vector that is given as $\vec{F} = 2\,\mathrm{N}\hat{\imath} + 2\,\mathrm{N}\hat{\jmath} + 1\,\mathrm{N}uk$.

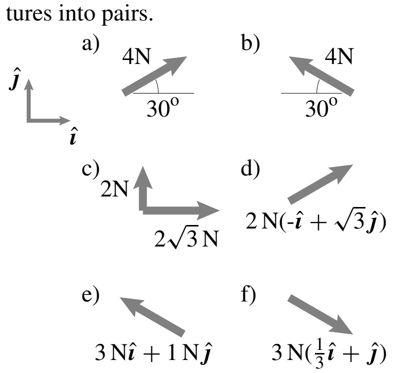
2.1.5 Represent the vector $\vec{r} = 5\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\hat{\jmath}$ in three different ways.

2.1.6 Which one of the following representations of the same vector $\vec{F}$ is wrong and why?



Filename:pfigure2-vec1-2
Problem 2.1.6

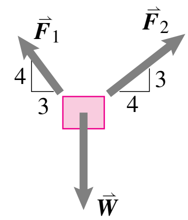
2.1.7 There are exactly two representations that describe the same vector in the following pictures. Match the correct pictures into pairs.



Filename:pfigure2-vec1-3
Problem 2.1.7

2.1.8 Find the sum of forces $\vec{F}_1 = 20\,\mathrm{N}\hat{\imath} - 2\,\mathrm{N}\hat{\jmath}$, $\vec{F}_2 = 30\,\mathrm{N}(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$, and $\vec{F}_3 = -20\,\mathrm{N}(-\hat{\imath} + \sqrt{3}\hat{\jmath})$.

2.1.9 The forces acting on a block of mass $m = 5\,\mathrm{kg}$ are shown in the figure, where $F_1 = 20\,\mathrm{N}$, $F_2 = 50\,\mathrm{N}$, and $W = mg$. Find the sum $\vec{F}$ $(= \vec{F}_1 + \vec{F}_2 + \vec{W})$?



Filename:pfigure2-vec1-6
Problem 2.1.9

2.1.10 Given that the sum of four vectors $\vec{F}_i, i = 1$ to 4, is zero, where $\vec{F}_1 = 20\,\mathrm{N}\hat{\imath}$, $\vec{F}_2 = -50\,\mathrm{N}\hat{\jmath}$, $\vec{F}_3 = 10\,\mathrm{N}(-\hat{\imath} + \hat{\jmath})$, find $\vec{F}_4$.

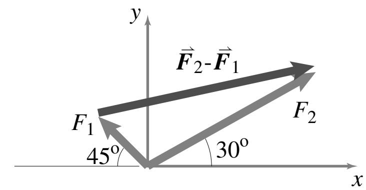
2.1.11 Three forces $\vec{F} = 2\,\mathrm{N}\hat{\imath} - 5\,\mathrm{N}\hat{\jmath}$, $\vec{R} = 10\,\mathrm{N}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$ and $\vec{W} = W\,\mathrm{N}\hat{\jmath}$ with $W > 0$, sum up to zero. Determine θ and W and draw the force vector $\vec{R}$ clearly showing its direction.

2.1.12 Given that $\vec{R}_1 = 1\,\mathrm{N}\hat{\imath} + 1.5\,\mathrm{N}\hat{\jmath}$ and $\vec{R}_2 = 3.2\,\mathrm{N}\hat{\imath} - 0.4\,\mathrm{N}\hat{\jmath}$, find $2\vec{R}_1 + 5\vec{R}_2$.

2.1.13 Find the magnitudes of the forces $\vec{F}_1 = 30\,\mathrm{N}\hat{\imath} - 40\,\mathrm{N}\hat{\jmath}$ and $\vec{F}_2 = 30\,\mathrm{N}\hat{\imath} + 40\,\mathrm{N}\hat{\jmath}$. Draw the two forces, representing them with their magnitudes.

2.1.14 Two forces $\vec{R} = 2\,\mathrm{N}(0.16\hat{\imath} + 0.80\hat{\jmath})$ and $\vec{W} = -36\,\mathrm{N}\hat{\jmath}$ act on a particle. Find the magnitude of the net force. What is the direction of this force?

2.1.15 In the figure shown, $F_1 = 100\,\mathrm{N}$ and $F_2 = 300\,\mathrm{N}$. Find the magnitude and direction of $\vec{F}_2 - \vec{F}_1$.

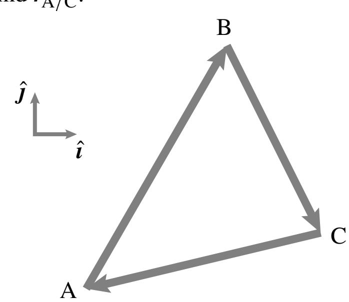


Filename:pfigure2-vec1-17
Problem 2.1.15

2.1.16 Two points A and B are located in the xy plane. The coordinates of A and B are (4 mm, 8 mm) and (90 mm, 6 mm), respectively.

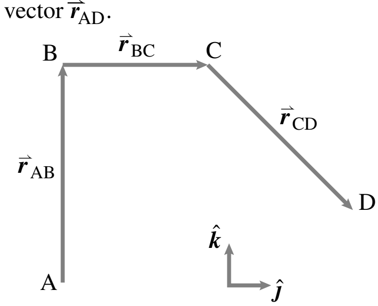
1. Draw position vectors $\vec{r}_A$ and $\vec{r}_B$.
2. Find the magnitude of $\vec{r}_A$ and $\vec{r}_B$.
3. How far is A from B?

2.1.17 Three position vectors are shown in the figure below. Given that $\vec{r}_{B/A} = 3\,\mathrm{m}(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$ and $\vec{r}_{C/B} = 1\,\mathrm{m}\hat{\imath} - 2\,\mathrm{m}\hat{\jmath}$, find $\vec{r}_{A/C}$.



Filename:pfigure2-vec1-7
Problem 2.1.17

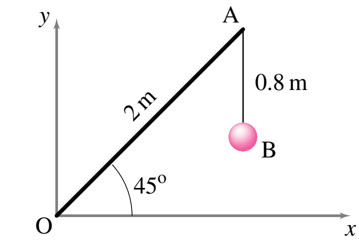
2.1.18 In the figure shown below, the position vectors are $\vec{r}_{AB} = 3\,\mathrm{ft}\hat{k}$, $\vec{r}_{BC} = 2\,\mathrm{ft}\hat{\jmath}$, and $\vec{r}_{CD} = 2(\hat{\jmath} + \hat{k})\,\mathrm{ft}$. Find the position vector $\vec{r}_{AD}$.



Filename:pfigure2-vec1-5
Problem 2.1.18

2.1.19 In the figure shown, a ball is suspended with a 0.8 m long cord from a 2 m long hoist OA.

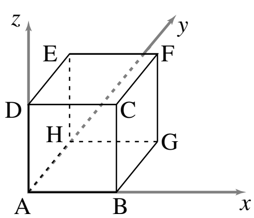
1. Find the position vector $\vec{r}_B$ of the ball.
2. Find the distance of the ball from the origin.



Filename:pfigure2-vec1-20
Problem 2.1.19

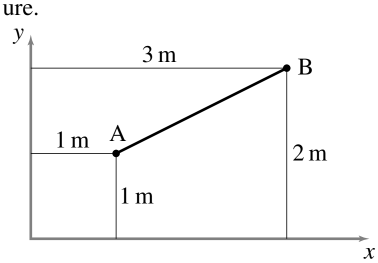
2.1.20 A cube of side 6 in is shown in the figure.

1. Find the position vector of point F, $\vec{r}_F$, from the vector sum $\vec{r}_F = \vec{r}_D + \vec{r}_{C/D} + \vec{r}_{F/c}$.
2. Calculate $|\vec{r}_F|$.
3. Find $\vec{r}_G$ using $\vec{r}_F$.



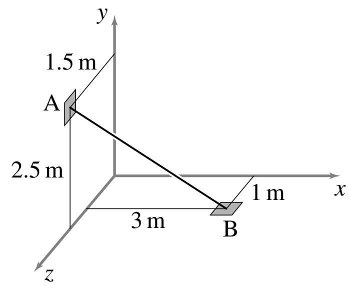
Filename:pfigure2-vec1-23
Problem 2.1.20

2.1.21 Find the unit vector $\hat{\lambda}_{AB}$, directed from point A to point B shown in the figure.



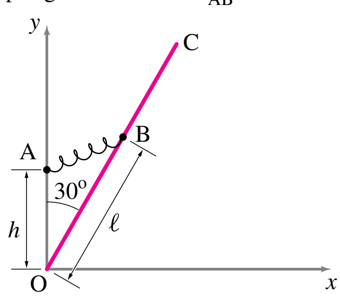
Filename:pfigure2-vec1-25
Problem 2.1.21

2.1.22 Find a unit vector along string BA and express the position vector of A with respect to B, $\vec{r}_{A/B}$, in terms of the unit vector.



Filename:pfigure2-vec1-26
Problem 2.1.22

2.1.23 In the structure shown in the figure, $\ell = 2\,\text{ft}, h = 1.5\,\text{ft}$. The force in the spring is $\vec{F} = k\vec{r}_{AB}$, where $k = 100\,\text{lbf/ft}$. Find a unit vector $\hat{\lambda}_{AB}$ along AB and calculate the spring force $\vec{F} = F\hat{\lambda}_{AB}$.



Filename:pfigure2-vec1-27
Problem 2.1.23

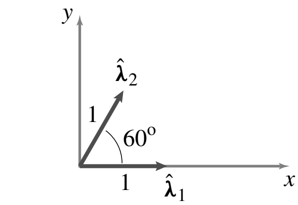
2.1.24 Express the vector $\vec{r}_A = 2\,\text{m}\hat{\imath} - 3\,\text{m}\hat{\jmath} + 5\,\text{m}\hat{k}$ in terms of its magnitude and a unit vector indicating its direction.

2.1.25 Let $\vec{F} = 10\,\text{lbf}\hat{\imath} + 30\,\text{lbf}\hat{\jmath}$ and $\vec{W} = -20\,\text{lbf}\hat{\jmath}$. Find a unit vector in the direction of the net force $\vec{F} + \vec{W}$, and express the the net force in terms of the unit vector.

2.1.26 Let $\hat{\lambda}_1 = 0.80\hat{\imath} + 0.60\hat{\jmath}$ and $\hat{\lambda}_2 = 0.5\hat{\imath} + 0.866\hat{\jmath}$.

1. Show that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are unit vectors.
2. Is the sum of these two unit vectors also a unit vector? If not, find a unit vector along the sum of $\hat{\lambda}_1$ and $\hat{\lambda}_2$.

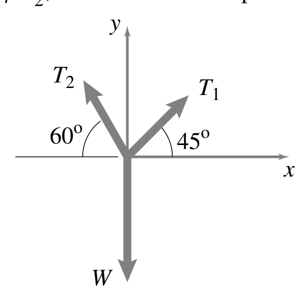
2.1.27 For the unit vectors $\hat{\lambda}_1$ and $\hat{\lambda}_2$ shown below, find the scalars α and β such that $\alpha\hat{\lambda}_2 - 3\hat{\lambda}_2 = \beta\hat{\jmath}$.



Filename:pfigure2-vec1-11
Problem 2.1.27

2.1.28 If a mass slides from point A towards point B along a straight path and the coordinates of points A and B are (0 in, 5 in, 0 in) and (10 in, 0 in, 10 in), respectively, find the unit vector $\hat{\lambda}_{AB}$ directed from A to B along the path.

2.1.29 In the figure shown, $T_1 = 20\sqrt{2}\,\text{N}, T_2 = 40\,\text{N}$, and W is such that the sum of the three forces equals zero. If W is doubled, find α and β such that $\alpha\vec{T}_1, \beta\vec{T}_2$, and $2\vec{W}$ still sum up to zero.

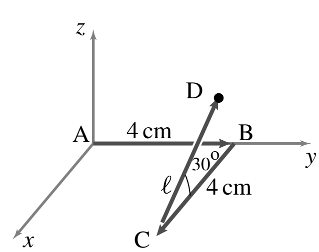


Filename:pfigure2-vec1-12
Problem 2.1.29

2.1.30 In the figure shown, rods AB and BC are each 4 cm long and lie along y and x axes, respectively. Rod CD is in the xz plane and makes an angle $\theta = 30^\circ$ with the x-axis.

1. Find $\vec{r}_{AD}$ in terms of the variable length ℓ.
2. Find ℓ and α such that

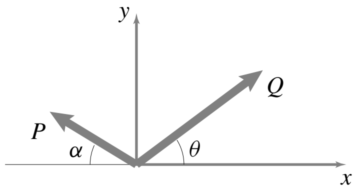
$$\vec{r}_{AD} = \vec{r}_{AB} - \vec{r}_{BC} + \alpha\hat{k}.$$



Filename:pfigure2-vec1-13
Problem 2.1.30

2.1.31 In Problem 2.1.30, find ℓ such that the length of the position vector $\vec{r}_{AD}$ is 6 cm.

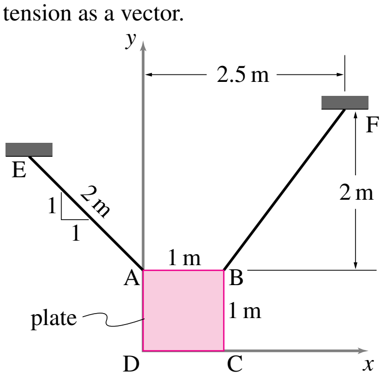
2.1.32 Let two forces $\vec{P}$ and $\vec{Q}$ act in the directions shown in the figure. You are allowed to change the direction of the forces by changing the angles α and θ while keeping the magnitudes fixed. What should be the values of α and θ if the magnitude of $\vec{P} + \vec{Q}$ is to be maximum?



Filename:pfigure2-vec1-18

Problem 2.1.32

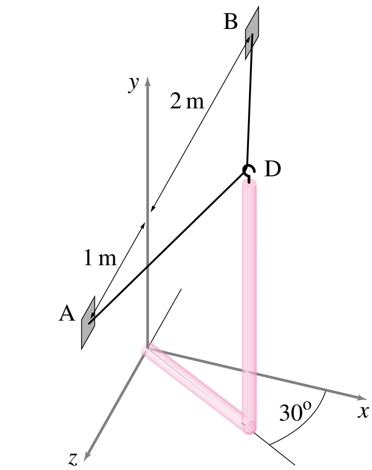
2.1.33 A 1 m × 1 m square board is supported by two strings AE and BF. The tension in the string BF is 20 N. Express this tension as a vector.



Filename:pfigure2-vec1-21

Problem 2.1.33

2.1.34 The top of an L-shaped bar, shown in the figure, is to be tied by strings AD and BD to the points A and B in the yz plane. Find the length of the strings AD and BD using vectors $\vec{r}_{AD}$ and $\vec{r}_{BD}$.

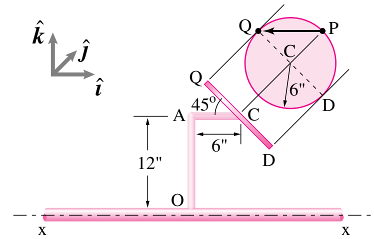


Filename:pfigure2-vec1-22

Problem 2.1.34

2.1.35 A circular disk of radius 6 in is mounted on axle x-x at the end an L-shaped bar as shown in the figure. The disk is tipped 45° with respect to the horizontal bar AC. Two points, P and Q, are marked on the rim of the disk; with CP directly into the page, and Q at the highest point above the center C. Taking the base vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ as shown in the figure ($\hat{\jmath}$ into the page), find

1. the relative position vector $\vec{r}_{Q/P}$,
2. the magnitude $|\vec{r}_{Q/P}|$.



Filename:pfigure2-vec1-24

Problem 2.1.35

2.1.36 Write the vectors $\vec{F}_1 = 30\,\text{N}\hat{\imath} + 40\,\text{N}\hat{\jmath} - 10\,\text{N}\hat{k}$, $\vec{F}_2 = -20\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$, and $\vec{F}_3 = -10\,\text{N}\hat{\imath} - 100\,\text{N}\hat{k}$ as a list of numbers (rows or columns). Find the sum of the forces using a computer.

2.1.37 Let $\alpha\vec{F}_1 + \beta\vec{F}_2 + \gamma\vec{F}_3 = \vec{0}$, where $\vec{F}_1, \vec{F}_2$, and $\vec{F}_3$ are as given in Problem 2.1.36. Solve for α, β, and γ using a computer.

2.1.38 Let $\vec{r}_n = 1\,\text{m}(\cos\theta_n\hat{\imath} + \sin\theta_n\hat{\jmath})$, where $\theta_n = \theta_0 - n\Delta\theta$. Using a computer generate the required vectors and find the sum

$$\sum_{n=0}^{44} \vec{r}_i, \quad \text{with } \Delta\theta = 1^\circ \quad \text{and} \quad \theta_0 = 45^\circ.$$

2.1.39 Find two non-zero and non-parallel vectors $\vec{A}$ and $\vec{B}$ so that $|\vec{A}+2\vec{B}| = 2|\vec{A}+\vec{B}|$. *

2.2 The dot product of two vectors

2.2.1 Find the dot product of $\vec{a} = 2\hat{\imath}+3\hat{\jmath}-\hat{k}$ and $\vec{b} = 2\hat{\imath} + \hat{\jmath} + 2\hat{k}$.

2.2.2 Find the dot product of $\vec{F} = 0.5\,\text{N}\hat{\imath} + 1.2\,\text{N}\hat{\jmath} + 1.5\,\text{N}\hat{k}$ and $\hat{\lambda} = -0.8\hat{\imath} + 0.6\hat{\jmath}$.

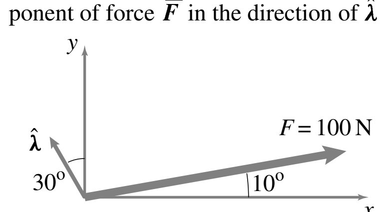
2.2.3 Find the dot product $\vec{F}\cdot\vec{r}$ where $\vec{F} = (5\hat{\imath}+4\hat{\jmath})\,\text{N}$ and $\vec{r} = (-0.8\hat{\imath}+\hat{\jmath})\,\text{m}$. Draw the two vectors and justify your answer for the dot product.

2.2.4 Two vectors, $\vec{a} = -4\sqrt{3}\hat{\imath} + 12\hat{\jmath}$ and $\vec{b} = \hat{\imath} - \sqrt{3}\hat{\jmath}$ are given. Find the dot product of the two vectors. How is $\vec{a}\cdot\vec{b}$ related to $|\vec{a}||\vec{b}|$ in this case?

2.2.5 Find the dot product of two vectors $\vec{F} = 10\,\text{lbf}\hat{\imath} - 20\,\text{lbf}\hat{\jmath}$ and $\hat{\lambda} = 0.8\hat{\imath} + 0.6\hat{\jmath}$. Sketch $\vec{F}$ and $\hat{\lambda}$ and show what their dot product represents.

2.2.6 The position vector of a point A is $\vec{r}_A = 30\,\text{cm}\hat{\imath}$. Find the dot product of $\vec{r}_A$ with $\hat{\lambda} = \frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}$.

2.2.7 From the figure below, find the component of force $\vec{F}$ in the direction of $\hat{\lambda}$.



Filename:pfigure2-vec1-33

Problem 2.2.7

2.2.8 Find the angle between $\vec{F}_1 = 2\,\text{N}\hat{\imath} + 5\,\text{N}\hat{\jmath}$ and $\vec{F}_2 = -2\,\text{N}\hat{\imath} + 6\,\text{N}\hat{\jmath}$.

2.2.9 Given $\vec{\omega} = 2\,\text{rad/s}\hat{\imath} + 3\,\text{rad/s}\hat{\jmath}$, $\vec{H}_1 = (20\hat{\imath} + 30\hat{\jmath})\,\text{kg m}^2/\,\text{s}$ and $\vec{H}_2 = (10\hat{\imath} + 15\hat{\jmath} + 6\hat{k})\,\text{kg m}^2/\,\text{s}$, find (a) the angle between $\vec{\omega}$ and $\vec{H}_1$ and (b) the angle between $\vec{\omega}$ and $\vec{H}_2$.

2.2.10 The unit normal to a surface is given as $\hat{n} = 0.74\hat{\imath} + 0.67\hat{\jmath}$. If the weight of a block on this surface acts in the $-\hat{\jmath}$ direction, find the angle that a 1000 N normal force makes with the direction of weight of the block.

2.2.11 Vector algebra. For each equation below state whether:

1. The equation is nonsense. If so, why?
2. Is always true. Why? Give an example.
3. Is never true. Why? Give an example.
4. Is sometimes true. Give examples both ways.

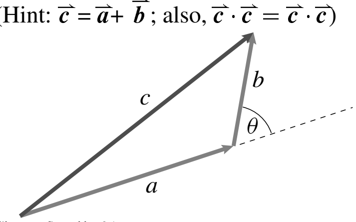
You may use trivial examples.

a) $\vec{A} + \vec{B} = \vec{B} + \vec{A}$
b) $\vec{A} + b = b + \vec{A}$
c) $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
d) $\vec{B}/\vec{C} = B/C$
e) $b/\vec{A} = b/A$
f) $\vec{A} = (\vec{A} \cdot \vec{B})\vec{B} + (\vec{A} \cdot \vec{C})\vec{C} + (\vec{A} \cdot \vec{D})\vec{D}$

2.2.12 Use the dot product to show 'the law of cosines'; i. e.,

$$c^2 = a^2 + b^2 + 2ab\cos\theta.$$

(Hint: $\vec{c} = \vec{a} + \vec{b}$; also, $\vec{c} \cdot \vec{c} = \vec{c} \cdot \vec{c}$)



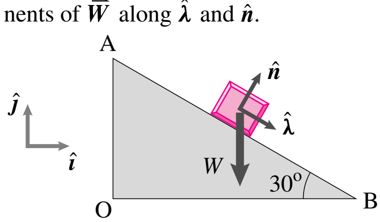
Filename:pfigure-blue-2-1
Problem 2.2.12

2.2.13 Find the direction cosines of $\vec{F} = 3\,\text{N}\hat{\imath} - 4\,\text{N}\hat{\jmath} + 5\,\text{N}\hat{k}$.

2.2.14 A force acting on a bead of mass m is given as $\vec{F} = -20\,\text{lbf}\hat{\imath} + 22\,\text{lbf}\hat{\jmath} + 12\,\text{lbf}\hat{k}$. What is the angle between the force and the z-axis?

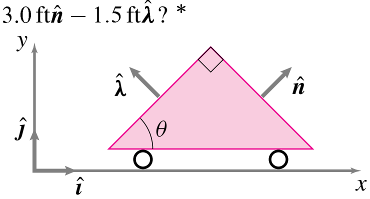
2.2.15 (a) Draw the vector $\vec{r} = 3.5\,\text{in}\hat{\imath} + 3.5\,\text{in}\hat{\jmath} - 4.95\,\text{in}\hat{k}$. (b) Find the angle this vector makes with the z-axis. (c) Find the angle this vector makes with the x-y plane.

2.2.16 In the figure shown, $\hat{\lambda}$ and $\hat{n}$ are unit vectors parallel and perpendicular to the surface AB, respectively. A force $\vec{W} = -50\,\text{N}\hat{\jmath}$ acts on the block. Find the components of $\vec{W}$ along $\hat{\lambda}$ and $\hat{n}$.



Filename:pfigure2-vec1-41
Problem 2.2.16

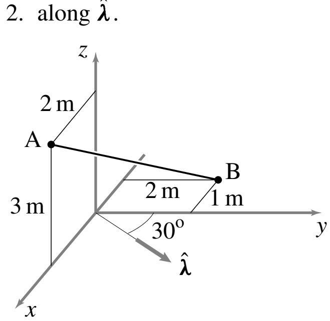
2.2.17 Express the unit vectors $\hat{n}$ and $\hat{\lambda}$ in terms of $\hat{\imath}$ and $\hat{\jmath}$ shown in the figure. What are the x and y components of $\vec{r} = 3.0\,\text{ft}\hat{n} - 1.5\,\text{ft}\hat{\lambda}$? *



Filename:efig1-2-27
Problem 2.2.17

2.2.18 From the figure shown, find the components of vector $\vec{r}_{AB}$ (you have to first find this position vector) along

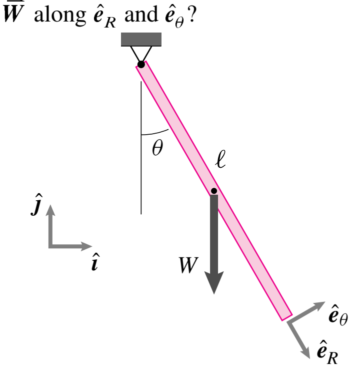
1. the y-axis, and
2. along $\hat{\lambda}$.



Filename:pfigure2-vec1-42
Problem 2.2.18

2.2.19 The net force acting on a particle is $\vec{F} = 2\,\text{N}\hat{\imath} + 10\,\text{N}\hat{\jmath}$. Find the components of this force in another coordinate system with basis vectors $\hat{\imath}' = -\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{\jmath}' = -\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}$. For $\theta = 30^\circ$, sketch the vector $\vec{F}$ and show its components in the two coordinate systems.

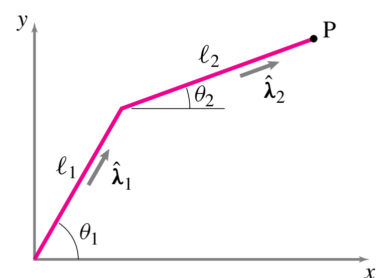
2.2.20 Find the unit vectors $\hat{e}_R$ and $\hat{e}_\theta$ in terms of $\hat{\imath}$ and $\hat{\jmath}$ with the geometry shown in the figure. What are the components of $\vec{W}$ along $\hat{e}_R$ and $\hat{e}_\theta$?



Filename:pfigure2-vec1-44
Problem 2.2.20

2.2.21 Write the position vector of point P in terms of $\hat{\lambda}_1$ and $\hat{\lambda}_2$ and

1. find the y-component of $\vec{r}_P$,
2. find the component of $\vec{r}_P$ along $\hat{\lambda}_1$.



Filename:pfigure2-vec1-45
Problem 2.2.21

2.2.22 Let $\vec{F}_1 = 30\,\text{N}\hat{\imath} + 40\,\text{N}\hat{\jmath} - 10\,\text{N}\hat{k}$, $\vec{F}_2 = -20\,\text{N}\hat{\jmath} + 2\,\text{N}\hat{k}$, and $\vec{F}_3 = F_{3_x}\hat{\imath} + F_{3_y}\hat{\jmath} - F_{3_z}\hat{k}$. If the sum of all these forces must equal zero, find the required scalar equations to solve for the components of $\vec{F}_3$.

2.2.23 A force $\vec{F}$ is directed from point A(3,2,0) to point B(0,2,4). If the x-component of the force is 120 N, find the y- and z-components of $\vec{F}$.

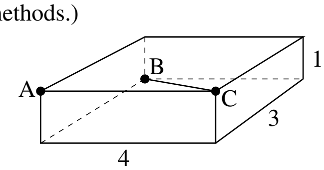
2.2.24 A vector equation for the sum of forces results into the following equation:

$$\frac{F}{2}(\hat{\imath} - \sqrt{3}\hat{\jmath}) + \frac{R}{5}(3\hat{\imath} + 6\hat{\jmath}) = 25\,\mathrm{N}\hat{\boldsymbol{\lambda}}$$

where $\hat{\boldsymbol{\lambda}} = 0.30\hat{\imath} - 0.954\hat{\jmath}$. Find two scalar equations by dotting both sides of the equation first with $\hat{\boldsymbol{\lambda}}$ and then with a vector orthogonal to $\hat{\boldsymbol{\lambda}}$.

2.2.25 Write a computer program (or use a canned program) to find the dot product of two 3-D vectors. Test the program by computing the dot products $\hat{\imath} \cdot \hat{\imath}, \hat{\imath} \cdot \hat{\jmath}$, and $\hat{\jmath} \cdot \hat{k}$. Now use the program to find the components of $\vec{F} = (2\hat{\imath} + 2\hat{\jmath} - 3\hat{k})\,\mathrm{N}$ along the line $\vec{r}_{\mathrm{AB}} = (0.5\hat{\imath} - 0.2\hat{\jmath} + 0.1\hat{k})\,\mathrm{m}$.

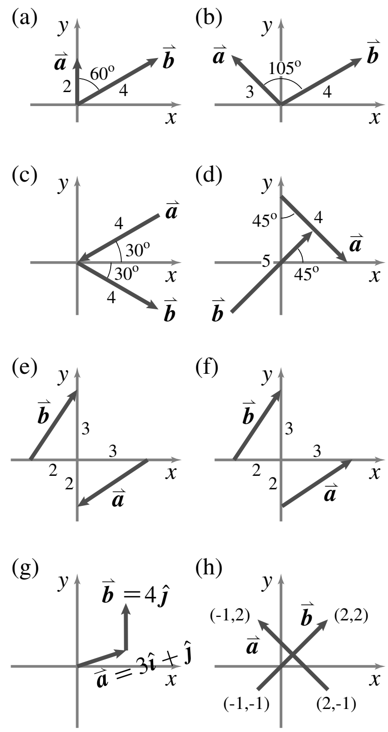
2.2.26 What is the shortest distance between the point A and the diagonal BC of the parallelepiped shown? (Use vector methods.)



Filename:pfigure-blue-2-3
Problem 2.2.26

2.3 Vector Cross Product

2.3.1 Find the cross product of the two vectors shown in the figures below from the information given in the figures.



Filename:pfigure2-vec2-1
Problem 2.3.1

2.3.2 Vector algebra. For each equation below state whether:

1. The equation is nonsense. If so, why?
2. Is always true. Why? Give an example.
3. Is never true. Why? Give an example.
4. Is sometimes true. Give examples both ways.

You may use trivial examples.

a) $\vec{B} \times \vec{C} = \vec{C} \times \vec{B}$

b) $\vec{B} \times \vec{C} = \vec{C} \cdot \vec{B}$

c) $\vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$

d) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$

2.3.3 What do you get when you cross a vector and a scalar? *

2.3.4 Carry out the following cross products in different ways and determine which method takes the least amount of time for you.

a) $\vec{r} = 2.0\,\mathrm{ft}\hat{\imath} + 3.0\,\mathrm{ft}\hat{\jmath} - 1.5\,\mathrm{ft}\hat{k}$; $\vec{F} = -0.3\,\mathrm{lbf}\hat{\imath} - 1.0\,\mathrm{lbf}\hat{k}$; $\vec{r} \times \vec{F} = ?$

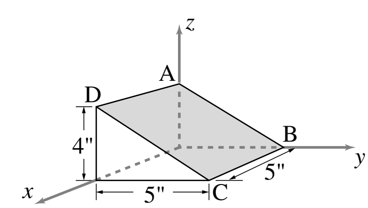
b) $\vec{r} = (-\hat{\imath} + 2.0\hat{\jmath} + 0.4\hat{k})\,\mathrm{m}$; $\vec{L} = (3.5\hat{\jmath} - 2.0\hat{k})\,\mathrm{kg\,m/s}$; $\vec{r} \times \vec{L} = ?$

c) $\vec{\omega} = (\hat{\imath} - 1.5\hat{\jmath})\,\mathrm{rad/s}$; $\vec{r} = (10\hat{\imath} - 2\hat{\jmath} + 3\hat{k})\,\mathrm{in}$; $\vec{\omega} \times \vec{r} = ?$

2.3.5 Cross Product program Write a program that will calculate cross products. The input to the function should be the components of the two vectors and the output should be the components of the cross product. As a model, here is a function file that calculates dot products in pseudo code.

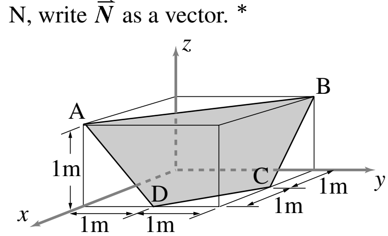
```
%program definition
z(1)=a(1)*b(1);
z(2)=a(2)*b(2);
z(3)=a(3)*b(3);
w=z(1)+z(2)+z(3);
```

2.3.6 Find a unit vector normal to the surface ABCD shown in the figure.



Filename:efig1-2-11
Problem 2.3.6

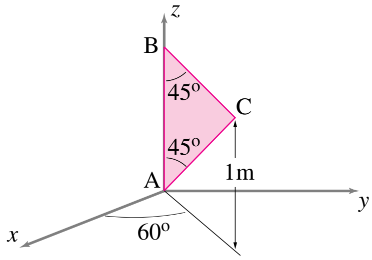
2.3.7 If the magnitude of a force $\vec{N}$ normal to the surface ABCD in the figure is 1000 N, write $\vec{N}$ as a vector. *



Filename:efig1-2-12
Problem 2.3.7

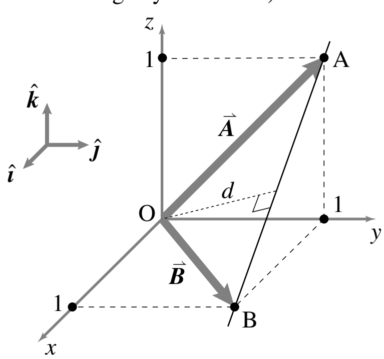
2.3.8 The equation of a surface is given as $z = 2x - y$. Find a unit vector $\hat{\boldsymbol{n}}$ normal to the surface.

2.3.9 In the figure, a triangular plate ACB, attached to rod AB, rotates about the z-axis. At the instant shown, the plate makes an angle of 60° with the x-axis. Find and draw a vector normal to the surface ACB.



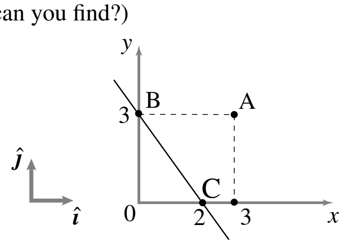
Filename:efig1-2-14
Problem 2.3.9

2.3.10 What is the distance d between the origin and the line AB shown? (You may write your solution in terms of $\vec{A}$ and $\vec{B}$ before doing any arithmetic). *



Filename:pfigure-blue-1-3
Problem 2.3.10

2.3.11 What is the perpendicular distance between point A and line BC shown? (There are at least 3 ways to do this using various vector products, how many ways can you find?)



Filename:pfigure-blue-2-2
Problem 2.3.11

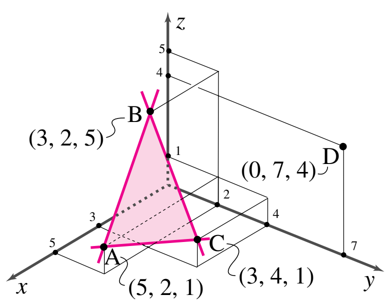
2.3.12 Given a force, $\vec{F}_1 = (-3\hat{\imath} + 2\hat{\jmath} + 5\hat{k})$ N acting at a point P whose position is given by $\vec{r}_{P/O} = (4\hat{\imath} - 2\hat{\jmath} + 7\hat{k})$ m, what is the moment about an axis through the origin O with direction $\hat{\lambda} = \frac{2}{\sqrt{5}}\hat{\jmath} + \frac{1}{\sqrt{5}}\hat{\jmath}$?

2.3.13 A, B, and C are located by position vectors $\vec{r}_A = (1, 2, 3)$, $\vec{r}_B = (4, 5, 6)$, and $\vec{r}_C = (7, 8, 9)$.

a) Use the vector dot product to find the angle BAC (A is at the vertex of this angle).
b) Use the vector cross product to find the angle BCA (C is at the vertex of this angle).
c) Find a unit vector perpendicular to the plane ABC.
d) How far is the infinite line defined by AB from the origin? (That is, how close is the closest point on this line to the origin?)
e) Is the origin co-planar with the points A, B, and C?

2.3.14 Points A, B, and C in the figure define a plane.

a) Find a unit normal vector to the plane. *
b) Find the distance from perpendicular distance from point D to this infinite plane. *
c) What are the coordinates of the point on the plane closest to point D? *
d) Is this point on or off the triangle used to define the plane?



Filename:pfigure-s95q2
Problem 2.3.14

2.3.15 What point on the line that goes through the points (1,2,3) and (7,12,15) is closest to the origin?

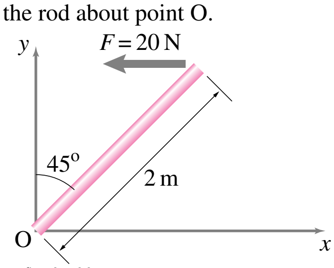
2.3.16 A regular tetrahedron is a triangular-based pyramid where all 6 edges have the same length ℓ. What is the perpendicular distance between a pair of non-touching edges? (There are many ways to solve this problem). *

2.3.17 Why did the chicken cross the road? *

2.4 Moment and Moment about an Axis

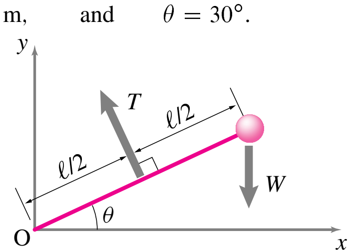
2.4.1 What is the moment $\vec{M}$ produced by a 20 N force F acting in the x direction with a lever arm of $\vec{r} = (16\,\text{mm})\hat{\jmath}$?

2.4.2 Find the moment of the force shown on the rod about point O.



Filename:pfigure2-vec2-2
Problem 2.4.2

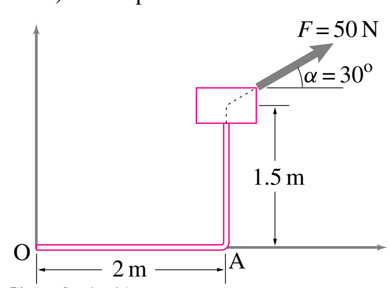
2.4.3 Find the sum of moments of forces $\vec{W}$ and $\vec{T}$ about the origin, given that $W = 100$ N, $T = 120$ N, $\ell = 4$ m, and $\theta = 30^\circ$.



Filename:pfigure2-vec2-3
Problem 2.4.3

2.4.4 Find the moment of the force

a) about point A
b) about point O.



Filename:pfigure2-vec2-4
Problem 2.4.4

2.4.5 The line of action of a force $\vec{F} = 20\,\text{N}\hat{\jmath} - 5\,\text{N}\hat{k}$ passes through a point A with coordinates (200 mm, 300 mm, -100 mm). What is the moment $\vec{M} (= \vec{r} \times \vec{F})$ of the force about the origin?

2.4.6 Drawing vectors and computing with vectors. In an xyz coordinate system, let point O be the origin. Two other points are specified: point A has xyz coordinates $(0m, 5m, 12m)$ and point B has xyz coordinates $(4m, 5m, 12m)$.

a) Make a neat sketch of the vectors OA, OB, and AB.

b) Find a unit vector in the direction of OA, call it $\hat{\boldsymbol{\lambda}}_{OA}$.

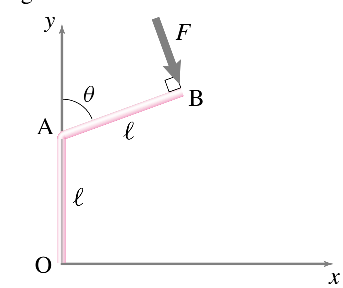
c) Find the force $\vec{F}$ which is 5 N in size and is in the direction of OA.

d) What is the angle between OA and OB?

e) What is $\vec{r}_{BO} \times \vec{F}$?

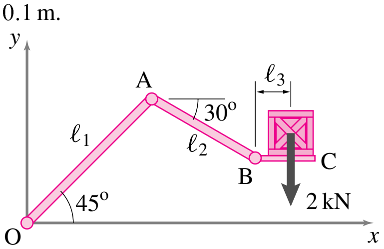
f) What is the moment of $\vec{F}$ about a line parallel to the z axis that goes through point B?

2.4.7 In the figure shown, OA = AB = 2 m. The force $F = 40$ N acts perpendicular to the arm AB. Find the moment of $\vec{F}$ about O, given that $\theta = 45°$. If $\vec{F}$ always acts normal to the arm AB, would increasing θ increase the magnitude of the moment? In particular, what value of θ will give the largest moment?



Filename:pfigure2-vec2-5
Problem 2.4.7

2.4.8 Calculate the moment of the 2 kNpayload on the robot arm about (i) joint A, and (ii) joint B, if $\ell_1 = 0.8$ m, $\ell_2 = 0.4$ m, and $\ell_3 = 0.1$ m.

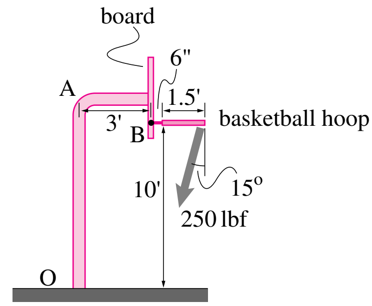


Filename:pfigure2-vec2-6
Problem 2.4.8

2.4.9 During a slam-dunk, a basketball player pulls on the hoop with a 250 lbf at point C of the ring as shown in the figure. Find the moment of the force about

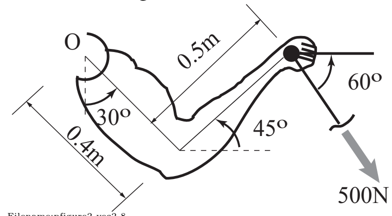
a) the point of the ring attachment to the board (point B), and

b) the root of the pole, point O.



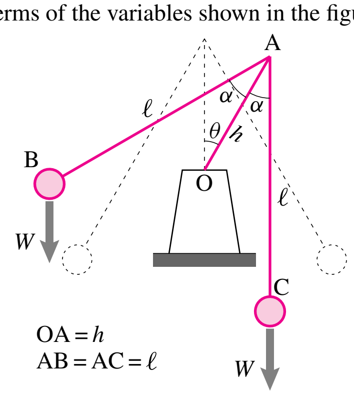
Filename:pfigure2-vec2-7
Problem 2.4.9

2.4.10 During weight training, an athlete pulls a weight of 500 N with his arms pulling on a handlebar connected to a universal machine by a cable. Find the moment of the force about the shoulder joint O in the configuration shown.



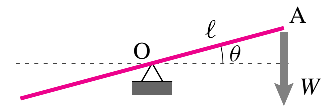
Filename:pfigure2-vec2-8
Problem 2.4.10

2.4.11 Find the sum of moments due to the two weights of the teeter-totter when the teeter-totter is tipped at an angle θ from its vertical position. Give your answer in terms of the variables shown in the figure.



Filename:pfigure2-vec2-9
Problem 2.4.11

2.4.12 Find the percentage error in computing the moment of $\vec{W}$ about the pivot point O as a function of θ, if the weight is assumed to act normal to the arm OA (a good approximation when θ is very small).



Filename:pfigure2-vec2-10
Problem 2.4.12

2.4.13 Vector Calculations and Geometry. The 5 N force $\vec{F}_1$ is along the line OA. The 7 N force $\vec{F}_2$ is along the line OB.

a) Find a unit vector in the direction OB. *

b) Find a unit vector in the direction OA. *

c) Write both $\vec{F}_1$ and $\vec{F}_2$ as the product of their magnitudes and unit vectors in their directions. *

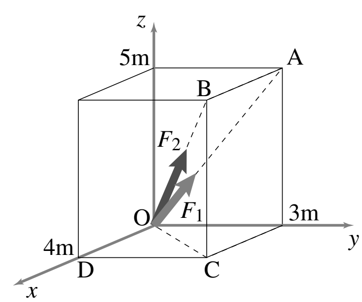
d) What is the angle AOB? *

e) What is the component of $\vec{F}_1$ in the x-direction? *

f) What is $\vec{r}_{DO} \times \vec{F}_1$? ($\vec{r}_{DO} \equiv \vec{r}_{O/D}$ is the position of O relative to D.) *

g) What is the moment of $\vec{F}_2$ about the axis DC? *

h) Repeat the last problem using either a different reference point on the axis DC or the line of action OB. Does the solution agree? [Hint: it should.] *



Filename:p1sp92
Problem 2.4.13

2.5 Solving vector equations

2.5.1 Consider the vector equation

$$a\vec{A} + b\vec{B} = \vec{C}$$

with $\vec{A}$, $\vec{B}$, and $\vec{C}$ given. For the cases below find a and b if possible. If there are multiple solutions give at least 2. If there are no solutions explain why.

a) $\vec{A} = \hat{\imath},\ \vec{B} = \hat{\jmath},$
$\vec{C} = 3\hat{\imath} + 4\hat{\jmath}$

b) $\vec{A} = \hat{\imath},\ \vec{B} = 2\hat{\imath},$
$\vec{C} = 3\hat{\imath}$

c) $\vec{A} = \hat{\jmath},\ \vec{B} = 2\hat{\jmath},$
$\vec{C} = 3\hat{\imath}$

d) $\vec{A} = \hat{\imath} + \hat{\jmath},\ \vec{B} = -\hat{\imath} + \hat{\jmath},$
$\vec{C} = 2\hat{\jmath}$

e) $\vec{A} = \hat{\imath} + 2\hat{\jmath},\ \vec{B} = 2\hat{\imath} + 3\hat{\jmath},$
$\vec{C} = 3\hat{\imath} + 4\hat{\jmath}$

f) $\vec{A} = \pi\hat{\imath} + e\hat{\jmath},\ \vec{B} = \sqrt{2}\hat{\imath} + \sqrt{3}\hat{\jmath},$
$\vec{C} = \hat{\imath}$

2.5.2 Consider the vector equation

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

with $\vec{A}$, $\vec{B}$, $\vec{C}$ and $\vec{D}$ given. For the cases below find a if possible, there is no need to find b and c.

a) $\vec{A} = \hat{\imath}, \vec{C} = \hat{k}$
$\vec{B} = \hat{\jmath}, \vec{D} = 3\hat{\imath} + 4\hat{\jmath} + 19\hat{k}$

b) $\vec{A} = 2\hat{\imath}, \vec{C} = 15\hat{\jmath} + 360\hat{k}$
$\vec{B} = 3\hat{\imath} + 4\hat{\jmath}, \vec{D} = 2\hat{\imath} + 17\hat{\jmath} + 37\hat{k}$

c) $\vec{A} = \hat{k}, \vec{C} = \hat{\imath} + \hat{\jmath} + \hat{k}$
$\vec{B} = \hat{\jmath} + \hat{k}, \vec{D} = \hat{\jmath}$

d) $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{C} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$
$\vec{B} = 2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}, \vec{D} = 4\hat{\imath} + 5\hat{\jmath} + 7\hat{k}$

e) $\vec{A} = \sqrt{1}\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k},$
$\vec{C} = \sqrt{7}\hat{\imath} + \sqrt{8}\hat{\jmath} + \sqrt{9}\hat{k},$
$\vec{B} = \sqrt{4}\hat{\imath} + \sqrt{5}\hat{\jmath} + \sqrt{6}\hat{k},$
$\vec{D} = -\hat{\imath} + \pi\hat{\jmath} + e\hat{k}$

2.5.3 In the problems below use matrix algebra on a computer to find a, b and c uniquely if possible. If not possible explain why not. You are given that

$$a\vec{A} + b\vec{B} + c\vec{C} = \vec{D}$$

and that

a) $\vec{A} = \hat{\imath}, \vec{B} = \hat{\jmath},$
$\vec{C} = \hat{k}, \vec{D} = -2\hat{\imath} + 5\hat{\jmath} + 10\hat{k}$

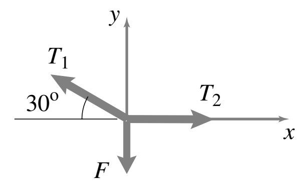
b) $\vec{A} = \hat{\imath} + \hat{\jmath}, \vec{B} = -\hat{\imath} + \hat{\jmath},$
$\vec{C} = \hat{k}, \vec{D} = 2\hat{\imath}$

c) $\vec{A} = \hat{\imath} + \hat{\jmath} + \hat{k}, \vec{B} = 2\hat{\imath} + \hat{\jmath} + \hat{k},$
$\vec{C} = \hat{\imath} + 2\hat{\jmath} + \hat{k}, \vec{D} = 2\hat{\imath} + \hat{\jmath} + \hat{k}$

d) $\vec{A} = 1\hat{\imath} + 2\hat{\jmath} + 3\hat{k}, \vec{B} = 4\hat{\imath} + 5\hat{\jmath} + 6\hat{k},$
$\vec{C} = 7\hat{\imath} + 8\hat{\jmath} + 9\hat{k}, \vec{D} = 10\hat{k}$

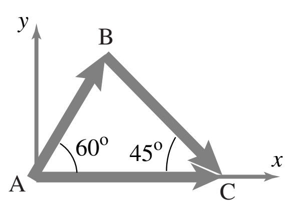
e) $\vec{A} = \sqrt{1}\hat{\imath} + \sqrt{2}\hat{\jmath} + \sqrt{3}\hat{k},\ \vec{B} = \sqrt{4}\hat{\imath} + \sqrt{5}\hat{\jmath} + \sqrt{6}\hat{k},$
$\vec{C} = \sqrt{7}\hat{\imath} + \sqrt{8}\hat{\jmath} + \sqrt{9}\hat{k},\ \vec{D} = \sqrt{10}\hat{k}$

2.5.4 The three forces shown in the figure are in equilibrium, *i.e.,* $\vec{T}_1 + \vec{T}_2 + \vec{F} = \vec{0}$. If $|\vec{F}| = 10\,\text{N}$, find tensions T_1 and T_2 (magnitudes of $\vec{T}_1$ and $\vec{T}_2$.



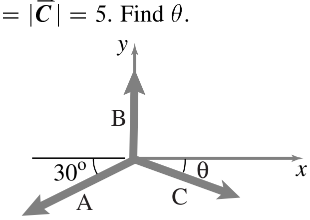
Filename:pfigure2-4-2Dp1
Problem 2.5.4

2.5.5 Points A, B, and C are located in the xy plane as shown in the figure. For position vectors, we can write, $\vec{r}_B + \vec{r}_{C/B} = \vec{r}_C$. Find $|\vec{r}_B|$ and $|\vec{r}_{C/B}|$ if $\vec{r}_C = 10\,\text{m}\,\hat{\imath}$.



Filename:pfigure2-4-2Dp2
Problem 2.5.5

2.5.6 Three vectors, $\vec{A}$, $\vec{B}$, and $\vec{C}$, (shown in the figure) are such that $\vec{A} + \vec{B} + \vec{C} = \vec{0}$. You are given that $A = |\vec{A}| = 8$ and $C = |\vec{C}| = 5$. Find θ.



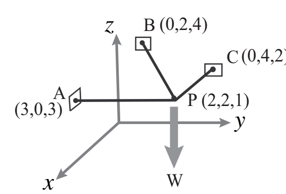
Filename:pfigure2-4-2Dp3
Problem 2.5.6

2.5.7 Let $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = \vec{0}$, where $\vec{F}_3 = 10\,\text{N}(\hat{\imath} - \hat{\jmath})$, and $\vec{F}_1/|\vec{F}_1| = 0.250\hat{\imath} + 0.968\hat{\jmath}$ and $\vec{F}_2/|\vec{F}_2| = -0.425\hat{\imath} - 0.905\hat{\jmath}$. Find $|\vec{F}_1|$ from a single scalar equation.

2.5.8 To evaluate the equation $\sum\vec{F} = m\vec{a}$ for some problem, a student writes $\sum\vec{F} = F_x\hat{\imath} - (F_y - 30\,\text{N})\hat{\jmath} + 50\,\text{N}\hat{k}$ in the xyz coordinate system, but $\vec{a} = 2.5\,\text{m/s}^2\hat{\imath}' + 1.8\,\text{m/s}^2\hat{\jmath}' - a_z\hat{k}'$ in a rotated $x'y'z'$ coordinate system. If $\hat{\imath}' = \cos 60^\circ\,\hat{\imath} + \sin 60^\circ\,\hat{\jmath}$, $\hat{\jmath}' = -\sin 60^\circ\,\hat{\imath} + \cos 60^\circ\,\hat{\jmath}$ and $\hat{k}' = \hat{k}$, find the scalar equations for the x', y', and z' directions.

2.5.9 A car travels straight north-east for a while on a dirt road that leads to a north-south highway. The car travels on the highway due north for a while. When the driver stops, the GPS system indicates that the car is 60 miles north and 30 miles east from the starting point. Find the distance travelled on the dirt road.

2.5.10 A particle is held at point P with the help of three strings PA, PB, and PC. Let the tensions in the three strings be $\vec{T}_A$, $\vec{T}_B$, and $\vec{T}_C$, respectively (so that $\vec{T}_A$ acts along line PA and so on). The equilibrium of the particle requires that $\vec{T}_A + \vec{T}_B + \vec{T}_C + \vec{W} = \vec{0}$ where $\vec{W} = -10\,\text{N}\,\hat{k}$ is the weight of the particle. Find the magnitudes of tensions in the three strings.



Filename:pfigure2-4-3D1
Problem 2.5.10

2.5.11 You are given that $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 5\,\text{kN}\ \hat{\jmath}$ where $\vec{F}_1 = (2\hat{\imath} - 3\hat{\jmath} + 4\hat{k})\,\text{kN}$, $\vec{F}_2 = (\hat{\imath} + 5\hat{k})\,\text{kN}$. Find the direction of $\vec{F}_3$ (An angle measured CCW from the $+x$ axis to the direction of positive $\vec{F}_3$.

2.5.12 A plane intersects the x, y, and z axis at 3,4, and 5 respectively. What point on the plane is in the direction $\hat{\imath} + 2\hat{\jmath} + 3\hat{\jmath}$ from the point (10,10,10)? (Find the x, y and z components of the point.).

These problems concern the solution of simultaneous equations. These could come from various vector equations.

2.5.13 Write the following equations in matrix form to solve for x, y, and z:

$$\begin{aligned} 2x - 3y + 5 &= 0, \\ y + 2\pi z &= 21, \\ \frac{1}{3}x - 2y + \pi z - 11 &= 0. \end{aligned}$$

2.5.14 Are the following equations linearly independent?

a) $x_1 + 2x_2 + x_3 = 30$

b) $3x_1 + 6x_2 + 9x_3 = 4.5$

c) $2x_1 + 4x_2 + 15x_3 = 7.5$.

2.5.15 Write computer commands (or a program) to solve for x, y and z from the following equations with r as an input variable. Your program should display an error message if, for a particular r, the equations are not linearly independent.

a) $5x + 2r\,y + z = 2$

b) $3x + 6y + (2r - 1)z = 3$

c) $2x + (r - 1)y + 3r\,z = 5$.

Find the solutions for $r = 3, 4.99$, and 5.

2.5.16 An exam problem in statics has three unknown forces. A student writes the following three equations (he knows that he needs three equations for three unknowns!) — one for the force balance in the x-direction and the other two for the moment balance about two different points.

a) $F_1 - \frac{1}{2}F_2 + \frac{1}{\sqrt{2}}F_3 = 0$

b) $2F_1 + \frac{3}{2}F_2 = 0$

c) $\frac{5}{2}F_2 + \sqrt{2}F_3 = 0$.

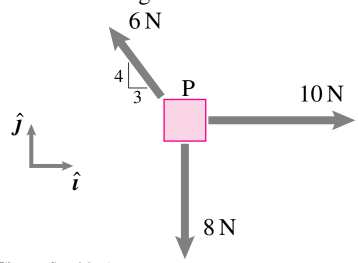
Can the student solve for F_1, F_2, and F_3 uniquely from these equations? *

2.5.17 What is the solution to the set of equations:

$$\begin{aligned} x + y + z + w &= 0 \\ x - y + z - w &= 0 \\ x + y - z - w &= 0 \\ x + y + z - w &= 2? \end{aligned}$$

2.6 Equivalent force systems and couples

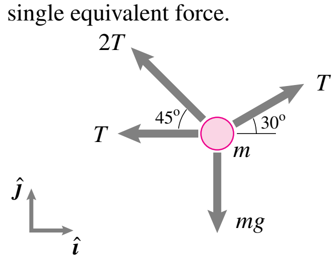
2.6.1 Find the net force on the particle shown in the figure.



Filename:pfigure2-3-rp1

Problem 2.6.1

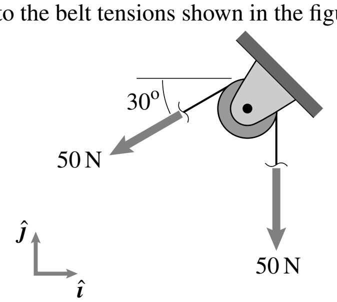
2.6.2 Replace the forces acting on the particle of mass m shown in the figure by a single equivalent force.



Filename:pfigure2-3-rp2

Problem 2.6.2

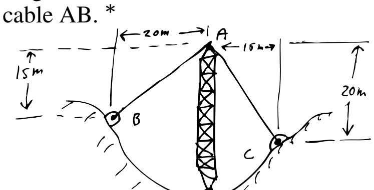
2.6.3 Find the net force on the pulley due to the belt tensions shown in the figure.



Filename:pfigure2-3-rp3

Problem 2.6.3

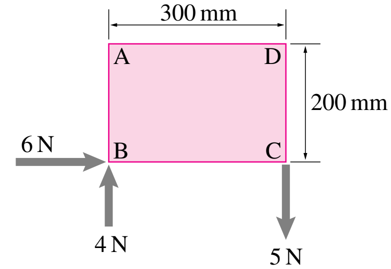
2.6.4 The net force on A from the two cables is a force that points down and has magnitude of 125 N. Find the tension in cable AB. *



Filename:S03quiz1atower

Problem 2.6.4

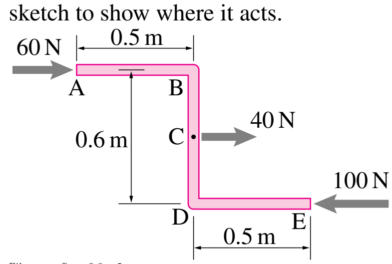
2.6.5 Replace the forces shown on the rectangular plate by a single equivalent force. Where should this equivalent force act on the plate and why?



Filename:pfigure2-3-rp4

Problem 2.6.5

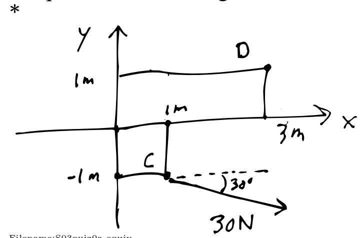
2.6.6 Three forces act on a Z-section ABCDE as shown in the figure. Point C lies in the middle of the vertical section BD. Find an equivalent force-couple system acting on the structure and make a sketch to show where it acts.



Filename:pfigure2-3-rp5

Problem 2.6.6

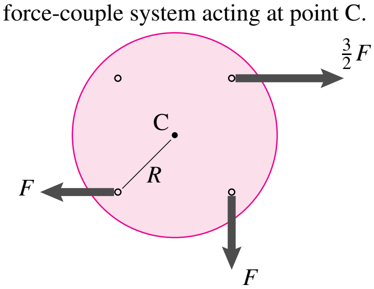
2.6.7 Find a force-couple system at D that is equivalent to the single force at C shown. *



Filename:S03quiz0a-equiv

Problem 2.6.7

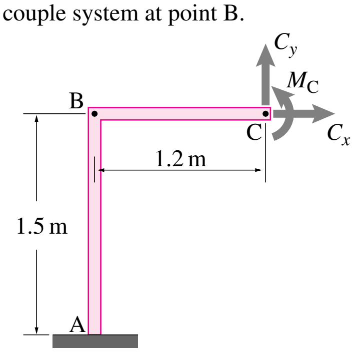
2.6.8 The three forces acting on the circular plate shown in the figure are equidistant from the center C. Find an equivalent force-couple system acting at point C.



Filename:pfigure2-3-rp6

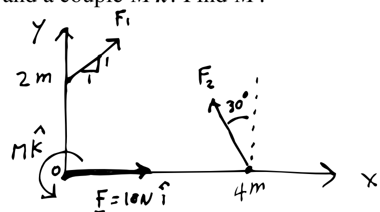
Problem 2.6.8

2.6.9 The forces and the moment acting on point C of the frame ABC shown in the figure are $C_x = 48\,\mathrm{N}$, $C_y = 40\,\mathrm{N}$, and $M_c = 20\,\mathrm{N{\cdot}m}$. Find an equivalent force couple system at point B.



Filename:pfigure2-3-rp7
Problem 2.6.9

2.6.10 The force system $(\vec{F}_1, \vec{F}_2)$ is equivalent to a force $\vec{F} = 10\hat{\imath}\,\mathrm{N}$ at the origin and a couple $M\hat{k}$. Find M. *

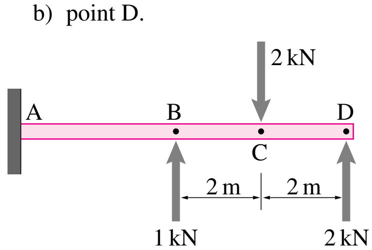


Filename:S03quiz0b-equiv
Problem 2.6.10

2.6.11 Find an equivalent force-couple system for the forces acting on the beam shown in the figure, if the equivalent system is to act at

a) point B,

b) point D.

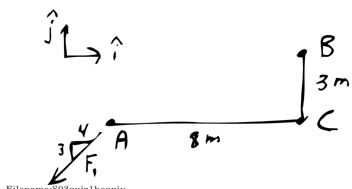


Filename:pfigure2-3-rp8
Problem 2.6.11

2.6.12 $\vec{F}_1$ acts at A and $\vec{F}_2 = 7\,\mathrm{N}\hat{\imath}$ acts at an unknown location. Together they are equivalent to a force $\vec{F}_B$ and moment $\vec{M}_B = 48\,\mathrm{N{\cdot}m}\hat{k}$ at B. Together they are also equivalent to a force $\vec{F}_C$ and moment $\vec{M}_C = 75\,\mathrm{N{\cdot}m}\hat{k}$ at C.

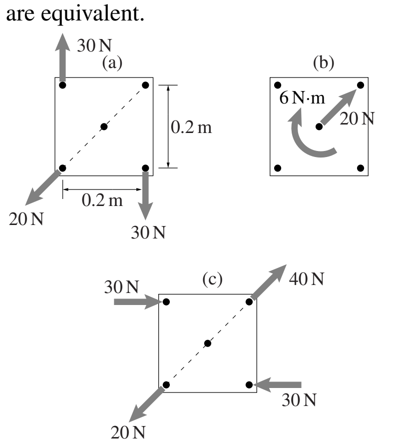
a) Find $\vec{F}_C$. *

b) Find the line of action of $\vec{F}_2$. *



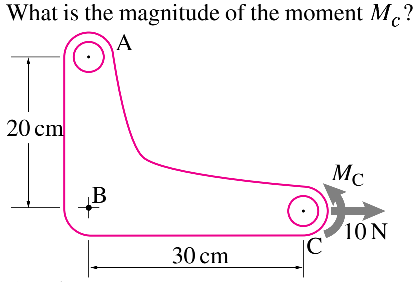
Filename:S03quiz1bequiv
Problem 2.6.12

2.6.13 The figure shows three different force-couple systems acting on a square plate. Identify which force-couple systems are equivalent.



Filename:pfigure2-3-rp9
Problem 2.6.13

2.6.14 The force and moment acting at point C of a machine part are shown in the figure where M_c is not known. It is found that if the given force-couple system is replaced by a single horizontal force of magnitude 10 N acting at point A then the net effect on the machine part is the same. What is the magnitude of the moment M_c?



Filename:pfigure2-3-rp10
Problem 2.6.14

2.6.15 2D. Assume a force system is equivalent to a force $\vec{F}_1 \neq \vec{0}$ and couple $\vec{M}_1 = M_1\hat{k}$ acting at point $\vec{r}_1$.

a) Find some point $\vec{r}_2$, and force $\vec{F}_2$ so that $\vec{F}_2$ acting at $\vec{r}_2$ is equivalent to $\vec{F}_1$ and $\vec{M}_1$ acting at $\vec{r}_1$. *

b) Find all possible wrenches (combinations of point location, force and moment) equivalent to the system with $\vec{F}_1$ and $\vec{M}_1$ acting at the point with position vector $\vec{r}_1$. *

c) Describe the situation in the special case when $\vec{F}_1 = \vec{0}$. *

2.6.16 3D. Assume a force system is equivalent to a force $\vec{F}_1$ and couple $\vec{M}_1$ acting at point with position vector $\vec{r}_1$.

a) Find a point P with position vector $\vec{r}_2$, so that an equivalent force system $\vec{F}_2$ and $\vec{M}_2$ acting at acting at P has $\vec{F}_2$ is parallel to $\vec{M}_2$. (Finding such a point, force and moment is called "reducing the force system to a *wrench*"). *

b) Find all possible wrenches (combinations of point location, force and moment) equivalent to the system with $\vec{F}_1$ and $\vec{M}_1$ acting at $\vec{r}_1$. *

Note, one special case with a slightly different result than the other cases is if $\vec{F}_1 = \vec{0}$, so it should be treated separately.

2.7 Center of mass and center of gravity

2.7.1 An otherwise massless structure is made of four point masses, m, $2m$, $3m$ and $4m$, located at coordinates (0, 1 m), (1 m, 1 m), (1 m, −1 m), and (0, −1 m), respectively. Locate the center of mass of the structure. *

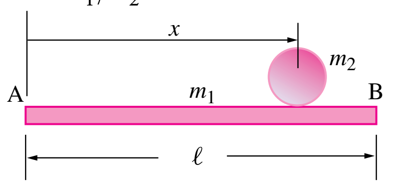
2.7.2 -3-D: The following data is given for a structural system modeled with five point masses in 3-D-space:

mass	**coordinates (in m)**
0.4 kg	(1,0,0)
0.4 kg	(1,1,0)
0.4 kg	(2,1,0)
0.4 kg	(2,0,0)
1.0 kg	(1.5,1.5,3)

Locate the center of mass of the system.

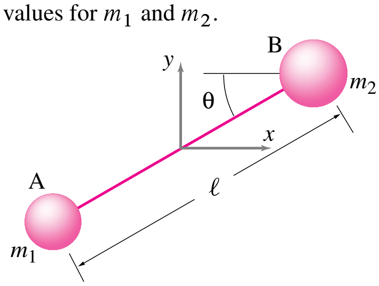
2.7.3 Write a computer program to find the center of mass of a point-mass-system. The input to the program should be a table (or matrix) containing individual masses and their coordinates. (It is possible to write a single program for both 2-D and 3-D cases, write separate programs for the two cases if that is easier for you.) Check your program on Problems 2.7.1 and 2.7.2.

2.7.4 A cylinder of mass m_2 and radius R rolls on a flat circular plate of mass m_1 and length ℓ. Let the position of the cylinder from the left edge of the plate be x. Find the horizontal position of the center of mass of the system as a function of x and a non-dimensional mass parameter $M = m_1/m_2$.



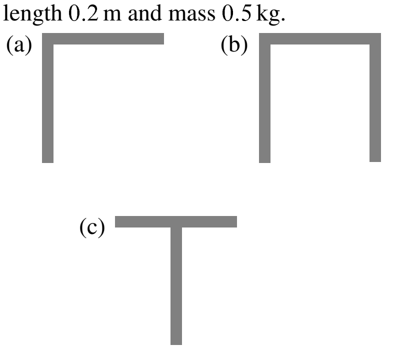
Filename:pfigure2-5-baranddisk
Problem 2.7.4

2.7.5 Two masses m_1 and m_2 are connected by a massless rod AB of length ℓ. In the position shown, the rod is inclined to the horizontal axis at an angle θ. Find the position of the center of mass of the system as a function of angle θ and the other given variables. Check if your answer makes sense by setting appropriate values for m_1 and m_2.



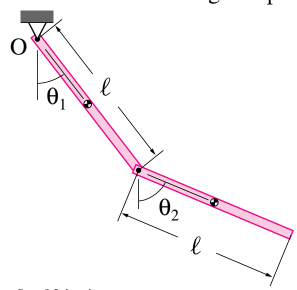
Filename:pfigure2-5-dumbbell
Problem 2.7.5

2.7.6 Find the center of mass of the following composite bars. Each composite shape is made of two or more uniform bars of length 0.2 m and mass 0.5 kg.



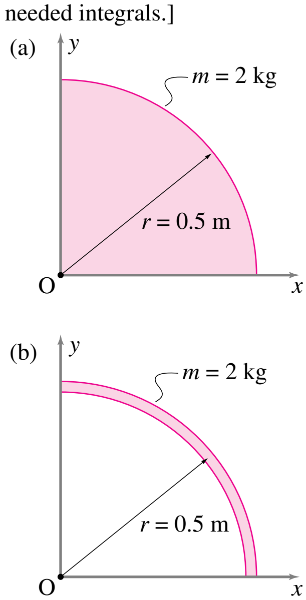
Filename:pfigure3-cm-rp7
Problem 2.7.6

2.7.7 A double pendulum consists of two uniform bars of length ℓ and mass m each. The pendulum hangs in the vertical plane from a hinge at point O. Taking O as the origin of a xy coordinate system, find the location of the center of gravity of the pendulum as a function of angles θ_1 and θ_2.



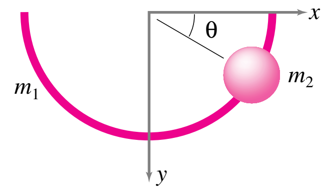
Filename:pfigure2-5-dpend
Problem 2.7.7

2.7.8 Find the center of mass of the following two objects [Hint: set up and evaluate the needed integrals.]



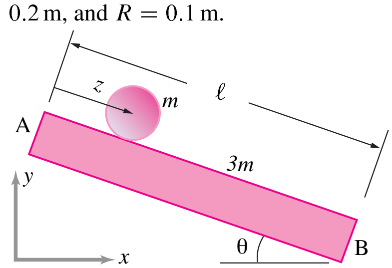
Filename:pfigure3-cm-rp8
Problem 2.7.8

2.7.9 A semicircular ring of radius $R = 1$ m and mass $m_1 = 0.1$ kg rests in the vertical plane. A bead of mass $m_2 = 0.25$ kg slides on the ring. Find the position of the center of mass of the ring-bead-system at an instant when $\theta = 30^\circ$. How does the center of mass position change as θ changes?



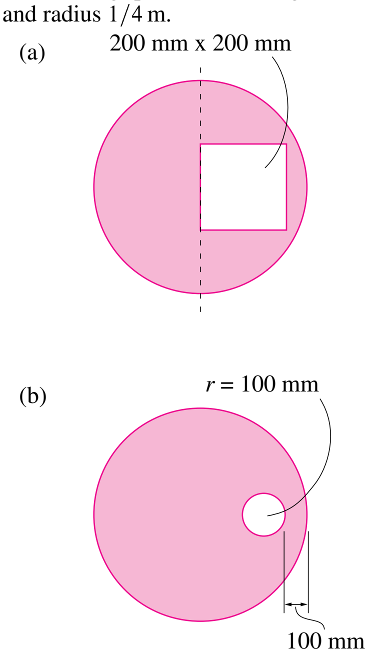
Filename:pfigure2-5-halfring
Problem 2.7.9

2.7.10 A uniform circular disk of mass m and radius R rolls on an inclined rectangular plate of mass $3m$ and dimensions $2R \times \ell$. When the plate is horizontal ($\theta = 0$), the left lower corner of the plate is at the origin of a fixed xy coordinate system. Find the coordinates of the center of mass of the system for $m = 1$ kg, $\ell = 1$ m, $z = 0.2$ m, and $R = 0.1$ m.



Filename:pfigure2-5-baranddisk2
Problem 2.7.10

2.7.11 Find the center of mass of the following plates obtained from cutting out a small section from a uniform circular plate of mass 1 kg (prior to removing the cutout) and radius 1/4 m.



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Problem 2.7.11

CHAPTER 3

Free-Body Diagrams

A free-body diagram is a sketch of the system to which you will apply the laws of mechanics, and all the non-negligible external forces and couples which act on it. The diagram indicates what material is in the system. The diagram shows what is, and what is not, known about the forces. Generally there is a force or moment component associated with any connection that causes or prevents a motion. Conversely, there is no force or moment component associated with motions that are freely allowed. Mechanics reasoning entirely rests on free body diagrams. Many student errors in problem solving are due to problems with their free body diagrams, so we give tips about how to avoid various common free-body diagram mistakes.

Contents

In mechanics a system is often called a *body* and when it is conceptually isolated it is *free*, as in *free* from its surroundings.

A *free-body diagram (FBD)* is a sketch of an isolated system and the external forces which act on it.

The laws of mechanics are applied using the forces shown on a free body diagram and not using any other forces. Thus, as we say again and again, drawing good free-body diagrams is essential for both statics and dynamics. The skills for drawing these diagrams are presented in the following sections.

Some basic mechanics assumptions

One way to understand something is to isolate it, see how it behaves on its own, and see how it responds to various stimuli. Then, when the thing is not isolated, you still think of it as isolated, but think of the effects of all its surroundings as stimuli. Reversing the point of view, we can also see the system's behavior as causing stimulus to other things around it, which themselves can be thought of as isolated and stimulating back, and so on.

This reductionist approach is used throughout the physical and social sciences. A tobacco plant is understood in terms of its response to light, heat flow, the chemical environment, insects, and viruses. The economy of Singapore is understood in terms of the flow of money and goods in and out of the country. And social behavior is regarded as being a result of individuals reacting to the sights, sounds, smells, and touch of other individuals and thus causing sights, sounds, smell and touch that the others react to in turn, *etc.*①

The "*free-body*" is a closed system. As in elementary thermodynamics we will only be concerned with so-called *closed* systems. A (closed) system, in mechanics, is a fixed collection of material. You can draw an imaginary boundary around a system, then in your mind paint all the atoms inside the boundary red, and then define the system as being the red atoms, no matter whether they later cross the original boundary markers or not. Thus mechanics depends on bits of matter as being durable and non-ephemeral. We assume that②

①**Closed systems in thermodynamics.**The isolated system approach to understanding is made most clear in thermodynamics courses. A system, usually a fluid, is isolated with rigid walls that allow no heat, motion or material to pass. Then, bit by bit, as the subject is developed, the response of the system to certain interactions across the boundaries is allowed. Eventually, enough interactions are understood that the system can be viewed as isolated even when in a useful context. The gas expanding in a refrigerator follows the same rules of heat-flow and work as when it was expanded in its 'isolated' container.

② **Open Systems.** The mechanics of *open systems*, where material crosses the system boundaries, is important in fluid mechanics. Such open fluid systems are first seen in some elementary dynamics problems (like rockets), where material is allowed to cross the system boundaries. But the equations governing these *open* systems are deduced from careful application of the more fundamental governing mechanics equations of *closed* systems. So we have to master the mechanics of closed systems first.

A given bit of matter in a system exists forever, has the same mass forever, and is always in that system.

Mechanics is based on the notion that any part of a system is itself a system and that all interactions between systems or subsystems have certain simple rules, most basically:

Force is the measure of mechanical interaction.

Thus a person can be moved by forces, but not by the sight of a tree falling towards them or the attractive smell of a flower. These things may cause, by rules that fall outside of mechanics, forces that move a person. When a person moves towards a flower or away from a falling tree she is moved by the force of the ground on her feet not by smell or fear. Finally,

The principle of "action and reaction": what one system does to another, the other does back to the first.

When a person accelerates away from a falling tree because of the force on the ground, her feet push equally hard on the ground the other way. Of course you think of this the other way around. To start running you make a quick action, pushing on the ground with your feet. The *reaction* force accelerates you. But what causes what is not the issue. Rather, if system A is pushed by B then B is pushed back equally by A. In kindergarten talk, 'it doesn't matter who started it.'

The rules above, which we call the zeroth③ laws of mechanics, imply that all the mechanical effects of the outside world on a system can be represented by a sketch of the system with arrows showing the forces of interaction. If we want to know how the system, in turn, affects some part of its surroundings we draw the opposite arrows on a sketch of the part.

3.1 Free-body diagrams: interactions, representing forces and partial FBDs

A *free-body diagram* is a sketch of the system of interest and the forces that act on the system. A free-body diagram precisely defines the system to which you are applying mechanics equations and the forces to be considered. Any reader of your calculations needs to see your free-body diagrams. To put it directly, if you want to be right and be seen as right, then ①

Draw a free-body diagram!

③ Why do we awkwardly number the most basic laws as zero? Because they are really more of an underlying assumption, a background concept, than a law. As a law they are a little imprecise since force has not yet been defined. You could take the latter two of these zeroth laws as an implicit and partial definition of force. The phrase "zeroth law" means "important implicit assumption". The third of the zeroth laws is usually called "Newton's third law."

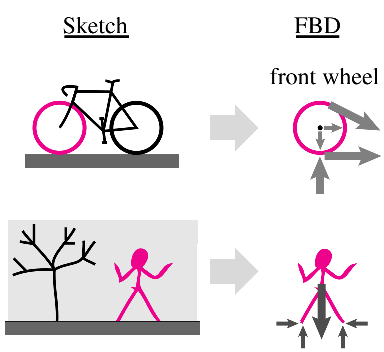

Figure 3.1: A sketch of a bicycle and a free-body diagram of the braked front wheel. A sketch of a person and a free-body diagram of the whole person.

① **Free-body Perkins.** In the 1950's Cornell professor Harold C. Perkins was nick named 'free-body Perkins'. Perkins would stop students in the hall and say "You! Come in my office! Draw a free-body diagram of this!" Students drew free-body diagrams to please persnickety Perkins. And so learned how to get mechanics problems right.

The concept of the free-body diagram is simple. In practice, however, drawing useful free-body diagrams takes some thought, even for those practiced at the art. Some basic tips are described below a few different ways.

How to draw a free-body diagram

We suggest the following procedure for drawing a free-body diagram, as shown schematically in fig. 3.4

1. **Define the system.** Define in your own mind to what system, or what collection of material, you would like to apply the laws of mechanics. This 'body' may be just a part of your overall system of interest. Figure 3.2 on page 151 shows some possible systems when considering plyers.
2. **Sketch the system.** Your sketch may include various cut marks to show how the 'body' is isolated from its environment. Imagine cutting the system free from its environment with a sharp scalpel or with a chain saw.
3. **Stare at each cut.** Look systematically at the picture at the places that the system interacts with material *not* shown in the picture, places where you made 'cuts'.
4. **Fool the body.** Use forces and torques to fool the system into thinking it has not been cut. For example, if the system is being pushed in a given direction at a given contact point where you have cut the system free, then show a force in that direction at that point. If a system is being prevented from rotating by a (cut) rod, then show a torque at that cut.
5. **Replace gravity with a force.** To show that you have cut the system from the earth's gravity force show the force of gravity on the system's center-of-mass or on the centers of mass of its parts.

What shows on a free-body diagram? What doesn't?

Here are some more details about the elements of a good free body diagram. Some of these are stylistic issues, but we think they help with problem solving.

- **The system.** A free-body diagram is a picture of the system for which you would like to apply linear or angular momentum balance (force and moment balance being special cases) or power balance. It shows the system isolated ('free') from its environment. That is, the free-body diagram does *not* show things that are near or touching the system of interest. See fig. 3.1.
- **The word 'body' means system.** A free-body diagram may show one or more particles, rigid objects, deformable objects, or parts thereof such as a machine, a component of a machine, or a part of a component of a machine. You can draw a free-body diagram of any collection

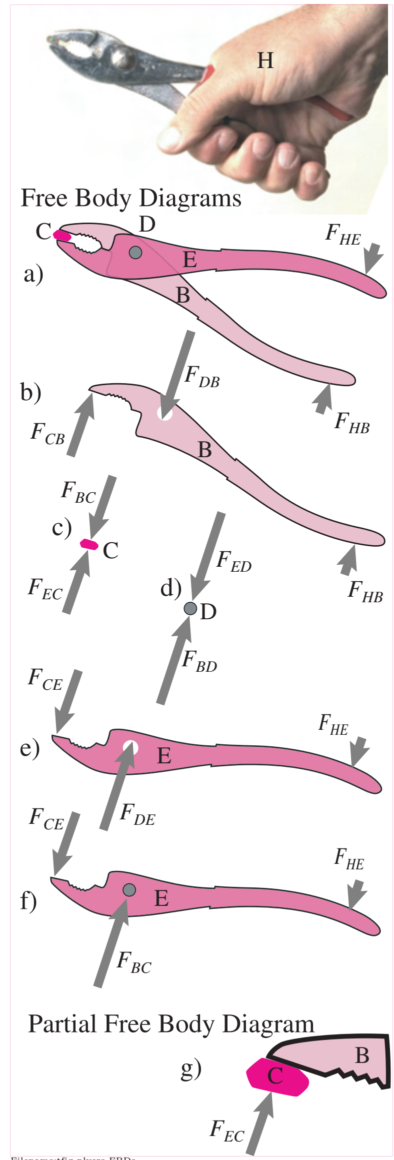

Figure 3.2: **Plyers crushing a pencil.** Some possible free body diagrams (FBDs), neglecting gravity. With four major parts (upper jaw, lower jaw, pin & pencil) there are 15 possible subsystems: **a)** Whole system; **b)** Upper jaw; **c)** Pencil; **d)** Connecting pin; **e)** Lower jaw; **f)** Lower jaw with connecting pin; and nine others (*e.g.,* plyers without pencil, upper jaw plus pin, both jaws plus pencil, *etc.*). For each system the external forces on that system are shown. **g)** A partial free body diagram, showing the force on the pencil and upper jaw. See fig. 3.6 on page 155 for some bad FBDs of the same system.

of material that you can identify. The word *body* connotes a standard object in some people's minds. In the context of free-body diagrams, 'body' means system. The body in a free-body diagram may be a subsystem of the overall system of interest. For a system of n parts there are $2^n - 1$ collections of parts. For the plyers of fig. 3.2 there are 4 parts and 15 possible FBDs (6 of which are shown).

- **Forces fool the system.** The free-body diagram of a system shows the forces and moments that the surroundings impose on the system. That is, since the only method of mechanical interaction that Nature has invented is force (and moment), the free-body diagram shows what it would take to mechanically fool the system if it were literally cut free. That is, the motion of the system would be totally unchanged if it were cut free and the forces shown on the free-body diagram were applied as a replacement for all external interactions.
- **Each force has a source and a target.** Every force shown on a FBD acts *on* the system (the body) and *from* another object according to some *rule*. For each force you should be able to name the target (the 'free body'), the source (*e.g.,* a contacting body) and the rule (*e.g.,* laws of gravity, a spring equation, the force sufficient to prevent interpenetration). Subscripts can help, such as F_{ED} indicating the force is from E and on D (See fig. 3.2).
- **Place forces at cuts.** The forces and moments are located on the free-body diagram at the points where they are applied. These places are where you made 'cuts' to *free* the *body*.
- **Motion is caused or prevented by forces.** At places where the outside environment causes or restricts translation of the isolated system, a contact force is drawn on the free-body diagram.
- **Rotation is caused or prevented by torques.** At connections to the outside world that cause or restrict rotation of the system a contact torque (or couple or moment) is drawn. Draw this moment outside the system for viewing clarity. Refer again to fig. 3.3 to see how the moment on the block due to the friction of the hinge is best shown outside the block.
- **Draw contact forces outside the body.** Draw the contact force outside the sketch of the system for viewing clarity. A block supported by a hinge with friction in fig. 3.3 illustrates how the reaction force on the block due to the hinge is best shown outside the block.
- **Draw body forces (*e.g.,* gravity forces) inside the body.** The free-body diagram shows the system cut free from the source of any *body forces* applied to the system. Body forces are forces that act on the inside of a body from objects outside the body. It is best to draw the body forces on the interior of the body, at the center-of-mass if that correctly represents the net effect of the body forces. Figure 3.3 shows the cleanest way to represent the gravity force on the uniform block acting at the center-of-mass. ②

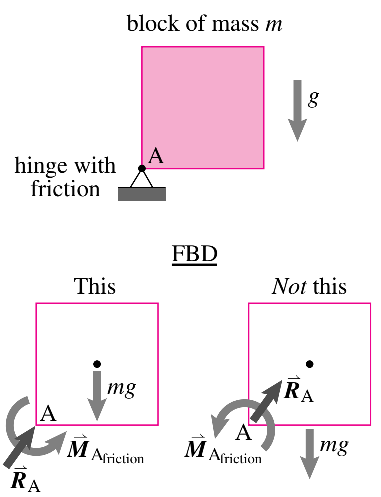

Figure 3.3: A uniform block of mass m supported by a hinge with friction in the presence of gravity. The free-body diagram on the right is correct, just less clear than the one on the left.

②In this book, the only body force we consider is gravity. For near-earth gravity, gravity forces show on the free-body diagram as a single force at the center of gravity, or as a collection of forces at the center of gravity of each of the system parts. For parts of electric motors and generators, not covered here in detail, electrostatic or electro-dynamic body forces also need to be considered.

- **Internal forces are not drawn.** The free-body diagram shows all external forces acting on the system but *no* internal forces — forces between objects within the body are not shown. See fig. 3.6 on page 155 for examples of what, despite temptation, *not* to do.
- **No velocity and no acceleration.** The free-body diagram shows nothing about the motion③. It shows: no "centrifugal force", no "acceleration force", and no "inertial force". (Of course for statics this is a non-issue because inertial terms are neglected for all purposes.) Repeating

Velocities, accelerations and inertial forces do not show on a free-body diagram ④ .

③**Warning.** A common error made by beginning dynamics students is to put velocity and/or acceleration arrows on the free-body diagram.

④ The prescription that you *not show inertial forces* is a white lie. Actually, in the D'Alembert approach to dynamics, a legitimate and intuitive approach for experts, one does show inertial forces on the free-body diagram. The D'Alembert approach is not followed in this book in any theory or examples because of the frequent sign errors and mind-confusions it causes in beginners (translation: "not allowed in homework or exams"). For those who are attracted to forbidden fruit, see box 9.4 on page 427.

How to draw forces on free-body diagrams

How you draw a force on a free-body diagram depends on

- How much you know about the force before your analysis. Do you know its direction? its magnitude? and
- Your choice of notation (which may vary from vector to vector within one free-body diagram). See page 46 for a description of the 'symbolic' and 'graphical' vector notations.

Some of the possibilities are shown in fig. 3.5 when

(a) Any $\vec{F}$ possible,

(b) the direction of $\vec{F}$ is known, and

(c) Everything about $\vec{F}$ is known.

In each case three different notations are shown.

Simplify using equivalent force systems

The concept of 'fooling' a system with forces is somewhat subtle. If the free-body diagram involves 'cutting' a rope what force should one show? A rope is made of many fibers so cutting the rope means cutting all of the rope fibers. Should one show hundreds of force vectors, one for each fiber that is cut? The answer is: yes and no. You would be correct to draw all of these hundreds of forces at the fiber cuts. But, since the equations that are used with any free-body diagram involve only the total force and total moment, you are also allowed to replace these forces with an equivalent force system (see section 2.6).

Any force system acting on a given free-body diagram can be replaced by an equivalent force and couple.

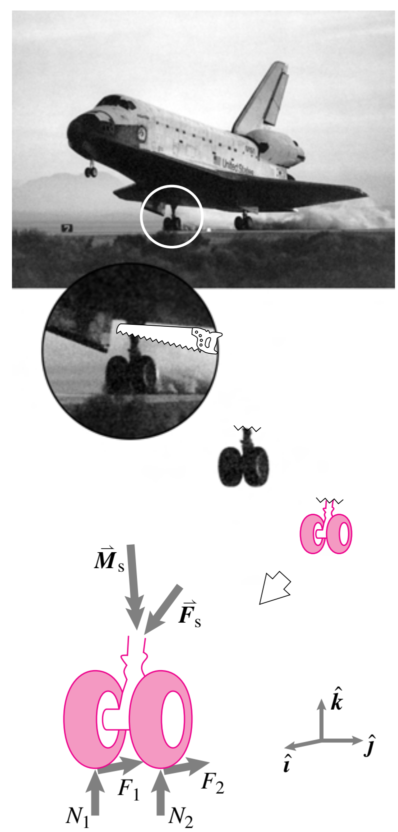

Figure 3.4: The process of drawing a FBD is illustrated by the sequence shown.

In the case of a rope, a single force directed nearly parallel to the rope and acting at about the center of the rope's cross section is equivalent to the force system consisting of all the fiber forces. In the case of an ideal rope, the force is exactly parallel to the rope and acts exactly at its center.

Similarly the force of the net effect of the distributed ground forces on a shoe is often represented by a single force at "the center of pressure".

Action and reaction

For some systems you will want to draw free-body diagrams of subsystems. For example, to study a machine, you may need to draw free-body diagrams of several of its parts; for a building, you may draw free-body diagrams of various structural components; and, for a biomechanics analysis, you may 'cut up' a human body (with your imagined scalpal). When separating a system into parts, you must take account of how the subsystems interact. Call the two touching parts of a machine $\mathcal{A}$ and $\mathcal{B}$. We then have that

If	$\mathcal{A}$	feels force	$\vec{\boldsymbol{F}}$	and couple	$\vec{\boldsymbol{M}}$	from $\mathcal{B}$,
then	$\mathcal{B}$	feels force	$-\vec{\boldsymbol{F}}$	and couple	$-\vec{\boldsymbol{M}}$	from $\mathcal{A}$.

To be precise we must make clear that $\vec{\boldsymbol{F}}$ and $-\vec{\boldsymbol{F}}$ have the same line of action.⑤

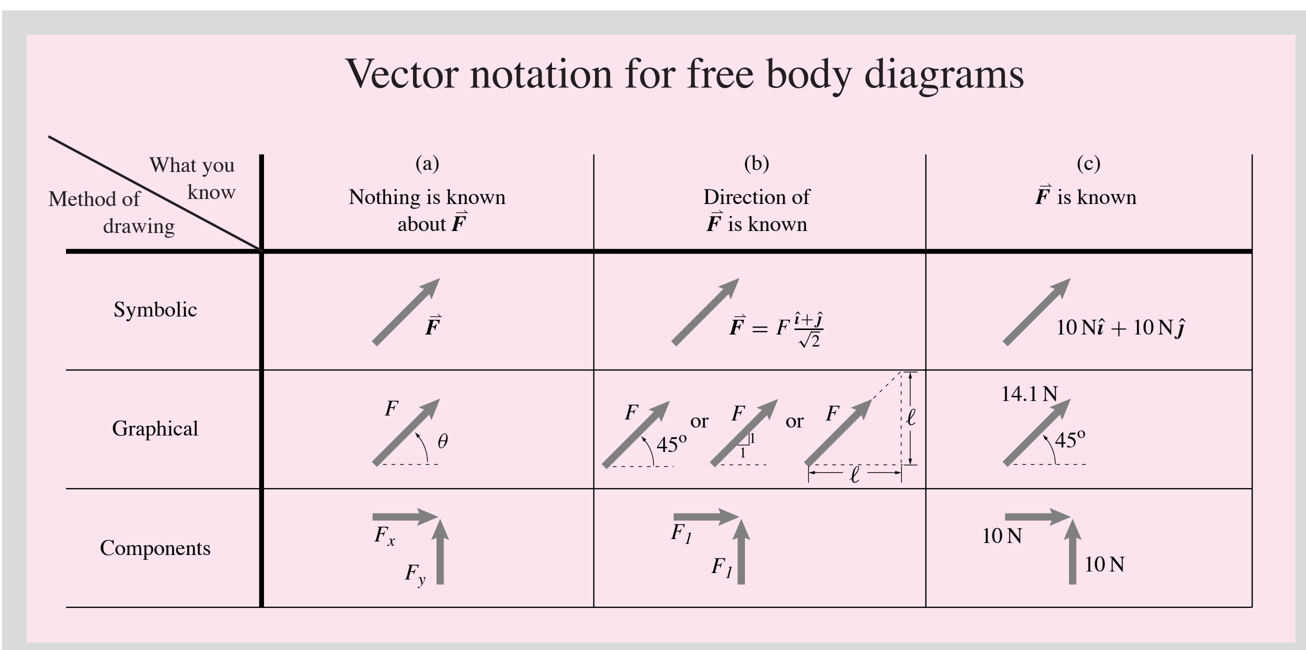

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Figure 3.5: The various ways of notating a force on a free-body diagram. In column (a) nothing is known and everything is variable. In column (b) the direction is known and the magnitude isn't. In column (c) Everything is known. In one free-body diagram different notations can be used for different forces, as needed or convenient. Other unusual cases can be extrapolated, such as if the magnitude is known and the direction is unknown.

The principle of action and reaction doesn't say anything about what force or moment acts on one object. It only says that the actor of a force and moment gets back the opposite force and moment.

It is easy to make mistakes when drawing free-body diagrams involving action and reaction. Box 3.2 on page 163 shows some correct and incorrect partial FBD's of interacting bodies $\mathcal{A}$ and $\mathcal{B}$. Use notation consistent with fig. 3.5 on page 154 for the action and reaction vectors.

Interactions

The way objects interact mechanically is by the transmission of a force or a set of forces. If you want to show the effect of body $\mathcal{B}$ on $\mathcal{A}$, in the most general case you can expect a force and a moment which are equivalent to the whole force system, however complex.

That is, the most general interaction of two bodies requires knowing

- Three numbers in two dimensions (two force components and one moment), and
- Six numbers in three dimensions (three force components and three moment components)

Often things often don't interact in this most general way and then fewer numbers are required.

Some of the common ways in which mechanical things interact, at least ideally, are described in the following sections. As you read this, refer also to first three columns of the summary table on page 1015. You should look frequently at this table until you have absorbed it. You will use the forces and moments on these connections again and again.

Constrained motion and free motion

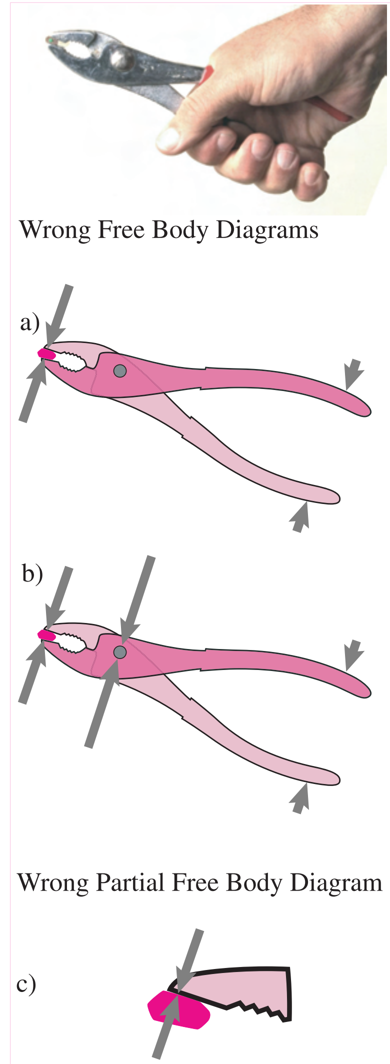
One general principle of interaction forces and moments concerns 'geometric' constraints.

> Wherever a *motion* of $\mathcal{A}$ is either caused *or* prevented by $\mathcal{B}$ there is a corresponding *force* shown at the interaction point on the free-body diagram of $\mathcal{A}$.

Similarly

> if $\mathcal{B}$ causes or prevents *rotation* there is a *moment* (or torque or couple) shown on the free-body diagram of $\mathcal{A}$ at the place of interaction.

The converse is also true. Many kinds of mechanical attachment gadgets are specifically designed to allow motion.



Filename:tfig-plyers-FBDs-bad

Figure 3.6: **The pencil is totally crushed by the plyers.** So it is natural to show the crushing force on a free body diagram of the plyers crushing the pencil (a). Of course there is a big force on the connecting pin between the plyers jaws also (b). **No!** Free body diagrams only show external forces on the system. For the systems shown in the sketches, (a) and (b) above, the forces on the pencil and on the pin and between the parts are internal forces and *do not* show on a free body diagram. Similarly a partial free body diagram (g) of part of the upper jaw and the pencil should *not* show the action and reaction pair (they are internal to the subsystem shown). See fig. 3.2 on page 151 for some good FBDs related to the plyers.

⑤The principle of action and reaction can be derived from the momentum balance laws by drawing free-body diagrams of little slivers of material. Nonetheless, in practice you can think of the principle of action and reaction as a basic law of mechanics. Newton did. The principal of action and reaction is "Newton's third law".

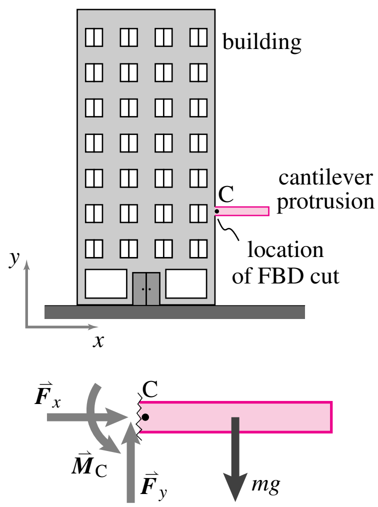


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Figure 3.7: **A rigid connection: a cantilever structure on a building.** At the point C where the cantilever structure is connected to the building all motions are restricted so every possible force needs to be shown on the free-body diagram cut at C.

> If an attachment allows free motion in some direction, a so called *degree of freedom*, then the free-body diagram shows no force in that direction. If the attachment allows free rotation about an axis then the free-body diagram shows no moment (couple or torque) about that axis.

You can think of each attachment point as having a variety of jobs to do. For every possible direction of translation and rotation, the attachment has to either allow free motion or restrict the motion. In every way that motion is restricted (or caused) by the connection a force or moment is required. In every way that motion is free there is no force or couple. Motion of body $\mathcal{A}$ is caused and restricted by forces and couples which act on $\mathcal{A}$. Motion is freely allowed by the absence of such forces and couples.

Here, demonstrating the ideas above, are some of the common connections.

Cuts at 'rigid' connections

Sometimes the body you draw in a free-body diagram is firmly attached to another. Figure 3.7 shows a cantilever structure on a building. The free-body diagram of the cantilever has to show all possible force and load components. Since we have used vector notation for the force $\vec{F}$ and the moment $\vec{M}_C$ we can be ambiguous about whether we are doing a two or three dimensional analysis.

Gravity is pointing down, so why do we show a horizontal reaction force at C? This is a reasonable question because a quick statics analysis shows that, for a stationary building and cantilever, that $\vec{F}_C$ must be vertical. There are two reasons to show the horizontal force anyway

1. Mechanics includes both statics *and* dynamics. In dynamics *the forces on a body do not add to zero*. In fact, we forgot to tell you, the building shown in fig. 3.7 happens to be accelerating rapidly to the right due to the motions of a violent earthquake occurring at the instant pictured in the figure.

2. Whether or not there is an earthquake, the attachment of the cantilever to the building at C in fig. 3.7 is surely intended to be rigid and prevent the cantilever from moving up or down (falling), and from moving sideways (and drifting into another building) and from rotating about point C. In most of the building's life, the horizontal reaction at C is small. But since the connection at C clearly prevents relative horizontal motion, it is probably best to draw a horizontal reaction force on the free-body diagram. Then the same free-body diagram is good during earthquakes and during more boring times.

   When you know a force is going to turn out to be zero, as for the sideways force in this example if treated as a statics problem, it is a matter of taste whether or not you show the sideways force on the free-body diagram (Box 3.1 on page 161 discusses just this issue). Our general advice is 'better safe then sorry'; if you don't *know* that a force or moment is going to turn out to be zero, leave it in the free-body diagram.

The situation with rigid connections, like the cantilever above, is shown more abstractly in both 3D and 2D in fig. 3.8.

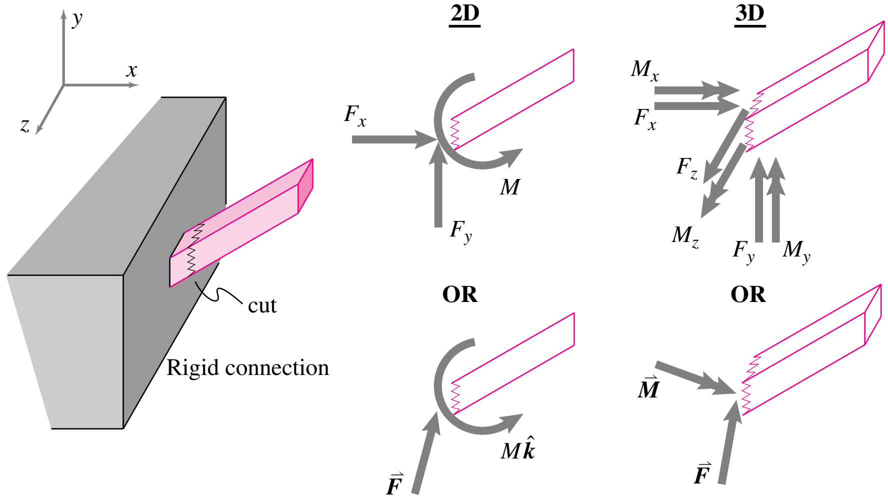

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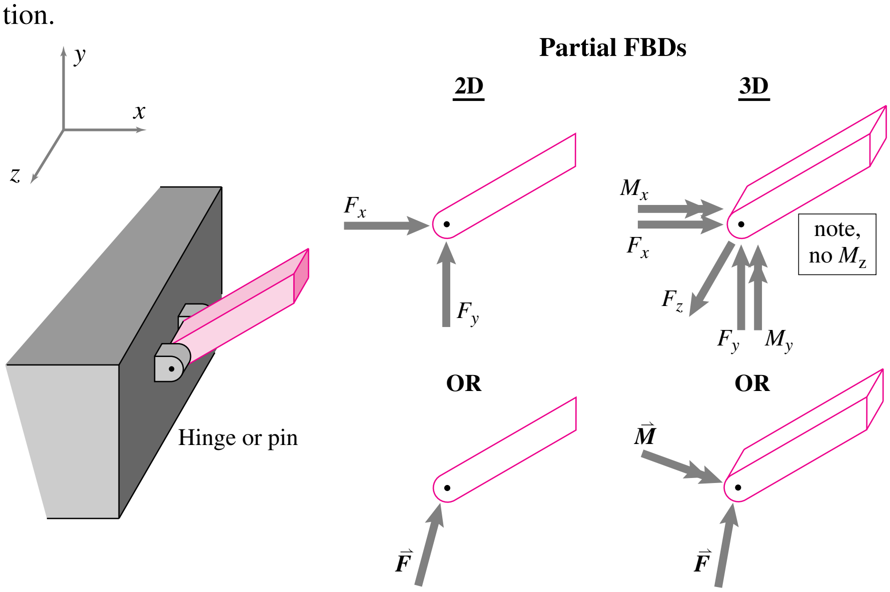
Figure 3.8: A rigid connection shown with partial free-body diagrams in two and three dimensions. One has a choice between showing the separate force components (top) or using the vector notation for forces and moments (bottom). The double head on the moment vector is optional.

Cuts at hinges

A hinge, shown in fig. 3.9, allows rotation and prevents translation. Thus, the free-body diagram of an object cut at a hinge shows *no* torque about the hinge axis but does show the force or its components which prevent translation.



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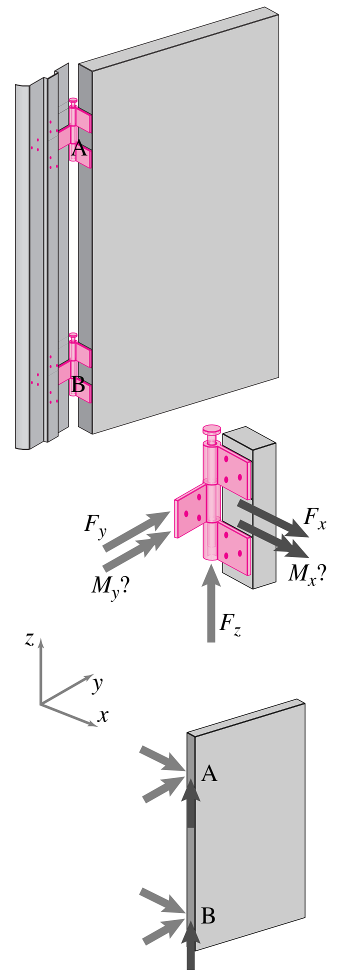

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Figure 3.10: **A door held by hinges.** One must decide whether to model hinges as proper hinges or as ball-and-socket joints. The partial free-body diagram of the door at the lower right neglects the couples at the hinges, effectively idealizing the hinges as ball-and-socket joints. This idealization is generally quite accurate since the rotations that each hinge might resist are already resisted by their being two connection points.

Figure 3.9: **A hinge with partial free-body diagrams in 2D and 3-D.** A hinge joint is also called a pin joint because it is sometimes built by drilling a hole and inserting a pin.

There is some ambiguity about how to model pin joints (hinges) in three dimensions. The ambiguity is shown with reference to a hinged door (fig. 3.10) and discussed in detail below. Clearly, one hinge, if the sole attachment, prevents rotation of the door about the x and y axes shown. So, it is natural to show a couple (torque or moment) in the x direction, M_x, and in the y direction, M_y. But, the hinge does not provide very stiff resistance to rotations in these directions compared to the resistance of the other hinge. That is, even if both hinges are modeled as ball-and-socket joints (see the next sub-section), offering no resistance to rotation, the door still cannot rotate about the x and y axes.

The stiffer constraint wins. If a connection between objects prevents relative translation or rotation that is already prevented by another stiffer connection, then the more compliant connection reaction is often neglected. Even without rotational constraints, the translational constraints at the hinges A and B restrict rotation of the door shown in fig. 3.10. Thus each of the two hinges are probably well modeled — that is, they will lead to reasonably accurate calculations of forces and motions — by ball-and-socket joints at A and B.

In 2-D a ball-and-socket joint is equivalent to a hinge or pin joint (with the axis of the hinge orthogonal to the page).

Bearing alignment. If two connections both do the same job, for example the two door hinges above, they might not do it exactly the same way. And the incompatibility can be a structural problem. Thus, for example, door hinges need to be well aligned in order that the door opening is free and to prevent large forces and moments of the hinges fighting each other.

Ball-and-socket joint

Sometimes one wishes to attach two objects in a way that allows no relative translation but for which all rotation is free. The device that is used for this purpose is called a 'ball-and-socket' joint. It is constructed by rigidly attaching a sphere (the ball) to one of the objects and rigidly attaching a partial spherical cavity (the socket) to the other object.

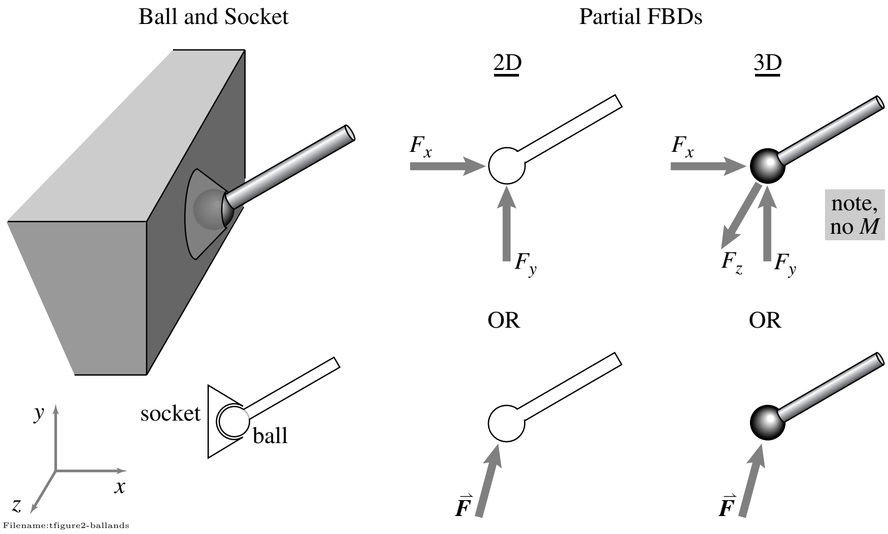


Figure 3.11: **A ball-and-socket joint** allows all relative rotations and no relative translations so reaction forces, but not moments, are shown on the partial free-body diagrams. In two dimensions a ball-and-socket joint is just like a pin joint. The top partial free-body diagrams show the reaction in component form. The bottom illustrations show the reaction in vector form.

The human hip joint is a ball-and-socket joint (See fig. 3.13). At the upper end of the femur bone is the femoral head, a sphere to within a few thousandths of an inch. The hip bone has a spherical cup that accurately fits the femoral head. The human hip joint is not so different from engineered ball and socket joints ⑥.

Car suspensions are constructed from a three-dimensional truss-like mechanism. Some of the parts need free relative rotation in three dimensions and thus use a joint called a 'ball joint' or 'rod end' that is a ball-and-socket joint.

Since the ball-and-socket joint allows all rotations, no moment is shown at a cut ball-and-socket joint. Since a ball-and-socket joint prevents relative translation in all directions, the possibility of force in any direction is shown.

String, rope, wires, and light chain

One way to keep a radio tower from falling over is with wire, as shown in fig. 3.12. If the weight of the wires seems small, and the wind resistance is negligible, it is common to assume they can only transmit forces along the line connecting their end points. Moments are not shown because ropes, strings, and wires are generally assumed to be so compliant in bending that the bending moments are negligible. For wires

> *tension* is the force pulling away from a free-body diagram cut⑦.

Figure 3.13: The ball part of an artificial hip joint. (Photo courtesy of Daniel Rutter, www.dansdata.com).

⑥**True story.** The Mann biomechanics lab at MIT put strain gauges in artificial hip joints, then surgically implanted these artificial joints in patients with bad bones to measure the hip forces (they measured contact pressure up to 18 Mpa≈ 2600 lbf/ in^2). Dicky at the MIT boat house said he wanted a ball-and-socket joint for the base of the mast of the sailboat he was building. "Oh" said Crispin of the Mann lab, "we have a hip joint we don't need", and gave Dicky an uninstrumented hip which dicky welded this and that for use in one of his boat projects.

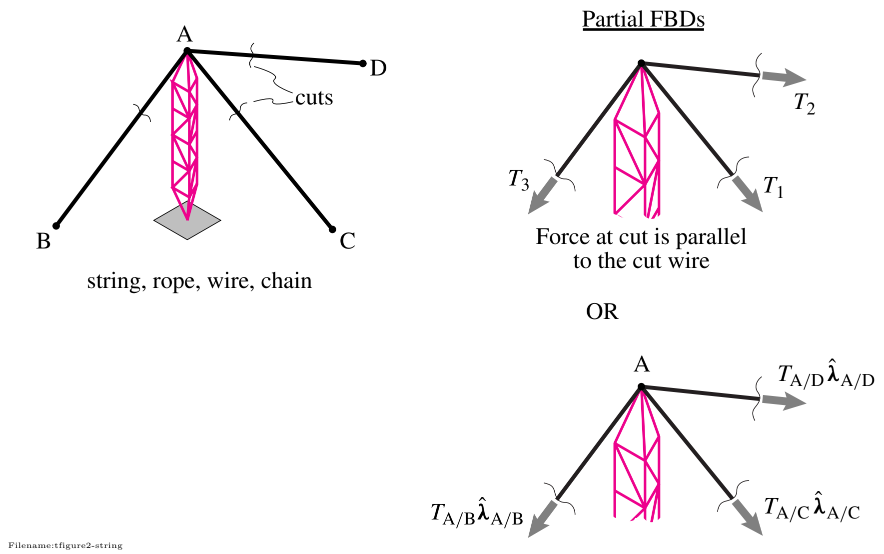


Figure 3.12: **A radio tower kept from falling with three wires.** A partial free-body diagram of the tower is drawn two different ways. The upper figure shows three tensions that are parallel to the three wires. The lower partial free-body diagram is more explicit, showing the forces to be in the directions of the $\hat{\boldsymbol{\lambda}}$s, unit vectors parallel to the wires.

All this talk about force, what is force?

Force is the measure of mechanical interaction. It is a vector. It obeys the principle of action and reaction. Using forces on free-body diagrams, with

I. constitutive laws, like $F = kx$ and $F = mg$ (see Pillar 1 on page 26) and

II. mechanics laws, like $\sum \vec{F}_i = \vec{0}$ or $\vec{F} = m\vec{a}$ (see Pillar 3 on page 29)

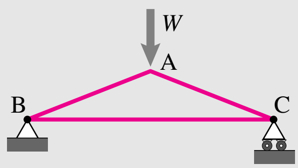
we make accurate predictions. What is force? Its that quantity, that miraculously, has all these properties. What is force really? Beyond this constellation of relations, force is really . . . never mind, that's just too deep.

Operationally, you can define force by how you can measure it. A force on a system can be measured by comparing its effect on the given system to

- A weight suspended by a string which goes over a pulley and is attached to the system of interest instead of the force.
- The effect of a calibrated spring on the system, or
- The effect, and would be hard to arrange in practice, of an accelerating mass connected by pulleys and strings to the system. Of course if you have some way of moving the force around without changing its magnitude, you can apply it to a mass and measure the acceleration it causes.
- Other contraptions that somehow show the effect of the questionable force on a suspended weight, a stretched spring or an accelerated mass.

3.1 How much mechanics reasoning should you use when you draw a free-body diagram?

Consider the simple symmetric truss with a load W in the middle, a pin support at the left and a roller support at the right.



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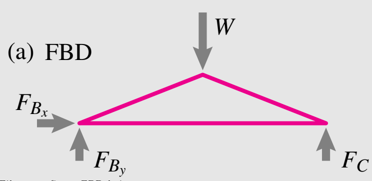
Here are various options for drawing a free-body-diagram of this truss.



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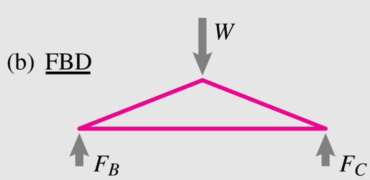
(a) The simple prescription is to draw an unknown force every place a motion is (caused or) prevented and an unknown torque where rotation is (caused or) prevented, as shown above. In particular, there is an unknown force restricting both horizontal and vertical motion at B.

Someone thinking ahead and noting that $F_{Bx} = 0$ might say that the free-body diagram in (a) is wrong. It is not. In FBD (a) force F_{Bx} is not specified because it is not known from just looking at the FBD cut at the pin. That $F_{Bx} = 0$ turns out to be zero is consistent with FBD (a) because $F_{Bx} = 0$ is not specified and thus could have any value, including zero.

As a rule, we favor FBD (a).



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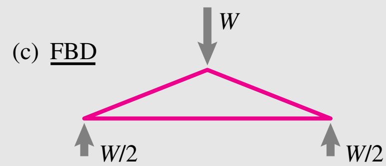
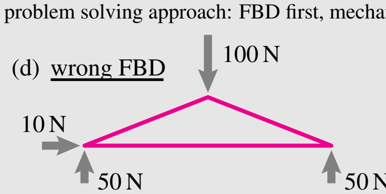
(b) A person who knows some statics will quickly deduce that the horizontal force at B is zero and thus draw the free-body diagram in figure (b). This is also correct, although somewhat violating the philosophy of drawing FBDs and later using mechanics reasoning.



Filename:tfigure-FBDchoices-c

(c) Thinking ahead even more one could draw the free body diagram above. All three free-body diagrams above are correct. In particular diagram (a) is correct even though F_{Bx} turns out to be zero and (b) is correct even though F_B turns out to be equal to F_C. FBD (c) has the most information in it, but also most violates the problem solving approach: FBD first, mechanics later.



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(d) In contrast, the free-body diagram above explicitly and incorrectly assigns a non-zero value to F_{Bx}, so it is wrong.

What to do? When in doubt we recommend following the naive rules yielding FBD (a). Then later use the force and momentum equations to find out more about the forces, *e.g.,* FBD (c). You might then never explitly draw FBD (c), as it will be implicit in your assignments of values to the forces (*e.g.,* $F_{Bx} = 0$). If you are confident about the anticipated results, it might be a time saver to use diagrams analogous to (b) or (c) but

Beware of:

- making assumptions that are not reasonable, rather than just being more naive and correct, and
- wasting time trying to think ahead when the force and momentum balance equations will tell all in the end anyway.

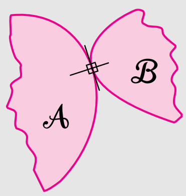
A common error is to sloppily think through the mechanics laws and then incorrectly eliminate, or over-specify, forces on a FBD.

Summary of free-body diagrams.

- Draw one or more clear free-body diagrams!
- Forces and moments on the free-body diagram show *all* mechanical interactions from outside the body.
- Every point on the boundary of a body has a force in every direction that motion is either being caused *or* prevented. Similarly with torques.
- If you do not know the direction of a force, use vector notation to show that the direction is yet to be determined.
- Leave off the free-body diagram forces that you think are negligible such as, possibly:
  - The force of air on small slowly moving bodies;
  - Forces that prevent motion that is already prevented by a much stiffer means (as for the torques at each of a pair of hinges);
  - See the table on page 1015 to see the forces at various connections.

3.2 Action and reaction on partial FBD's

Imagine bodies $\mathcal{A}$ and $\mathcal{B}$ are interacting and that you want to draw separate free-body diagrams (FBD's) of each.



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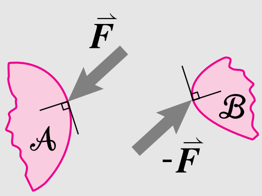
Part of the FBD of each shows the interaction force. The FBD of $\mathcal{A}$ shows the force of $\mathcal{B}$ on $\mathcal{A}$ and the FBD of $\mathcal{B}$ shows the force of $\mathcal{A}$ on $\mathcal{B}$. To illustrate the concept, we show partial FBD's of both $\mathcal{A}$ and $\mathcal{B}$ using the principle of action and reaction. Items (a - d) are correct and items (e - g) are wrong. See sample 2.1 on page 52 for related comments on vector notation.

Correct partial FBD's

(a)



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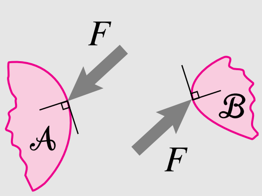
(a) These are good partial FBD's. the action and re-action vectors ($\vec{F}$ and $-\vec{F}$) are equal in magnitude, opposite in sign, and applied on the same line of action. Because the symbolic notation takes precedence (see page 46)the direction and length of the drawn arrows, although drawn nicely here, are irrelevant.

(b)



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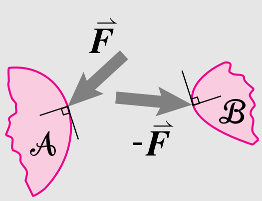
(b) These partial FBD's are also good since the opposite arrows multiplied by equal magnitude F produce net vectors that are equal and opposite.

(c)



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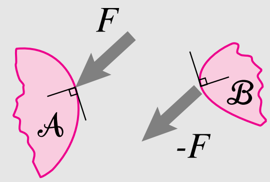
(c) The partial FBD's may *look* wrong, and they are impractically misleading and not advised. But technically they are okay because we take the vector notation to have precedence over the drawing inaccuracy.

(d)

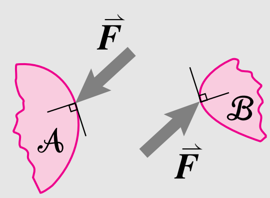


Filename:tfigure2-actionreaction-d

(d) The partial FBD's may look wrong but since no vector notation is used, the forces should be interpreted as in the direction of the drawn arrows and multiplied by the shown scalars. Since the same arrow is multiplied by F and $-F$, the net vectors are actually equal and opposite.

Wrong partial FBD's

(e)



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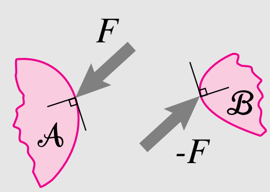
(e) These partial FBD's are wrong because the vector notation $\vec{F}$ takes precedence over the drawn arrows. So the drawing shows the *same* force $\vec{F}$ acting on both $\mathcal{A}$ and $\mathcal{B}$, rather than the opposite force.

(f)



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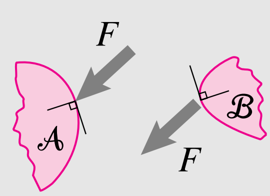
(f) Because the opposite arrow is multiplied by the negative scalars, the partial FBD's here show the *same* force acting on both $\mathcal{A}$ and $\mathcal{B}$. Treating a double-negative as a negative is a common mistake.

(g)



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(g) These partial FBD's are obviously wrong since they again show the same force acting on $\mathcal{A}$ and $\mathcal{B}$. These FBD's would represent *the principle of double action* which applies to laundry detergents but not to mechanics.

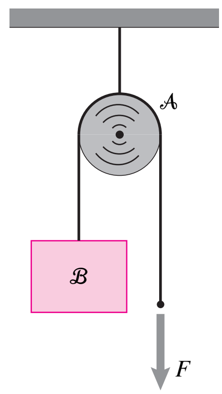

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Figure 3.14:

SAMPLE 3.1 A mass and a pulley. A block of mass m is held up by applying a force F through a massless pulley as shown in the figure. Assume the string to be massless. Draw free-body diagrams of the mass and the pulley separately and as one system.

Solution The free-body diagrams of the block and the pulley are shown in Fig. 3.15. Since the string is massless and we assume an ideal massless pulley, the tension in the string is the same on both sides of the pulley. Therefore, the force applied by the string on the block is simply F. When the mass and the pulley are considered as one system, the force in the string on the left side of the pulley doesn't show because it is internal to the system.

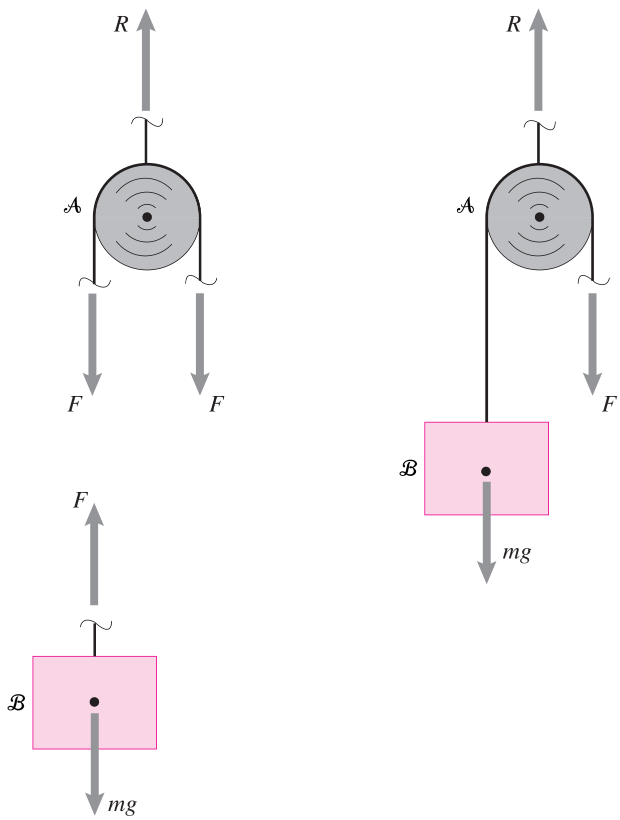

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Figure 3.15: The free-body diagrams of the mass, the pulley, and the mass-pulley system. Note that for the purpose of drawing the free-body diagram we need not show that we know that $R = 2F$. Similarly, we could have chosen to show two different rope tensions on the sides of the pulley and reasoned that they are equal as is done in the text.

SAMPLE 3.2 **Forces in strings.** A block of mass m is held in position by strings AB and AC as shown in Fig. 3.16. Draw a free-body diagram of the block and write the vector sum of all the forces shown on the diagram. Use a suitable coordinate system.

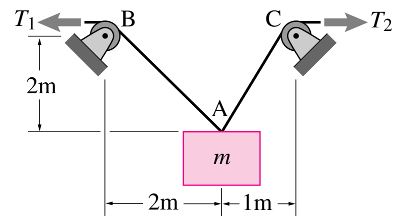

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Figure 3.16:

Solution To draw a free body diagram of the block, we first *free* the block. We cut strings AB and AC very close to point A and show the forces applied by the cut strings on the block. We also isolate the block from the earth and show the force due to gravity. (See Fig. 3.17.)

To write the vector sum of all the forces, we need to write the forces as vectors. To write these vectors, we first choose an xy coordinate system with basis vectors $\hat{\imath}$ and $\hat{\jmath}$ as shown in Fig. 3.17. Then, we express each force as a product of its magnitude and a unit vector in the direction of the force. So,

$$\vec{T}_1 = T_1\hat{\lambda}_{AB} = T_1\frac{\vec{r}_{AB}}{|\vec{r}_{AB}|},$$

where $\vec{r}_{AB}$ is a vector from A to B and $|\vec{r}_{AB}|$ is its magnitude. From the given geometry,

$$\begin{aligned} \vec{r}_{AB} &= -2\,\mathrm{m}\hat{\imath} + 2\,\mathrm{m}\hat{\jmath} \\ \Rightarrow \qquad \hat{\lambda}_{AB} &= \frac{2\,\cancel{\mathrm{m}}(-\hat{\imath}+\hat{\jmath})}{\sqrt{2^2+2^2}\,\cancel{\mathrm{m}}} = \frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}). \end{aligned}$$

Thus,

$$\vec{T}_1 = T_1\frac{1}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}).$$

Similarly,

$$\begin{aligned} \vec{T}_2 &= T_2\frac{1}{\sqrt{5}}(\hat{\imath}+2\hat{\jmath}) \\ m\vec{g} &= -mg\hat{\jmath}. \end{aligned}$$

Now, we write the sum of all the forces:

$$\begin{aligned} \sum\vec{F} &= \vec{T}_1 + \vec{T}_2 + m\vec{g} \\ &= \left(-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}}\right)\hat{\imath} + \left(\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg\right)\hat{\jmath}. \end{aligned}$$

The $\hat{\imath}$ and $\hat{\jmath}$ components of the net force depend on the values of the scalars (magnitudes) T_1, T_2 and mg.

$$\boxed{\sum\vec{F} = \left(-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}}\right)\hat{\imath} + \left(\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg\right)\hat{\jmath}}$$

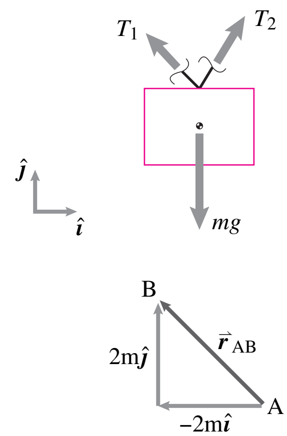

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Figure 3.17: Free-body diagram of the block and a diagram of the vector $\vec{r}_{AB}$.

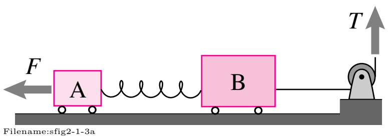

Figure 3.18: Two carts connected by a massless spring

SAMPLE 3.3 Two bodies connected by a massless spring. Two carts A and B are connected by a massless spring. The carts are pulled to the left with a force F and to the right with a force T as shown in Fig. 3.18. Assume the wheels of the carts to be massless and frictionless. Draw free body diagrams of

- cart A,
- cart B, and
- carts A and B together.

Solution The three free body diagrams are shown in Fig. 3.19 (a) and (b). In Fig. 3.19 (a) the force F_s is applied by the spring on the two carts. Why is this force the same on both carts? In Fig. 3.19(b) the spring is a part of the system. Therefore, the forces applied by the spring on the carts and the forces applied by the carts on the spring are internal to the system. Therefore these forces do not show on the free body diagram.

Note that the normal reaction of the ground can be shown either as separate forces on the two wheels of each cart or as a resultant reaction.

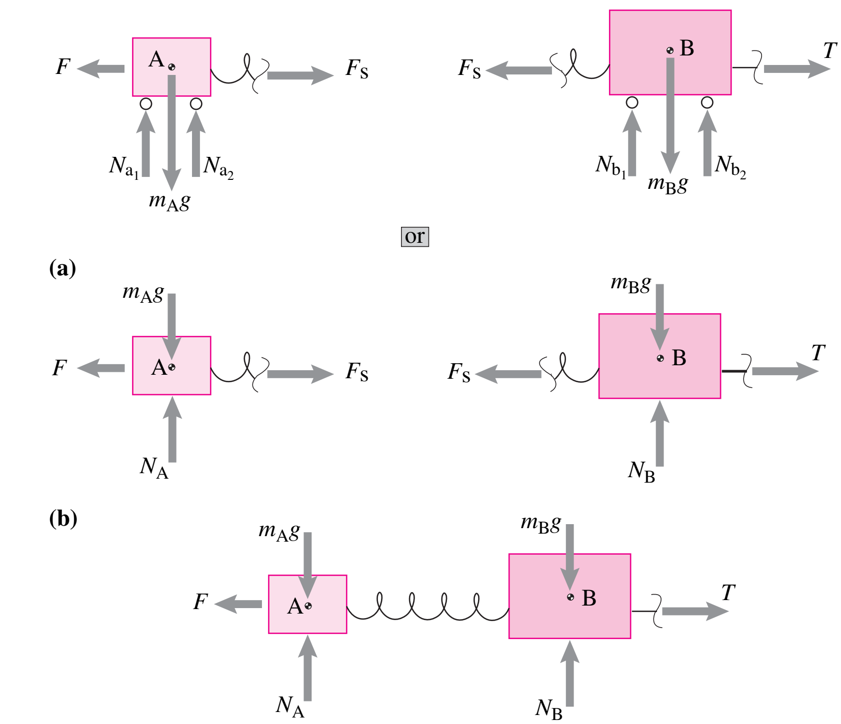

Figure 3.19: Free body diagrams of (a) cart A and cart B separately and (b) cart A and B together

SAMPLE 3.4 **Two carts connected by pulleys.** The two masses shown in Fig. 3.22 have frictionless bases and round frictionless pulleys. The inextensible massless cord connecting them is always taut. Mass A is pulled to the left by force F and mass B is pulled to the right by force P as shown in the figure. Draw free body diagrams of each mass.

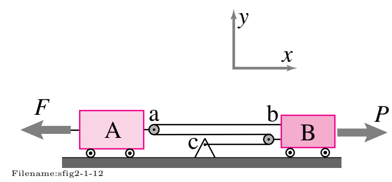

Figure 3.22: Two carts connected by massless pulleys.

Solution Let the tension in the cord be T. Since the pulleys and the cord are massless, the tension is the same in each section of the cord. This equality is clearly shown in the Free body diagrams of the two masses below.

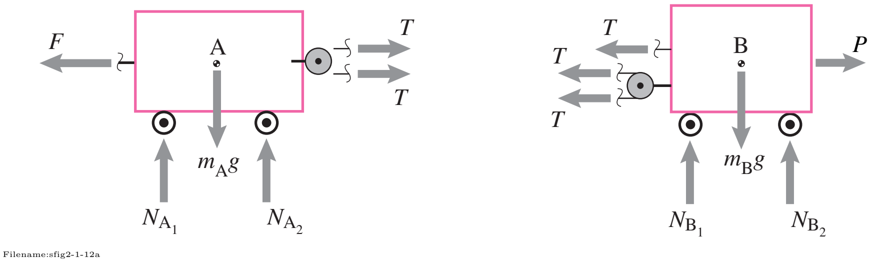

Figure 3.20: Free body diagrams of the two masses.

Comments: We have shown unequal normal reactions on the wheels of mass B. In fact, the two reactions would be equal only if the forces applied by the cord on mass B satisfy a particular condition. Can you see what condition must be satisfy for, say, $N_{A_1} = N_{A_2}$.

[Hint: think about the moment balance about the center-of-mass A.]

SAMPLE 3.5 **Structures with pin connections.** A horizontal force T is applied on the structure shown in the figure. The structure has pin connections at A and B and a roller support at C. Bars AB and BC are rigid. Draw free body diagrams of each bar and the structure including the spring.

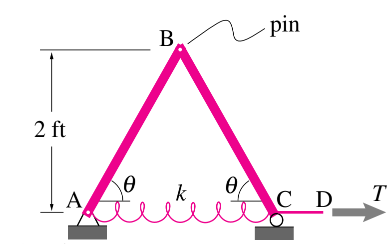

Figure 3.23:

Solution The free body diagrams are shown in figure 3.21. Note that there are both vertical and horizontal forces at the pin connections because pins restrict translation in any direction. At the roller support at point C there is only vertical force from the support (T is an externally applied force).

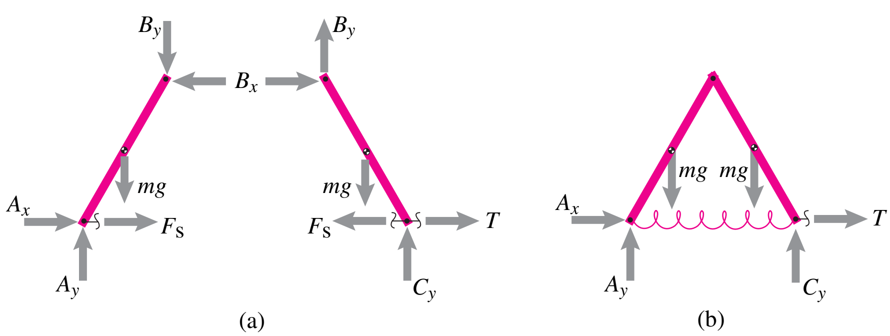

Figure 3.21: Free body diagrams of (a) the individual bars and (b) the structure as a whole.

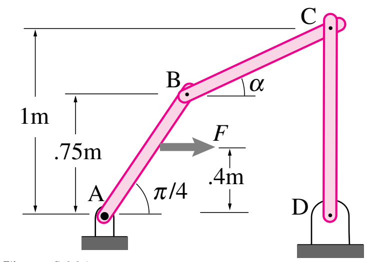

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Figure 3.24: A four bar linkage. Force F is the only load. Pin B has non-negligible friction, the other pins have negligible friction.

SAMPLE 3.6 The **four-bar linkage** shown is pushed to the right with a force F. Pins A, C & D have negligible friction but joint B is rusty and resists rotation (has non-negligible friction). Draw free body diagrams of each of the bars separately and of the whole structure. Use consistent notation for the interaction forces and moments. Clearly mark the action-reaction pairs. Neglect gravity.

Solution An ideal pin resists any relative translation of the pinned parts by exerting forces on them. We could draw three separate free-body diagrams of the pin, the first part, and the second part. Usually, however, we let the pin be a part of one of the objects and just draw two free body diagrams.

Because rusty joint B resists relative rotation we also show a moment at point B acting on rods AB and BC.

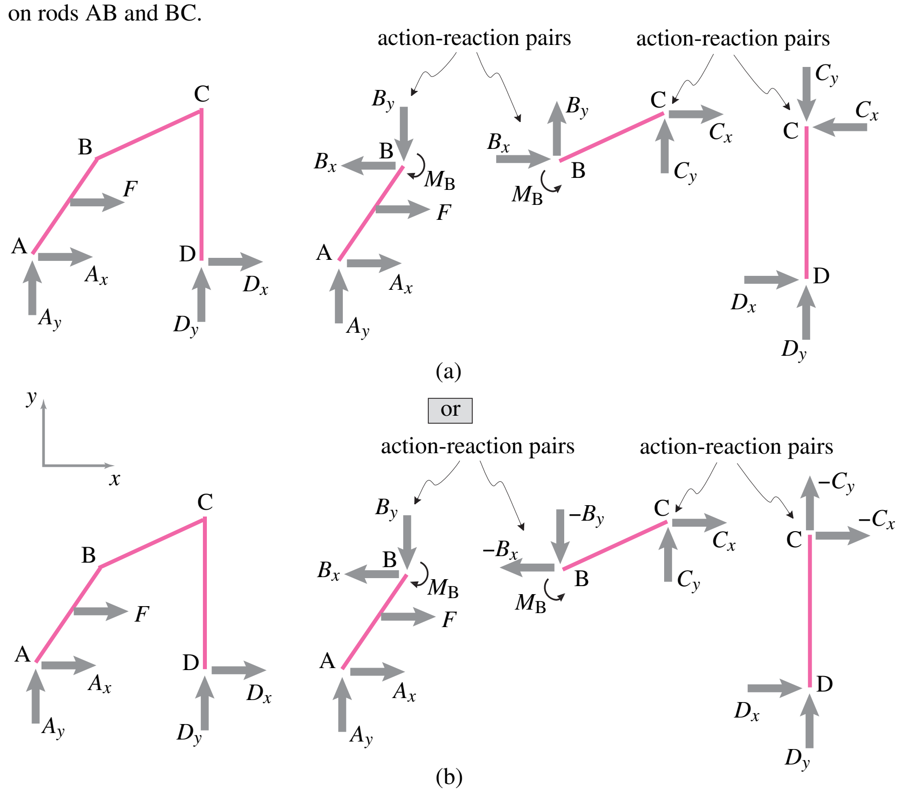

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Figure 3.25: Style 1, components: Free body diagrams of the structure and the individual bars. The forces shown in (a) and (b) are the same.

Figure 3.25 shows the forces in terms of their x- and y-components. The directions of the force components are shown by the arrows and the magnitude is labeled as A_x, A_y, etc. Here we use the word 'magnitude' to mean a scalar, positive or negative. Therefore, a force, shown as an arrow in the positive x-direction with magnitude A_x, is the same as that shown as an arrow in the negative x-direction with 'magnitude' $-A_x$. Thus, the free body diagrams in Fig. 3.25(a) show exactly the same forces as in Fig. 3.25(b).

In Fig. 3.26, we show the forces by an arrow in an arbitrary direction. The corresponding labels represent their magnitudes. The angles represent the unknown directions of the forces.

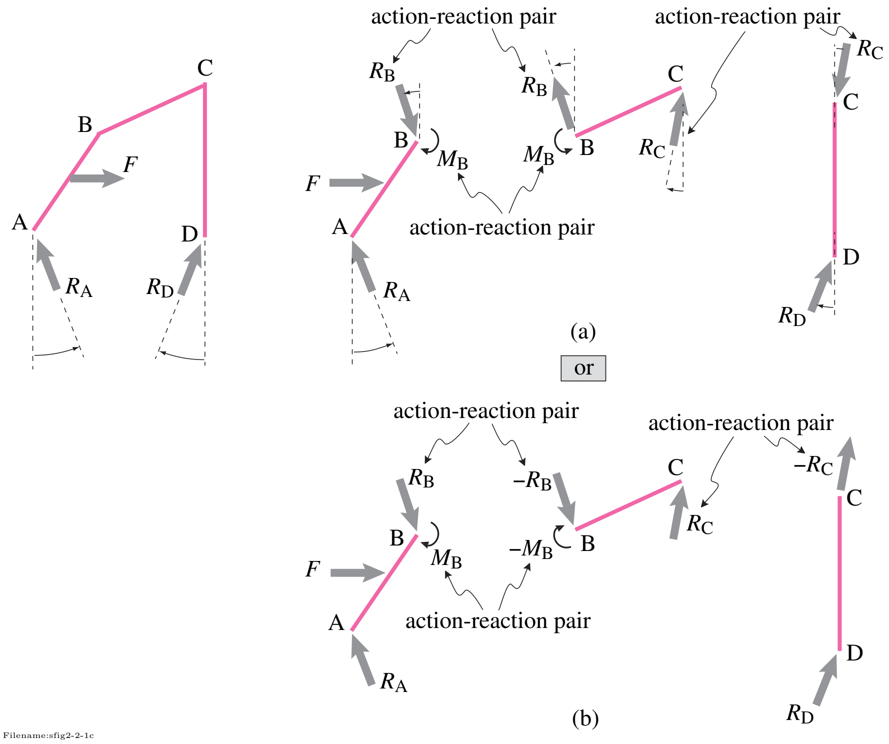

Figure 3.26: Style 2, magnitude times drawn unit vector: The forces in (a) have arcs indicating angles which would show the directions of the forces. The FBDs in (b) are more informal and not recommended. Why not? There is no visible notation showing the directions of the forces.

In Fig. 3.27, we show yet another way of drawing and labeling the free body diagrams, where the forces are labeled as vectors.

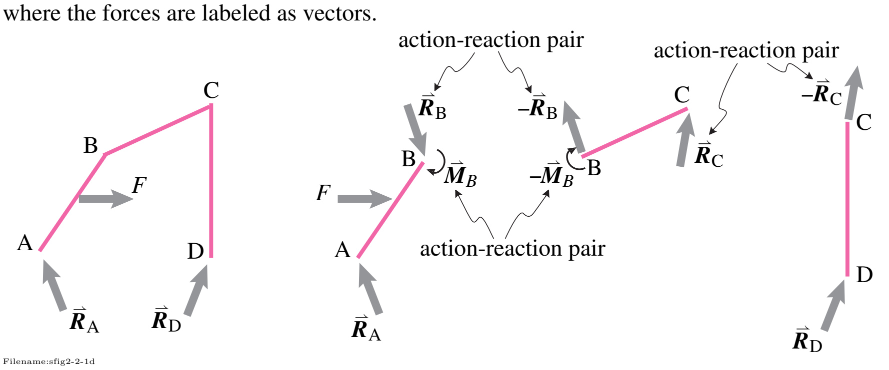

Figure 3.27: Style 3, vector notation: The label of a forces as vectors indicates both their magnitude and direction. The arrows are arbitrary and merely indicate that a force or a moment acts at those locations.

Note: Bar CD is a two-force member. We could have used that to simplify all of the free body diagrams above by showing equal and opposite forces at C and D that were parallel to the line CD.

3.2 Contact: Sliding, friction, and rolling

The primary mechanical interaction between intermediate-sized objects, say much smaller than the earth and much larger than an atom, is through contact ①. Contact between two bodies restricts their possible motions and causes forces on the bodies. Things cause contact forces on each other when and where they touch. Some contact situations are modeled in standard ways that we have discussed, including contact at a pin-joint, ball-and-socket, hinge, weld, and tied string.

① In common engineering, the most-often encountered *non*-contact force is gravity, especially the gravity force from the earth on terrestrial objects like cars, buildings, and rain drops.

Here we consider objects that press against each other in ways not necessarily well-idealized with one of these standard mechanical connections. More specifically, we need rules for finding the forces during sliding and rolling contact . There are many candidate descriptions for friction and rolling. They vary in their conceptual simplicity, their ease of use in analytical or numerical calculations, and their accuracy and applicability. We will present the simplest rules, describe some of the short-comings and then give some guidance towards more sophisticated rules.

Contact laws are all rough approximations

A rule for finding the forces of interaction in terms of the bodies' positions and velocities is called, as mentioned in Chapter 1, a *constitutive law* or *constitutive relation* (see particularly page 26). Generally people categorize contact as being one of the three major types: friction, rolling, or collision②. Discussion of collisional free body diagrams is discussed in the chapters on dynamics.

② In practice it is not always clear how to make the distinctions between sliding and rolling or between sliding and collision. But at least for a first pass it is a useful conceptual distinction to think of sliding, rolling, and collisions as three different kinds of contact.

We must emphasize at the outset:

> Constitutive laws for contact are rough approximations. For at least some of the quantities of interest. theory and practice typically differ by 5-50%.

Equations for contact are of a lower class than the momentum equations. For most engineering purposes the momentum balance equations are extremely accurate, with error of well less than a part per billion. Newton's law of gravitational attraction is a similarly accurate law. And the laws of Euclidean (non-Riemanian) geometry and calculus (the kinds you studied) are also extremely accurate (See chapter 1.2 page 31). Less accurate are the laws for springs and dashpots. But still, accuracies of one part per thousand are possible for measuring spring stiffness and perhaps parts per hundred for some dashpot constants.

But the laws for the contact interactions of solids are much less accurate. Not only can't you know the coefficient of friction between any two pieces of steel with any certainty, you also can't even trust the *concept* of a coefficient of friction to be very accurate. It is easy to forget this inaccuracy in contact laws because you will see contact-force equations in books. Once we see an equation in print, we are too-easily tempted into believing it is 'true.' So a common mistake amongst beginning engineers is to use contact constitutive equations with confidence, as if accurate. Rather, all contact simple equations only a rough approximation at best.

Friction

When two objects are in contact and one is sliding with respect to the other, we call the force which resists this sliding *friction*. Frictional contact is usually assumed to be either 'lubricated' or 'dry.' When bodies are in lubricated contact they are not in real contact at all, a thin layer of liquid or gas separates them. Most of the metal to metal contact in a car engine is so lubricated. The contact of the car tires with the road is 'dry' unless the car is 'hydroplaning' on worn-smooth tires on a very wet road. The friction forces in lubricated contact are very small compared forces of unlubricated contact. There is no quick way to estimate these small lubricated slip forces. The accurate estimation of lubricated friction forces requires use of lubrication theory, a part of fluid mechanics. For many purposes lubricated friction forces are neglected. We now drop discussion of lubricated friction forces because they are often negligible and because estimating them is a more advanced topic.

Dry friction forces are not small and thus cannot be sensibly neglected in mechanics problems involving sliding contact. The simplest model for friction forces is called *Coulomb's law of friction* or just *Coulomb friction*. But, use of even this law is full of subtleties.

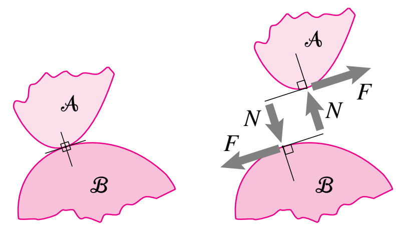

Figure 3.28: Two bodies in contact. The forces between them satisfy the law of action and re-action. It is often convenient to decompose the force of interaction into a part F tangent to the surface of interaction and a part N perpendicular to the surface of interaction.

'Smooth' and 'Rough' are common misnomers for low-friction and high-friction

As a simplification when we think friction is not important we sometimes neglect it by setting $\mu = \phi = 0$. In many books this neglect is named "perfectly smooth". Smooth surfaces separated by a little fluid (say water between your feet and the bathroom tile, or oil between pieces of a bearing) do slide easily by each other. And even without a lubricant sometimes slipping can be reduced by smoothing a surface. But making a surface progressively smoother does not diminish the friction to zero. Rather, extremely smooth surfaces sometimes have anomalously high friction (extremely clean flat surfaces can even bond to each other). In general, there is no reliable correlation between smoothness and low friction.

Similarly many books use the phrase "perfectly rough" to mean perfectly high friction ($\mu \to \infty$ and $\phi \to 90^{\circ}$) and hence that no slip is allowed. This is misleading twice over. First, as just stated, rougher surfaces do not reliably have more friction than smooth ones. Second, even when $\mu \to \infty$ slip can proceed in some situations (see, for example, box 4.6 on page 215).

We use the phrase *frictionless* or *negligible friction* to mean that there is no tangential force component. We use the phrase *no slip* to mean that no tangential motion is allowed and that there is some unknown tangential force. So

We do not use the words smooth and rough in this book to indicate low and high friction.

Coulomb friction

Coulomb's law of friction, also attributed sometimes to Amonton and sometimes to DaVinci, is summarized by the simple equation:

$$F = \mu N. \tag{3.1}$$

This equation, like many other simple equations, needs some descriptive words to be useful.

First of all the direction of the force F on body $\mathcal{A}$ is in the opposite direction of the slip velocity of $\mathcal{A}$ relative to $\mathcal{B}$. By the principle of action and reaction we deduce that the force on body $\mathcal{B}$ is in the opposite direction. This force is also opposite to the relative slip velocity of $\mathcal{B}$ relative to $\mathcal{A}$. That is, F resists relative motion between $\mathcal{A}$ and $\mathcal{B}$.

The friction force F is proportional to the normal force N with the proportionality constant μ. The constant μ is assumed to be independent of the area of contact between bodies $\mathcal{A}$ and $\mathcal{B}$. In the simplest renditions of Coulomb's law μ is assumed to be independent of slip distance, slip velocity, time of contact, etc. When contacting bodies are not sliding the role of friction changes somewhat. In some sense the friction still resists slip, in fact it is the presence of the friction force that prevents slip. But another way to think of friction is that it puts an upper limit on the size of the force of interaction between two bodies which seem stuck to each other. The friction force must be less than or equal to μN in magnitude during contact.

$$|F| \leq \mu N \tag{3.2}$$

All of the discussion above can be summarized with the following equations for the friction force

$$\overbrace{\vec{\boldsymbol{F}}_{\text{on } \mathcal{A} \text{ from } \mathcal{B}}}^{\text{The friction force}} = \frac{-\mu \overbrace{\vec{\boldsymbol{v}}_{\mathcal{A}/\mathcal{B}}}^{\text{Relative slip velocity.}}}{|\vec{\boldsymbol{v}}_{\mathcal{A}/\mathcal{B}}|} N \qquad \text{during slip}$$

$$\underbrace{|\vec{\boldsymbol{F}}_{\text{on } \mathcal{A} \text{ from } \mathcal{B}}|}_{\text{The magnitude of the friction force}} \leq \underbrace{\mu N}_{\text{An upper bound on the friction force}} \qquad \text{during stationary contact}$$

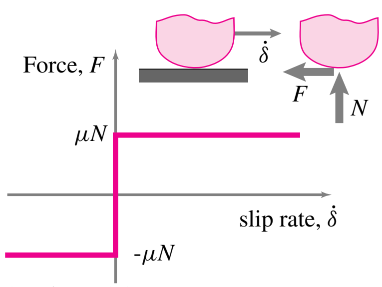

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Figure 3.29: **Coulomb friction.** The relation between friction force F and relative slip rate $\dot{\delta}$ is described by the dark line. Since there is a jump from $-\mu N$ to μN in the friction force when the slip rate goes from negative to positive the relation is not a proper mathematical function between F and $\dot{\delta}$. Instead the relation is a curve in the $F, \dot{\delta}$ plane.

For two-dimensional problems where slip can only be in one direction (or the opposite) this pair of functions describes the dark line in the friction graph of fig. 3.29 in which $\dot{\delta}$ is the speed of relative slip.

The simplest friction law, the one we use in this book, uses a single constant coefficient of friction μ. Almost always $.05 \le \mu \le 1.2$ and more commonly $.2 \le \mu \le 1$. We do not distinguish the static coefficient μ_s from the dynamic coefficient μ_d or μ_k. That is $\mu = \mu_s = \mu_k = \mu_d$ for our purposes. We promote the use of this simplest law for a few reasons.

- All friction laws used are quite approximate, no matter how complex. Unless the distinction between static and dynamic coefficients of friction is essential to the engineering calculation, using $\mu_s \neq \mu_k$ doesn't add to the calculation's usefulness.
- The concept of a static coefficient of friction that is larger than a dynamic coefficient is, it turns out, not well defined if bodies have more than one point of contact, which they often do have. (See page 987.)
- Students learning mechanics are often confused about friction. Because the more complex friction laws are of questionable accuracy and usefulness anyway, it seems time is better spent understanding the simplest friction laws.

See page 989 for more discussion of the pros and cons of the Coulomb-friction approximation.

In summary, the simple model of friction we use is:

> Friction resists relative slipping motion. *During slip the friction force opposes relative motion and has magnitude* $F = \mu N$. When there is no slip the magnitude of the friction force F cannot be determined from the friction law but it cannot exceed μN, that is $F \le \mu N$.

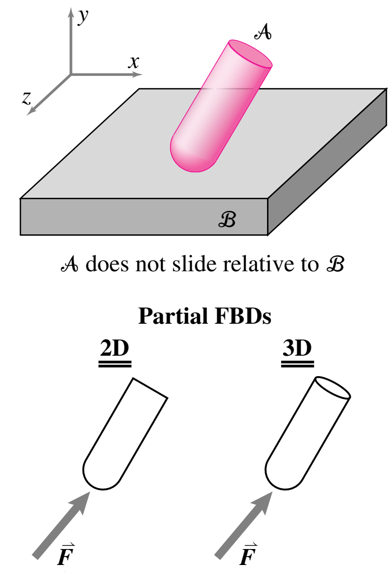

Figure 3.30: Object $\mathcal{A}$ does not slide relative to the plane $\mathcal{B}$.

Friction angle

Sometimes people describe the friction coefficient with a *friction angle* ϕ rather than the coefficient of friction (see fig. 3.31). The friction angle is the angle between the net interaction force (normal force plus friction force) and the normal to the sliding surface when slip is occurring. The relation between the friction coefficient μ and the friction angle ϕ is

$$\tan\phi \equiv \mu.$$

The use of ϕ or μ to describe friction are equivalent. Which you use is a matter of taste and convenience. Sometimes analytic formulas in problems come out simpler looking with one or the other of μ and ϕ used to describe the friction.

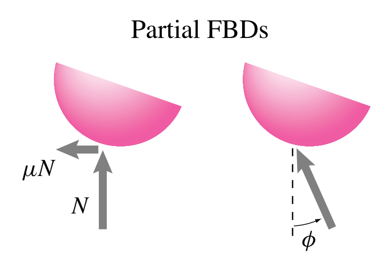

Figure 3.31: Two ways of characterizing friction: the friction coefficient μ and friction angle ϕ.

Rolling contact

An idealization for the non-skidding contact of balls, wheels, and the like is *pure rolling*.

> *Objects $\mathcal{A}$ and $\mathcal{B}$ are in pure rolling contact when their (relatively convex) contacting points have equal velocity. They are not slipping, separating, or interpenetrating.*

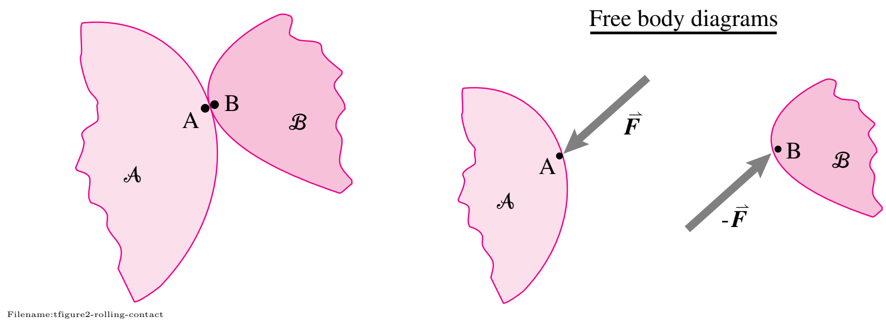

Figure 3.32: Rolling contact: Points of contact on adjoining bodies have the same velocity, $\vec{v}_A = \vec{v}_B$.

Most often, we are interested in cases where the contacting bodies have some non-zero relative angular velocity — a ball sitting *still* on level ground may be technically in rolling contact, but not interestingly so.

The simplest common example is the rolling of a round wheel on a flat surface in two dimensions. See fig. 3.33.

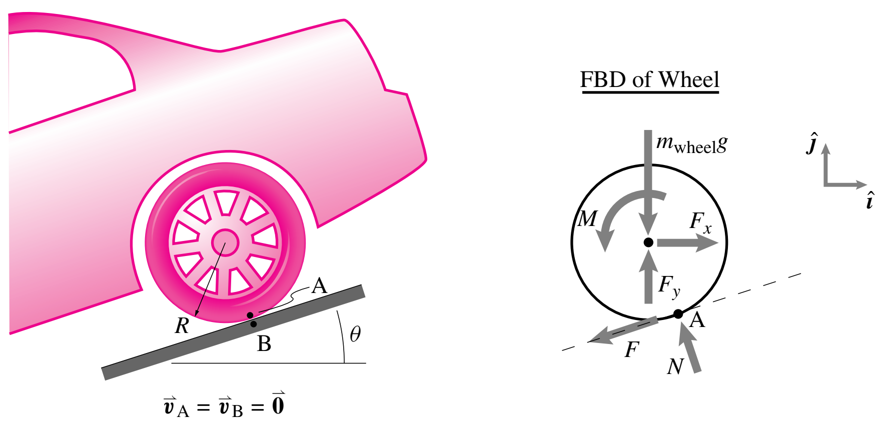

Figure 3.33: Pure rolling of a round wheel on a flat slope in two dimensions.

In practice, there is often confusion about the direction and magnitude of the force F shown in the free body diagram in fig. 3.33. Here is a recipe:

1.) Draw F as shown in any direction which is tangent to the surface.

2.) Solve the statics or dynamics problem and find the scalar F. (It may turn out to be a negative, which is fine.)

3.) Check that rolling is really possible; that is, that slip would not occur. If the force is greater than the frictional strength, $|F| > \mu N$, the assumption of rolling contact is not appropriate. In this case, you must assume that $F = \mu N$ or $F = -\mu N$ and that slip occurs; then, re-solve the problem.

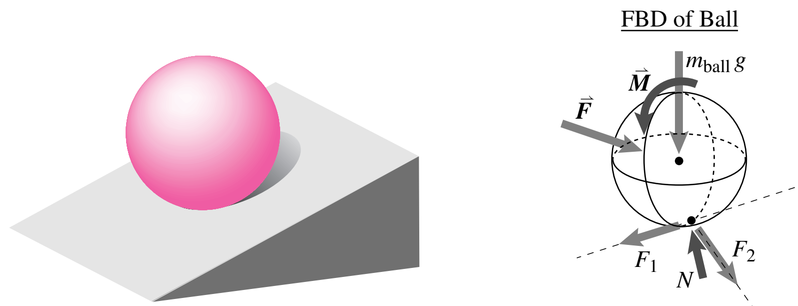

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Figure 3.34: **Rolling ball in 3-D.** The force $\vec{F}$ and moment $\vec{M}$ are applied loads from, say, wind, gravity, and any attachments. N is the normal reaction and F_1 and F_2 are the in plane components of the frictional reaction. One must check the no-slip condition, $\mu^2 N^2 \geq F_1^2 + F_2^2$.

In three-dimensional rolling contact, we have a free body diagram that again looks like a free-body diagram for non-slipping frictional contact. Consider, for example, the ball shown in fig. 3.34. For the friction force to be less than the friction coefficient times the normal force, we have the *no slip condition*

$$\sqrt{F_1^2 + F_2^2} \;\leq\; \mu N \quad \text{or} \quad F_1^2 + F_2^2 \;\leq\; \mu^2 N^2$$

Rolling is just a special case of frictional contact. It is the case where bodies contact at a single point (or on a line, as with cylinders) and have relative rotation yet have no relative velocity at their contacting points.

Rolling resistance

Non-ideal rolling contact includes provision for *rolling resistance*. This resistance is simply represented by either moving the location of the point of contact force or by a contact couple. Rolling resistance leads to subtle questions which we skip here③.

③Note that the tangential forces in fig. 3.35 and fig. 3.2 are not rolling resistance.

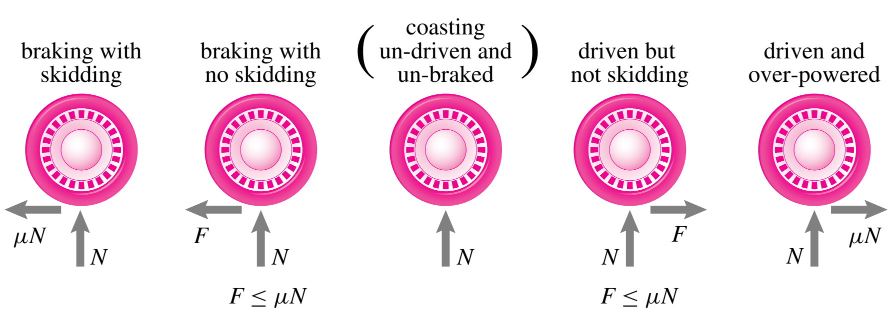

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Figure 3.35: Partial free body diagrams of wheel in a braking or accelerating car that is pointed and moving to the right. The force of the ground on the tire is shown. But, for simplicity, the forces of the axle, gravity, and brakes on the wheel are not shown (that's why its a partial FBD). An ideal point-contact wheel is assumed. There is no 'rolling resistance' here.

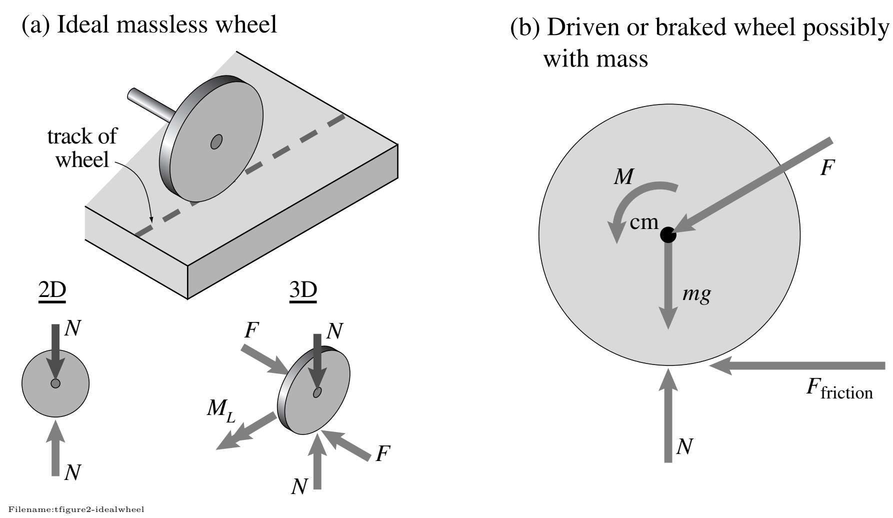

Figure 3.36: An ideal wheel is round, massless, rigid, undriven, and rolls on flat rigid ground with no rolling resistance. Free body diagrams of ideal undriven wheels are shown in two and three dimensions. The force F shown in the three-dimensional picture is perpendicular to the path of the wheel. The lateral moment M_L keeps the wheel from falling over sideways. (b) 2D free body diagram of a wheel with mass, possibly driven or braked. If the wheel has mass but is not driven or braked the figure is unchanged but for the moment M being zero.

Ideal wheels

An ideal wheel is an approximation of a real wheel. It is a sensible approximation if the mass of the wheel is negligible, bearing friction is negligible, and rolling resistance is negligible. Free body diagrams of undriven ideal wheels in two and three dimensions are shown in fig. 3.36. This idealization is rationalized in chapter 4 in box 4.6 on page 215. Note that if the wheel is not massless, the 2-D free body diagram looks more like the one in fig. 3.36b with $F_{\text{friction}} \leq \mu N$.

Extended contact

When things touch each other over an extended region, like the block on the plane of fig. 3.37a, it is not clear what forces to put where on the free body diagram. On the one hand one imagines reality to be somewhat reflected by millions of small forces as in fig. 3.37b which may or may not be divided into normal (n_i) and frictional (f_i) components. But one generally is not interested in such detail, and even if interested one cannot find it easily.

A simple approach is to replace the detailed force distribution with a single equivalent force, as shown in fig. 3.37c broken into components. The location of this force is not relevant for some problems. ④

If one wants to make clear that the contact forces serve to keep the block from rotating, one may replace the contact force distribution with a pair of contacts at the corners as in fig. 3.37d.

Collisional free-body diagrams

As noted earlier, there are special conventions for drawing free-body diagrams of objects that are in the process of colliding. These we treat in the relevant dynamics portions of the book.

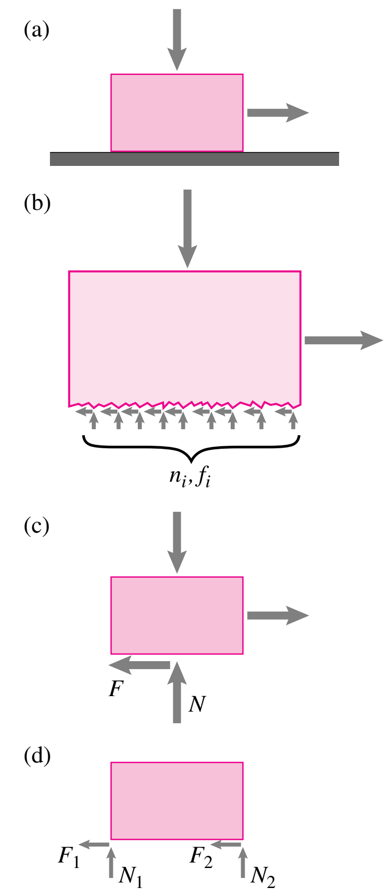

Figure 3.37: The contact forces of a block on a plane can be sensibly modeled in various ways.

④In 3D, contact force distributions cannot always be replaced with an equivalent force at an appropriate location (see section 2.6). A couple may be required. Nonetheless, many people often make the approximation that a contact force distribution can be replaced by a force at an appropriate location. For example, this is the "center of pressure" approach used to describe the location of an imagined-equivalent ground force on a robot's foot. This approximation neglects any frictional resistance to twisting about the normal to the contact plane.

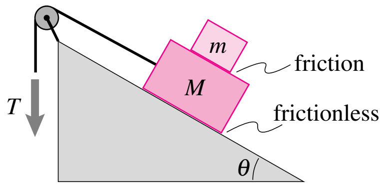

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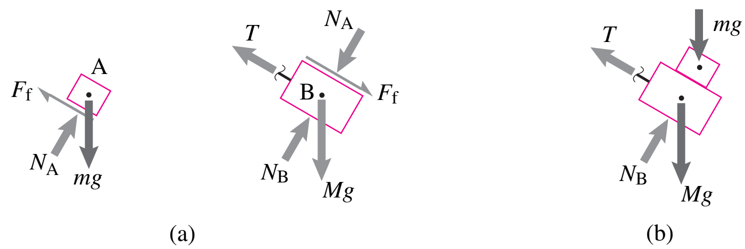
Figure 3.38: Two blocks held in place on an inclined surface

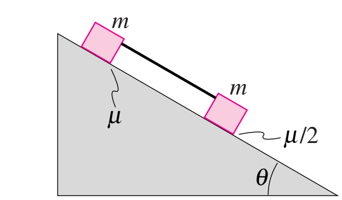
SAMPLE 3.7 Stacked blocks at rest on an inclined plane. Blocks A and B with masses m and M, respectively, rest on a frictionless inclined surface with the help of force T as shown in Fig. 3.38. There is friction between the two blocks. Draw free body diagrams of each of the two blocks separately and a free body diagram of the two blocks as one system.

Solution The three free body diagrams are shown in Fig. 3.40 (a) and (b). Note the action and reaction pairs between the two blocks; the normal force N_{A} and the friction force F_{f} between the two bodies A and B. If we consider the two blocks together as a system, then the forces N_{A} and F_{f} do not show on the free body diagram of the system (See Fig. 3.40(b)), because now they are internal to the system.



Filename:sfig2-1-4b

Figure 3.40: Free body diagrams of (a) block A and block B separately and (b) blocks A and B together.

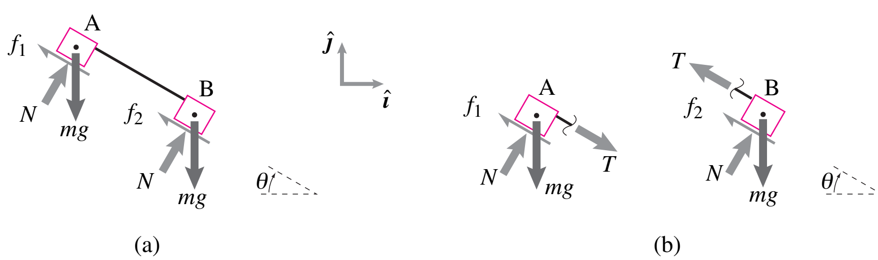


Filename:sfig2-1-15

Figure 3.39: Two blocks slide down a frictional inclined plane. The blocks are connected by a light rigid rod.

SAMPLE 3.8 Two blocks slide down a frictional inclined plane. Two blocks of identical mass but different material properties are connected by a massless rigid rod. The system slides down an inclined plane which provides different friction to the two blocks. Draw free body diagrams of the two blocks separately and of the system (two blocks with the rod).

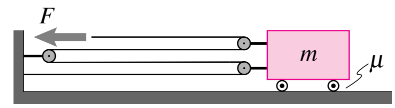
Solution The Free body diagrams are shown in Fig. 3.41. Note that the friction forces on the two blocks are different because the coefficients of friction are different for the two blocks. The normal reaction of the plane, however, is the same for each block (why?).



Filename:sfig2-1-15a

Figure 3.41: Free body diagrams of (a) the two blocks and the rod as a system and (b) the two blocks separately.

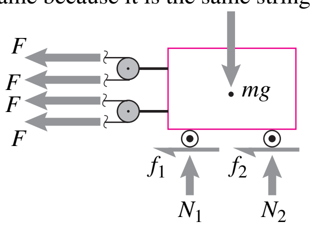
SAMPLE 3.9 Massless pulleys. A force F is applied to the pulley arrangement connected to the cart of mass m shown in Fig. 3.44. All the pulleys are massless and frictionless. The wheels of the cart are also massless but there is friction between the wheels and the horizontal surface. Draw a free body diagram of the cart, its wheels, and the two pulleys attached to the cart, all as one system.



Filename:sfig2-1-7a

Figure 3.44: A cart with pulleys

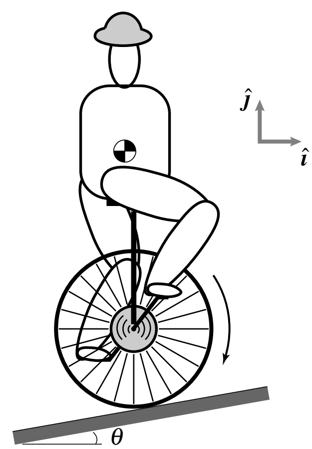
Solution The free body diagram of the cart system is shown in Fig. 3.42. The force in each part of the string is the same because it is the same string that passes over all the pulleys.



Filename:sfig2-1-7b

Figure 3.42: Free-body diagram of the cart.

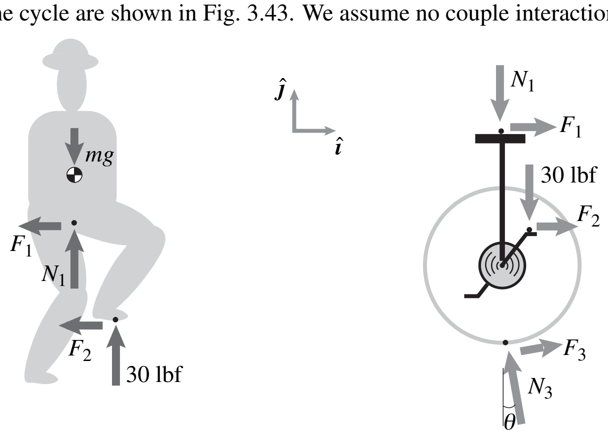
SAMPLE 3.10 A unicyclist in action. A unicyclist weighing 160 lbs exerts a force on the front pedal with a vertical component of 30 lbf at the instant shown in figure 3.45. The rear pedal barely touches the other foot. Assume the wheel and the frame are massless. Draw free body diagrams of the cyclist and the cycle. Make other reasonable assumptions if required.



Filename:sfig2-1-8

Figure 3.45: The unicyclist

Solution Let us assume, there is friction between the seat and the cyclist and between the pedal and the cyclist's foot. Let's also assume a 2-D analysis. The free body diagrams of the cyclist and the cycle are shown in Fig. 3.43. We assume no couple interaction at the seat.



Filename:sfig2-1-8a

Figure 3.43: Free-body diagram of the cyclist and the cycle.

Problems for Chapter 3

Free body diagrams

3.1 Interactions, Partial FBDs

Preparatory Problems

3.1.1 How does one know what forces and moments to use in

a) the statics force balance and moment balance equations? *

b) the dynamics linear momentum balance and angular momentum balance equations? *

3.1.2 In a free body diagram of a whole man standing with his right hand extended how do you show the force of his right arm on his body? *

3.1.3 Reproduce the first column of the table in fig. 3.5 on page 154 for the force acting on your right foot from the ground as you step on a stair.

3.1.4 Reproduce the second column of the table in fig. 3.5 on page 154 for a force in the direction of $3\hat{\imath} + 4\hat{\jmath}$ but with unknown magnitude.

3.1.5 Reproduce the third column of the table in fig. 3.5 on page 154 for a 50 N force in th direction of the vector $3\hat{\imath} + 4\hat{\jmath}$.

More-Involved Problems

3.1.6 Simple massless pulleys. Draw free body diagrams of

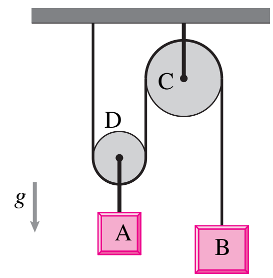
a) mass A with a little bit of rope

b) mass B with a little bit of rope

c) Pulley B with three bits of rope

d) Pulley C with three bits of rope

e) The system consisting of everything below the ceiling



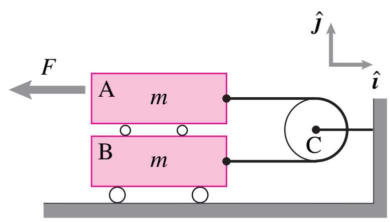
Filename:pfig3-1-twopulleys
Problem 3.1.6

3.1.7 For the block and pulley arrangement shown in the figure, assume negligible friction at the wheels. Draw the free body diagrams of

a) the upper mass A,

b) the lower mass B, and

c) the pulley C.

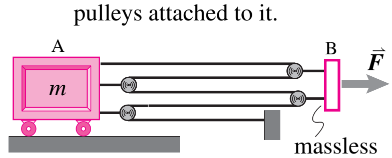


Filename:pfig3-1-2blocksandpulley
Problem 3.1.7

3.1.8 Multiple pulleys. A goods container of mass m is pulled to the right using a force $\vec{F}$ and the pulley arrangement shown in the figure. Draw the free body diagram of

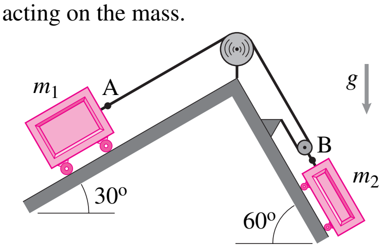
a) the massless block B, and

b) the container A along with the two pulleys attached to it.



Filename:pfig3-1-4pulleys
Problem 3.1.8

3.1.9 Pulleys on inclined planes. Draw the free body diagram of mass m_2 and write the expression for each force vector acting on the mass.



Filename:pfig3-blocks-pulleys-incline
Problem 3.1.9

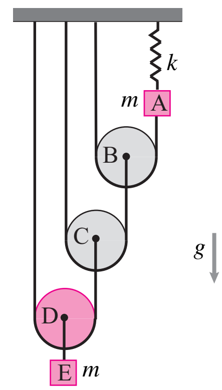
3.1.10 Nested pulleys. In the nested arrangement of pulleys shown in the figure, assume that all the pulleys are massless. Draw the free body diagram of

a) mass A,

b) mass E, and

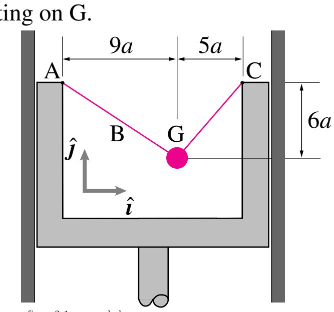
c) pulley D.

Write the expression for the net force on pulley D.



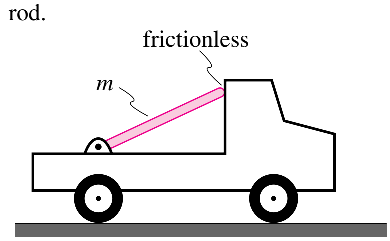
Filename:pfig3-1-3pulleys
Problem 3.1.10

3.1.11 A point mass m at G is attached to a piston by two inextensible cables. There is gravity. Draw a free body diagram of the mass with a little bit of the cables and write the vector expression for the net force acting on G.



Filename:pfigure2-1-suspended-mass
Problem 3.1.11

3.1.12 A uniform rod of mass m rests in the back of a flatbed truck as shown in the figure. Draw a free-body diagram of the rod.



Filename:pfig2-2-rp5
Problem 3.1.12

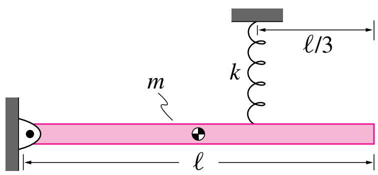
3.1.13 A spring mounted pinned rod. The uniform rigid rod shown in the figure hangs in the vertical plane with the support of the spring shown. In this position the spring is stretched by Δs from its rest length.

a) Draw a free body diagram of the spring.

b) Draw a free body diagram of the rod.

c) Write an expression for the net force on the rod.

d) Write the expression for the net moment on the rod about its center of mass.

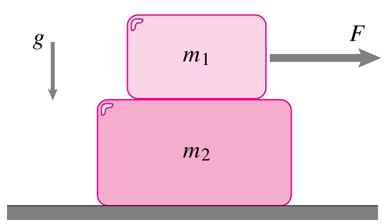


Filename:pfig2-1-rp1
Problem 3.1.13

3.1.14 Two stacked blocks sliding without friction. Two frictionless blocks sit stacked on a frictionless surface. A force F is applied to the top block.

a) Draw a free body diagrams of each block separately.

b) Draw a free body diagram of the two blocks together.



Filename:ch2-3
Problem 3.1.14

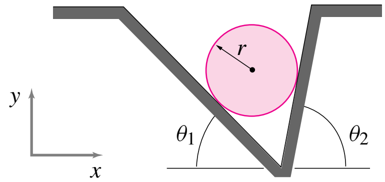
3.1.15 A disk in a frictionless groove. A disc of mass m sits in a wedge shaped groove. There is gravity but no friction.

a) Draw a free body diagram of the disk.

b) Write vector expressions for the reaction forces from the two walls.

c) Write the expression for the net force on the disk.

d) What is the net moment on the disk about its mass center?



Filename:ch2-5
Problem 3.1.15

3.1.16 FBD of an arm throwing a ball. An arm throws a ball up. A crude model of an arm is that it is made of four rigid bodies (shoulder, upper arm, forearm and a hand) that are connected with hinges. At each hinge there are muscles that apply torques between the links. Draw a FBD of

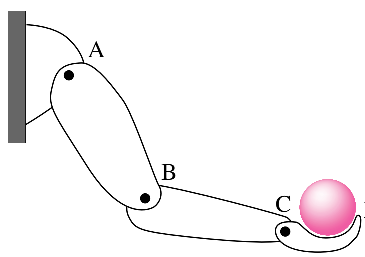
a) the system consisting of the whole arm (three parts, but not the shoulder) and the ball.

b) the ball,

c) the hand, and

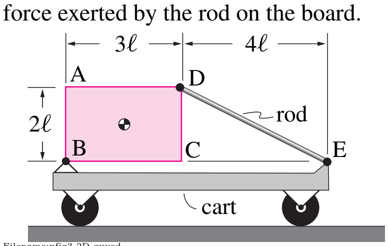
d) the fore-arm,

e) the upper arm,



Filename:efig2-1-23
Problem 3.1.16

3.1.17 A uniform rectangular board of mass m sits on a cart supported by a rod on one corner and a pin the diagonally opposite corner as shown in the figure. Draw a free body diagram of the board and write an appropriate vector expression for the force exerted by the rod on the board.



Filename:pfig3-2D-guyed
Problem 3.1.17

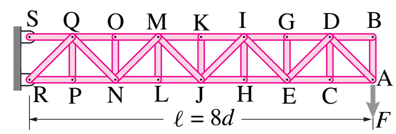
3.1.18 Cantilevered truss A cantilever truss, shown in the figure, is made up of identical horizontal and vertical bars of length d. A vertical force F is applied at point A. The truss is pinned to the wall at joints S and R.

a) Draw a free body diagram of the entire truss.

b) Cut the truss to the right of bar GE by cutting rods GD, EC and ED, and draw a free body diagram of that portion of the truss to the right of bar GE.

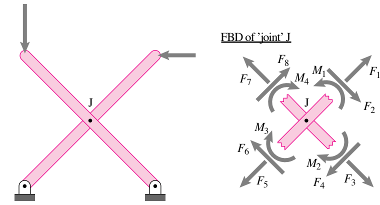
c) Draw a free body diagram of bar IE.

d) Draw a free body diagram of the joint at I with a small length of the bars protruding from I.



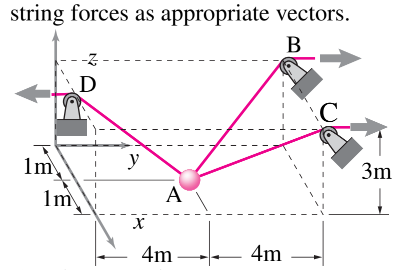
Filename:pfig3-1-cantilevertruss
Problem 3.1.18

3.1.19 An X structure. Two rods are pinned together in the middle to form a structure in the shape of 'X' as shown in the figure. A free body diagram of the joint J with a little bit of the bars near J is shown. Draw free body diagrams of each bar and of the whole structure. *



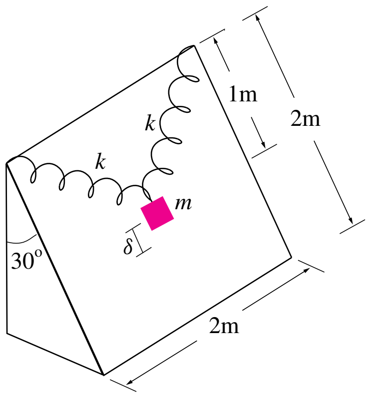
Filename:pfigure-xstructure
Problem 3.1.19

3.1.20 The strings connected to winches at B, C, and D hold up the mass $m = 3\,\text{kg}$ at A. The relevant dimensions are shown in the figure. There is gravity. Draw a free body diagram of the mass and express the string forces as appropriate vectors.



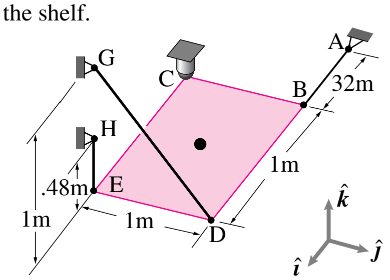
Filename:pfigure2-1-3D-pulley-fbd
Problem 3.1.20

3.1.21 Mass on inclined plane. A block of mass m rests on a frictionless inclined plane. It is supported by two stretched springs. The mass is pulled down the plane by an amount δ and released. Draw a FBD of the mass just after it is released.



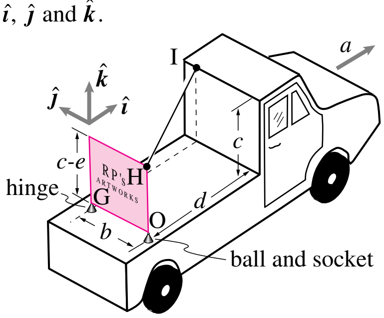
Filename:efig2-1-24
Problem 3.1.21

3.1.22 Hanging a shelf. A shelf with negligible mass supports a 0.5 kgmass at its center. The shelf is supported at one corner with a ball and socket joint and the other three corners with strings. Draw a FBD of the shelf.



Filename:ch3-14
Problem 3.1.22

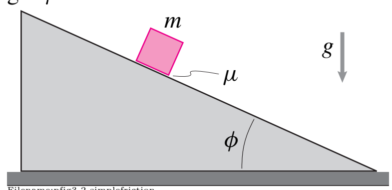
3.1.23 Billboard in the back of a truck. Draw a free body diagram of the billboard sitting in the back of the truck as shown. Write all forces shown on your diagram as appropriate vectors in terms of unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$.



Filename:pfigure3-5-2
Problem 3.1.23

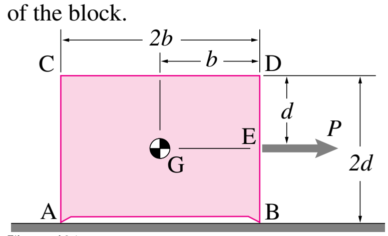
3.2 Contact and friction

3.2.1 A block on an inclined plane. A block of mass m sits on an inclined plane. The coefficient of friction between the block and the plane is μ. Draw the free body diagram of the block and write the expression for the force(s) applied by the incline on the block in terms of incline angle ϕ.



Filename:pfig3-2-simplefriction
Problem 3.2.1

3.2.2 A block of mass m sits on a surface supported at points A and B. A horizontal force P acts at point E. There is gravity. The block is sliding to the right. The coefficient of friction between the block and the ground is μ. Draw a free body diagram of the block.

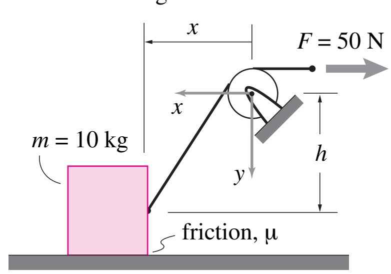


Filename:ch2-1
Problem 3.2.2

3.2.3 A block sliding on level ground. A block of mass 10 kg is pulled by an inextensible cable over the pulley.

a) Assuming the block remains on the floor, draw a free diagram of the block. (There are various correct answers depending how you model the interaction of the bottom of the block with the ground. See fig. 3.37 on page 177)

b) Draw a free body diagram of the pulley with a little bit of the cable extending to both sides.

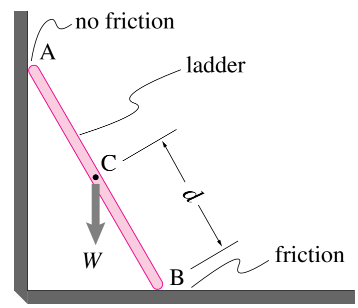


Filename:pfig3-2-block-pulley
Problem 3.2.3

3.2.4 A ladder standing still. A ladder of mass m rests against a frictionless wall and a floor with more than enough friction to prevent slip. There is gravity.

a) Draw a free body diagram of the ladder.

b) What is force on the ladder at point B? Find the direction of this force at B assuming the coefficient of friction to be μ and the ladder to be in a state of impending slip. Does the direction of the net force at B depend on the relative positions of A and B (again, assuming impending slip)?

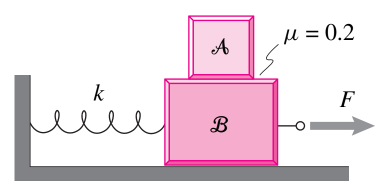


Filename:pfigure4-single-ladder
Problem 3.2.4

3.2.5 For the system shown in the figure draw free-body diagrams of each mass separately and of the system of two blocks.

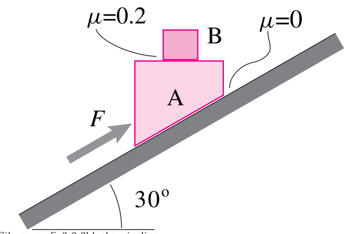
a) Assume there is friction with coefficient μ. At the time of interest block $\mathcal{B}$ is sliding to the right and block $\mathcal{A}$ is sliding to the left relative to $\mathcal{B}$.

b) Assume there is so much friction that neither block slides.



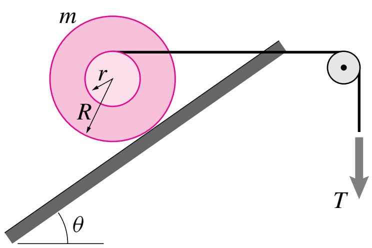
Filename:pfig2-1-rp7
Problem 3.2.5

3.2.6 Two blocks A and B, with mass m_A and m_B respectively, are held on an inclined plane as shown in the figure. Draw a free body diagram of block A and find the net force acting on the block.



Filename:pfig3-2-2blocksonincline
Problem 3.2.6

3.2.7 A spool rolling up an inclined plane. Draw a free body diagram of the spool shown, including a bit of the rope. Assume the spool does not slid on the ramp.

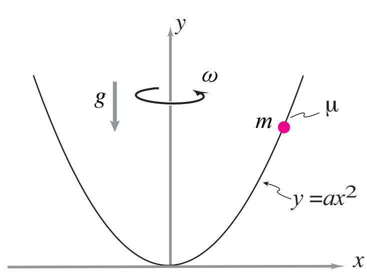


Filename:pfigure4-single-spool
Problem 3.2.7

3.2.8 A bead riding on a rotating wire. A bead of mass m is free to slide on a wire bent in the shape of a parabola. The coefficient of friction between the wire and the bead is μ. The wire rotates about the y-axis. A snapshot during the motion captures the wire in the xy-plane. At this instant, assume the position of the bead to be (x, y).

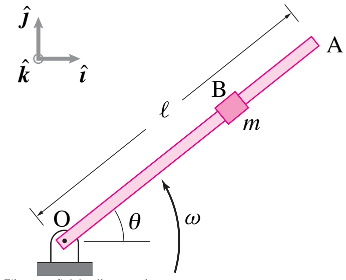
a) Draw a free body diagram of the bead.

b) Write the vector expression for the force on the bead from the wire. [Hint: you need to find the normal vector to the wire at the position of the bead.]



Filename:pfig3-2-bead-on-wire
Problem 3.2.8

3.2.9 A collar sliding up a rotating rod. A uniform rod OA of negligible mass rotates in the plane about point O. A collar B of mass m slides on the rod but faces friction with coefficient $\mu = 0.1$. At the instant shown, draw the free body diagram of the rod and the collar separately evaluating the force of their interaction as explicitly as possible. Ignore gravity.

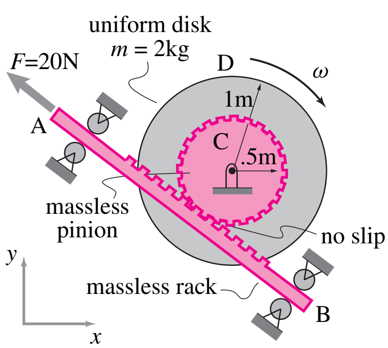


Filename:pfig3-2-collar-on-rod
Problem 3.2.9

3.2.10 A rack and a pinion. In the rack and pinion arrangement shown in the figure, the pinion C is welded to the disk D. The rack is pulled up with a force F. As a result the pinion rotates clockwise along with disk D. Assume that the rack and the pinion mesh perfectly together and that there is no slip between them. You can include or ignore gravity.

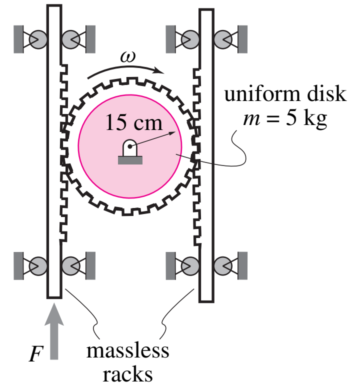
a) Draw a free body diagram of the rack.

b) Draw a free body digram of the pinion along with the disk D.



Filename:pfig3-2-inclinedrackandpinion
Problem 3.2.10

3.2.11 Two racks with one pinion A pinion of mass 5 kg and radius 15 cm messes with two massless racks. The left rack is pushed up with force F making the pinion rotate clockwise. Assuming no slipping between the meshing teeth, draw the free body diagrams of each rack and the pinion separately. Ignore gravity.



Filename:pfig3-2-tworacksandpinion
Problem 3.2.11

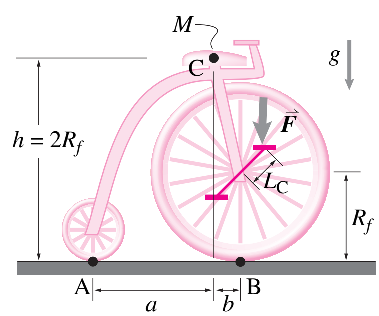
3.2.12 A bicycle with unequal wheels. For the bicycle shown in the figure, assume the mass of the bicycle (and possibly the rider) to be a point mass located at C. A vertical downward force F is applied on the front pedal.

a) Draw a free body diagram of the front wheel.

b) Draw a free body diagram of the back wheel.

c) Draw a free body diagram of the entire bicycle.

d) What assumptions have you made in modeling the interaction force of the ground with the wheels?



Filename:pfig3-2-weirdbicycle
Problem 3.2.12

CHAPTER 4

Statics of one object

Equilibrium of one object is defined by the balance of forces and moments. For a particle, force balance tells all. But for an extended object, moment balance is also useful. There are special shortcuts for an objects that has exactly two or exactly three forces acting on it. If friction forces are relevant the possibility of motion needs to be taken into account. Many real-world problems are not statically determinate and thus yield either only partial solutions, or yield full solutions after you have made extra assumptions.

Contents

The goal here is to find unknown aspects of the forces acting on one part of a machine or structure. Such a part is also called an 'object' or 'body'. By 'unknown' we mean 'unknown at the outset' or 'you-need-to-do-mechanics-calculations-to-find'. Most often 'unknowns' are tensions in ropes or rods, contact forces where one part presses and rubs against another, and the force on an object at a point of connection to another object. We will also find 'unkown' forces and moments that one part of an object applies to another part of the same object. Finally we might also find the 'unknown' direction or point of application of a force that has *a priori* known magnitude.

Needed skills

Throughout this and all later chapters you need mastery of the vector and free body diagram skills and concepts from chapters 2 and 3.

Statics is a subset of dynamics

Statics is the mechanics of things that don't move. But everything does move, at least a little. So strictly speaking dynamics is always the applicable subject. For many practical problems, however, statics is a good approximation of dynamics, very good. With little loss of accuracy, sometimes very little loss, and a great saving of effort, usually a great saving, statics can be used instead of dynamics. Statics is a useful model. Even for a fast moving system, say an accelerating car, statics calculations are appropriate for many of the parts. Although statics is a subset of dynamics (See box 4.2 on page 193) typical engineers do more statics calculations than dynamics calculations. Statics is the core of structural and strength analysis. Statics is the central tool used to predict when a structure or part will or will not break. Finally, Statics is good preparation for Dynamics ①.

Here, and for all of statics, we neglect the role of inertia in small motions. We assume static equilibrium.

① Ironically, for some people the main benefit from learning dynamics is the side effect of better mastery of the generally-more-useful statics.

Two dimensional and three dimensional mechanics

The world we live in is three dimensional and the theory of mechanics is a three dimensional theory. But three dimensions are harder to understand than two. So most learning and engineering analysis is done in two dimensions. You can't critically judge the degree of simplification this involves until you understand 3D mechanics. But you aren't ready to learn 3D mechanics until you understand 2D mechanics. We escape this catch-22 by being casual about the precise meaning of the 2D world view. For now we think of a cylinders and spheres as circles, of boxes as rectangles, and of cars as things with two wheels (one in front, one in back).

Static equilibrium in a nutshell

The basic idea for all of statics is this: if the forces on a system (i.e., the forces showing on a free body diagram of the system) satisfy eqs. (Ic) and (IIc) (inside cover) the system is said to be in *static equilibrium* or just *in equilibrium*②.

② To be precise, static equilibrium requires that the system and all subsystems, all billion gazillion of them (all the different ways you could cut a piece out of your system), satisfy the equilibrium conditions. However, for simplicity at this point in the book, we don't concern ourselves much with subsystems, just with a single whole object.

> A system is in static equilibrium if the applied forces and moments add to zero.

Another way to say this is that

> A system in static equilibrium satisfies the linear and angular momentum balance neglecting the inertial ($m\vec{a}$) terms.

A final alternative description of statics is:

> The full collection of forces on a system in static equilibrium are equivalent to (see Section 2.6 on page 112) a zero force and a zero couple.

The statics story is now, in-principle, complete. You have the tools (vectors and free body diagrams) and you know the basic facts (the definition of statics, above). These are enough. But we'll guide you through some of the subtleties, warn you away from common misconceptions, and teach you some of the tricks of the trade. You will see that the simply-stated laws of statics (above) allow you to accurately calculate things that most people who have not studied statics only vaguely understand.

4.1 Static equilibrium of a particle

What is a particle?

The word particle usually means something small. In mechanics a *particle* is an object for which we don't worry about rotation, or the tendency of forces to cause rotation. A particle may or may not be small. Besides, smallness is in the eyes of the beholder. For some purposes a galaxy is well-modeled as a particle and for others a molecule is too big to be thought of as a particle. Big or small, the particle model of a system is defined by the lack of attention paid to the moment balance equations ①. Either moment balance is trivially satisfied or you can find what you need without worrying about how it is satisfied ②.

① When trying to understand the motion of a galaxy in a cluster of galaxies, for example, the overall displacement (translation) through space is well-described by modeling the galaxy as a particle. Although the galaxy may rotate and distort in interesting ways, one can ignore the equations that describe that rotation and distortion when looking at the galaxy's overall average translation. Similarly when looking at the motion of an accelerating car, or a block on sliding on ramp or a part sliding on a rod, one may learn enough about the forces using a particle model without worrying about how various forces do or do not cause or prevent rotation or distortion.

②Equations are 'satisfied' when the right side is equal to the left.

For statics of a particle force-balance tells all: $$\sum_{\substack{\text{All forces}\\ \text{on the FBD}}} \vec{F}_i = \vec{0} \qquad (Ic)$$

In two dimensions this equilibrium equation makes up 2 independent scalar equations (2 components of the net force vector). In 3 dimensions we get 3 independent scalar equations. So we expect to be able to solve for 2 unknown quantities in 2D particle mechanics, and 3 in 3D.

The statics-of-a-particle recipe

For particle statics we work with a simplified form of the general recipe from the inside back cover.

1) **Draw a free body diagram (FBD) of the part of interest.** Use knowledge of the contact conditions (see Chapter 3) to draw known and unknown aspects of the forces appropriately (see fig. 3.5 on page 154);
2) **Set the sum of the forces on the FBD to zero: $\sum \vec{F}_i = \vec{0}$.** ('Equilibrium', 'force balance', or 'linear momentum balance in statics');
3) **Solve the equations for unkowns.** Use vector manipulation skills (Chapter 2) to solve the force balance equation for unknowns of interest.

Scalar mechanics

In scalar, as opposed to vector, mechanics people sometimes like to take the dot product of Eqn. (Ic) with unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ and write the three scalar component equations ③.

$$\sum F_x = 0, \qquad \sum F_y = 0, \qquad \text{additionally, in 3D} \qquad \sum F_z = 0.$$

Although you can do most any problem plowing through with the 'component' or scalar approach, often there are shortcuts or insights that depend on a more vectorial view.

③ If people who think in terms of scalar mechanics actually took these dot products they would be doing vector mechanics. Rather, people who do scalar mechanics sometimes think of the component equations as fundamental.

1D statics of a particle

Let's call the one dimension of interest the x direction. The key governing equation is

$$\sum F_x = 0.$$

You could call the special direction y, z, x' or s if you like and then use, say $\sum F_z = 0$. The next two simple examples pretty much cover 1D particle statics.

Filename:tfigure-particle-1Dstatics

Figure 4.1: Equilibrium of a particle in 1D.

Example: Balance of two forces
For the particle in fig. 4.1, force balance gives

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad 10\,\text{N}\hat{\imath} - F\hat{\imath} = \vec{0}.$$

Either by equating x components of both sides, or equivalently dotting both sides with $\hat{\imath}$, we get $F = 10\,\text{N}$. Or, we could have just done scalar mechanics,

$$\sum F_x = 0 \quad \Rightarrow \quad 10\,\text{N} - F = 0 \quad \Rightarrow \quad F = 10\,\text{N}.$$

Most often we have to contend with forces which don't show up until we draw a free body diagram.

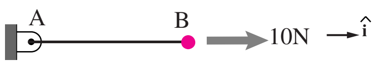


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Figure 4.2: Particle held by a string.

Example. Force pulling on a string. For the particle in fig. 4.2 the quantity of interest, the tension in the cable, doesn't show in the sketch. We need to draw a free body diagram of the particle which means cutting the string. This FBD is shown in fig. 4.1, where $F = T_{\text{AB}}$ represents the tension in cable AB. So force balance gives $T_{\text{AB}} = F = 10\,\text{N}$.

2D statics of a particle

The situation is less trivial when we go to 2D.

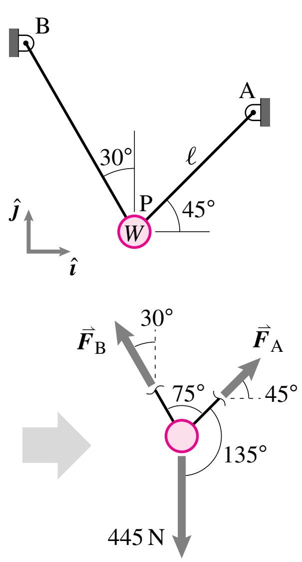

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Figure 4.3: Weight hung from two lines with associated free body diagram.

Example. A 100 pound (445 N) weight hangs from 2 lines in fig. 4.3. We cut the strings, draw a free body diagram and add the forces to get

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad 445\,\text{N}(-\hat{\jmath}) + \underbrace{F_A \frac{(\hat{\imath} + \hat{\jmath})}{\sqrt{2}}}_{\vec{F}_A} + \underbrace{F_B(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})}_{\vec{F}_B} = \vec{0}. \tag{4.1}$$

This can be solved various ways (see below)to get $F_A = 230.3\,\text{N}$ and $F_B = 325.8\,\text{N}$.

Although moment balance is technically superfluous in particle mechanics, when the forces are concurrent moment balance can be used as a shortcut.

How to solve vector statics equations?

Method 1) Pull out x and y components of the vectors to get 2 equations in 2 unkowns;

$$\sum F_x = 0 \quad \Rightarrow \quad F_A/\sqrt{2} - F_B/2 = 0$$
$$\sum F_y = 0 \quad \Rightarrow \quad F_A/\sqrt{2} + F_B\sqrt{3}/2 - 445N = 0.$$

Method 2) Equivalently, dot both sides of the equation with $\hat{\imath}$ and $\hat{\jmath}$ to get 2 equations in 2 unknowns;

$$\{_{\text{eqn. (4.1)}}\} \cdot \hat{\imath} \quad \Rightarrow \quad 0 = F_A/\sqrt{2} - F_B/2$$
$$\{_{\text{eqn. (4.1)}}\} \cdot \hat{\imath} \quad \Rightarrow \quad 0 = F_A/\sqrt{2} + F_B\sqrt{3}/2 - 445N.$$

Method 3) dot both sides with a vector orthogonal to r_{AP} to get one equation in F_B, similarly dot with a vector orthogonal to $\vec{r}_{\text{BP}}$ to get one equation in F_A,

$$\{_{\text{eqn. (4.1)}}\} \cdot (-\hat{\imath} + \hat{\jmath}) \quad \Rightarrow \quad 0 = -445N + F_B(\frac{1}{2} + \frac{\sqrt{3}}{2})$$

$$\{_{\text{eqn. (4.1)}}\} \cdot (\sqrt{3}\hat{\imath} + \hat{\jmath}) \quad \Rightarrow \quad 0 = -445N + F_A\frac{\sqrt{3}+1}{\sqrt{2}}$$

Method 4) cross both sides with $\vec{r}_{AP}$ to get one equation in F_B, similarly cross with r_{BP} to get F_A.

$$\vec{0} = \left(-\ell\hat{\imath}/\sqrt{2} - \ell\hat{\jmath}/\sqrt{2}\right) \times \left(445\,\text{N}(-\hat{\jmath}) + F_A\frac{(\hat{\imath}+\hat{\jmath})}{\sqrt{2}} + F_B(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})\right) \quad \Rightarrow \quad 0 \quad = -445N + F_B(\frac{1}{2} + \frac{\sqrt{3}}{2})$$

$$\vec{0} = \left(d\hat{\imath}/2 - d\sqrt{3}\hat{\jmath}/2\right) \times \left(445\,\text{N}(-\hat{\jmath}) + F_A\frac{(\hat{\imath}+\hat{\jmath})}{\sqrt{2}} + F_B(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})\right) \quad \Rightarrow \quad 0 \quad = -445N + F_A\frac{\sqrt{3}+1}{\sqrt{2}}.$$

See section 2.5 starting on page 92 for more discussion about how to solve vector equations.

Another approach, mathematically equivalent to method 4 above, is to use moment balance.

Example: Moment balance
Consider again fig. 4.3. Moment balance about point A gives

$$\sum \vec{M}_{\text{A}} = \vec{0} \qquad \Rightarrow \qquad \vec{r}_{\text{P/A}} \times (445\,\text{N}(-\hat{\jmath})) + \vec{r}_{\text{P/A}} \times \left(F_B(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})\right) + \vec{0} = \vec{0}.$$

Evaluating the cross products one way or another one again gives $F_B = 325.8\,\text{N}$. Similarly moment balance about B gives $F_A = 230.3\,\text{N}$.

Example: A kite.
A kite flying steadily in a breeze is roughly in static equilibrium. The three forces acting on it are from the air, pushing the kite downwind and up; from gravity, pulling the kite down; and from the string pulling the kite upwind and down. The three forces must add to zero.

A funny thing about kites is that they only stay up because you pull them down.

Whether force or moment balance is used, for concurrent force systems we only have two independent scalar equilibrium equations in 2D, and three in 3D .

Frictionless contact

As discussed in Chapter 3.1, engineered parts which slide often have bearings or lubrication which minimize the sliding resistance. To simplify analyses, that remaining resistance is often neglected and we model the contact as 'frictionless' ($\mu = 0$). This means the interaction force is normal to the contacting surfaces.

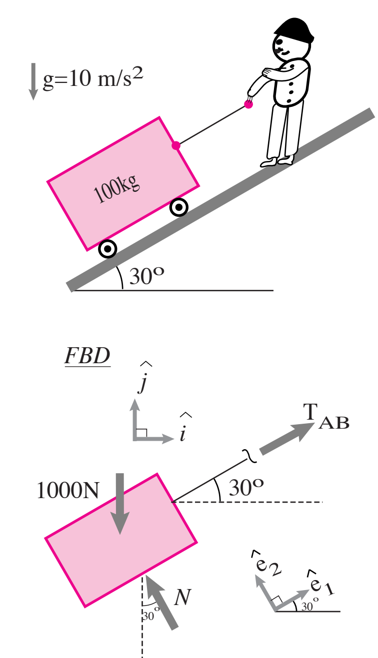

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Figure 4.4: A wagon pulled uphill and associated free body diagram. Because we are doing particle mechanics we combine the two forces on the wheels into the single resultant N. Further, for particle mechanics we need not worry about where that N is applied.

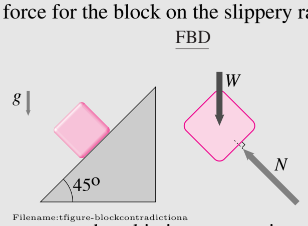
4.1 Existence and uniqueness

This is a mathematical aside for those interested in fine points.

Sometimes equations have no solutions and solutions are said to *not exist*. For example no solutions *exist* for the equations $x+2y=7, 2x+4y=15$ (subtracting twice the first equation from the second shows the contradiction that $0=1$). Sometimes equations have *non-unique* solutions: the equation $x+y=1$ has many solutions including $(x,y)=(1,0)$, $(x,y)=(0,1)$, $(x,y)=(10,-9)$ *etc.*

Although the words *existence* and *uniqueness* have a mathematically abstract irrelevant-to-the-real-world ring to them, they are relevant to real-world mechanics. In a class or in engineering practice you will likely run into such an *il-posed* problem.

Example: A block on a frictionless sloped ramp. Using statics find the normal force for the block on the slippery ramp below.

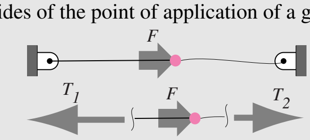


At a glance you can see that this is not a statics problem so there is no way to get a statics solution. Even without intuition, the force balance equations show that there is no value of N that can make the force vectors add to zero.

In practice, such contradictions could be more subtle. In sec. 5.5 there are a few examples where even the best experts couldn't intuitively see that there are no solutions. In engineering practice issues of existence are quickly found once one builds a prototype (which falls apart) or a detailed calculation (and the computer coughs).

Sometimes statics problems have more than one solution. In contrast to the relatively rare 'existence' issues above, issues of uniqueness are extremely common in practice. Perhaps annoyingly common.

Example: Particle held by two strings. Find the tension in the strings to the sides of the point of application of a given load F.



Force balance along the strings gives us one equation for the two unknown tensions.

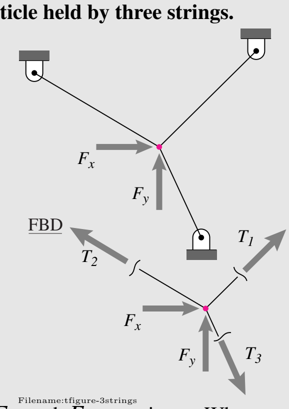
$$\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \qquad \Rightarrow \qquad -T_1 + F + T_2 = 0$$

No other force balance or moment balance equation gives more information. For any given F this equation has many solutions. The pair (T_1, T_2) could be $(F, 0)$ or $(0, -F)$ $(2F, F)$ $(F + 7\,\text{N}, 7\,\text{N})$, etc.

Of course if you tie strings together like this and apply a force there is some actual tension in each string; reality, at any instant in time, is unique (as far as we know). For example, if you had tied the strings loosely together the right string gets slack and has $T_2 = 0$ and thus $T_1 = F$. But it takes an extra assumption of this nature to get a unique solution.

And just because you can make an assumption that leads you to a unique solution doesn't mean that assumption corresponds well to reality. You might assume your friend had tied the strings together loosely (and thus calculate $T_2 = 0$ and $T_1 = F$) when she had really tied them together tightly (and so really, say, $T_2 = 30\,\text{N}$ and $T_1 = F + 30\,\text{N}$) Here is the same idea in 2D.

Example: Particle held by three strings.



Assume that F_x and F_y are given. What are the three tensions. Planar force balance gives two equations for the 3 unknown tensions. As in the previous example these equations have many solutions.

If you assume a) that one string goes slack and b) that no string can carry compression, then this problem has a unique answer. But you would have to know at the start that the strings were not tied tightly.

One could also make an example with 4 strings holding a particle in 3D. All of these examples allow a 'one parameter family of solutions'; specifying one number (say that the tension in cable 2 is zero) determines the other tensions. We could have more non-uniquenes than that by holding a particle with 3 strings in 2D, 4 strings in 2D, or 5 strings in 3D. And even more non-uniqueness with even more strings. Sometimes the problem can lead to a unique solution with a simple reasonable physical assumption, and sometimes not.

Counting equations and unknowns All of the examples above could be picked out as problematic by counting equations and unknowns. For the block on the ramp we had two equations for the one unknown N. Whereas for the string problems we had more unknowns than equations. In summary,

- If you have more equations than unknowns existence is likely to be an issue; you probably can't find any solutions.
- If you have more unknowns than equations then uniqueness *is* an issue; any solution you find will be non-unique.

But there are 'exceptional' cases for which equation counting does not tell all about existence and uniqueness, as discussed in detail in the advanced truss section 5.5 (see the lower right corner of the 2×2 tables there).

Example: Pull a wagon uphill
See fig. 4.4. From the free body diagram we have

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -1000\,\mathrm{N}\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{e}}_2 + T_{\mathrm{AB}}\hat{\boldsymbol{e}}_1 = \vec{\mathbf{0}}. \tag{4.2}$$

where $\hat{\boldsymbol{e}}_1 = \cos(30^\circ)\hat{\boldsymbol{\imath}} + \sin(30^\circ)\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_2 = -\sin(30^\circ)\hat{\boldsymbol{\imath}} + \cos(30^\circ)\hat{\boldsymbol{\jmath}}$. N and T_{AB} are unkown forces. Here are two ways to solve for the unknowns.
Method I. Substitute the expressions for $\hat{\boldsymbol{e}}_1$ and $\hat{\boldsymbol{e}}_2$ above into eqn. (4.2), extract x and y components to get 2 equations in two unknowns which you can solve to get $T_{\mathrm{AB}} = 500\,\mathrm{N}$ and $N = 500\sqrt{3}\,\mathrm{N}$ (note the font confusion that the force quantity N and unit N have different meanings).
Method II. Using well chosen dot products can simplify the algebra. Take the dot products of both sides of eqn. (4.2) with $\hat{\boldsymbol{e}}_1$ and then with $\hat{\boldsymbol{e}}_2$, to get two scalar equations. Dotting eqn. (4.2) with $\hat{\boldsymbol{e}}_1$ eliminates terms orthogonal to $\hat{\boldsymbol{e}}_1$, namely $N\hat{\boldsymbol{e}}_2$. And dotting eqn. (4.2) with $\hat{\boldsymbol{e}}_2$ 'kills' the $T_{\mathrm{AB}}\hat{\boldsymbol{e}}_1$ term. So the two equations each have only one unknown. See page 98 for more discussion of this method.

Three dimensional particle mechanics

The basic idea is the same in 3D as in 2D.

Example: One unknown force.
Assume 3 known forces and one unknown force $\vec{\boldsymbol{F}}$ are acting on a particle (fig. 4.5). Then from force balance

$$\begin{aligned}
\vec{\mathbf{0}} &= \sum \vec{\boldsymbol{F}}_i \\
\Rightarrow \quad \vec{\mathbf{0}} &= (36\,\mathrm{lbf}\hat{\boldsymbol{\imath}} - 16\,\mathrm{lbf}\hat{\boldsymbol{\jmath}}) + (-52\,\mathrm{lbf}\hat{\boldsymbol{k}} + 5\,\mathrm{lbf}\hat{\boldsymbol{\imath}}) \\
&\quad + (-42\,\mathrm{lbf}\hat{\boldsymbol{k}} + 20\,\mathrm{lbf}\hat{\boldsymbol{\imath}} - 16\,\mathrm{lbf}\hat{\boldsymbol{\jmath}}) + \vec{\boldsymbol{F}} \\
\Rightarrow \quad \vec{\boldsymbol{F}} &= (-61\hat{\boldsymbol{\imath}} + 32\hat{\boldsymbol{\jmath}} + 94\hat{\boldsymbol{\jmath}})\ \mathrm{lbf}.
\end{aligned}$$

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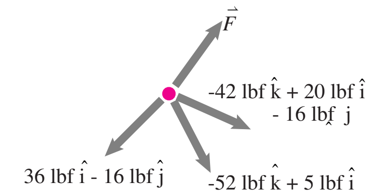

Figure 4.5: One unknown force $\vec{\boldsymbol{F}}$.

The new difficulties in 3D particle mechanics are

- Visualization in 3D. (So practice making and reading 3D drawings.); and
- The vector force-balance equation is 3D and thus equivalent to 3 scalar equations. Solving these is at the upper boundary of what most people can do reliably by hand or even with a non-programmable calculator. So methods that reduce the complexity of the solution are useful, as is the ability to set up the resulting equations on a computer or programmable calculator.

Hint: If the direction of a force is given (possibly implicitly) express the force as a scalar times a unit vector: $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\lambda}}$. (See top row, middle column of fig. 3.5 on page 154.)

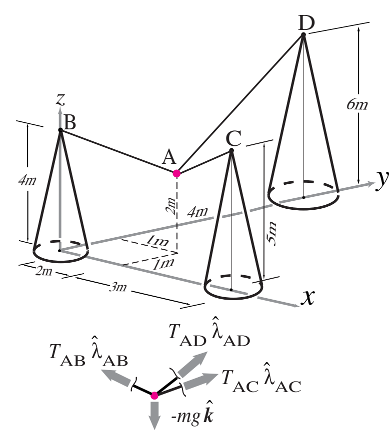

Figure 4.6: Mass A suspended by 3 ropes and a free body diagram of the mass.

Example: Particle held by 3 ropes.
Say $m = 100\,\mathrm{kg}$ and $g = 10\,\mathrm{N/kg}$ in fig. 4.6. Force balance gives

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad T_{\mathrm{AB}}\hat{\boldsymbol{\lambda}}_{\mathrm{AB}} + T_{\mathrm{AC}}\hat{\boldsymbol{\lambda}}_{\mathrm{AC}} + T_{\mathrm{AD}}\hat{\boldsymbol{\lambda}}_{\mathrm{AD}} - mg\hat{\boldsymbol{k}} = \vec{\mathbf{0}} \tag{4.3}$$

which is a 3D vector equation in 3 unknowns (3=3, good). The $\hat{\boldsymbol{\lambda}}$'s in eqn. (4.6) are known because, for example,

$$\hat{\boldsymbol{\lambda}}_{\mathrm{AB}} = \frac{\vec{\boldsymbol{r}}_{\mathrm{AB}}}{|\vec{\boldsymbol{r}}_{\mathrm{AB}}|} = \frac{\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}}{|\vec{\boldsymbol{r}}_{\mathrm{B}} - \vec{\boldsymbol{r}}_{\mathrm{A}}|}$$

and the position vectors are given in the picture. To get to a numerical answer for the the tensions you can use many methods such as (see Sample 4.4 on page 197)

1. Brute force by hand.
2. Systematically set up matrix equations for solution by some means.
3. Set up and solve equations on a computer.
4. Use an appropriate dot product to extract one equation in one unknown.
5. Use moment about an axis to extract one equation in one unknown.

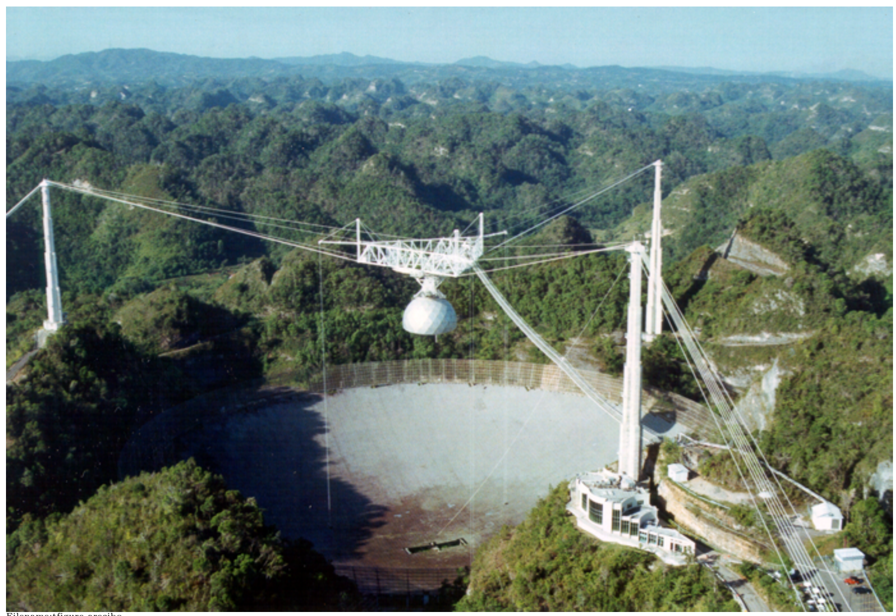

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Figure 4.7: "Suspended 450 feet above the reflector is the 900 ton platform. Similar in design to a bridge, it hangs in midair on eighteen cables, which are strung from three reinforced concrete towers. One is 365 feet high, and the other two are 265 feet high." *Courtesy of the NAIC-Arecibo Observatory, a facility of the NSF.*

4.2 The simplification of dynamics to statics

The bit of theory here is for people interested in the appropriateness of the statics model for systems that move. It will not help you with learning statics skills or doing statics homework problems.

The mechanics equations in the front inside cover are accurate enough for everything that 99.99% of engineers will ever encounter. The statics subset covers special case that apply less exactly to many things. But exactly enough for, say, 90% of the engineering mechanics calculations in the world. In statics we set the right hand sides of equations I and II to zero. We think that these terms are small enough, compared to other included terms, that they can be counted as zero. The neglected terms involve mass times acceleration and are called the *inertial terms*. Thus we replace the linear and angular momentum balance equations with their simplified statics forms

$$\sum_{\substack{\text{All external}\\\text{forces}}} \vec{F} = \vec{0} \quad \text{and} \quad \sum_{\substack{\text{All external}\\\text{torques}}} \vec{M}_{\mathrm{C}} = \vec{0} \quad (Ic, IIc).$$

which are sometimes called the *force balance* and *moment balance* equations and together are called the *equilibrium* equations. The forces to be summed (added) are the ones you see on a free body diagram. The torques that are summed are those due to the same forces (by means of $\vec{r}_{\mathrm{i}/\mathrm{C}} \times \vec{F}_{\mathrm{i}}$) plus applied couples (force systems with zero resultant that have been replaced on the FBD with equivalent couples). The approximating assumption ('the model', see page 33) of an object being in static equilibrium is that the forces mediated by an object are much larger than the forces needed to accelerate it.

Detailed estimation of the errors from neglecting dynamics terms is a dynamics problem, so we can't fully address it here. But you can do a rough check by making sure that the mass times acceleration is a small fraction of typical forces you find from your statics analysis. This inertial term comes from the total of all the forces. So the approximation in statics is that the total of the forces is much less than any of the individual forces. If

$$\sum_{\substack{\text{All forces}\\\text{on the part}}} \vec{F}_i \ll F_{\text{typical}}$$

then statics is probably a good model; the forces are more canceling each other out, balancing each other, than causing acceleration. You can then figure out how these forces cancel each other, that is, you can do statics.

The statics equations are often accurate-enough for engineering purposes for

- Things that a normal person would call "still" such as a building or bridge on a calm day, and a sleeping person;
- Things that move with little acceleration, such as a tractor plowing a field and most of the parts in a smooth-flying airplane; and
- Parts that mediate the forces needed to accelerate more massive parts, such as gears in a transmission, the rear wheel of an accelerating bicycle, the strut in the landing gear of an airplane, and the individual structural members of a building swaying in an earthquake.

If your statics calculations make a bad prediction one of the possible errors is your neglect of dynamic terms. If the machine or structure seems relatively still it is more likely, however, that inaccuracies in your statics calculation come from in-accurate estimates of material properties (friction coefficient, failure strength, etc) or from mis-estimation of geometric features (a dimension, clearance, angle, etc).

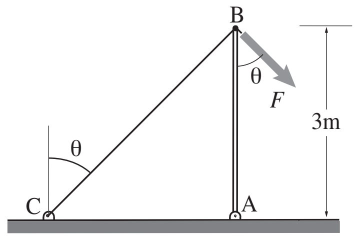

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Figure 4.8:

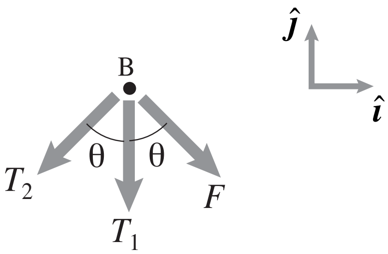

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Figure 4.9: The free body diagram of the pin at B. Although we can see that rod AB is in compression, we use tension as positive and let the solution take care of the proper sign.

④Note that the tension in rod AB, T_1 turns out to be negative, that is, rod AB is in *compression*. This is expected since the other two forces at B, F and T_2, are pushing on rod AB. The equality of F and T_2 is also expected from symmetry.

⑤How do we find the normal vector $\hat{\boldsymbol{n}}$? Well, we know the direction of $\vec{\boldsymbol{T}}_2$, which is $\hat{\boldsymbol{\lambda}} = -\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$. We can draw a unit vector $\hat{\boldsymbol{n}}$ normal to $\hat{\boldsymbol{\lambda}}$ and find its components from geometry (see figure below), or we can set $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}$ which guarantees $\hat{\boldsymbol{n}}$ to be normal to $\hat{\boldsymbol{\lambda}}$. In either case we get $\hat{\boldsymbol{n}} = \cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$.

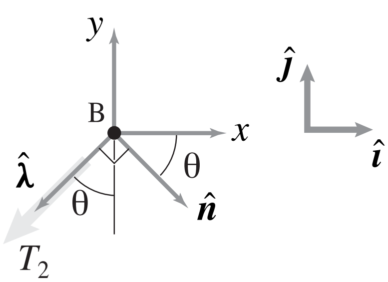

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Figure 4.10: Finding a unit vector $\hat{\boldsymbol{n}}$ normal to the direction $\hat{\boldsymbol{\lambda}}$ of T_2.

SAMPLE 4.1 Equilibrium of a pin. Two rods, AB and BC, are pinned together at point B and to the ground as shown in the figure. A force $F = 100$ N is applied at point B. Given that $\theta = 45^\circ$, find the tension in the two rods.

Solution The free-body diagram of the pin at B is shown in fig. 4.9 where T_1 and T_2 are the tensions in rod AB and BC respectively. The static equilibrium of the pin at B requires that

$$\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}} \quad \Rightarrow \quad \vec{\boldsymbol{F}} + \vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{T}}_2 = \vec{\boldsymbol{0}}$$

$$\text{or} \qquad F(\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) - T_1\hat{\boldsymbol{\jmath}} + T_2(-\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) = 0. \tag{4.4}$$

This is one vector equation in 2D in two unknowns T_1 and T_2. We can solve for the unknowns in various ways.

Method-1: Separate out scalar equations in x and y directions. The force equilibrium equation, eqn. (4.4), gives us two independent scalar equations in the x and y directions:

$$\begin{aligned} \sum F_x = 0 \quad &\Rightarrow \quad F\sin\theta - T_2\sin\theta = 0 \\ \sum F_y = 0 \quad &\Rightarrow \quad -F\cos\theta - T_1 - T_2\cos\theta = 0. \end{aligned}$$

Solving these two equations simultaneously, we get

$$\begin{aligned} T_2 &= F = 100\,\text{N} \\ T_1 &= -(F + T_2)\cos\theta \\ &= -2F\cos\theta = -141.4\,\text{N}. \end{aligned}$$

$$\boxed{T_1 = -141.4\,\text{N}, \qquad T_2 = 100\,\text{N}}$$

④

Method-2: Dot the equation with appropriate vectors. The goal here is to dot the vector equation with appropriate vectors that give us one scalar equation in one unknown. Here, $\vec{\boldsymbol{T}}_1$ acts in the $-\hat{\boldsymbol{\jmath}}$ direction; therefore, dotting the equation with $\hat{\boldsymbol{\imath}}$ gets rid of $\vec{\boldsymbol{T}}_1$ and results in a scalar equation involving only T_2:

$$\begin{aligned} [\text{eqn. (4.4)}] \cdot \hat{\boldsymbol{\imath}} \quad &\Rightarrow \quad F\sin\theta - T_2\sin\theta = 0 \\ &\Rightarrow \quad T_2 = F = 100\,\text{N}. \end{aligned}$$

Similarly, to get rid of $\vec{\boldsymbol{T}}_2$, dot the equation with a vector $\hat{\boldsymbol{n}}$ normal to $\vec{\boldsymbol{T}}_2$, *i.e.,* with $\hat{\boldsymbol{n}} = \cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}$:⑤

$$\begin{aligned} [\text{eqn. (4.4)}] \cdot \hat{\boldsymbol{n}} \quad &\Rightarrow \quad [-T_1\hat{\boldsymbol{\jmath}} + F(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}}) \cdot (\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}})] = 0 \\ &\Rightarrow \quad T_1\sin\theta + F(\cos^2\theta + \sin^2\theta) = 0 \\ &\Rightarrow \quad T_1 = -\frac{F}{\sin\theta} = -\frac{100\,\text{N}}{1/\sqrt{2}} = -141.4\,\text{N} \end{aligned}$$

These are the same values of T_1 and T_2, as they must, obtained by Method-1.

Method-3: Use matrix equation and solve by hand or on a computer. The two scalar equations obtained from eqn. (4.5) can be written in the matrix form as

$$\begin{bmatrix} 0 & \sin\theta \\ 1 & \cos\theta \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} F\sin\theta \\ -F\cos\theta \end{pmatrix}.$$

Using $F = 100\,\text{N}$ and $\theta = 45^\circ = \pi/4$, and solving the above matrix equation (see Sample 2.28 on page 108 and Sample 2.30 on page 110), we get

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} -141.4 \\ 100 \end{pmatrix} \text{N}$$

which is, of course, the same result as we got above.

SAMPLE 4.2 **A mass held in equilibrium by unequal strings in 2D.** A 10 kg block m hangs from strings AB and AC in the vertical plane as shown in the figure. Find the tension in the strings.

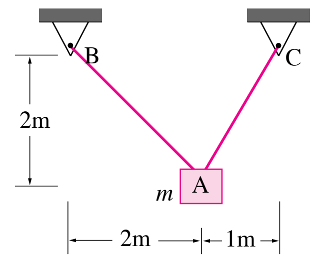

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Figure 4.11:

Solution The free body diagram of the block is shown in figure 4.12. The equation of force balance, $\sum \vec{F} = \vec{0}$, gives

$$\text{or} \qquad T_1 \hat{\boldsymbol{\lambda}}_{\mathrm{AB}} + T_2 \hat{\boldsymbol{\lambda}}_{\mathrm{AC}} - mg\,\hat{\boldsymbol{j}} = \vec{\mathbf{0}}, \tag{4.5}$$

where $\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}$ and $\hat{\boldsymbol{\lambda}}_{BC}$ are unit vectors in the AB and AC directions:

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_{AB} &= \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \frac{-2\,\mathrm{m}\hat{\boldsymbol{\imath}} + 2\,\mathrm{m}\hat{\boldsymbol{j}}}{2\sqrt{2}\,\mathrm{m}} = \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{j}}) \\
\hat{\boldsymbol{\lambda}}_{\mathrm{AC}} &= \frac{\vec{\boldsymbol{r}}_{AC}}{|\vec{\boldsymbol{r}}_{AC}|} = \frac{1\,\mathrm{m}\hat{\boldsymbol{\imath}} + 2\,\mathrm{m}\hat{\boldsymbol{j}}}{\sqrt{5}\,\mathrm{m}} = \frac{1}{\sqrt{5}}(\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{j}}).
\end{aligned}$$

Substituting in eqn. (4.5) and rearranging terms, we have

$$\left(-\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}}\right)\hat{\boldsymbol{\imath}} + \left(\frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg\right)\hat{\boldsymbol{j}} = \vec{\mathbf{0}}.$$

Separating x and y components of this equation, we get the scalar equations

$$\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad -\frac{T_1}{\sqrt{2}} + \frac{T_2}{\sqrt{5}} = 0 \\
\sum F_y = 0 \quad &\Rightarrow \quad \frac{T_1}{\sqrt{2}} + \frac{2T_2}{\sqrt{5}} - mg = 0.
\end{aligned}$$

Solving these two equations simultaneously we get,

$$T_1 = \frac{\sqrt{2}}{3}mg = 46.24\,\mathrm{N} \quad \text{and} \quad T_2 = \frac{\sqrt{5}}{3}mg = 73.12\,\mathrm{N}.$$

$$\boxed{T_1 = 46.24\,\mathrm{N},\ T_2 = 73.12\,\mathrm{N}}$$

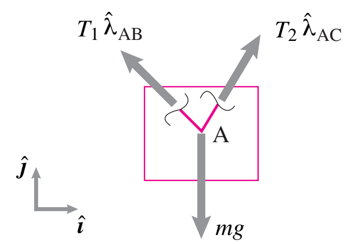

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Figure 4.12:

Note:

Solving for T_1 and T_2 If you are comfortable with vector algebra, then solving for T_1 and T_2 from eqn. (4.5) is quite easy. Let us say, we find two unit vectors $\hat{\boldsymbol{n}}_{AB}$ and $\hat{\boldsymbol{n}}_A C$ normal to unit vectors $\hat{\boldsymbol{\lambda}}_{AB}$ and $\hat{\boldsymbol{\lambda}}_{AC}$, respectively. Then dotting eqn. (4.5) with $\hat{\boldsymbol{n}}_{AB}$ and $\hat{\boldsymbol{n}}_A C$, one at a time, we can solve for T_1 and T_2 in one step:

$$T_2 = \frac{mg\,\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{n}}_{AB}}{\hat{\boldsymbol{\lambda}}_{AC} \cdot \hat{\boldsymbol{n}}_{AB}}, \quad \text{and} \quad T_1 = \frac{mg\,\hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{n}}_{AC}}{\hat{\boldsymbol{\lambda}}_{AB} \cdot \hat{\boldsymbol{n}}_{AC}}.$$

For computing the values, we need to carry out the dot products. Noting that $\hat{\boldsymbol{n}}_{AB} = \frac{1}{\sqrt{2}}(i + j)$ and $\hat{\boldsymbol{n}}_{AC} = \frac{1}{\sqrt{5}}(-2\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{j}})$ (you can write these vectors by looking at $\hat{\boldsymbol{\lambda}}_{AB}$ and $\hat{\boldsymbol{\lambda}}_{AC}$), we can carry out the dot product and get the values of T_1 and T_2.

Matrix equation The two scalar equations obtained from eqn. (4.5) can be written in the matrix form as

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 0 \\ mg \end{pmatrix}.$$

Using $mg = (10\,\mathrm{kg}) \cdot (9.81\,\mathrm{m/s^2}) = 98.1\,\mathrm{N}$, and solving the above matrix equation (see Sample 2.28 on page 108 and Sample 2.30 on page 110), we get

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} = \begin{pmatrix} 46.24 \\ 73.12 \end{pmatrix}\,\mathrm{N}$$

which is, of course, the same result as we got above.

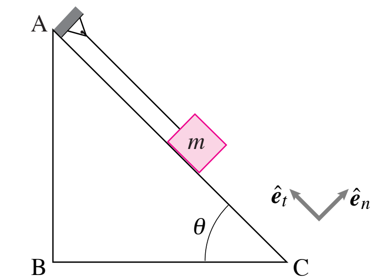

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Figure 4.13: A mass-particle on an inclined plane.

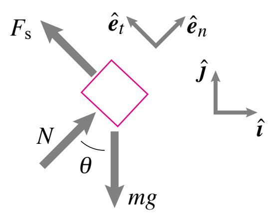

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Figure 4.14: Free-body diagram of the mass and the geometry of force vectors.

SAMPLE 4.3 A single string holding a mass on a frictionless incline. A block of mass m rests on a frictionless inclined plane with the help of a string that connects the mass to a fixed support at A. Find the force in the string.

Solution The free-body diagram of the mass is shown in Fig. 4.14. The string force F_s and the normal reaction of the plane N are unknown forces. The force balance equation, $\sum \vec{F} = \vec{0}$, is

$$\vec{F}_s + \vec{N} + m\vec{g} = \vec{0}.$$

We can express the forces in terms of their components in various ways and then dot the vector equation with appropriate unit vectors to get two independent scalar equations. For example, we write the force balance equation using mixed basis vectors $\hat{e}_t$ and $\hat{e}_n$, and $\hat{\imath}$ and $\hat{j}$:

$$F_s\hat{e}_t + N\hat{e}_n - mg\hat{j} = \vec{0}. \tag{4.6}$$

We can now find F_s directly by taking the dot product of the above equation with $\hat{e}_t$ since the other unknown N is in the $\hat{e}_n$ direction and $\hat{e}_n \cdot \hat{e}_t = 0$:

$$\begin{aligned} \{\text{eqn. (4.6)}\} \cdot \hat{e}_t \quad &\Rightarrow \quad F_s - mg\overbrace{(\hat{j} \cdot \hat{e}_t)}^{\sin\theta} = 0 \\ &\Rightarrow \quad F_s = mg\sin\theta. \end{aligned}$$

$$\boxed{F_s = mg\sin\theta}$$

• • •

Note: We can also find N from a single equation by taking the dot product of eqn. (4.6) with $\hat{n}$:

$$\begin{aligned} \{\text{eqn. (4.6)}\} \cdot \hat{e}_n \quad &\Rightarrow \quad N - mg\overbrace{(\hat{j} \cdot \hat{e}_n)}^{\cos\theta} = 0 \\ &\Rightarrow \quad N = mg\cos\theta. \end{aligned}$$

Scalar approach: We resolve all forces into their $\hat{e}_t$ and $\hat{e}_n$ components and then sum the forces. Here, F_s is along the plane and therefore, has no component perpendicular to the plane. Force N is perpendicular to the plane and therefore, has no component along the plane. We resolve the weight mg into two components: (1) $mg\cos\theta$ perpendicular to the plane (along $\hat{e}_n$) and (2) $mg\sin\theta$ along the plane (along $\hat{e}_t$). Now we can sum the forces.

$$\begin{aligned} &\sum F_t = 0 \quad \Rightarrow \quad F_s - mg\sin\theta = 0; \\ \text{and} \quad &\sum F_n = 0 \quad \Rightarrow \quad N - mg\cos\theta = 0 \end{aligned}$$

which, of course, is essentially the same as the equations obtained above.

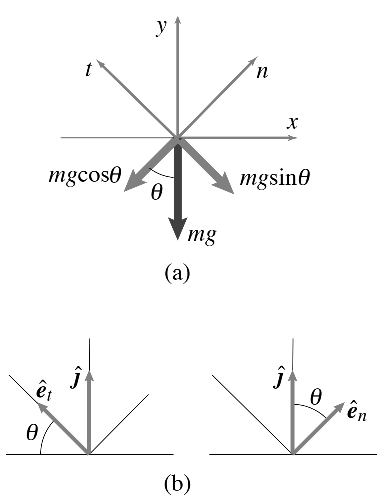

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Figure 4.15: (a) Components of mg along t and n directions. (b) The mixed basis dot products: $\hat{j} \cdot \hat{e}_t = \sin\theta$ and $\hat{j} \cdot \hat{e}_n = \cos\theta$

SAMPLE 4.4 **A particle in 3D.** A particle of mass 1 kg is attached to two strings tied at points C and D shown in the figure. Another string, AB, attached to the particle, passes over a pulley and is used to hold the particle in equilibrium under gravity such that it loses contact with the ground at point A. Find the tension in string AB.

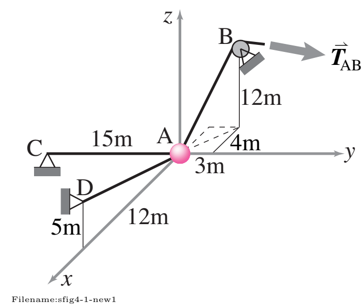


Figure 4.16:

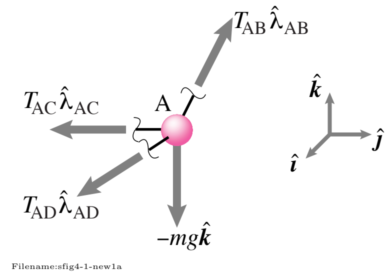


Figure 4.17:

Solution The free-body diagram of the particle is shown in fig. 4.17. Assuming the tensions in strings AB, AC, and AD to be T_{AB}, T_{AC}, and T_{AD} respectively, we can represent the string forces acting on the particle as $T_{AB}\hat{\boldsymbol{\lambda}}_{AB}$, $T_{AC}\hat{\boldsymbol{\lambda}}_{AC}$, and $T_{AD}\hat{\boldsymbol{\lambda}}_{AD}$, where the $\hat{\boldsymbol{\lambda}}$'s are the unit vectors along the strings.

The force balance on the particle gives us

$$T_{AB}\hat{\boldsymbol{\lambda}}_{AB} + T_{AC}\hat{\boldsymbol{\lambda}}_{AC} + T_{AD}\hat{\boldsymbol{\lambda}}_{AD} - mg\hat{\boldsymbol{k}} = 0. \tag{4.7}$$

This is the equation we need to solve to find T_{AB}. We show various methods below that you can use to get T_{AB}.

1. **Brute force (by hand).**

   From the given figure, the unit vectors are:

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_{AB} &= \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2+3^2+12^2}} = -\frac{4}{13}\hat{\boldsymbol{\imath}} + \frac{3}{13}\hat{\boldsymbol{\jmath}} + \frac{12}{13}\hat{\boldsymbol{k}} \\
\hat{\boldsymbol{\lambda}}_{AC} &= -\hat{\boldsymbol{\jmath}} \\
\hat{\boldsymbol{\lambda}}_{AD} &= \frac{12\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{k}}}{\sqrt{12^2+5^2}} = \frac{12}{13}\hat{\boldsymbol{\imath}} + \frac{5}{13}\hat{\boldsymbol{k}}.
\end{aligned}$$

   Substituting these vectors in eqn. (4.7) and equating the x, y and z components of the equation to zero separately, we get

$$\begin{aligned}
-\frac{4}{13}T_{AB} + \frac{12}{13}T_{AD} &= 0 \\
\frac{3}{13}T_{AB} - T_{AC} &= 0 \\
\frac{12}{13}T_{AB} + \frac{5}{13}T_{AD} &= mg.
\end{aligned} \tag{4.8}$$

   We can solve the three equations simultaneously to get

$$T_{AB} = \frac{39}{41}mg, \quad T_{AC} = \frac{9}{41}mg, \quad \text{and} \quad T_{AD} = \frac{13}{41}mg.$$

   Substituting $m = 1\,\text{kg}$ and $g = 9.81\,\text{m/s}^2$, we get the required values.

$$\boxed{T_{AB} = 9.33\,\text{N}, \quad T_{AC} = 2.15\,\text{N}, \quad T_{AD} = 3.11\,\text{N}}$$

2. **Systematically set up matrix equations.** Eqn. 4.7 can be written in matrix form as

$$\underbrace{\begin{bmatrix} \left[\hat{\boldsymbol{\lambda}}_{AB}\right]'_{xyz} & \left[\hat{\boldsymbol{\lambda}}_{AC}\right]'_{xyz} & \left[\hat{\boldsymbol{\lambda}}_{AD}\right]'_{xyz} \end{bmatrix}}_{3\times 3 \text{ matrix}} \begin{bmatrix} T_{AB} \\ T_{AB} \\ T_{AB} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix}$$

   where $\left[\hat{\boldsymbol{\lambda}}_{AB}\right]'_{xyz}$ is a column of 3 numbers, namely the x, y, and z components of $\hat{\boldsymbol{\lambda}}_{AB}$; similarly for the other two columns of the 3×3 matrix. This matrix equation is

⑥ Pseudo-code:

```
Let m=1, 9=9.81
A = [ -4/13 0 12/13
      3/13 -1 0
      12/13 0 5/13 ]
b = [ 0 0 m*g]'
solve A*T = b for T
```

then ready to hand to a calculator or computer for a matrix solution. Thus, eqn. (4.8) can be written as,

$$\begin{bmatrix} -4/13 & 0 & 12/13 \\ 3/13 & -1 & 0 \\ 12/13 & 0 & 5/13 \end{bmatrix} \begin{pmatrix} T_{\mathrm{AB}} \\ T_{\mathrm{AC}} \\ T_{\mathrm{AD}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ mg \end{pmatrix}.$$

Using the pseudo code shown on the side ⑥ we solve the equations on a computer and get,

```
T = [9.33   2.15   3.11]
```

which is the solution that we obtained above by hand calculation.

3. **Computer solution.** All the math can be handed to a computer by a sequence of commands like this, working from the knowns to the unknowns (see page 16), all in consistent units:

```
% Get all the knowns into the computer
  rA = [0 0 0]' ;   rB = [-4 3 12]'
  rC = [0 -15 0]' ;   rD = [12 0 5]'
  m = 1 ;  g = 9.81 ;
% Make relative position vectors
  rAB = rB- rA;   rAC = rC - rA;  rAD = rD - rA
% Make unit vectors
  lamdaAB = rAB/magnitude(rAB);
  lamdaAC = rAC/magnitude(rAC);
  lamdaAD = rAD/magnitude(rAD);
% Set up and solve the matrix equation
  M = [lamdaAB  lamdaAC  lamdaAD]  %3x3 matrix
  F = [0 0 mg]'   %column vector of known force
  solve {M T = F} for T
```

The column of numbers `T` will be the tensions in the 3 cables. Using this pseudo-code on a computer, we get `T = [9.33   2.15   3.11]` again.

4. **Be tricky to get one equation in one unknown.** Since we are interested only in T_{AB}, we can get rid of the terms we don't know or care about. ⑦ The vector $\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}$ is orthogonal to both $\vec{\boldsymbol{r}}_{\mathrm{AC}}$ and $\vec{\boldsymbol{r}}_{\mathrm{AD}}$, so it is orthogonal to $\hat{\boldsymbol{\lambda}}_{\mathrm{AC}}$ and $\hat{\boldsymbol{\lambda}}_{\mathrm{AD}}$. So taking the dot product of both sides of eqn. (4.7) with $\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}$, we get

$$(\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}) \cdot \left\{ T_{\mathrm{AB}}\hat{\boldsymbol{\lambda}}_{\mathrm{AB}} + T_{\mathrm{AC}}\hat{\boldsymbol{\lambda}}_{\mathrm{AC}} + T_{\mathrm{AD}}\hat{\boldsymbol{\lambda}}_{\mathrm{AD}} - mg\hat{\boldsymbol{k}} \right\} = (\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}) \cdot \vec{\boldsymbol{0}}$$

$$\left((\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}) \cdot \hat{\boldsymbol{\lambda}}_{\mathrm{AB}} \right) T_{\mathrm{AB}} = mg \left((\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}) \cdot \hat{\boldsymbol{k}} \right)$$

$$T_{\mathrm{AB}} = \frac{mg\,(\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}) \cdot \hat{\boldsymbol{k}}}{(\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}}) \cdot \hat{\boldsymbol{\lambda}}_{\mathrm{AB}}}.$$

Since, $\vec{\boldsymbol{r}}_{\mathrm{AC}} \times \vec{\boldsymbol{r}}_{\mathrm{AD}} = (-15\hat{\boldsymbol{\jmath}})\,\mathrm{m} \times (12\hat{\boldsymbol{\imath}} + 5\hat{\boldsymbol{k}})\,\mathrm{m} = (180\hat{\boldsymbol{k}} - 75\hat{\boldsymbol{\imath}})\,\mathrm{m}^2$, substituting this cross product and other known quantities, we get

$$\begin{aligned} T_{\mathrm{AB}} &= \frac{mg \cdot 180}{180 \cdot 12/13 + 75 \cdot 4/13} \\ &= 0.95\,mg \\ &= 9.33\,\mathrm{N}. \end{aligned}$$

$$T_{\mathrm{AB}} = 9.33\,\mathrm{N}$$

⑦ Another way of doing this is by taking **Moment about an axis.** This approach is similar in spirit to the previous approach. Instead of the equilibrium eqn. (4.7) we could have used moments about axis CD to 'kill off' the tensions in ropes AC and AD (they have no moment about that axis), like this,

$$\sum M_{\text{axis CD}} = 0$$

$$\vec{\boldsymbol{r}}_{\mathrm{CD}} \cdot \left\{ \vec{\boldsymbol{r}}_{\mathrm{DA}} \times \left\{ T_{\mathrm{AB}}\hat{\boldsymbol{\lambda}}_{\mathrm{AB}} - mg\hat{\boldsymbol{k}} \right\} \right\} = 0$$

$$T_{\mathrm{AB}} = \frac{mg\vec{\boldsymbol{r}}_{\mathrm{CD}} \cdot \left(\vec{\boldsymbol{r}}_{\mathrm{DA}} \times \hat{\boldsymbol{k}} \right)}{\vec{\boldsymbol{r}}_{\mathrm{CD}} \cdot \left(\vec{\boldsymbol{r}}_{\mathrm{DA}} \times \hat{\boldsymbol{\lambda}}_{\mathrm{AB}} \right)}.$$

Again we have found one equation for one unknown, T_{AB}. All the quantities on the right can be evaluated give T_{AB}.

4.2 Equilibrium of one object

For particle statics we used that the forces acting on an object in equilibrium have no net push or pull; the forces add to zero. Now we will use that the forces have no tendency to cause rotation; the moments add to zero. These are not two in a long list of facts about equilibrium, but the whole story. As stated on the inside front cover and this chapter's introduction (page 186)

An object is in static equilibrium if and only if the force balance and moment balance equations hold.

$$\underbrace{\sum_{\substack{\text{All external}\\\text{forces}}} \vec{F}_i = \vec{0}}_{\text{force balance}} \qquad \text{and} \qquad \underbrace{\sum_{\substack{\text{All external}\\\text{torques}}} \vec{M}_{i/C} = \vec{0}}_{\text{moment balance}} \qquad \text{(Ic,IIc)}$$

The total force system acting on the object is then equivalent to a zero force and zero moment acting at C.

By supplementing the force-balance equation with moment balance we can determine more about the forces that act on an object.

Rigid-body statics

To start with one often thinks of the object of interest is one piece, for example a whole car, a wheel, a person, a limb, a chair or a derrick. We often think of such an object as rigid, meaning that the object's shape and size only change negligibly due to the forces of interest. Thus the phrase *rigid-body mechanics*. Actually, however, the equilibrium equations, force and moment balance, apply just as well things to all things with little acceleration, whether or not they are stiff and solid. For a first pass at the subject, one thinks of applying the principles of statics to single rather-solid simply-defined objects. And such will be our main initial concern in this section. But really the delineation of an 'object' is up to you. And in later chapters we apply the same statics equations to clearly-non-rigid systems like water and rope. For statics the only concern is the delineation of the system at the instant of interest.

Once you know its shape, whether an object is rigid or not is irrelevant for statics.

The reference point C in moment balance.

The moment balance equation is calculated by calculating the moments of forces relative to a point C using

$$\vec{M}_{\text{i}} = \vec{r}_{\text{i}/\text{C}} \times \vec{F}_i .$$

C is any convenient point, possibly the origin O of your coordinate system. C is not a special point. As discussed in Section 2.6 if a force system is equivalent to zero force and zero couple at C it is equivalent to a zero force and zero couple at any and every point D, E, Q, *etc.*

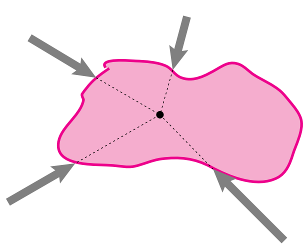
Example. As you sit still reading, gravity is pulling you down and forces from the floor on your feet, the chair on your seat, and the table on your elbows hold you up. All of these forces add to zero. The net moment of these forces about the front-left corner of your desk adds to zero. And the net moment of these forces about the mole near your left elbow is also zero.

The freedom to use any point you like for moment balance provides and oft-used shortcut.

Number of equations and number of unknowns

In two dimensions the equilibrium equations make up 3 independent scalar equations. These could be:

- 2 components of force balance and the one non-trivial component of moment balance ; or
- moment balance about any two points and force balance in any direction (except in the direction orthogonal to the line connecting the two moment-balance points).
- moment balance about 3 points (any three points not on a straight line suffice).



Filename:tfigure-particleequilib

Figure 4.18: A set of forces acting *concurrently* on an object. All force lines of action intersect at one point.

Note that moment balance necessarily is part of the equilibrium equations, but that force balance can be finessed. With one 2D free body diagram the equilibrium equations can be solved to find three unknown scalars, for example,

- The magnitudes of three forces whose directions are known *a priori*; or
- One unknown force vector (two components, or angle and magnitude) and one unknown magnitude; or
- Some other list of three scalars associated with the forces on the free body diagram. Besides force components and magnitudes these could include a force angle θ, a friction coefficient μ, or the location of force application.

Once you have three independent equations any additional equations you write, say moment about still another point, contains no new information①. In some problems the forces shown on a free body diagram automatically

① A fourth equilibrium equation may superficially look different from an equation already written, but it could be derived from the other equations.

satisfy one or more of the equilibrium equations; in making the drawing you may have implicitly solved some equilibrium equations. The equilibrium equations then offer less new information, and sometimes none at all (see 2-force bodies below).

In 3 dimensions the equilibrium equations make up 6 independent scalar equations. Most directly these are 3 components of force and 3 components of moment. But there are many combinations of equilibrium equations that yield 6 independent scalar equations.

Special cases: concurrent forces, two-force bodies, three-force bodies

We now discuss some special loading situations for which there are special insights or problem-solution tricks. In principle you don't need to know any of them because force balance and moment balance spell out the whole statics story. In practice it is best to know these special cases.

Concurrent forces

In the special case when the lines of actions of all applied forces intersect at one point, moment balance is trivially satisfied (because none of the forces has a moment about the intersection point). Such a system of forces is called *concurrent* (fig. 4.18) and the particle model is particularly appropriate[2]. In such a case the 2D equilibrium equations only provide two independent scalar equations and one can only use them to solve for two unknown scalars. In 3D one gets three independent scalar equations for a concurrent force system.

One-force body

Lets first treat "one-force" bodies. Consider a finite body with only one force acting on it. Assume it is in equilibrium. Force balance says that the sum of forces must be zero. So that force must be zero.

> If only one force is acting on a body in equilibrium that force is zero.

That was too easy. But a count to 3 wouldn't feel complete if it didn't start at 1.

Two-force body

When only two forces act on an object the situation is also simplified, though not so drastically as the case with one force. An object with only two forces acting on it is called a *two-force body* or *two-force member*.

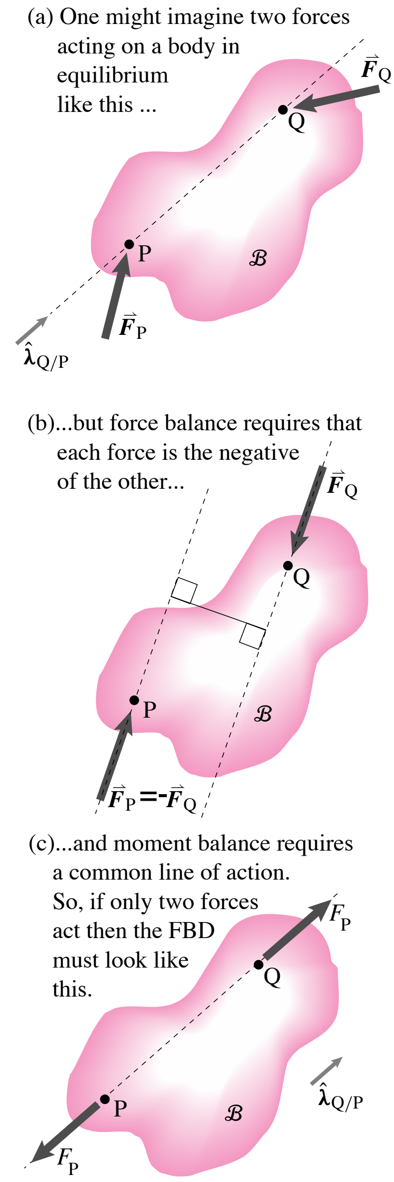


Figure 4.19: (a) Two forces acting on a body $\mathcal{B}$. (b) force balance implies that the forces are equal in magnitude and opposite in direction ($\vec{F}_P = -\vec{F}_Q$. (c) moment balance implies that the forces are colinear. Body $\mathcal{B}$ is a two-force member; the equilibrium equations imply that the two forces must be equal in magnitude, opposite in direction, and collinear. If the free body diagram shows equal-in-magnitude, opposite-in-direction and collinear forces then the equilibrium equations add no new information.

② If forces are not concurrent the particle model may still be useful, as demonstrated in the previous section.

> If a body in static equilibrium is acted on by two forces, then those forces are equal in magnitude, opposite in direction, and have a common line of action (the line connecting the two points of application).

This result is shown in fig. 4.19 and explained in box 4.3. If you recognize a two-force body you can draw it in a free body diagram as in fig. 4.19c and the equations of force and moment balance provide no new information. The two-force-body shortcut is especially useful for systems with several parts some of which are two-force members. Springs, dashpots, struts, and strings are generally idealized as two-force bodies.

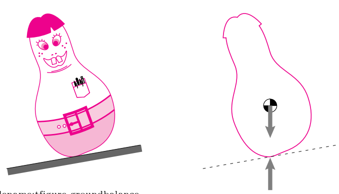


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Figure 4.20: **An object balancing on one point of support** is a *two-force body* so has to have its center-of-mass over the support point.

Example: Tower and strut
Consider an accelerating cart (fig. 4.22) holding up massive tower AB which is pinned at A and braced by the light strut BC. The rod BC qualifies as a two-force member. The rod AB does not because it has three forces and is also not in static equilibrium (non-negligible accelerating mass). Thus, the free body diagram of rod BC shows the two equal and opposite colinear forces at each end parallel to the rod and the tower AB does not.

Example: Logs as bearings
Consider the ancient egyptian dragging a big stone fig. 4.21. If the stone and ground are flat and rigid, and the log is round, rigid and much lighter than the stone we are led to the free body diagram of the log shown. *With these assumptions there can't be any resistance to rolling.* Note that this effectively frictionless rolling occurs no matter how big the friction coefficient between the contacting surfaces. That the egyptian got tired comes from logs not being perfectly round, the ground or stone not being perfectly flat, and, most importantly, the ground, log or stone not being perfectly rigid. In any case it takes effort to pick up the logs in the back and move them to the front.

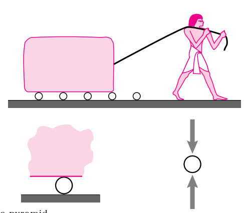

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Figure 4.21: **Dragging a heavy stone on rolling round logs.** If there is only point contact between the logs and the ground and stone, the each log is a *two-force body*. If the logs are also round there is no resistance to forward motion. The situation with ball bearings is identical.

Example: Plyers
The plyers of fig. 3.2a on page 151, when considered as a whole (with the pencil they are squeezing), are a two-force body. Thus $F_{HE} = F_{EH}$ and these two forces must act on a common line. Assuming the forces are large enough that gravity can be neglected, and the motions are slow enough that statics is accurate, the person has no choice but to apply the forces in this way.

Example: One point of support
If an object with weight is supported at just one point (fig. 4.20), that point must be directly above or below the center-of-mass. Why? The gravity forces are equivalent to a single force at the center-of-mass. The body is then a two force body. Since the direction of the gravity force is down, the support point and center-of-mass must be above one another.

Similarly,

> If a body is suspended from one point, the center of gravity must be directly above or below that point.

Three-force body

If a body in equilibrium has only three forces on it, the equilibrium equations again restrict the forces in a geometrically describable manner. The simplification is not as great as for two-force bodies but is remarkably useful for both calculation and intuition. In box 4.4 on page 205 moment balance about various axes is used to prove that

> If exactly three forces act on a body (2D and 3D) the body is in equilibrium only if
> 1. the three force vectors are coplanar,
> 2. and either
>   a) have lines of action which intersect at a single point (ie, they are concurrent), or
>   b) they are parallel.

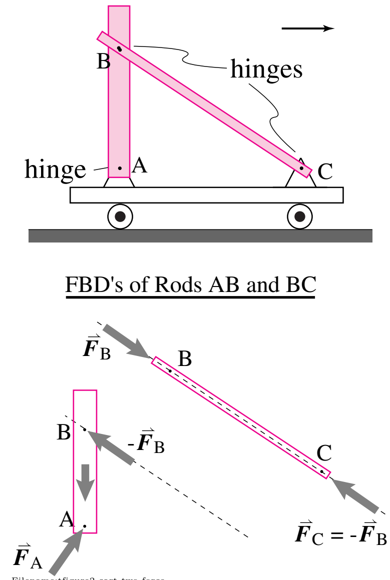

Figure 4.22: A cart is propelled by forces not showing. We are concerned about light strut BC and heavy tower AB. The strut is probably well-modeled as a *two-force body*.

One could imagine three random forces acting on a body. But, for equilibrium they must be coplanar and either concurrent or parallel. Unlike the case for 2-force bodies where the 2-force-body conditions imply the satisfaction of all equilibrium equations, for 3-force bodies planar concurrency still leaves two independent equilibrium equations possibly unsatisfied (for both 2D and 3D). That is, one still needs the equations of force balance in the plane (or, in the special case of three parallel forces, one scalar force balance and one moment balance equation).

Example: Hanging book box

A box with a book inside is hung by two strings so that it is in equilibrium on when level. The system is a *three-force body* so the lines of action of the two strings must intersect on the vertical line that goes through the center-of-mass of the box/book system.

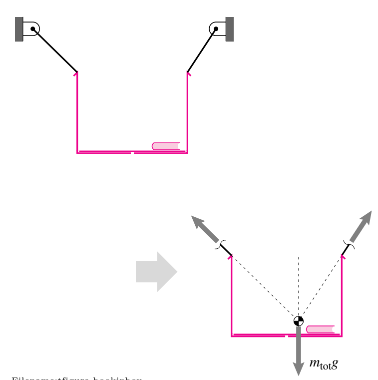

Figure 4.23: **A book is in a box** which hangs from two ropes: a *three-force body*.

Example: Which way do the forces go?

The maximum angle between pairs of forces in a 3-force body can be (a) greater than, (b) equal to, or (c) less than 180° (see figure below). In each case we can know something about the directions of the forces. Call the point of force concurrency D.

(a) Forces spread over more than 180°. Force balance perpendicular to the middle force implies that the outer two forces are both directed from D or both directed away from D. Force balance in the direction of the middle force shows that it has to have the opposite sense than the outer forces. If the others are pushing in then it is pulling away. If the outer forces are pulling away than it is pushing in.

(b) Forces spread exactly 180°. Force balance in the direction perpendicular to line ADC shows that the odd force must be zero. The other forces must obviously oppose each other.

(c) Forces spread over more than 180°. Force balance perpendicular to the force at C shows that the other two forces must both pull away towards D or both push in. Then force balance along C shows that all three forces must have the same sense. All three forces are pulling away from D or all three are pushing in.

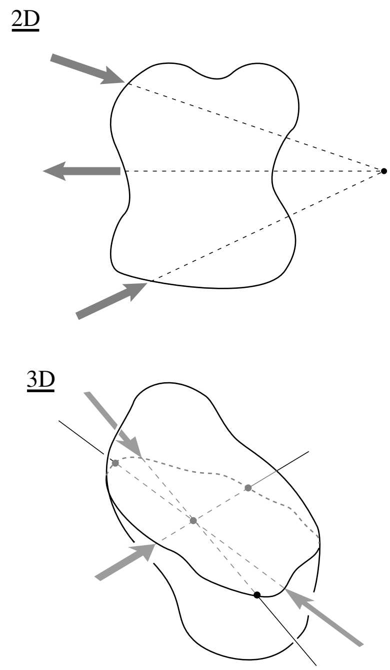

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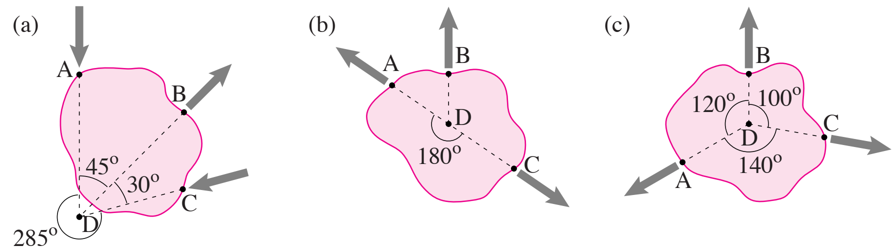
Figure 4.24: If exactly 3 forces act on a body the lines of action of the forces intersect at a single point and are coplanar. The point of intersection does not have to lie within the body. A special case is when that point is at infinity and the three forces are parallel.



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The idealized massless pulley

Both real machines and mechanical models are built of various building blocks. One of the standards is a pulley. We often draw pulleys schematically something like in fig. 4.25a which shows that we believe that the tension in a string, line, cable, or rope that goes around an ideal pulley is the same on both sides, $T_1 = T_2 = T$. An ideal pulley is

(i.) Round,

(ii.) Has frictionless bearings,

(iii.) Has negligible inertia, and

(iv.) Is wrapped with a line which only carries forces along its length.

We now show that these assumptions lead to the result that $T_1 = T_2 = T$. First, look at a free body diagram of the pulley with a little bit of string at both ends (fig. 4.25b). Since we assume the bearing has no friction, the interaction between the pulley bearing shaft and the pulley has no component tangent to the bearing.

4.3 Two-force bodies

Here we derive the ubiquitously-used result that if only two forces act on a body the two forces must be equal in magnitude, opposite in direction, and on a common line of action. You can (and will) use this result even if you do not master the reasoning in this box. But learning this reasoning may help your intuition.

Consider the free body diagram of a body $\mathcal{B}$ in fig. 4.19a. Forces $\vec{F}_P$ and $\vec{F}_Q$ are acting on $\mathcal{B}$ at points P and Q. Let's apply the equilibrium equations. First, we have that the sum of all forces on the body are zero,

$$\sum_{\substack{\text{All external}\\ \text{forces}}} \vec{F} = \vec{0}$$

$$\vec{F}_P + \vec{F}_Q = \vec{0} \quad \Rightarrow \quad \vec{F}_P = -\vec{F}_Q.$$

Thus, the two forces must be equal in magnitude and opposite in direction. So, thus far, we can conclude that the forces must be parallel as shown in fig. 4.19b. But the forces still seem to have a net turning effect, thus still violating the concept of static equilibrium. The sum of all external torques on the body about any point are zero. So, summing moments about point P, we get,

$$\sum_{\substack{\text{All external}\\ \text{torques}}} \vec{M}_{/P} = \vec{0}$$

$$\vec{r}_{Q/P} \times \vec{F}_Q = \vec{0} \qquad (\vec{F}_P \text{ produces no torque about } P)$$

$$|\vec{r}_{Q/P}| \left(\hat{\lambda}_{Q/P} \times \vec{F}_Q\right) = \vec{0} \qquad \left(\hat{\lambda}_{Q/P} = \frac{\vec{r}_{Q/P}}{|\vec{r}_{Q/P}|} = -\frac{\vec{r}_{P/Q}}{|\vec{r}_{P/Q}|}\right)$$

So $\vec{F}_Q$ has to be parallel to the line connecting P and Q. Similarly, taking the sum of moments about point Q, we get

$$-\hat{\lambda}_{Q/P} \times \vec{F}_Q = \vec{0}$$

and $\vec{F}_P$ also must be parallel to the line connecting P and Q. So, not only are $\vec{F}_P$ and $\vec{F}_Q$ equal and opposite, they are collinear as well since they are parallel to the axis passing through their points of action (see fig. 4.19c).

To find the relation between tensions, we apply angular momentum balance (equation II) about point O

$$\left\{\sum \vec{M}_{\rm O} = \dot{\vec{H}}_{\rm O}\right\} \cdot \hat{k}. \tag{4.9}$$

Evaluating the left hand side of eqn. 4.9

$$\begin{aligned}\sum \vec{M}_O \cdot \hat{k} &= R_2 T_2 - R_1 T_1 + \underbrace{\text{bearing friction}}_{0} \\ &= R(T_2 - T_1), \text{ since } R_1 = R_2 = R.\end{aligned}$$

Because there is no friction, the bearing forces acting perpendicular to the round bearing shaft have no moment about point O (see also the short example on page 113). Because the pulley is round, $R_1 = R_2 = R$.

When mass is negligible, dynamics reduces to statics.

Putting these assumptions and results together gives

$$\begin{aligned}&\left\{\sum \vec{M}_{\rm O} = \vec{0}\right\} \cdot \hat{k} \\ \Rightarrow\ & R(T_2 - T_1) = 0 \\ &\Rightarrow T_1 = T_2\end{aligned}$$

Thus, the tensions on the two lines of an ideal massless pulley are equal.

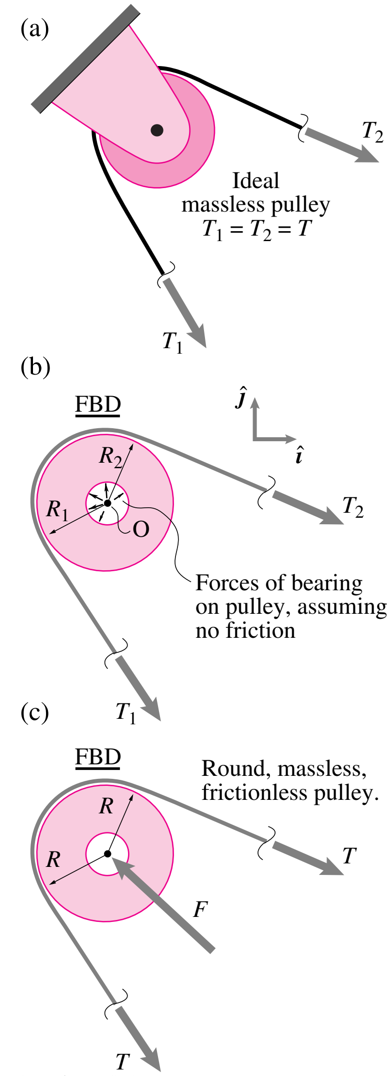

Figure 4.25: (a) An ideal massless pulley, (b) FBD of idealized massless pulley, detailing the frictionless bearing forces and showing forces at the cut strings, (c) final FBD after analysis.

Lopsided pulleys are not often encountered, so it is usually satisfactory to assume round pulleys. But, in engineering practice, the assumption of frictionless bearings is often suspect. In dynamics, you may not want to neglect pulley mass.

Lack of equilibrium as a sign of dynamics

Surprisingly, statics calculations often give useful information about dynamics. If, in a given problem, you find that forces or moments cannot be balanced this is a sign that the related physical system will accelerate in the direction of imbalance (See the example 'block on ramp' on page 212). For more about non-existance of a statics solution, see box 4.1 on page 190.

4.4 Three-force bodies

Here is a brief derivation of the result for three force bodies. The derivation is not needed for problem solving. However understanding the derivation may help build intuition.

Consider a body in static equilibrium with just three forces on it; $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ acting at $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$. Taking moment balance about the axis through points at $\vec{r}_2$ and $\vec{r}_3$ implies that the line of action of $\vec{F}_1$ must pass through that axis. Similarly, for equilibrium to hold, the line of action of $\vec{F}_2$ must intersect the axis through points at $\vec{r}_1$ and $\vec{r}_3$ and the line of action of $\vec{F}_3$ must intersect the axis through $\vec{r}_1$ and $\vec{r}_2$. So, the lines of action of all three forces are in the plane defined by the three points of action and the lines of action of $\vec{F}_2$ and $\vec{F}_3$ must intersect. Taking moment balance about this point of intersection implies that $\vec{F}_1$ has line of action passing through the same point. A special case is when $\vec{F}_1$, $\vec{F}_2$, and $\vec{F}_3$ are parallel and have a common plane of action (equivalent to the concurrency point being at infinity).

Linearity and superposition

For a given geometry the equilibrium equations are *linear*: If you know a set of forces that is in equilibrium and you also know a second set of forces that is in equilibrium, then the sum of the two sets is also in equilibrium.

Example: A bicycle wheel

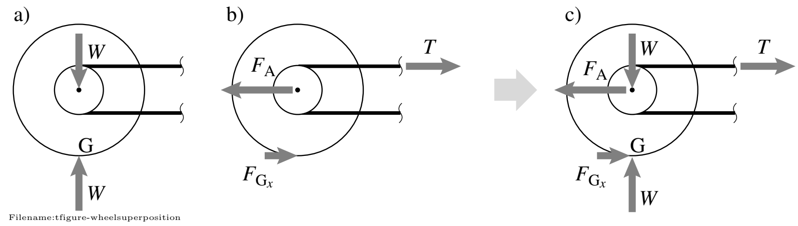

Figure 4.26: A bicycle wheel.

The free body diagram of an ideal massless bicycle wheel with a vertical load is shown in (a) above. The same wheel driven by a chain tension but with no weight is shown in equilibrium in (b) above. The sum of these two load sets (c) is therefor in equilibrium.

That you can add solutions of linear equations is called the *principle of superposition*, also called the principle of superimposition③. The principle of superposition provides a useful shortcut for some mechanics problems.

③**Superimposition.** Here's a bad pun to help your memory. When talkative Sam comes over you get bored. When hungry Sally comes over you reluctantly go get a snack for her. When Sam and Sally come over together you get bored *and* reluctantly go get a snack. Each one of them is imposing. When they come over together their effects add. They are super imposing.

4.5 Moment balance about 3 points is sufficient in 2D

This is a theoretical aside showing that moment balance can totally replace force balance.

In 2D one can solve any statically determinate problem using moment balance about any 3 non-colinear points. Force balance adds no information.

Here we show the math behind this useful trick. The derivation here is only for logical completeness, it does not help with problem solving.

Consider two points A and B. Moment balance about these two points gives

$$\sum \vec{r}_{i/A} \times \vec{F}_i = \vec{0} \quad \text{and} \quad \sum \vec{r}_{i/B} \times \vec{F}_i = \vec{0}.$$

Subtracting one of these equations from the other gives:

$$\begin{aligned}
\sum \vec{r}_{i/A} \times \vec{F}_i - \sum \vec{r}_{i/B} \times \vec{F}_i &= \vec{0} \\
\sum (\vec{r}_{i/A} - \vec{r}_{i/B}) \times \vec{F}_i &= \vec{0} \\
\sum ((\vec{r}_i - \vec{r}_A) - (\vec{r}_i - \vec{r}_B)) \times \vec{F}_i &= \vec{0} \\
\sum (\vec{r}_B - \vec{r}_A) \times \vec{F}_i &= \vec{0} \\
(\vec{r}_B - \vec{r}_A) \times \sum \vec{F}_i &= \vec{0}
\end{aligned}$$

Dotting both sides with a vector $\hat{k}$ normal to the plane we get (recalling the mixed triple product identity from page 71 in Section 2.3 that $(\vec{A} \times \vec{B}) \cdot \vec{C} = (\vec{C} \times \vec{A}) \cdot \vec{B}$) we can re-arrange terms to get

$$\begin{aligned}
\left((\vec{r}_B - \vec{r}_A) \times \sum \vec{F}_i\right) \cdot \hat{k} &= \vec{0} \cdot \hat{k} \\
\left(\hat{k} \times (\vec{r}_B - \vec{r}_A)\right) \cdot \sum \vec{F}_i &= 0.
\end{aligned}$$

Thus moment balance about the points A and B implies force balance in the direction $\hat{k} \times (\vec{r}_B - \vec{r}_A)$. This is force balance in the direction normal to the line AB (and in the plane).

Now consider a third point C. By the same reasoning moment balance about B and C implies force balance in the direction orthogonal to BC. So long as BC is not parallel to AB then we have force balance in two independent directions. So

$$\sum \vec{F}_i = \vec{0}.$$

The result only goes sour if the two directions are parallel, which occurs when two of the points A, B, and C are on a line. If A, B, and C are not on a line, moment balance about them implies force balance. So use of moment balance replaces the force balance equilibrium equations.

Moment balance about convenient points A, B, and C can simplify the equilibrium equations if the points are picked so that, by inspection, some forces have no moment.

SAMPLE 4.5 Find force F for equilibrium of the angle shown in the figure. The dimensions of the angle are $d = 0.3\,\text{m}$ and $a = 0.2\,\text{m}$.

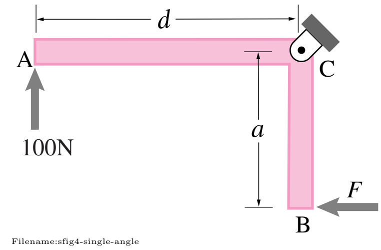

Figure 4.27:

Solution The free-body diagram of the angle is shown in Fig. fig. 4.28. Since we are interested in force F, we can write the scalar moment balance equation (in $\hat{\boldsymbol{k}}$ direction) about point C (and thus get rid of the other unknown force $\vec{\boldsymbol{R}}$):

$$\begin{aligned} -Fa - (100\,\text{N})d &= 0 \\ \Rightarrow \quad F &= -(100\,\text{N})\frac{d}{a} \\ &= -100\,\text{N} \cdot \frac{0.3\,\text{m}}{0.2\,\text{m}} \\ &= -150\,\text{N}. \end{aligned}$$

$\vec{\boldsymbol{F}} = -(150\,\text{N})\hat{\boldsymbol{\imath}}$

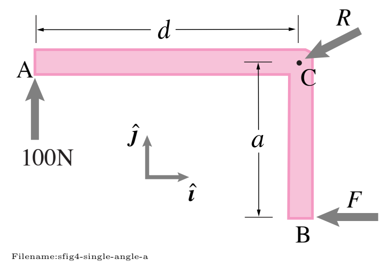

Figure 4.28:

SAMPLE 4.6 Consider the angle shown in the figure with the applied forces. Can the angle be in equilibrium for some value of F? Explain.

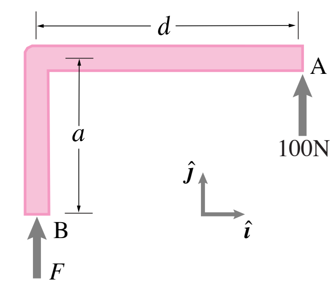

Figure 4.29:

Solution Let us assume that the angle is in equilibrium. Then the forces acting on the angle must satisfy the force and moment balance equations. Now the force balance in the $\hat{\boldsymbol{\jmath}}$ direction gives

$$\begin{aligned} F + (100\,\text{N}) &= 0 \\ \Rightarrow \quad F &= -100\,\text{N}. \end{aligned}$$

The moment balance about point A gives

$$\begin{aligned} Fd &= 0 \\ \Rightarrow \quad F &= 0 \end{aligned}$$

Thus,

$$-100\,\text{N} = 0$$

which is a contradiction. Thus the angle cannot be in equilibrium with the applied forces.

It is easy to see that no matter which way F acts (up or down), it cannot simultaneously balance the applied force at A and its moment. If $F = 100\,\text{N}$ acts upwards at B, the angle will accelerate up because it has a net force in the $\hat{\boldsymbol{\jmath}}$ direction. If $F = 100\,\text{N}$ acts downwards at B, the two equal and opposite forces at A and B produce a net moment on the angle and therefore the angle will start spinning about the $\hat{\boldsymbol{k}}$ direction. In fact, no matter what the value or direction of F is, as long as it acts at point B, the angle cannot be in equilibrium. This is because the angle, as given, is a two force body, and for equilibrium, the two applied forces must be equal, opposite and colinear.

Equilibrium not possible.

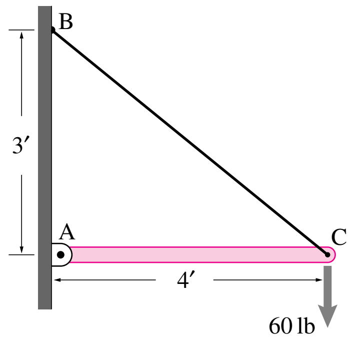

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Figure 4.30:

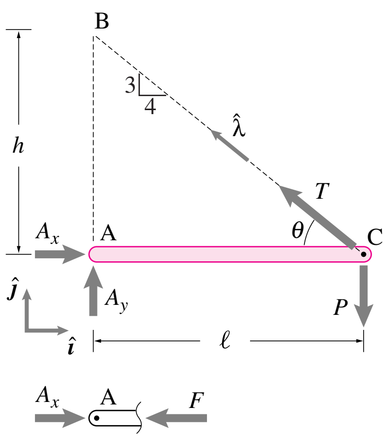

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Figure 4.31:

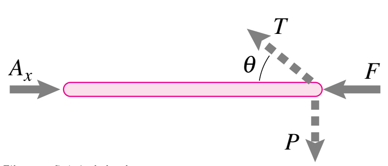

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Figure 4.32:

SAMPLE 4.7 A bar as a 2-force body: A 4 ft long horizontal bar AC supports a load of 60 lbf at one end and is pinned to a wall at the other end. The bar is also supported by a string BC as shown in the figure. Find the forces applied by the pin and the string on the bar.

Solution Let us do this problem two ways — using equilibrium equations without much thought, and using those equations with some insight.

The free-body diagram of the bar is shown in Fig. 4.31. The moment balance about point A, $\sum \vec{M}_A = 0$, gives

$$
\begin{aligned}
\vec{r}_{C/A} \times T\hat{\boldsymbol{\lambda}} + \vec{r}_{C/A} \times (-P\,\hat{\jmath}) &= \vec{0} \\
\underbrace{\ell\hat{\imath} \times T(-\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})}_{\ell T \sin\theta \hat{k}} + \underbrace{\ell\hat{\imath} \times (-P\,\hat{\jmath})}_{-\ell P \hat{k}} &= \vec{0} \\
(T\ell\sin\theta - P\ell)\hat{k} &= \vec{0} \qquad (4.10)\\
\Rightarrow \qquad T = \frac{P}{\sin\theta} = \frac{60\,\text{lbf}}{3/5} &= 100\,\text{lbf}.
\end{aligned}
$$

The force equilibrium, $\sum \vec{F} = 0$, gives

$$(A_x - T\cos\theta)\hat{\imath} + (A_y + T\sin\theta - P)\hat{\jmath} = \vec{0} \qquad (4.11)$$

Separating out x and y components of this equation, we get

$$
\begin{aligned}
A_x = T\cos\theta = (100\,\text{lbf})\cdot\frac{4}{5} &= 80\,\text{lbf} \\
A_y = P - T\sin\theta &= 0
\end{aligned}
$$

where the last equation, $A_y = P - T\sin\theta = 0$ follows from eqn. (4.10). Thus, the force in the rod is $\vec{A} = (80\,\text{lbf})\hat{\imath}$, *i.e.,* a purely compressive force, and the tension in the string is 100 lbf.

$$\boxed{\vec{A} = (80\,\text{lbf})\hat{\imath}, \quad T = 100\,\text{lbf}}$$

Alternate Solution: From the free-body diagram of the rod (see fig. 4.32), we realize that the rod is a *two-force body*, since the forces act at only two points of the body, A and C. The reaction force at A is a single force $\vec{A}$, and the forces at end C, the tension $\vec{T}$ and the load $\vec{P}$, sum up to a single net force, say $\vec{F}$. So, now using the fact that the rod is a two-force body, the equilibrium equation requires that $\vec{F}$ and $\vec{A}$ be equal, opposite, and colinear (along the longitudinal axis of the bar). Thus,

$$\vec{A} = -\vec{F} = -F\hat{\imath}.$$

Now,

$$
\begin{aligned}
\vec{F} &= \vec{P} + \vec{T} \\
-F\hat{\imath} &= -P\,\hat{\jmath} + T\sin\theta\,\hat{\jmath} - T\cos\theta\,\hat{\imath} \qquad (4.12)
\end{aligned}
$$

Separating out x and y components of this equation, we get

$$
\begin{aligned}
-F + T\cos\theta &= 0 \qquad (4.13)\\
P - T\sin\theta &= 0. \qquad (4.14)
\end{aligned}
$$

Solving these two equations simultaneously, we get $T = P/\sin\theta = 100\,\text{lbf}$ and $F = T\cos\theta = 80\,\text{lbf}$. The answers, of course, are the same.

SAMPLE 4.8 A bottle holder: A clever design of a bottle holder (a plank with a hole) is shown in the figure. Note that the holder is not fixed to the support; it stands freely, but only when the bottle is in. Assume that the mass of the bottle is 1 kg and that the center-of-mass of the bottle is at 3/5th of its length ($h = 35\,\text{cm}$) from the neck support point. The bottle in its rest position is slightly tipped down ($\alpha = 15^{\circ}$). Assuming the mass of the stand to be negligible and $\ell = 30\,\text{cm}$, find the angle θ of the stand so that the bottle and the stand can stand together as shown.

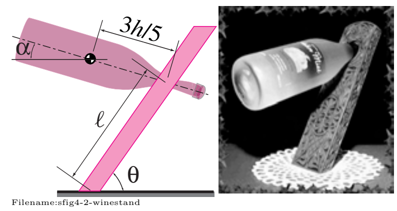

Figure 4.33:

Solution Let us draw the free-body diagram of the bottle and the stand together as one system. The forces acting are shown in fig. 4.34. Since the only forces acting on the system are R and mg, they must be equal, opposite and colinear. Thus the line of action of the weight, mg, must pass through the center of the stand's footprint. From the given geometry, then, we must have,

$$
\begin{aligned}
\ell\cos\theta &= \frac{3h}{5}\cos\alpha \\
\Rightarrow \quad \theta &= \cos^{-1}\left(\frac{3h}{5\ell}\cos\alpha\right) \\
&= \cos^{-1}\left(\frac{3\cdot 35\,\text{cm}}{5\cdot 30\,\text{cm}}\cos 15^{\circ}\right) \\
&= 47.5^{\circ}.
\end{aligned}
$$

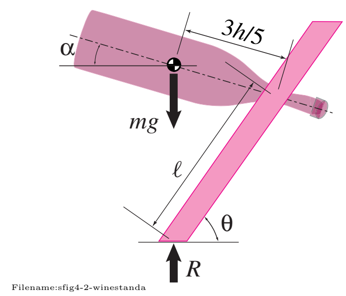

Figure 4.34:

$\boxed{\theta = 47.5^{\circ}}$

Note: The latitude in design of the angle θ depends on the width of the base of the stand. The two forces acting on the system must be colinear and must pass through the base. Therefore, a wider base (perhaps at the expense of elegance) provides more freedom for the forces to move sideways, giving a range of θ and α for design. (see fig. 4.35.)

Figure 4.35: A bottle holder with a wider base.

SAMPLE 4.9 Reactions at fixed ends. For the bent bar shown in the figure, find the reaction forces at the fixed end for $F = 10\,\text{kN}$.

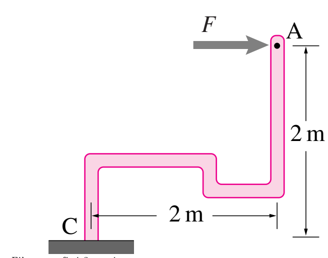

Figure 4.36:

Solution The free-body diagram of the rod is shown in fig. 4.37. Note that in addition to the reaction force $\vec{R}$, there is a reaction moment $\vec{M} = M\hat{k}$ acting on the rod because of the fixed support.

The force balance equation, $\sum\vec{F} = \vec{0}$, gives us

$$
\begin{aligned}
F\hat{\imath} + \vec{R} &= \vec{0} \\
\Rightarrow \quad \vec{R} &= -F\hat{\imath} = -(10\,\text{kN})\hat{\imath}.
\end{aligned}
$$

Now, we can write the moment balance equation about point C, $\sum\vec{M}_C = \vec{0}$, to give

$$
\begin{aligned}
M\hat{k} - Fd\hat{k} &= \vec{0} \\
\Rightarrow \quad M &= Fd = (20\,\text{kN}\cdot\text{m}).
\end{aligned}
$$

$\boxed{\vec{R} - (2\,\text{kN})\hat{\imath},\ \vec{M} = (20\,\text{kN}\cdot\text{m})\hat{k}}$

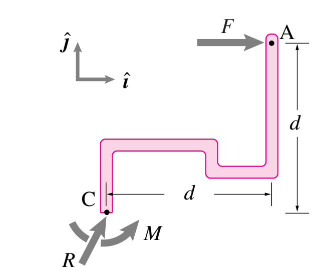

Figure 4.37:

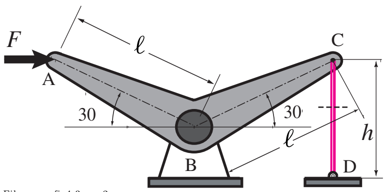

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Figure 4.38:

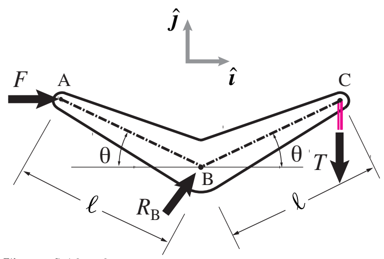

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Figure 4.39: Free body diagram of the structure with rod CD cut just below C.

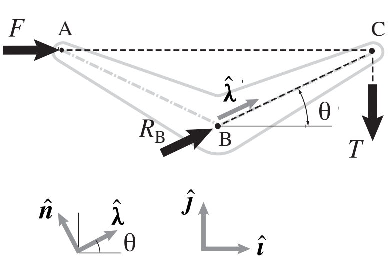

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Figure 4.40: The rocker arm is a three force body and, therefore, the three forces have to be concurrent (their lines of action intersect at one point, C). Thus the support reaction $\vec{\boldsymbol{R}}_B$ acts in the $\hat{\boldsymbol{\lambda}}$ direction. From geometry, $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$. A unit vector normal to $\hat{\boldsymbol{\lambda}}$ is, therefore, $\hat{\boldsymbol{n}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$ (you can guess it from geometry or find it from $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}$).

SAMPLE 4.10 Consider the structure (a rocker arm) shown in the figure. Assume that bar CD can only take axial load (tension or compression). If a horizontal force, $F = 2\,\text{kN}$ is applied at point A, what is the tension in rod CD?

Solution Let T be the tension in the rod (although intuitively you can see that the rod must be under compression). Then, the free-body diagram of the rocker arm ABC is as shown in fig. 4.39. We need to find T. We can do so using either moment balance or force balance as shown below.

Method-1: Using moment balance The easiest way to solve this problem is to apply moment balance, $\sum \vec{\boldsymbol{M}}_{\text{B}} = \vec{\boldsymbol{0}}$, about point B. Taking moments about this point gets rid of the unknown reaction force R_B and relates T to F directly:

$$\vec{\boldsymbol{r}}_{\text{A/B}} \times \vec{\boldsymbol{F}} + \vec{\boldsymbol{r}}_{\text{C/B}} \times \vec{\boldsymbol{T}} = \vec{\boldsymbol{0}}$$

We can evaluate the cross products vectorially or use the scalar form of the moment calculation (force times the lever arm) to give

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{A/B}} \times \vec{\boldsymbol{F}} &= -F\ell\sin\theta\hat{\boldsymbol{k}} \\
\vec{\boldsymbol{r}}_{\text{C/B}} \times \vec{\boldsymbol{T}} &= -T\ell\cos\theta\hat{\boldsymbol{k}}.
\end{aligned}$$

So, the scalar moment balance equation in the $\hat{\boldsymbol{k}}$ direction is

$$\begin{aligned}
-F\ell\sin\theta - T\ell\cos\theta &= 0 \\
\Rightarrow \quad T &= -F\tan\theta.
\end{aligned}$$

Now substituting the given values, $F = 2\,\text{kN}$, and $\theta = 30^{\circ}$, we get

$$T = -(2\,\text{kN}) \cdot (\tan 30^{\circ}) = -1.15\,\text{kN}.$$

Thus the rod is under compression, not tension. It is also clear from the picture that if we push at A, ABC will try to rotate clockwise about B, thus pushing down on the rod at C.

$$\boxed{T = -1.15\,\text{kN}}$$

Method-2: Using force balance We can also use the force balance equation, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ to find T. However, force balance will involve two unknown forces T and R. The force balance gives

$$\begin{aligned}
\vec{\boldsymbol{F}} + \vec{\boldsymbol{T}} + \vec{\boldsymbol{R}}_B &= 0 \\
\text{or} \qquad F\hat{\boldsymbol{\imath}} - T\hat{\boldsymbol{\jmath}} + R_B\hat{\boldsymbol{\lambda}} &= 0 \qquad (4.15)
\end{aligned}$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector in the direction of $\vec{\boldsymbol{R}}_B$ and is not known yet. However, we know that the rocker arm is a three force body, and therefore, all the three forces must be concurrent (they cannot be parallel here). From geometry it is clear that the lines of action of all the three forces must pass through point C. This realization immediately gives us the direction of $\vec{\boldsymbol{R}}_B$, that is, $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$. So, now we can write out eqn. (4.15), separate out x and y components and solve the two scalar equations simultaneously to find both T and R_B. But we are not interested in finding R_B. So why not get use an appropriate dot product with eqn. (4.15) to get rid of R_B and get one scalar equation relating T to F. Let $\hat{\boldsymbol{n}}$ be normal to $\hat{\boldsymbol{\lambda}}$. Thus, $\hat{\boldsymbol{n}} = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$. Now, with $\hat{\boldsymbol{n}}$ gives

$$\begin{aligned}
[\text{eqn. } (4.15)] \cdot \hat{\boldsymbol{n}} \qquad &\Rightarrow \qquad F\underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}}}_{-\sin\theta} - T\underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{n}}}_{\cos\theta} + R_B\underbrace{\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{n}}}_{0} = 0 \\
&\Rightarrow \qquad -F\sin\theta - T\cos\theta = 0 \\
&\Rightarrow \qquad T = -F\tan\theta = -1.15\,\text{kN}
\end{aligned}$$

as obtained by moment balance.

4.3 Equilibrium with frictional contact

Contacting objects are prevented from passing through each other by pressing against each other. Generally there is also some frictional resistance to relative slip. We have neglected friction so far for simplicity and because the neglect of friction is a reasonable approximation for some lubricated contact problems. On the other extreme, in some situations we have assumed that friction so well resists slip that we assumed 'no slip' and that frictional contact acts like a hinge or weld. Either way, with friction negligibly small, or reliably large, we have not worried about it.

However, for some purposes friction forces are not reasonably neglected during slip. Or, when there is no slip, sometimes we have to worry about whether the frictional bond is strong enough to prevent slip.

Although slip means motion and motion sounds like dynamics (contradicting the premise of statics), there are many situations where there is enough motion for friction to be important but not so much acceleration that inertial terms ($m\vec{a}$) are important.

How friction forces are represented on free body diagrams was discussed in Section 3.1 which you should review before proceeding further here. We will now consider friction forces in equilibrium conditions.

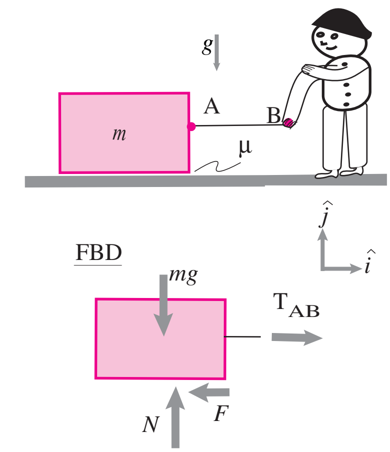


Figure 4.41: Pulling a block on a frictional ground, with related FBD. The complicated distribution of normal forces and friction forces on the block from the ramp have been replaced with the equivalent pair N and F. For particle mechanics we don't worry about where N and F are applied.

For simplicity, and because of the relatively high accuracy to complexity ratio, we consider only Coulomb friction with a single coefficient of friction μ①.

①If you have studied friction before, our approximation is that $\mu = \mu_{static} = \mu_{dynamic}$ adequately captures the complex and hard to quantify reality of frictional forces

Example: Drag a block with friction.
Consider the block with friction on a slope (fig. 4.41). You want to pull it slowly to the right with rod AB. Say $m = 100\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, and $\mu = 0.3$.
Force balance, using the forces on the free body diagram gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{AB}\hat{\imath} + N\hat{\jmath} - F\hat{\imath} = \vec{0} \tag{4.16}$$

This, with the friction relation $F = \mu N$, is 3 scalar equations in T_{AB}, F and N with solution $N = mg = 1000\,\mathrm{N}$, $F = \mu mg = 300\,\mathrm{N}$, and $T_{AB} = \mu mg = 300\,\mathrm{N}$.

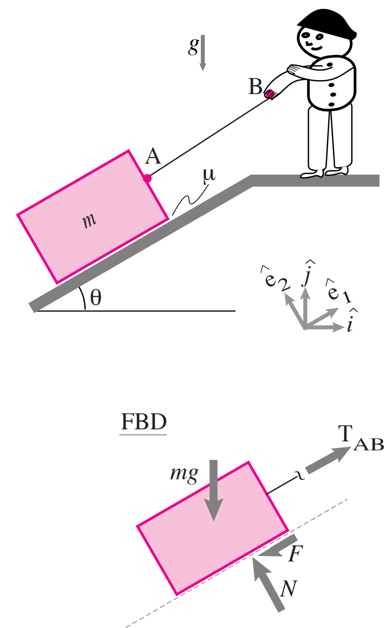
Example: Drag a block on a ramp with friction.
Consider the block with friction on a slope (fig. 4.42). You want to hold it with rod AB. Maybe you want to (i) slide it up slowly, or (ii) down slowly or (iii) hold it still. Say $m = 100\,\mathrm{kg}$, $g = 10\,\mathrm{m/s^2}$, $\theta = 45^\circ$, and $\mu = 0.3$.
Force balance, using the forces on the free body diagram, gives:

$$\sum \vec{F}_i = \vec{0} \quad \Rightarrow \quad -mg\hat{\jmath} + T_{AB}\hat{e}_1 + N\hat{e}_2 - F\hat{e}_1 = \vec{0} \tag{4.17}$$

This, with the friction relation, is 3 scalar equations in T_{AB}, F and N.

Summing forces in the rope direction and normal to the plane we get:

$$\begin{aligned} \{(\text{Eqn. } 4.17)\}\cdot\hat{e}_1 \quad &\Rightarrow \quad -mg\sin\theta + T_{AB} - F = 0 \\ \{(\text{Eqn. } 4.17)\}\cdot\hat{e}_2 \quad &\Rightarrow \quad mg\cos\theta + N = 0 \end{aligned} \tag{4.18}$$



Filename:tfigure-particle-blockslope

Figure 4.42: Pulling a block up a frictional slope with related FBD. As in the previous example, the pair N and F represent the net normal and frictional force.

② **Caution:** A common mistake amongst beginners is to assume the equation $F = \mu N$ applies when there is friction. Rather, if the friction is preventing slip F could be anything so long as $|F| \leq \mu N$. And if the slip is opposite in direction from that implicitly assumed in the free body diagram then $F = -\mu N$ (see case (ii) in the example above).

③ We could be tricky and get a single equation for the scalar T_{AB} by dotting both sides of eqn. (4.17) with a vector orthogonal to the resultant of $N\hat{e}_2 - F\hat{e}_1$. For the case of uphill sliding such a vector would be $\hat{e}_1 + \mu\hat{e}_2$.

or, for the quantities given $N = (100\,\text{kg})(10\,\text{m/s}^2)(\cos 45^\circ) = 1414\,\text{N}$. We assume that F and N are related by friction described with the standard Coulomb's friction model②:

i) $F = \mu N$ if the block is sliding up;
ii) $F = -\mu N$ if the block is sliding down; or
iii) $-\mu N \leq F \leq \mu N$ if the block is not sliding.

Solving eqn. (4.17) with the friction relations gives③.

i) $T_{AB} = mg\,(\mu\cos\theta + \sin\theta)$ if the block is sliding up;
ii) $T_{AB} = mg\,(-\mu\cos\theta + \sin\theta)$ if the block is sliding down;
Note that if $\tan\theta < \mu$ then $T_{AB} < 0$ and it then takes a push to slide down; or
iii) $mg\,(-\mu\cos\theta + \sin\theta) \leq T_{AB} \leq mg\,(\mu\cos\theta + \sin\theta)$ if the block is not sliding.
If $\tan\theta < \mu$ then $T_{AB} = 0$ is amongst the solutions for, so no sliding and the block can sit still on the slope with no pull on the rope.

Note that the tension T_{AB} scales with mg. So doubling m or g doubles all the forces in all of these answers, as you might guess from dimensional considerations. The mathematically-abstract-sounding issues of existence and uniqueness often show up in friction problems. For example, sometimes there is no statics solution (non-existence).

Example: Block on ramp.
A statics problem without a solution. A block with coefficient of friction $\mu = .5$ is in static equilibrium sliding steadily down a 45° ramp (fig. 4.43). *Not!* If there is constant velocity motion then statics would apply. But the forces in the free body diagram cannot add to zero (since the resulant of the friction and normal force is tipped up and to the left and thus cannnot be parallel to the vertical gravity force). The assumptions are not consistent with statics (actually this is a dynamics problem, the block accelerates down the ramp). If you saw a block just sitting there on a ramp, then you can be sure that the slope and friction coefficient are not those given above.

Friction problems might be studied with a particle model, as above, or also with moment balance.

Example: Dragged block as an extended body.
This is a repeat of the first example on page 211. One might wonder if the dragging causes an uneven distribution of force up on the block. Does the block dragging back, for example, cause a bigger pressure on the back? As a simple model assume all the ground force is at the front and back edge of the block. Force balance gives basically the same information as for the particle model, namely that:

$$N_C + N_D = W \quad \text{and} \quad \underbrace{F_C}_{\mu N_C} + \underbrace{F_D}_{\mu N_D} = T_{AB} \quad \Rightarrow \quad T_{AB} = \mu W.$$

One can find more with moment balance about any point you like, say C, with force balance gives

$$\sum M_{/C} = 0 \quad \Rightarrow \quad N_D = \frac{W}{2} + \frac{\mu h W}{2\ell} \quad \text{and} \quad N_C = \frac{W}{2} - \frac{\mu h W}{2\ell}$$

So there is more pressure on the front than back. This difference goes away if the either the friction or the height of the string attachment vanish.

Conditional contact, consistency, and contradictions

There is a natural hope that a subject will reduce to the solution of some well defined equations. For better and worse, things are not always this simple. For better because it means that the recipes are still not so well defined that computers can easily steal the subject of mechanics from people. For worse because it means you have to think hard to do some mechanics problems.

One source of these difficulties is the conditional nature of the equations that govern contact. For example:

- The ground pushes up on something to prevent interpenetration *if* the pushing is positive, *otherwise* the ground does not push up.
- The force of friction opposes motion and has magnitude μN *if* there is slip, *otherwise* the force of friction is something less than μN in magnitude.
- The distance between two points is kept from increasing by the tension in the string between them *if* the tension is positive, *otherwise* the tension is zero.

These conditions are, implicitly or explicitly, in the equations that govern these interactions. One does not always know which of the alternative contact conditions, if either, apply when one starts a problem. Sometimes multiple possibilities need to be checked④.

On a FBD at every point of frictional contact

- *If* the direction of slip or impending slip is known, *either*
  - Draw a normal force N and a friction force $F = \mu N$ opposing the relative slip, *or*
  - Draw a single force R at an angle ϕ from the normal of the contact in the direction which resists slip (with $\tan\phi = \mu$)
- *If* there is no slip, *either*
  - Draw a normal force N and tangential force F *or*
  - Draw a single force vector $\vec{\boldsymbol{R}}$ with unknown components
- *If* you don't know whether of not there is slip, *first*
  - Guess that there is no slip *then*
  - Solve the equilibrium equations, *then*
    - ∗ *If* $F \leq \mu N$: you guessed right and have found a solution to both the equilibrium and friction equations.
    - ∗ *If* $F > \mu N$: you guessed wrong and have to guess that there is slip in one direction (guess which), *then*
      - · see if you can solve the equilibrium equations, if not *then*
      - · assume slip in the opposite direction and try to solve the equilibrium equations, if you can't *then*
      - · the problem has no solution

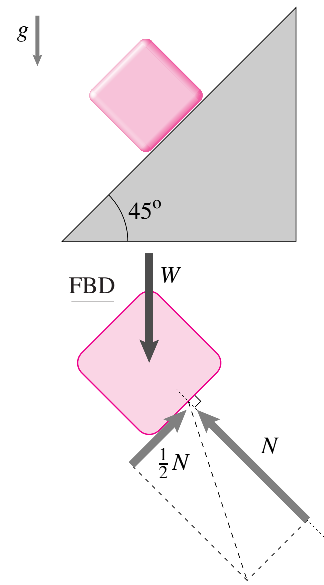


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Figure 4.43: Block on steep ramp and related FBD.

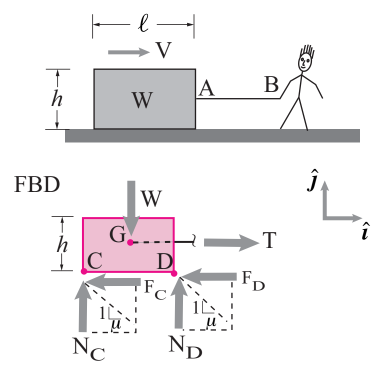

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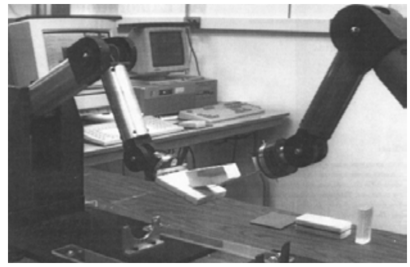
Figure 4.44: Dragging a block, taking account (in a simple way) the distribution of contact forces from the ground. Assume slip to the right is occuring.

④ All of the combinatorics in this recipe follows from the inability of our mathematics to deal easily with relations between variables with the step shape of Coulomb friction (fig. 3.29 on page 172).

Example: Robot hand



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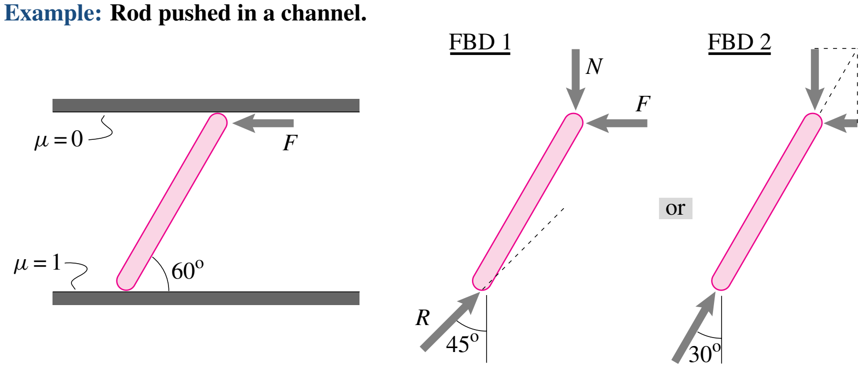
Roboticist Michael Erdmann has designed a palm manipulator that manipulates objects without squeezing them. The flat robot palms just move around and the object consequently slides. Determining whether the object slides on one the other or possibly on both hands in a given movement is a matter of case study. The computer checks to see if the equilibrium equations can be solved with the assumption of sticking or slipping at one or the other contact.

Once you find a solution to a problem with friction there remains the possibility of multiple solutions, in this case for different reasons than the usual static indeterminacy. The following problem shows a case where a statics problem has multiple solutions due to friction effects.

Example: Rod pushed in a channel.



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A light rod is just long enough to make a 60° angle with the walls of a channel. One channel wall is frictionless and the other has $\mu = 1$. What is the force needed to keep it in equilibrium in the position shown? If we assume it is sliding we get the first free body diagram. The forces shown can only be in equilibrium if all the forces are zero. So a solution is that the rod slides in equilibrium with no force. If we assume that the block is not sliding the friction force on the lower wall can be at any angle between $\pm 45^\circ$. Thus we have equilibrium with the second FBD for arbitrary positive F. This is a second set of solutions. A rod like this is said to be *self locking* in that it can hold arbitrary force F without slipping. That we have found freely slipping solutions with no force and jammed solutions with arbitrary force corresponds physically to one being able to easily slide a rod like this down a slot and then have it totally jamb. Some rock-climbing equipment depends on such self-locking and easy release.

Statically indeterminate problems

When there are two or more points of frictional contact and there is no slip nor impending slip then static indeterminacy is likely.

Example: Chair with friction
If we assume Coulomb friction at the chair feet we know that

$$|F_A| \le \mu N_A \qquad \text{and} \qquad |F_B| \le \mu N_B$$

The equilibrium equations tell us (assuming for simplicity that W acts in the middle of the chair):

$$F_A + F_B = 0, N_A = N_B = W/2.$$

Putting these equations together we find that

$$-W/2 \le F_A \le W/2 \qquad \text{and} \qquad F_B = -F_A$$

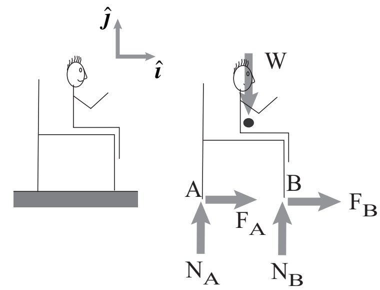

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Figure 4.45: A chair with friction at the feet.

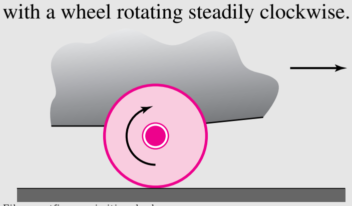
4.6 Undriven wheels and two force bodies

One often hears whimsical reverence for the "invention of the wheel." Now, using elementary mechanics, we can gain some appreciation for this revolutionary way of sliding things.

Without a wheel the force it takes to drag something is about μW. Since μ ranges between about .1 for teflon, to about .6 for stone on ground, to about 1 for rubber on pavement, you need to pull with a force that is on the order of a half of the full weight of the thing you are dragging.

You have seen how rolling on round logs cleverly take advantage of the properties of two-force bodies (page 202). But that good idea has the major deficiency of requiring that logs be repeatedly picked up from behind and placed in front again.

The simplest wheel design uses a dry "journal" bearing consisting of a non-rotating shaft protruding through a near close fitting hole in the wheel. Here is shown part of a cart rolling to the right with a wheel rotating steadily clockwise.



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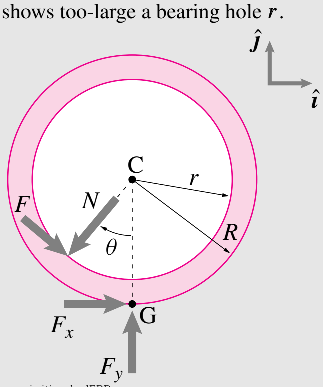
To figure out the forces involved we draw a free body diagram of the wheel. We neglect the wheels weight because it is generally much smaller than the forces it mediates. To make the situation clear the picture shows too-large a bearing hole r.



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The force of the axle on the wheel has a normal component N and a frictional component F. The force of the ground on the wheel has a part holding the cart up F_y and a part along the ground F_x which will surely turn out to be negative for a cart moving to the right. If we take the wheel dimensions to be known and also the vertical part of the ground reaction force F_y we have as unknowns N, F, θ and F_x. To find these we could use the friction equation for the sliding bearing contact

$$F = \mu N;$$

force balance

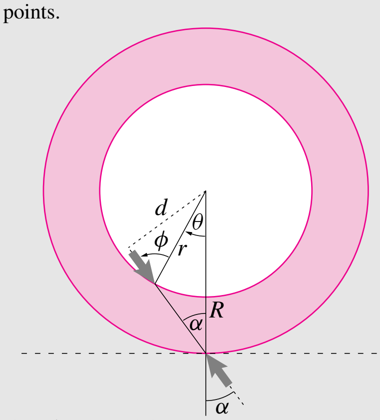
$$F_x\hat{\imath} + F_y\hat{\jmath} + N(-\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) + F(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) = \vec{\mathbf{0}},$$

which could be reduced to 2 scalar equations by taking components or dot products; and moment balance about C, which we calculate with forces and perpendicular distances as

$$Fr + F_x R = 0.$$

Of key interest is finding the force resisting motion F_x. With some mathematical manipulation we could solve the 4 scalar equations above for any of F_x, N, F, and θ in terms of r, R, F_y, and μ. We follow a more intuitive approach instead.

As modeled, the wheel is a two-force body so the free body diagram shows equal and opposite colinear forces at the two contact points.



Filename:tfigure-primitivewheelFBD2

(continued...)

and no more. That is, all we can tell that both are within the friction limits and that the horizontal forces cancel each other.

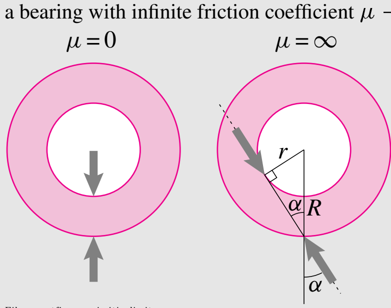
> If a free body diagram shows two forces with a common line of action, like the friction forces F_A and F_B on the chair above, the laws of statics might only find their sum, but otherwise can't untangle them.

Only if there is independent information, as would be the case if we knew the chair was sliding to the right (which it clearly isn't in this static example), could we find the friction forces.

4.6 Undriven wheels and two force bodies (continued)

The friction angle ϕ describes the friction between the axle and wheel (with $\tan\phi = \mu$). The angle α describes the *effective* friction of the wheel. This is not the friction angle for sliding between the wheel and ground which is assumed to be larger (if not, the wheel would skid and not roll), probably much larger. The *specific resistance* or the *coefficient of rolling resistance* or the *specific cost of transport* is $\mu_{\text{eff}} = \tan\alpha$. (If there was no wheel, and the cart or whatever was just dragged, the specific resistance would be the friction between the cart and ground $\mu_{\text{eff}} = \mu$.)

Although we can solve for α in terms of μ or ϕ let's first consider two extreme cases: one is a frictionless bearing and the other is a bearing with infinite friction coefficient $\mu \to \infty$ and $\phi \to 90^\circ$.



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In the case that the wheel bearing has no friction we satisfyingly see clearly that there is no ground resistance to motion. The case of infinite friction is perhaps surprising. Even with infinite friction we have that

$$\sin\alpha = \frac{r}{R}.$$

Thus if the axle has a diameter of 10 cm and the wheel of 1 m then $\sin\alpha$ is less than .1 no matter how bad the bearing material. For such small values we can make the approximation $\mu_{\text{eff}} = \tan\alpha \approx \sin\alpha$ so that the effective coefficient of friction is .1 or less no matter what the bearing friction.

> The genius of the wheel design is that it makes the effective friction less than r/R no matter how bad the bearing friction.

Going back to the two-force body free body diagram we can see that

$$\Rightarrow \quad \overbrace{r\sin\phi}^{d} = \overbrace{R\sin\alpha}^{d}$$
$$\Rightarrow \quad \sin\alpha = \frac{r}{R}\sin\phi. \qquad (*)$$

From this formula we can extract the limiting cases discussed previously ($\phi = 0$ and $\phi = 90^\circ$). We can also plug in the small angle approximations ($\sin\alpha \approx \tan\alpha$ and $\sin\phi \approx \tan\phi$) if the friction coefficient is low to get

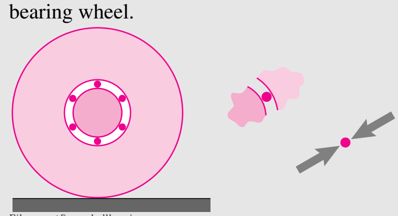
$$\mu_{\text{eff}} \approx \mu\frac{r}{R}.$$

The effective friction is the bearing friction attenuated by the radius ratio. Or, we can use the trig identity $\sin = \sqrt{1+\tan^{-2}}^{\,-1}$ to solve the exact equation (*) for

$$\mu_{\text{eff}} = \mu\frac{r}{R}\left(\frac{1}{\sqrt{1+\mu^2(1-r^2/R^2)}}\right),$$

where the term in parenthesis is always less than one and close to one if the sliding coefficient in the bearing is low.

Finally we combine the genius of the wheel with the genius of the rolling log and invent a wheel with rolling logs inside, a ball bearing wheel.



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Each ball is a two force body and thus only transmits radial loads. It's as if there were no friction on the bearing and we get a specific resistance of zero, $\mu_{\text{eff}} = 0$. Of course real ball bearings are not perfectly smooth or perfectly rigid, so its good to keep r/R small as a back up plan even with ball bearings.

By this means some wheels have effective friction coefficients as low as about .003. The force it takes to drag something on wheels can be as little as one three hundredth the weight.

SAMPLE 4.11 A block on a ramp sliding down or up. Consider a block of mass $m = 10\,\text{kg}$ pushed up by the force F on the ramp as shown in the figure. The coefficient of friction between the ramp and the block is $\mu = 0.7$.

1. Let $\theta = 60^\circ$ and $\alpha = 0^\circ$. Assuming that the block slides steadily downhill, find the tension in the string.
2. Let $\theta = 30^\circ$ and $\alpha = 30^\circ$. If the applied force $F = 20\,\text{N}$, find the force of friction on the block.
3. Let $\theta = 60^\circ$ and $\alpha = 30^\circ$. If the applied force $F = 10\,\text{N}$, find the force of friction on the block.
4. For $\theta = 30^\circ$ and $\alpha = 30^\circ$, what will be the required tension in the string to make the block just about slide up the slope? Express your answer in terms of the weight of the block.

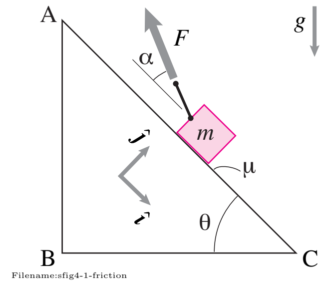


Figure 4.46:

Solution The free-body diagram of the block is shown in fig. 4.47. We have assumed that the friction force acts upwards along the inclined plane. The direction of the friction force can be up or down depending on the direction of sliding. We will let the equilibrium equation tell us which way the friction force acts in a particular case. In fig. 4.47, we also use rotated unit vectors $\hat{\imath}$ and $\hat{\jmath}$, parallel and perpendicular to the inclined plane, respectively. This is just to make calculations easier. We can use these basis vectors in any orientation to suit our convenience.

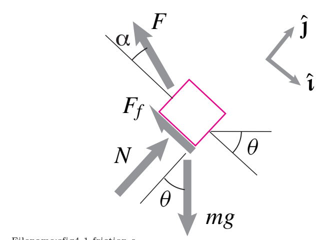


Figure 4.47: Free body diagram of the block. The friction force $\vec{F}_f$ may be known or unknown depending on whether the block is sliding or not. Under downhill sliding, $F_f = \mu N$, whereas for no sliding $|F_f| \leq \mu N$.

The force balance equation for the static equilibrium of the block gives

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad \vec{F} + \vec{N} + \vec{F}_f + \vec{W} = \vec{0}. \tag{4.19}$$

$$\Rightarrow \quad F(-\cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}) + N\hat{\jmath} - F_f\hat{\imath} + mg(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) = \vec{0} \tag{4.20}$$

Now depending on what is given and what is unknown, we can manipulate this vector equation to find what we want.

1. **Block sliding down:** If the block slides down steadily or very slowly, we can use the static equilibrium equation written above with $\vec{F}_f = -\mu N\hat{\imath}$ (that is, the friction force is known. This is the case of sliding friction and the friction force is maximum possible). Substituting this value of $\vec{F}_f$ and separating out the $\hat{\imath}$ and $\hat{\jmath}$ components of eqn. (4.20), we get

$$-F\cos\alpha - \mu N + mg\sin\theta = 0 \tag{4.21}$$
$$F\sin\alpha + N - mg\cos\theta = 0. \tag{4.22}$$

Adding μ times eqn. (4.22) to eqn. (4.21) in order to get rid of N, and rearranging terms, we get

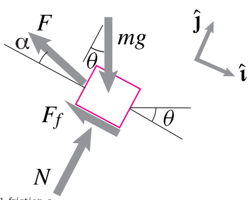
$$F(\cos\alpha - \mu\sin\alpha) = mg(\sin\theta - \mu\cos\theta)$$
$$\Rightarrow \quad F = \frac{\sin\theta - \mu\cos\theta}{\cos\alpha - \mu\sin\alpha}mg. \tag{4.23}$$

Substituting $\alpha = 0^\circ$, $\theta = 60^\circ$, and $\mu = 0.7$ in eqn. (4.23), we get

$$F = (\sin 60^\circ - 0.7\cdot\cos 60^\circ)\,mg = 0.52\,mg. = 51\,\text{N}.$$

$$\boxed{\vec{F} = -(51\,\text{N})\hat{\imath}}$$

2. **Block sliding or not sliding – not known:** Now, we are given that $F = 20\,\text{N}$, $\alpha = 30^\circ$, and $\theta = 30^\circ$. We do not know if the block is sliding or not. So, let us assume static equilibrium in the given configuration and solve for the friction force F_f. Then, we will check if it satisfies friction law for static equilibrium ($|F_f| \leq \mu N$).



Filename:sfig4-1-friction-c

Figure 4.48: Free body diagram of the block. The friction force $\vec{F}_f$ is not known because we do not know if the block is sliding or not.

Substituting $\vec{F}_f = -F_f\,\hat{\imath}$ in eqn. (4.20) and separating out the $\hat{\imath}$ and $\hat{\jmath}$ components of the equation, we get

$$\begin{aligned} -F\cos\alpha - F_f + mg\sin\theta &= 0 \\ F\sin\alpha + N - mg\cos\theta &= 0 \end{aligned}$$

which are easily solved for F and N to give

$$\begin{aligned} F_f &= mg\sin\theta - F\cos\alpha \\ N &= mg\cos\theta - F\sin\alpha. \end{aligned}$$

Substituting the given values of F, θ, and α, we get

$$\begin{aligned} F_f &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\sin 30^\circ - 20\,\mathrm{N}\cdot\cos 30^\circ = 31.73\,\mathrm{N} \\ N &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\cos 30^\circ - 20\,\mathrm{N}\cdot\sin 30^\circ = 74.96\,\mathrm{N}. \end{aligned}$$

Now, the maximum possible value of friction force is $\mu N = 0.7\cdot 74.96\,\mathrm{N} = 52.47\,\mathrm{N}$. Thus, $|F_f| < \mu N$, and therefore, our assumption of static equilibrium is valid. This equilibrium requires that $F_f = 31.73\,\mathrm{N}$.

$$\boxed{\vec{F}_f = -(31.73\,\mathrm{N})\hat{\imath}}$$

3. **Block sliding or not sliding – not known, again:** In this case, $F = 10\,\mathrm{N}$, $\alpha = 30^\circ$, and $\theta = 60^\circ$. Again, assuming static equilibrium, we do exactly the same calculations as above (in fact, use the same expressions) and substituting the given values, we get

$$\begin{aligned} F_f &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\sin 60^\circ - 10\,\mathrm{N}\cdot\cos 30^\circ = 76.3\,\mathrm{N} \\ N &= 10\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot\cos 60^\circ - 10\,\mathrm{N}\cdot\sin 30^\circ = 44\,\mathrm{N}. \end{aligned}$$

Here, $|F_f| > \mu N$. Clearly, F_f is not less than or equal to μN, and therefore, our assumption of static equilibrium is not valid. In fact, the given parameters of the problem will make the block accelerate downhill — a problem of dynamics. However, the friction force remains constant, at its maximum $\vec{F}_f = \mu N = 33.88\,\mathrm{N}$ once the sliding starts, accelerating or not.

$$\boxed{\vec{F}_f = -33.88\,\mathrm{N}\hat{\imath}}$$

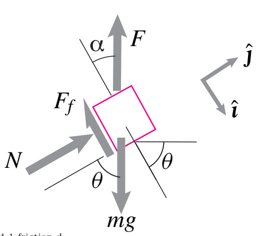


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Figure 4.49: Free body diagram of the block. The friction force $\vec{F}_f$ is not known again because we do not know if the block is sliding or not.

4. **Block just about to slide upwards:** If the block is about to slide upwards, then the friction force must act downwards as shown in fig. 4.50. We also know the magnitude, $F_f = \mu N$ because it is the case of impending slip. Now the force balance equations in the $\hat{\imath}$ and $\hat{\jmath}$ directions are:

$$\begin{aligned} -F\cos\alpha + \mu N + mg\sin\theta &= 0 \\ F\sin\alpha + N - mg\cos\theta &= 0 \end{aligned}$$

Eliminating N from the two equations, we get F in terms of mg and substituting $\alpha = 30^\circ$, and $\theta = 60^\circ$, we get the desired value:

$$\begin{aligned} F &= \frac{\sin\theta + \mu\cos\theta}{\cos\alpha + \mu\sin\alpha}\,mg \\ &= \frac{1.216}{1.216}\,mg = mg. \end{aligned}$$

$$\boxed{F = mg}$$

Does the answer make sense? Yes, it does. For the given θ and α, the string tension is vertical. If it balances the weight of the block, the normal force goes to zero and so does the friction force. The block is then ready to slide up if the tension increases by any tiny amount.

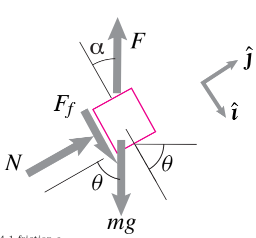

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Figure 4.50: Free body diagram of the blockabout to slip upwards. The friction force $\vec{F}_f$ is completely known because of impending slip, $\vec{F}_f = \mu N\hat{\imath}$.

SAMPLE 4.12 **How much friction does the cylinder need?** A cylinder of mass m sits between an incline and a vertical wall as shown in the figure. There is no friction between the wall and the cylinder but there is friction between the incline and the cylinder. Take the coefficient of friction to be μ and the angle of incline with the horizontal to be θ. Find the force of friction on the cylinder from the incline.

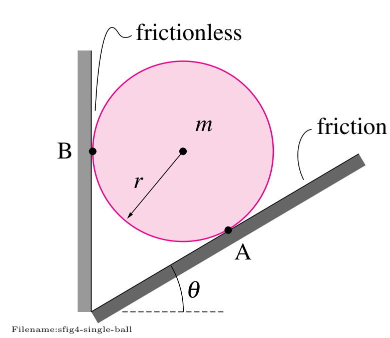


Figure 4.51:

Solution The free body diagram of the cylinder is shown in fig. 4.52. We need to find the force of friction F_s.

Note that the normal reaction of the vertical wall, N, the force of gravity, mg, and the normal reaction of the incline, R, all pass through the center C of the cylinder. So, if we do moment balance about point C, $\sum \vec{M}_C = \vec{0}$, none of these forces will appear in the equation since their moment about C is zero. Therefore, to find F_s, we should use the moment balance equation about point C. Noting that F_s acts along the inclined plane, its normal distance (*lever arm*) from point C is simply r, the radius of the cylinder, we have,

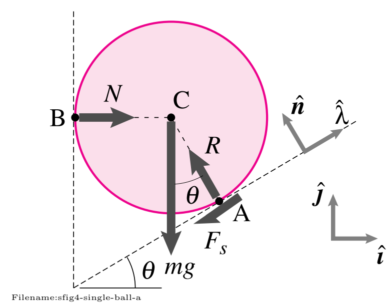


Figure 4.52:

$$\sum \vec{M}_C = \vec{0} \qquad \Rightarrow \qquad rF_s(-\hat{k}) = \vec{0}$$
$$\Rightarrow \qquad F_s = 0.$$

Thus the force of friction on the cylinder is zero! Note that F_s is independent of θ, the angle of incline. Thus, irrespective of what the angle of incline is, in the static equilibrium condition, there is no force of friction on the cylinder.

$$\boxed{F_s = 0}$$

Note: The cylinder here is a three force body since there are three forces acting on it — two contact forces (at A and B) and one gravity force. Therefore, for equilibrium, all the three forces must intersect at a single point. Now, lines of action of the gravity force and the normal reaction at B intersect at the center C of the cylinder. Therefore, the line of action of the contact force at A also must pass through the center. This is clearly not possible if the contact force is not normal to the incline (see the candidate contact forces marked by the dashed gray arrows in fig. 4.53. If there is any non-zero friction force at A, the contact force (the resultant of the normal reaction and the friction force) at A will be tipped away from the normal, thus making its line of action miss the center of the cylinder and, therefore, violate equilibrium condition.

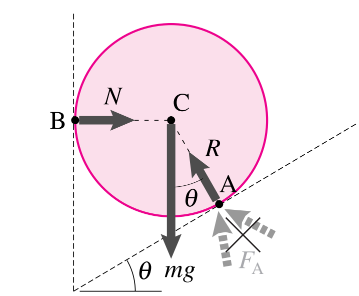
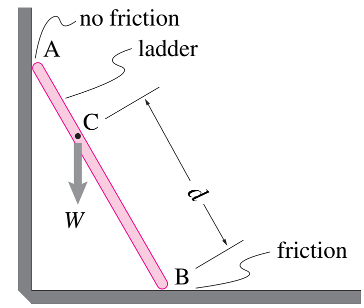
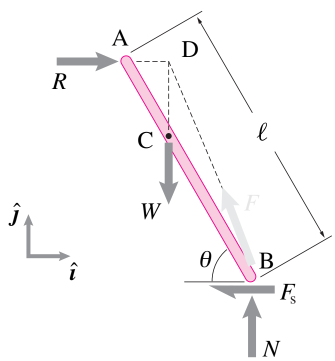


Figure 4.53:



Filename:sfig4-single-ladder

Figure 4.54:



Filename:sfig4-single-ladder-a

Figure 4.55: The free body diagram of the ladder indicates that it is a three force body. Since the direction of the forces acting at points A and C are known (the normal, horizontal reaction at A and the vertical gravity force at C), it is easy to find the direction of the net ground reaction at B — it must pass through point D. The ground reaction F at B (shown by light grey arrow) can be decomposed into a normal reaction N and a horizontal reaction (the force of friction) F_s at B.

SAMPLE 4.13 Will the ladder slip? A ladder of length $\ell = 4\,\text{m}$ rests against a wall at $\theta = 60^\circ$. Assume that there is no friction between the ladder and the vertical wall but there is friction between the ground and the ladder with $\mu = 0.5$. A person weighing 700 N starts to climb up the ladder.

1. Can the person make it to the top safely (without the ladder slipping)? If not, then find the distance d along the ladder that the person can climb safely. Ignore the weight of the ladder in comparison to the weight of the person.
2. Does the "no slip" distance d depend on θ? If yes, then find the angle θ which makes it safe for the person to reach the top.

Solution

1. The free-body diagram of the ladder is shown in fig. 4.55. There is only a normal reaction $\vec{R} = R\hat{\imath}$ at A since there is no friction between the wall and the ladder. The force of friction at B is $\vec{F}_s = -F_s\hat{\imath}$ where $F_s \leq \mu N$. To determine how far the person can climb the ladder without the ladder slipping, we take the critical case of impending slip. In this case, $F_s = \mu N$. Let the person be at point C, a distance d along the ladder from point B. We need to find d and check if $d < \ell$ (cannot make it to point A).

   From moment balance about point B, $\sum \vec{M}_B = \vec{0}$, we find

$$\begin{aligned} \vec{r}_{A/B} \times \vec{R} + \vec{r}_{C/B} \times \vec{W} &= \vec{0} \\ -R\ell \sin\theta \hat{k} + Wd \cos\theta \hat{k} &= \vec{0} \\ \Rightarrow \quad R &= W\frac{d\cos\theta}{\ell\sin\theta}. \end{aligned} \tag{4.24}$$

   From force equilibrium, we get

$$(R - \mu N)\hat{\imath} + (N - W)\hat{\jmath} = \vec{0}. \tag{4.25}$$

   Dotting eqn. (4.25) with $\hat{\jmath}$ and $\hat{\imath}$, respectively, we get

$$\begin{aligned} N &= W \\ R &= \mu N = \mu W. \end{aligned}$$

   Substituting this value of R in eqn. (4.24) we get

$$\begin{aligned} \mu W &= W\frac{d\cos\theta}{\ell\sin\theta} \\ \Rightarrow \quad d &= \mu\ell\tan\theta \qquad (4.26) \\ &= 0.5\cdot(4\,\text{m})\cdot\tan 60^\circ = 3.46\,\text{m}. \end{aligned}$$

   Thus, the ladder is about to slip when the person is at $d = 3.56\,\text{m}$. But, $d < \ell$, therefore, the person cannot make it to the top of the ladder safely.

$$\boxed{d = 3.46\,\text{m}}$$

2. The "no slip" distance d depends on the angle θ via the relationship in eqn. (4.26). The person can climb the ladder safely up to the top if

$$\tan\theta = \frac{1}{\mu} \quad \Rightarrow \quad \theta = \tan^{-1}(\mu^{-1}) = 63.43^\circ.$$

   Thus, any reasonable angle $\theta \geq 64^\circ$ will allow the person to climb up to the top safely.

$$\boxed{\theta \geq 64^\circ}$$

SAMPLE 4.14 Will it tip or will it slide? Whether or not a box of a given width and height will slide or tip over on an inclined plane depends on the slope of the plane and the coefficient of friction. For a given slope θ, find the relationship between the coefficient of friction μ and the aspect ratio of the box, $\gamma = b/h$ for impending tipping.

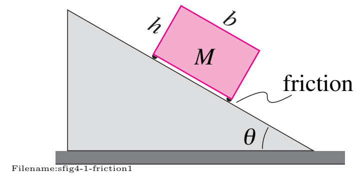


Figure 4.56:

Solution Let us imagine that we put the box on a flat surface and then slowly start tilting the surface up with respect to the horizontal. At some slope, the box will either tip over or slide. Just before the instant the box starts to tip over or slide, it is in static equilibrium. The magnitude of the friction force at the contact points is $|F| \leq \mu N$ where N is the magnitude of the normal force at the contact, and the equality holds only in the case of impending slip. That is, if the box is about to slip, then $F = \mu N$ at each contact point.

The free body diagram of the box is shown in fig. 4.57. Let us first write the equations of static equilibrium assuming there is no impending slip.

The force balance in the $\hat{\imath}$ and $\hat{\jmath}$ directions (see fig. 4.60) gives

$$F_A + F_B = mg\sin\theta \tag{4.27}$$

$$N_A + N_B = mg\cos\theta. \tag{4.28}$$

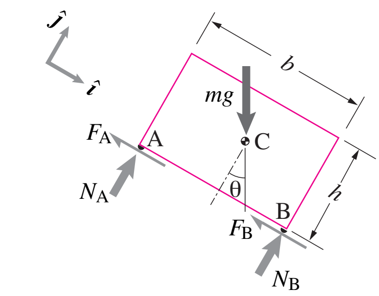


Figure 4.57:

The moment equilibrium about the center-of-mass, $\sum \vec{M}_C = \vec{0}$, in the $\hat{k}$ direction gives

$$N_B \cdot \frac{b}{2} - N_A\frac{b}{2} - (F_A + F_B)\frac{h}{2} = 0. \tag{4.29}$$

Substituting $F_A + F_B = mg\sin\theta$ from eqn. (4.27) in eqn. (4.29), and solving eqns. (4.28) and (4.29) simultaneously, we get

$$N_A = \frac{1}{2}mg\left(\cos\theta - \frac{h}{b}\sin\theta\right), \text{ and } \quad N_B = \frac{1}{2}mg\left(\cos\theta + \frac{h}{b}\sin\theta\right).$$

If the box were to tip over (about point B), the support forces at A will go to zero (because of loss of contact). Thus, for impending tipping,

$$N_A = 0 \quad \Rightarrow \quad \cos\theta - \frac{h}{b}\sin\theta = 0 \quad \Rightarrow \quad \tan\theta = \frac{b}{h} = \gamma.$$

Thus, the condition for impending tipping is

$$\tan\theta = \gamma. \tag{4.30}$$

This condition, however, does not guarantee that the box *will* tip over. In fact, it may start sliding before it tips over. We need to check if sliding condition is met before eqn. (4.30) is satisfied. In other words, we need to check the value of friction forces and make sure that $|F_A + F_B| \leq \mu(N_A + N_B)$. Thus, for no slipping,

$$F_A + F_B \leq \mu(N_A + N_B) \quad \Rightarrow \quad mg\sin\theta \leq \mu mg\cos\theta \quad \Rightarrow \quad \tan\theta \leq \mu.$$

Using this condition (with equality) in eqn. (4.30), we get the *critical* condition for tipping:

$$\gamma = \mu.$$

$$\gamma = \tan\theta \leq \mu$$

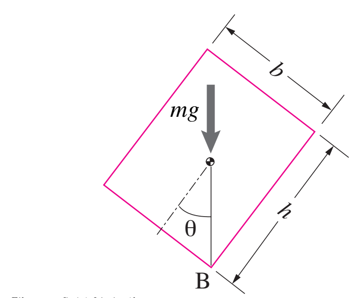


Figure 4.58: The impending tipping condition $\tan\theta = \gamma \equiv b/h$, provided $\tan\theta \leq \mu$.

You may know this condition geometrically as the line of action of the weight of the box must pass through B and beyond for tipping over (see fig. 4.58).

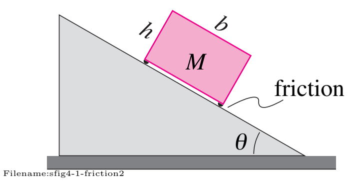


Figure 4.59:

SAMPLE 4.15 How big does the friction force get? Consider the box on the inclined plane of Sample 4.14 again. The box has aspect ratio $\gamma = b/h$. The coefficient of friction is μ. Imagine that the angle θ of the inclined plane can be varied. How does the force of friction on the box vary with θ? How does the maximum value of this force depend on μ?

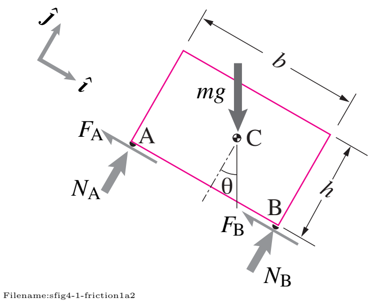


Figure 4.60:

Solution If we imagine the inclined plane to be not inclined ($\theta = 0$) but horizontal and the box to be just sitting there, the force of friction on the box has to be zero. As we tilt the plane up ($\theta > 0$), the friction force starts increasing. It increases up to the point of impending slip unless the box tips over before that. Assuming that the aspect ratio of the box prevents it from tipping (see Sample 4.14), we can determine the maximum value up to which the friction force rises before the box starts slipping.

From Sample 4.14, we know that the total friction force $F_s = F_A + F_B = mg\sin\theta$. Thus the normalized friction force (as a fraction of the weight of the block), F_s/mg is

$$\frac{F_s}{mg} = \sin\theta.$$

Thus the total friction force varies as sine of the ramp angle. Howver, this variation is valid only upto the maximum value of the friction force (μN) when the block starts sliding. The critical angle at which this maximum is attained is $\theta_{\text{slip}} = \tan^{-1}\mu \equiv \phi$ (friction angle). Thus,

$$\left.\frac{F_s}{mg}\right|_{\max} = \sin\phi.$$

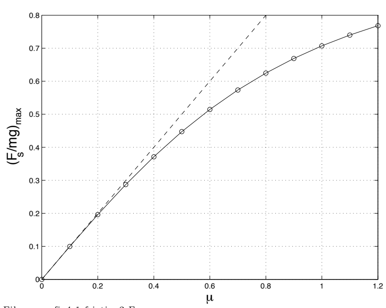


Figure 4.61: The maximum net friction force on the box as a function of the friction coefficient μ. Note that the relationship is almost linear for most practical values of μ.

Figure 4.61 shows how the maximum normalized friction force varies with μ. Note that for lower values of μ (which covers most practical values of μ), the relationship is almost linear. Thus, $|F_s/mg| \approx \mu$ for $\mu \leq 0.5$.

$$\boxed{\frac{F_s}{mg} = \sin\theta, \qquad \left.\frac{F_s}{mg}\right|_{\max} = \sin\phi}$$

What happens to the friction force after it attains the maximum value $F_s = mg\sin\phi$? For a given ramp angle, the friction force remains constant and the box slides.

SAMPLE 4.16 A spool of mass $m = 2\,\text{kg}$ rests on an incline as shown in the figure. The inner radius of the spool is $r = 200\,\text{mm}$ and the outer radius is $R = 500\,\text{mm}$. The coefficient of friction between the spool and the incline is $\mu = 0.4$, and the angle of incline $\theta = 60°$.

1. Which way does the force of friction act, up or down the incline?
2. What is the required horizontal pull T to balance the spool on the incline?
3. Is the spool about to slip?

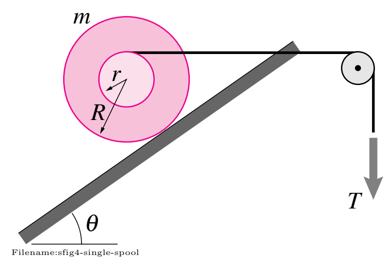


Figure 4.62:

Solution

1. The free-body diagram of the spool is shown in fig. 4.63. Note that the spool is a 3-force body. Therefore, in static equilibrium all the three forces — the force of gravity mg, the horizontal pull T, and the incline reaction F — must intersect at a point. Since T and mg intersect at the top of the inner drum (point B), the reaction force $\vec{F}$ of the incline must be along the direction AB. Now the incline reaction $\vec{F}$ is the vector sum of two forces — the normal (to the incline) reaction N and the friction force F_s (along the incline). The normal reaction force N passes though the center C of the spool. Therefore, the force of friction F_s must point up along the incline to make the resultant $\vec{F}$ point along AB.

Up the incline

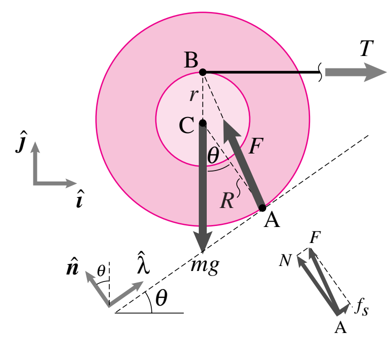


Figure 4.63:

2. We need to find the tension T in the string. From the free body diagram, we see that the force equilibrium will involve $\vec{T}$ along with another unknown force $\vec{F}$, the reaction of the incline. On the other hand, if we do moment balance about point A, we can get rid of $\vec{F}$ and get one scalar equation involving T and mg, giving T in terms of mg. So, writing the moment equilibrium equation about point A,

$$\sum \vec{M}_A = \vec{0},$$

we get

$$\vec{r}_{C/A} \times (-mg\,\hat{\jmath}) + \vec{r}_{B/A} \times (T\,\hat{\imath}) = \vec{0}. \tag{4.31}$$

These cross products can be easily evaluated by using the scalar form of the moment of a force—the product of force and the lever arm. Thus the moment of mg is $mg \cdot R\sin\theta$ and the moment of T is $-T \cdot (r + R\cos\theta)$ about point A in the $\hat{k}$ direction. Thus the scalar form of the moment balance equation gives

$$\begin{aligned} mgR\sin\theta &= T(R\cos\theta + r) \\ \Rightarrow \quad T &= mg\,\frac{\sin\theta}{\cos\theta + r/R} \\ &= 2\,\text{kg}\cdot 9.81\,\text{m/s}^2 \cdot \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2} + \frac{.2\,\text{m}}{.5\,\text{m}}} \\ &= 18.88\,\text{N}. \end{aligned}$$

$T = 18.88\,\text{N}$

Alternatively,
We can also evaluate the net moment on the spool, given by eqn. (4.31), using direct cross products of vectors in the equation. We can use mixed basis vectors ($\hat{\imath}$, $\hat{\jmath}$, $\hat{\lambda}$, and $\hat{n}$) as shown in fig. 4.63. Since,

$$\vec{r}_{C/A} = R\hat{n} \quad \text{and} \quad \vec{r}_{B/A} = R\hat{n} + r\hat{\jmath},$$

we have,

$$\begin{aligned}\vec{r}_{C/A} \times (-mg\hat{\boldsymbol{\jmath}}) &= -mgR(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\jmath}}) \\ \vec{r}_{B/A} \times T\hat{\boldsymbol{\imath}} &= T[R(\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\imath}}) + r(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\imath}})].\end{aligned}$$

Now, from the geometry of the basis vectors (see fig. 4.63), we have,

$$\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\jmath}} = -\sin\theta\hat{\boldsymbol{k}} \quad \text{and} \quad \hat{\boldsymbol{n}} \times \hat{\boldsymbol{\imath}} = -\cos\theta\hat{\boldsymbol{k}}.$$

Therefore,

$$\begin{aligned}\vec{r}_{C/A} \times (-mg\hat{\boldsymbol{\jmath}}) &= mgR\sin\theta\hat{\boldsymbol{k}}, \\ \text{and} \qquad \vec{r}_{B/A} \times T\hat{\boldsymbol{\imath}} &= -TR\cos\theta\hat{\boldsymbol{k}} - r\hat{\boldsymbol{k}}.\end{aligned}$$

Hence, eqn. (4.31) becomes

$$mgR\sin\theta\hat{\boldsymbol{k}} - TR\cos\theta\hat{\boldsymbol{k}} - r\hat{\boldsymbol{k}} = \vec{\boldsymbol{0}}.$$

Dotting both sides of this equation with $\hat{\boldsymbol{k}}$, we get the scalar equation

$$mgR\sin\theta = T(R\cos\theta + r)$$

which is the same equation as obtained above using moment lever arms.

3. To find if the spool is about to slip, we need to find the force of friction $|F_s|$ and see if it satisfies the condition of impending slip: $F_s = \mu N$. The force balance on the spool, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ gives

$$T\hat{\boldsymbol{\imath}} - mg\hat{\boldsymbol{\jmath}} + F_s\hat{\boldsymbol{\lambda}} + N\hat{\boldsymbol{n}} = \vec{\boldsymbol{0}} \qquad (4.32)$$

where $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ are unit vectors along the incline and normal to the incline, respectively. Dotting eqn. (4.32) with $\hat{\boldsymbol{\lambda}}$ we get

$$\begin{aligned}F_s &= -T(\underbrace{\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{\lambda}}}_{\cos\theta}) + mg(\underbrace{\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{\lambda}}}_{\sin\theta}) \\ &= -T\cos\theta + mg\sin\theta \\ &= -18.88\,\text{N}(1/2) + 19.62\,\text{N}(\sqrt{3}/2) \\ &= 7.55\,\text{N}.\end{aligned}$$

Similarly, we compute the normal force N by dotting eqn. (4.32) with $\hat{\boldsymbol{n}}$:

$$\begin{aligned}N &= -T(\hat{\boldsymbol{\imath}} \cdot \hat{\boldsymbol{n}}) + mg(\hat{\boldsymbol{\jmath}} \cdot \hat{\boldsymbol{n}}) \\ &= T\sin\theta + mg\cos\theta \\ &= 18.88\,\text{N}(\sqrt{3}/2) + 19.62\,\text{N}(1/2) \\ &= 26.16\,\text{N}.\end{aligned}$$

Now we find that $\mu N = 0.4(26.16\,\text{N}) = 10.46\,\text{N}$ which is greater than $F_s = 7.55\,\text{N}$. Thus $F_s < \mu N$, and therefore, the spool is not about to slip.

Not about to slip

4.4 Internal forces

The vague concept of 'forces inside' a structure is in superficial conflict with the subject of mechanics. Mechanics equations only concern the forces *on* an object shown in a free body diagram;'internal forces' have no place on a free body diagram and thus no place in mechanics.

> **Example: Pulling on the ends of a rope; nothing internal**
> Consider two people pulling apart the frayed rope of fig. 4.64a. A free body diagram of the rope is shown in fig. 4.64b. The laws of mechanics use the *external* forces on an isolated system. These are the forces that show on a free body diagram. For the rope these are the forces at the ends. The free body diagram does not include internal forces. Thus nothing about the 'internal forces' at the fraying part of the rope shows up in the mechanics equations describing the rope.
>
> Mechanics has nothing to say about so called 'internal forces' and thus nothing to say about the rope breaking in the middle. 'Internal forces' are meaningless in mechanics. The section title describes a non-existent subject.

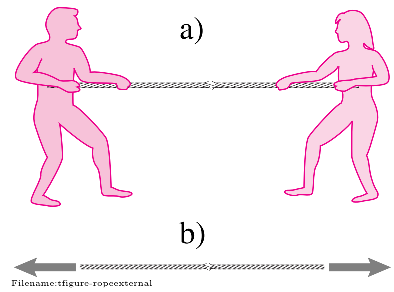


Figure 4.64: a) Two people pulling on a rope that is likely to break in the middle, b) A free body diagram of the rope.

Something's wrong. The problem is somewhat one of language: 'internal forces' are not really internal and they are not really forces!

'Internal forces' represent external forces on a smaller body

On page 25 we advertised mechanics as being useful for predicting when things will break. And our intuitions strongly tell us that there is something about the forces *in* the rope that make it break. Yet mechanics equations are based on the forces that show on free body diagrams. And free body diagrams only show external forces. How can we use mechanics based on external forces to describe the 'forces' inside a body? We use an idea whose simplicity hides its incredible utility:

> You cut the body, and what was inside is now on the outside of a smaller body.

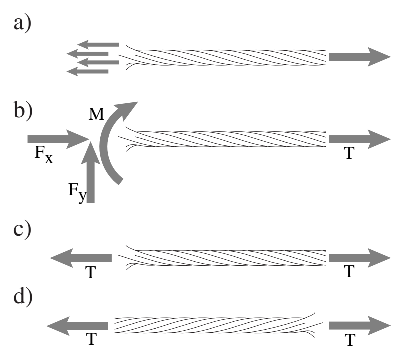


Figure 4.65: a) free body diagram of the right part of the rope, b)the same free body diagram, with the force distribution at the cut replaced with an equivalent force couple system, c) further simplified by using the laws of mechanics, and d) a free body diagram of the left portion of the rope.

In the case of the rope, we cut it in the middle. Then we fool the rope into thinking it wasn't cut using forces (remember, 'forces are *the* measure of mechanical interaction'), one force, say, at each fiber that is cut. Then we get the free body diagram of fig. 4.65a. We can simplify this to the free body diagram of fig. 4.65b because we know that every force system is equivalent to a force and couple at any point, in this case the middle of the rope. If we apply the equilibrium conditions to this cut rope we see that

$$
\begin{array}{rcrcl}
\text{Sum of vertical forces is zero} & \Rightarrow & F_y & = & 0 \\
\text{Sum of horizontal forces is zero} & \Rightarrow & F_x & = & -T \\
\text{Sum of moments about the cut is zero} & \Rightarrow & M & = & 0.
\end{array}
$$

Thus we get the simpler free body diagram of fig. 4.65c as you probably already guessed without using the equilibrium equations explicitly.

Tension

We have just derived the concept of 'tension in a rope' also sometimes called the 'axial force'. The tension is the pulling force on a free body diagram of the cut rope. If we had used the same cut for a free body diagram of the left half of the rope we would see the free body diagram of fig. 4.65d. Either by the principle of action and reaction, or by the equilibrium equations for the left half of the rope, you see also a tension T. The force vector is the opposite of the force vector on the right half of the rope. So it doesn't make sense to talk about the tension force vector *in* the rope since different (opposite) force vectors manifest themselves on the two sides of the cut ($-T\hat{\boldsymbol{\imath}}$ on the left end of the right half and $T\hat{\boldsymbol{\imath}}$ on the right end of the left half). Instead we talk about the scalar tension T which expresses the force vector at the cut as

$$\vec{\boldsymbol{F}} = T\hat{\boldsymbol{\lambda}}$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector pointing out from the free body diagram cut. Because $\hat{\boldsymbol{\lambda}}$ switches direction depending on which half rope you are looking at, the same scalar T works for both pieces.

The *tension* in a rope, cable, or bar is the amount of force pulling out on a free body diagram of the cut rope, cable, or bar. Tension is a scalar.

Internal 'forces' are not force vectors

Note our abuse of language: force is a vector, tension is an 'internal force' and tension is a scalar. What we call 'internal forces' are not really forces. We can't talk about the internal force vector at a point in the string because there are two different vectors for each cut, one for left half of string and one for the right. An 'internal force' isn't a force vector. Rather it is a quantity from which we can find a force vector once we have made a cut and picked which side of the cut we care about. We use this confusing language because of its firm place in the engineering workplace.

①Calling tension a scalar is a deception for pedagogical purposes. The best representation of 'internal forces' is with tensors which are too mathematically advanced for this book. But it is fun to notice that the concept of a tensor, something prominent in Einstein's theory of general relativity for example, has its origin in tension, our object of study here. Note the non-coincidental similarity of the words *tensor* and *tension*. What is a tensor? Loosely, a tensor is a quantity that helps you find a vector (the force at a cut) once you are told another vector (the unit vector pointing outwards from the cut). [Aside for hyper-experts: The relation between the tension tensor and tension scalar can be expressed by the dyadic representation $\underline{\underline{\boldsymbol{T}}} = T\hat{e}_1\hat{e}_1$.]

The common phrase *internal force* means 'a scalar with dimensions of force from which you can find the force on one side of a free body diagram cut'.

① Summarizing:

- Internal forces are not internal. Rather they describe the forces on the boundary of a smaller system that has a free body diagram cut that is inside the system of previous interest.
- Internal forces are not force vectors. Rather they are scalars from which you can find the force vector acting on one side of a free body diagram cut.

What is the strength of a structural piece?

Getting back to the question of whether or not the rope will break, we can now characterize the rope by the tension it can carry. A $10kN$ cable can carry a tension of $10,000\,\mathrm{N}$ all along its length. This means a free body diagram of the rope, cut anywhere along its length, could show forces up to but not bigger than $10,000\,\mathrm{N}$. If the rope is frayed it may break at, say, a tension of $2,000\,\mathrm{N}$, meaning a free body diagram with a cut at the fray can only show forces up to $2,000\,\mathrm{N}$.

Note that tension is not always positive. A negative tension (negative pulling out from the ends) is also called a positive compression (positive pushing in at the ends). For ropes we don't see much negative tension, the rope bends with just a hint of compression. But for metal and wood bars, and bones, compression is as important as tension.

Shear force and bending moment

To characterize the strength of more than just 2-force bodies we need to generalize the concept of tension. The main idea, which was emphasized in Chapter 3, is this:

> You can make a free body diagram cut anywhere on any body no matter how it is loaded.

As for tension, we define internal forces in terms of the forces (and moments) that show up on a free body diagram cut. Again we consider things (bars) that are rather longer than they are wide or thick because

- Long narrow pieces are commonly used in construction of buildings, machines, plants and animals.
- Internal forces in long narrow things are easier to understand than in bulkier objects.

For now we limit ourselves to 2D statics. At an arbitrary cut we can find the force and moment on the remaining piece in the same manner as in Section 4.2. And we could look at the x and y components of the force. Fine. The problem is that the force and moment we find do not just depend on the cut, but on which body we look at. One one side of the cut a force and moment act, on the other body on the other side of the cut, the opposite force and moment act. Another problem with xy components is that they don't necessarily line up with the natural directions for the structural part. So, for the purposes of thinking about internal forces we break the force into two components (see fig. 4.66) lined up with the part. And we measure the internal forces with scalars that are the same for both sides of the cut:

- The *tension* T is the scalar part of the force directed along the bar assumed positive when pulling away from the free body diagram cut.

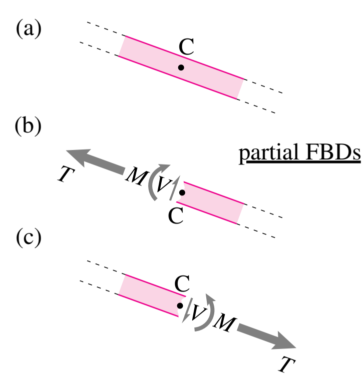


Figure 4.66: a) A piece of a structure, loads not shown; b) a partial free body diagram of the right part of the bar; c) a partial free body diagram of the left part of the bar.

- The *shear force* V is the force perpendicular to the bar (tangent to the free body diagram cut). Our sign convention is that shear is positive if it tends to rotate the cut object clockwise. An equivalent statement of the sign convention is that shear is positive if down on cuts at the right of a bar and positive if up on a cut on the left of bar (and to the right on top and to the left on the bottom).

Since we are just doing 2D problems now, the moment is always in the out-of-plane (typically $\hat{\boldsymbol{k}}$) direction.

- *The bending moment* M is the scalar part of the bending moment. The sign convention is that for a smiling beam (fig. 4.67): A clockwise ($-\hat{\boldsymbol{k}}$) couple is positive on a left cut and a counterclockwise ($\hat{\boldsymbol{k}}$) couple is positive on a right cut②.

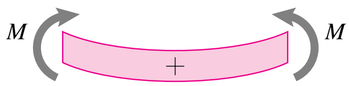


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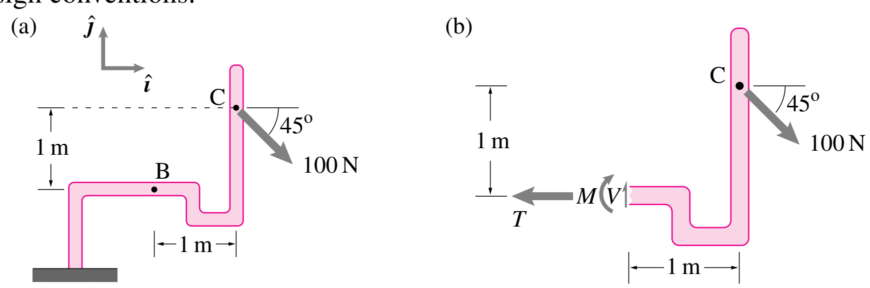
Figure 4.67: The smiling beam sign convention for bending moment. For a horizontal beam, moments which tend to make the beam smile (curve up) are called positive.

②Note that neither V nor T changes if you rotate your paper until the picture is upside down. However, the definition for the sign convention for M has the disadvantage that the bending moment does changes sign if you turn your paper upside down. (This ambiguity can only be avoided if one picks a favored side or end of the beam).

The tension T, shear V, and bending moment M on fig. 4.66 follows these sign conventions.



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Example: Internal forces in a bent rod

The internal forces at B can be found by making a free body diagram of a portion of the structure with a cut at B.

$$\begin{array}{ccccc} \text{Sum of vertical forces is zero} & \Rightarrow & V & = & (100/\sqrt{2})\,\text{N} \\ \text{Sum of horizontal forces is zero} & \Rightarrow & T & = & (100/\sqrt{2})\,\text{N} \\ \text{Sum of moments about the cut at B is zero} & \Rightarrow & M & = & -100\sqrt{2}\,\text{N m}. \end{array}$$

You may have noticed that we did get ahead of ourselves and use the concept of tension in a rope or rod as a source of loading with known direction on a particle and rigid body. We will use the concept of tension extensively in our analysis of trusses. Calculating how internal forces vary from point to point in a structure is picked up in Section 7 on page 377.

SAMPLE 4.17 A structure is made up of two bars – a thick bent bar ABC and a thin bar CE. Point C is halfway between B and D, $\ell = 0.8\,\text{m}$ and $\theta = 60^\circ$. Bar ABC is pulled up by a force $F = 500\,\text{N}$ at point A.

1. Find the internal forces in the bar ABC just to the right of point B.
2. Find the force in bar CE at the section s-s shown in the figure.

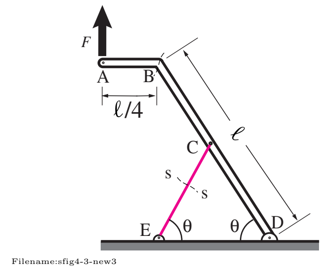

Figure 4.68:

Solution We cut the bar ABC at point B. The free-body diagram of the left part AB is shown in fig. 4.69. The internal forces acting at the cut section are tension T, shear force V and the bending moment M. From force balance of part AB in x and y directions, we have

$$T = 0, \quad \text{and} \quad V = F = 500\,\text{N}.$$

From the moment balance about point B, we have

$$M - F\ell/4 = 0 \quad \Rightarrow \quad M = F\ell/4 = 100\,\text{N·m}.$$

$$\boxed{T = 0, \quad V = 500\,\text{N}, \quad M = 100\,\text{N·m}}$$

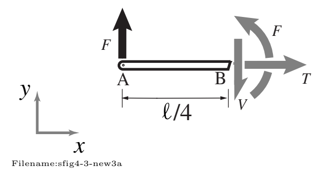

Figure 4.69:

For finding the tension in rod CE at the given section, we cut the rod at s-s and draw the free-body diagram of the structure along with the upper part of the rod attached at point C. The tension in bar CE is T and the reaction of the support at pin D is R. We need to find T.

We can write the moment balance equation about point D, $\sum \vec{M}_\text{D} = \vec{0}$, so that the unknown force R (that we are not interested in) disappears from the equation:

$$\vec{r}_\text{A/D} \times \vec{F} + \vec{r}_\text{C/D} \times \vec{T} \quad = \quad \vec{0}.$$

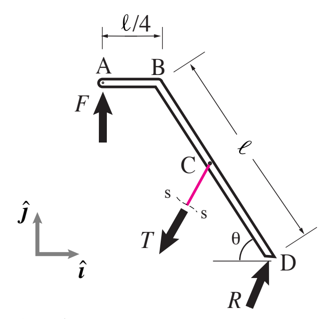

Figure 4.70:

The moments of F and T about point D can be easily evaluated using the scalar formula 'force times the lever arm' (see fig. 4.71). Thus, the moment balance equation in $\hat{k}$ direction is:

$$\begin{aligned} -F\ell(1/4 + \cos\theta) + T\frac{\ell}{2}\sin 2\theta &= 0 \\ \Rightarrow \quad T = \frac{2(1/4 + \cos\theta)}{\sin 2\theta} F. \end{aligned}$$

Substituting the given values, $F = 500\,\text{N}$ and $\theta = 60^\circ$, we get

$$T = 866\,\text{N}.$$

$$\boxed{T = 866\,\text{N}}$$

Note: Evaluation of the moment equation about point D using vectors and cross products is as follows. Since $\vec{r}_\text{A/D} = \vec{r}_\text{A/B} + \vec{r}_\text{B/D} = -\frac{\ell}{4}\hat{\imath} + \ell(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$, $\vec{r}_\text{C/D} = \frac{\ell}{2}(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$, $\vec{F} = F\hat{\jmath}$, and $\vec{T} = T(-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})$,

$$\vec{r}_\text{A/D} \times \vec{F} = -F\left(\frac{\ell}{4} + \ell\cos\theta\right)\hat{k}, \text{ and } \vec{r}_\text{C/D} \times \vec{T} = T\ell\cos\theta\sin\theta\hat{k}.$$

Therefore, the moment balance equation is

$$-F\ell(1/4 + \cos\theta)\hat{k} + T\frac{\ell}{2}\sin 2\theta\hat{k} = \vec{0}.$$

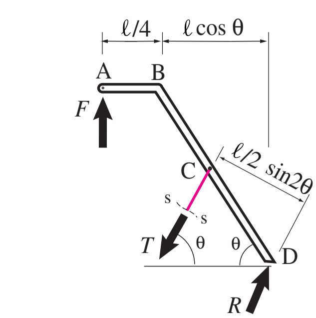

Figure 4.71: Moments of F and T about point D can be evaluated by multiplying the forces with their respective lever arms $\ell/4 + \ell\cos\theta$ and $\ell/2\sin 2\theta$, respectively.

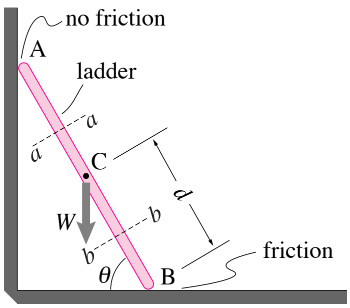

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Figure 4.72:

SAMPLE 4.18 A ladder of length $2d = 4\,\text{m}$ rests against a wall as shown. A person of weight $W = 700\,\text{N}$ stands at C. Assume that the ladder does not slip. Neglecting the weight of the ladder, find the internal forces in the ladder at sections a-a and b-b, at mid points of AC and AB, respectively. (See Sample 4.13.)

Solution To find the internal forces at the indicated sections, we need to cut the ladder at those sections, one at a time, draw the free body diagram of each part and carry out the force and moment balance equations. A little anticipation shows that we will need the support reactions at A and B in our calculations. So, let us first determine the support reactions. The free-body diagram of the ladder is shown in fig. 4.73. The moment balance about point B in $\hat{\boldsymbol{k}}$ direction gives

$$-R(2d\sin\theta) + W(d\cos\theta) = 0 \quad \Rightarrow \quad R = \frac{W}{2}\frac{\cos\theta}{\sin\theta}.$$

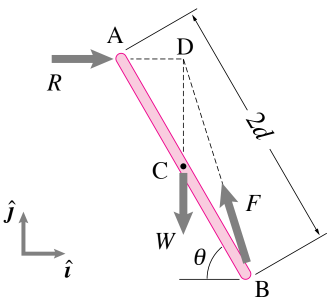

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Figure 4.73:

The force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$R\hat{\imath} - W\hat{\jmath} + \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}} \quad \Rightarrow \quad \vec{\boldsymbol{F}} = -R\hat{\imath} + W\hat{\jmath}.$$

Substituting the given values of $\theta(60^\circ)$ and $W(700\,\text{N})$, we get,

$$\vec{\boldsymbol{R}} = (202\,\text{N})\hat{\imath}, \quad \text{and} \quad \vec{\boldsymbol{F}} = (-202\hat{\imath} + 700\hat{\jmath})\,\text{N}.$$

Section a-a: Now, we cut the ladder at a-a and draw the free-body diagram of the upper part of the ladder as shown in fig. 4.74. The force balance for this part gives

$$\begin{aligned} T\hat{\boldsymbol{\lambda}} - V\hat{\boldsymbol{n}} + R\hat{\imath} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad T &= -R(\hat{\imath}\cdot\hat{\boldsymbol{\lambda}}) = -R\cos\theta \\ \text{and} \quad V &= R(\hat{\imath}\cdot\hat{\boldsymbol{n}}) = R\sin\theta. \end{aligned}$$

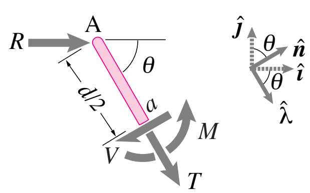

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Figure 4.74: Free body diagram of the upper part of the ladder cut at section a-a. Note that $\hat{\boldsymbol{\lambda}} = \cos\theta\hat{\imath} - \sin\theta\hat{\jmath}$ and $\hat{\boldsymbol{n}} = \sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$.

Substituting the numerical values of R and θ, we get $T = -101\,\text{N}$ and $V = 175\,\text{N}$. Now, the moment balance equation about a (the cut) gives

$$M - R(d/2)\sin\theta = 0 \quad \Rightarrow \quad M = (1/2)Rd\sin\theta$$

which, with numerical values, gives $M = 175\,\text{N·m}$.

$$T = -101\,\text{N}, \quad V = 175\,\text{N}, \quad M = 175\,\text{N·m}$$

Section b-b: Now we consider the internal forces at section b-b. We cut the ladder at the given section. We can consider the free-body diagram of the upper part or the lower part of the ladder to find the internal forces. Considering the upper part, (see fig. 4.75) we get, from force balance,

$$T\hat{\boldsymbol{\lambda}} - V\hat{\boldsymbol{n}} + R\hat{\imath} - W\hat{\jmath} = \vec{\boldsymbol{0}}$$

which, as the analysis above, gives

$$\begin{aligned} T &= -R(\hat{\imath}\cdot\hat{\boldsymbol{\lambda}}) + W(\hat{\jmath}\cdot\hat{\boldsymbol{\lambda}}) = -R\cos\theta + W(-\sin\theta) = -707\,\text{N} \\ V &= R(\hat{\imath}\cdot\hat{\boldsymbol{n}}) - W(\hat{\jmath}\cdot\hat{\boldsymbol{n}}) = R\sin\theta - W\cos\theta = -175\,\text{N}. \end{aligned}$$

Similarly, the scalar moment balance equation about point b gives

$$M - R\frac{3d}{2}\sin\theta + W\frac{d}{2}\cos\theta = 0 \quad \Rightarrow \quad M = 175\,\text{N·m}.$$

$$T = -707\,\text{N}, \quad V = -175\,\text{N}, \quad M = 175\,\text{N·m}$$

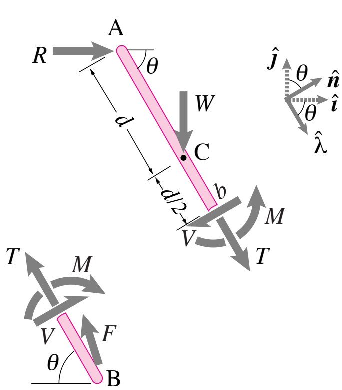

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Figure 4.75:

4.5 3D statics of one part

The structures and machines we study are most-often adequately modeled as 2D. But sometimes 2D analysis is too crude. Sometimes the 3D analysis gives an answer we could have found accurately enough with a 2D model, and sometimes a 2D model is inadequate. Here we use 3D statics to find various unknown aspects of forces acting on one part. By learning the 3D approach you can get a better sense of when to use a 2D model (which is most of the time for most engineers).

3D statics is conceptually the same as i2D: draw a free body diagram and use the force and moment balance equations. However, the geometry can be more of a challenge, the moment balance equation becomes a full vector equation (instead of just having one non-zero component it has three)①, and the number of scalar equations from one free body diagram increases from 3 to 6. In 3D issues related to static-determinacy arise more often and more subtly.

①For 2D problems we used the phrase 'moment about a point' to be short for 'moment about an axis in the z direction that passes through the point. In 3D moment about a point is a 3-component vector.

The statics-of-a-3D-object recipe

Our recipe here:

1) **Draw a free body diagram (FBD)** of the part of interest.
   Use knowledge of the contact conditions (see Chapter 3) to draw known and unknown aspects of the forces appropriately (seefig. 3.5 on page 154) [hint: use of the form $F\hat{\boldsymbol{\lambda}}$ is often appropriate];
2) **Write equilibrium equations** in terms of the forces (and couples) shown on the FBD;
3) **Solve the equilibrium equations** for unkowns.

The brute-force approach to statically determinate problems

A problem is *statically determinate* when all as-yet-unknown forces can be found using the equilibrium equations. In 3D statics this generally means that the two vector equilibrium equations

$$\sum \vec{\boldsymbol{F}}_i = \vec{\boldsymbol{0}} \qquad \text{and} \qquad \sum \vec{\boldsymbol{M}}_{\mathrm{i/C}} = \vec{\boldsymbol{0}}$$

(where C is any one point that you chose) make up 6 independent scalar equations which you can solve for 6 unknown aspects of the applied forces (say the magnitudes of 6 forces whose directions are known *a priori*)②.

②Most elementary text-book problems are statically determinate. Unfortunately most real-world problems, when you first model them, are not statically determinate.

Alternative equation sets

In 2D single-part statics we noted various alternative to using vector force balance and moment about one point (see page 200). Similarly, here there are also an infinite number of true equilibrium equations, for example

- $\left(\sum \vec{F}_i\right) \cdot \hat{\boldsymbol{\lambda}} = 0$ where $\hat{\boldsymbol{\lambda}}$ is a vector in any direction you please. ; and
- $\left(\sum \vec{M}_{/C}\right) \cdot \hat{\boldsymbol{\lambda}} = 0$. This is moment balance about an axis through C in the $\hat{\boldsymbol{\lambda}}$ direction.

From these there are various ways to extract 6 independent scalar equations, including:

- Cartesian components of force balance and moment balance about any point C: $\sum F_x = 0, \sum F_y = 0, \sum F_z = 0, \sum M_{Cx} = 0, \sum M_{Cy} = 0$, and $\sum M_{Cz} = 0$. This always works, although it does not necessarily minimize algebra.
- Force balance in any 3 non-coplanar directions and moment balance about point C resolved in any three non-coplanar directions.
- Moment balance about 6 independent axes. There seems to be no simple description of indendent axes but for that they give independent equilibrium equations ③. Practically speaking, six moment-about-an-axis equations are likely to be independent if not too many axes are parallel with each other, not too many are coplanar, and not too many intersect at one point.

③ One can test the sufficiency of the equations by this check: if a force at the origin and a couple are the only forces applied to a system, do the equations demand that they must both be zero.

In any case force balance contributes at most 3 independent equations and moment balance can contribute up to 6 (thus rendering force balance a non-essential tool).

Solving 6 equations in 6 unknowns, or even setting up such for computer solution, is relatively time consuming and error prone. Thus one looks for shortcuts when one can, namely:

Useful shortcuts:

- Use moment balance about an axis that intersects, or is parallel to, as many unknown force lines-of-action as possible (thus those forces do not show up in that equilibrium equation);
- Use force balance in a direction orthogonal to as many of the unknown forces as possible (so those forces don't show up in that equation).

Special loadings

Two- and three-force bodies

The concepts of two-force (page 201) and three-force (page 203) bodies are identical in 3D.

- If there are only two forces applied to a body in equilibrium they must be equal and opposite and acting along the line connecting the points of application. The full set of six equations tell you no more.
- If there are only three force applied to a body they must all be in the plane of the points of application and the three forces must have lines of action that intersect at one point. The three equations of force balance are an additional restriction on these three forces.

There are other special loadings where the equilibrium equations offer less than 6 independent equations:

- **2D**. If all of the forces have **lines of action in one plane** then there are only three independent scalar equations and thus one can solve for 3 unknowns. For example, if all the forces lie in the xy plane then automatically $\sum F_{iz} = 0$, $\sum M_{ix/C} = 0$, and $\sum M_{iy/C} = 0$.
- **Concurrent forces**. If all the lines of action intersect in one point, say D, then $\sum \vec{M}_{/\mathrm{D}} = \vec{0}$ is automatically satisfied and only the 3 equations of force balance are independent.
- If all the forces are **parallel** in, say the $\hat{k}$ direction then force balance in the $\hat{\imath}$ and $\hat{\jmath}$ directions as well as moment balance about any axis in the $\hat{k}$ direction are automatically satisfied and there are only three independent equilibrium equataions (say $\sum F_z = 0$, $\sum M_x = 0$ and $\sum M_y = 0$.

What does it mean for a problem to be '2D'?

The world we live in is three dimensional, all the objects to which we wish to study mechanically are three dimensional, and if they are in equilibrium they satisfy the three-dimensional equilibrium equations. How then can an engineer justify doing 2D mechanics? There are a variety of overlapping justifications.

- The 2D equilibrium equations are a subset of the 3D equations. In both 2D and 3D, $\sum F_x = 0$, $\sum F_y = 0$, and $\sum \vec{M}_{/0} \cdot \hat{k} = 0$. So, if when doing 2D mechanics, one just neglects the z component of any applied forces and the x and y components of any applied couples, one is doing correct 3D mechanics, just not all of 3D mechanics. If the forces or conditions of interest to you are contained in the 2D equilibrium equations then 2D mechanics is really 3D mechanics, ignoring equations you don't need.
- If the xy plane is a plane of symmetry for the object and any applied loading, then the three dimensional equilibrium equations not covered by the two dimensional equations, are automatically satisfied. For a

car, say, the assumption of symmetry implies that the forces in the z direction will automatically add to zero, and the moments about the x and y axis will automatically be zero.

- If the object is thin and there are constraint forces holding it near the xy plane, and these constraint forces are not of interest, then 2D statics is also appropriate. This last case is caricatured by all the poor mechanical objects you have drawn so. They are conceptually constrained to lie in your flat paper by invisible slippery glass in front of and behind the paper.

"Internal forces" in 3D

At a free body diagram cut on a long narrow structural piece in 2D there showed two force components, tension and shear, and one scalar moment. In 3D such a cut shows a force $\vec{F}$ and a moment $\vec{M}$ each with three components. If one picks a coordinate system with the x axis aligned with the bar at the cut, the concept of tension remains the same. Tension is the force component along the bar.

$$T = F_x = \vec{F} \cdot \hat{\imath}.$$

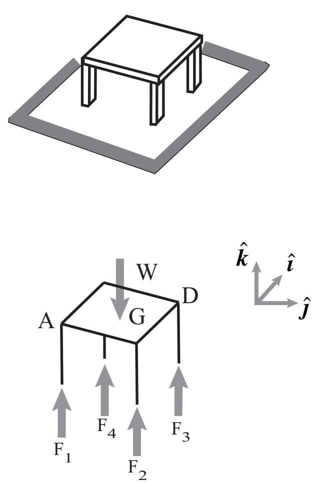
The two other force components, F_x and F_y, are two components of shear. The net shear force is a vector in the plane orthogonal to $\hat{\imath}$.

The new concept, often called *torsion* is the component of $\vec{M}$ along the axis:

$$\text{torsion} = M_x = \vec{M} \cdot \hat{\imath}$$

Torsion is the part of the moment that twists the shaft.

The remaining part of the $\vec{M}$, in the yz plane, is the bending moment. It has two components M_x and M_y.



Filename:tfigure-4legtable

Figure 4.76: A symmetric 4 leg table on a flat floor. Statics is insufficient to find the 4 forces of the ground on the table. The full weight could be carried by either diagonal pair of legs.

The preponderance of statically indeterminate problems

Unfortunately the real world does not often present problems which are at first blush statically determinate. The statics equations are relevant and provide useful information, they are just not sufficient for finding all unknowns of interest. Finding the forces depends on knowing the deformation properties of the structures as well as details of their initial state.

Example: Four-leg furniture

Take the table, chair or bed you are now interacting with. It probably has 4 legs. To keep it simple imagine the legs are on a slippery (negligible-friction) floor and the table is symmetric (left-right and front-back). What are the forces of the floor on the legs? The most we can get from the statics equations is that

$$R_1 = R_3, \quad R_2 = R_4, \qquad \text{and} \qquad R_1 + R_2 = W/2.$$

If we insist that there is no glue between the floor and table then $R_1 \geq 0, R_2 \geq 0, R_3 \geq 0, R_4 \geq 0$. But we still can't find the reactions. Here is the variety of solutions:

$$\begin{array}{llll}
 & R_1 = R_3 = W/2 & \text{and} & R_2 = R_4 = 0 \\
\text{or} & R_1 = R_3 = 0 & \text{and} & R_2 = R_4 = W/2 \\
\text{or} & R_1 = R_3 = W/4 & \text{and} & R_2 = R_4 = W/4 \\
\text{or} & R_1 = R_3 = C & \text{and} & R_2 = R_4 = W/2 - C \\
 & & & (\text{with } C \text{ anything in the interval } \; 0 \le C \le W/2).
\end{array}$$

It takes more than just statics to find the forces. One has to know the exact initial shape of the table and floor and how the table and floor 'give' in response to loads.

The lack of static determinacy of a table is not merely an academic curiosity. If you measured the forces of the floor on your table legs they could well differ noticeably from $W/4$ each. Once friction is taken into account the situation is near hopeless.

Example: Statically determinate stool
Is it even possible to make a stool in 3 dimensions that is statically determinate? Here's one way. Give it three legs. One leg can have a point frictional contact (3 reaction components), one leg can have a wheel (2 reaction components) and one can be frictionless (like with a castored wheel, 1 reaction component). $3+2+1 = 6$.

In general it is hard to hold an object in place in three dimensions in a statically determinate manner. Here are some other ways (besides the unusual stool above):

- with six rods that have ball-and-socket joints at both the object-end and at the ground-end. The rods need to have a variety of orientations and attachment points (this ideas is used in a 'Stewart Platform').
- With one ball-and-socket joint and three rods.
- A 3 leg stool with three wheels (at the contact points one can draw a line in the direction normal to rolling, the three such lines must not intersect at a point).
- With one hinge and one two-force-member rod.
- With one axially sliding hinge and two rods.
- With a single welded connection.

Given that many things are held in place in a manner that seems statically indeterminate what what can one do in practice? A common approach is to remove reaction components that you think are relatively unimportant. Some examples:

- A door held by two hinges. That's 10 reaction components. Usually one replaces, in the analysis, the hinges with ball-and-socket joints. That makes 6 unknown reaction components but is still statically indeterminate no matter what the loading (the force along the line connecting the joints cannot be decomposed into parts acting at each joint). So one joint is allowed to slide along the nominal hinge axis.
- 4 leg furniture. Counting friction there are 12 reaction components. If side loads are not an issue than we can assume-away friction. Thus we

have only 4 reaction components for 3 equations (see table example above). We can get a unique solution by assuming the forces share the symmetry of the table (thus $F_1 = F_2$).

Given this sad state of affairs in 3D it is easy to see why engineers often resort to the more-easily-made determinate 2D world for their models and analyses.

SAMPLE 4.19 3-D moment at the support: A 'T' shaped cantilever beam is loaded as shown in the figure. Find all the support reactions at A.

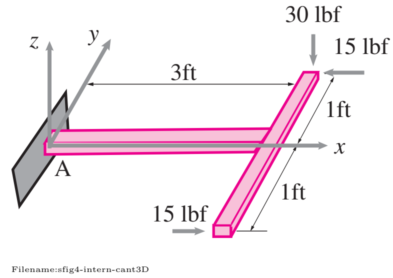


Figure 4.77:

Solution The free-body diagram of the beam is shown in Fig. 4.78. Note that the forces acting on the beam can produce in-plane as well as out of plane moments. Therefore, we show the unknown reactions $\vec{R}$ and $\vec{M}_A$ as general 3-D vectors at A. The moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, gives

$$\vec{M}_A + \vec{r}_{C/A} \times (\vec{F}_1 + \vec{F}_2) + \vec{r}_{D/A} \times \vec{F}_3 = \vec{0}.$$

$$\begin{aligned}\Rightarrow \quad \vec{M}_A &= (\vec{r}_{B/A} + \vec{r}_{C/B}) \times (\vec{F}_1 + \vec{F}_2) + (\vec{r}_{B/A} + \vec{r}_{D/B}) \times \vec{F}_3 \\ &= (\ell\hat{\imath} + a\hat{\jmath}) \times (-F_1\hat{k} - F_2\hat{\imath}) + (\ell\hat{\imath} - a\hat{\jmath}) \times F_3\hat{\imath}.\end{aligned}$$

But $F_3 = -F_2 = F$ (say). Therefore,

$$\begin{aligned} &= (\ell\hat{\imath} + a\hat{\jmath}) \times (-F_1\hat{k} - F\hat{\imath}) + (\ell\hat{\imath} - a\hat{\jmath}) \times F\hat{\imath} \\ &= F_1\ell\hat{\jmath} - F_1 a\hat{\imath} - 2Fa\hat{k} \\ &= 30\,\text{lbf} \cdot 3\,\text{ft}\hat{\jmath} - 30\,\text{lbf} \cdot 1\,\text{ft}\hat{\imath} - 2(30\,\text{lbf} \cdot 1\,\text{ft})\hat{k} \\ &= (-30\hat{\imath} + 90\hat{\jmath} - 60\hat{k})\,\text{lb·ft}.\end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned}\vec{R} &= -\vec{F}_1 - \vec{F}_2 - \vec{F}_3 \\ &= -\vec{F}_1 - \vec{F} + \vec{F} \\ &= -(-F_1\hat{k}) = F_1\hat{k} \\ &= 30\,\text{lbf}\hat{k}.\end{aligned}$$

$$\boxed{\vec{A} = 30\,\text{lbf}\hat{k}, \qquad \text{and} \qquad \vec{M}_A = (-30\hat{\imath} + 90\hat{\jmath} - 60\hat{k})\,\text{lb·ft}}$$

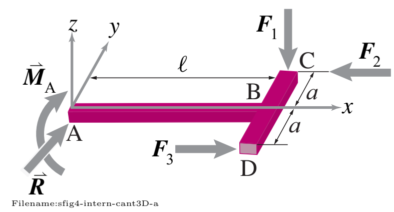


Figure 4.78: Free body diagram of the cantilever.

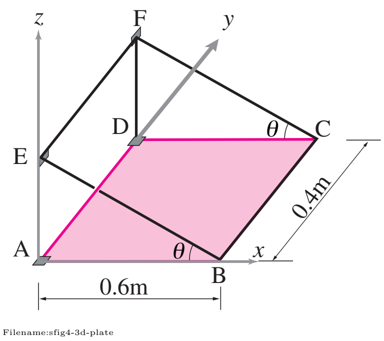


Figure 4.79:

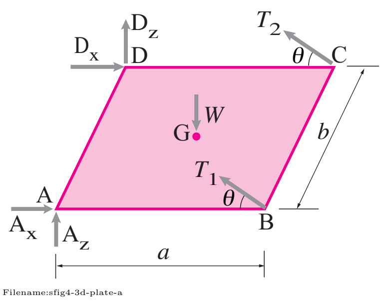


Figure 4.80:

SAMPLE 4.20 An unsolvable problem? A $0.6\,\mathrm{m} \times 0.4\,\mathrm{m}$ uniform rectangular plate of mass $m = 4\,\mathrm{kg}$ is held horizontal by two strings BE and CF and linear hinges at A and D as shown in the figure. The plate is loaded uniformly with books of total mass $6\,\mathrm{kg}$. If the maximum tension the strings can take is $100\,\mathrm{N}$, how much more load can the plate take?

Solution The free-body diagram of the plate is shown in fig. 4.80. Note that we model the hinges at A and D with no resistance in the y-direction. Since the plate has uniformly distributed load (including its own weight), we replace the distributed load with an equivalent concentrated load $\vec{\boldsymbol{W}}$ acting vertically through point G.

The various forces acting on the plate are

$$\vec{\boldsymbol{W}} = -W\hat{\boldsymbol{k}},\ \vec{\boldsymbol{T}}_1 = T_1\hat{\boldsymbol{\lambda}}_{\mathrm{BE}},\ \vec{\boldsymbol{T}}_2 = T_2\hat{\boldsymbol{\lambda}}_{\mathrm{CF}},\ \vec{\boldsymbol{A}} = A_x\hat{\boldsymbol{\imath}} + A_z\hat{\boldsymbol{k}},\ \vec{\boldsymbol{D}} = D_x\hat{\boldsymbol{\imath}} + D_z\hat{\boldsymbol{k}}.$$

Here, $\hat{\boldsymbol{\lambda}}_{\mathrm{BE}} = \hat{\boldsymbol{\lambda}}_{\mathrm{CF}} = -\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{k}} = \hat{\boldsymbol{\lambda}}$ (let). Now, we apply moment equilibrium about point A, *i.e.*, $\sum \vec{\boldsymbol{M}}_{\mathrm{A}} = \vec{\boldsymbol{0}}$.

$$\vec{r}_{\mathrm{B}} \times \vec{\boldsymbol{T}}_1 + \vec{r}_{\mathrm{C}} \times \vec{\boldsymbol{T}}_2 + \vec{r}_{\mathrm{G}} \times \vec{\boldsymbol{W}} + \vec{r}_{\mathrm{D}} \times \vec{\boldsymbol{D}} = \vec{\boldsymbol{0}} \tag{4.33}$$

where,

$$\begin{aligned}
\vec{r}_{\mathrm{B}} \times \vec{\boldsymbol{T}}_1 &= a\hat{\boldsymbol{\imath}} \times T_1\hat{\boldsymbol{\lambda}} = -aT_1\sin\theta\,\hat{\boldsymbol{\jmath}} \\
\vec{r}_{\mathrm{C}} \times \vec{\boldsymbol{T}}_2 &= (a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) \times T_2\hat{\boldsymbol{\lambda}} = T_2 b\sin\theta\hat{\boldsymbol{\imath}} - T_2 a\sin\theta\,\hat{\boldsymbol{\jmath}} + T_2 b\cos\theta\hat{\boldsymbol{k}} \\
\vec{r}_{\mathrm{G}} \times \vec{\boldsymbol{W}} &= \frac{1}{2}(a\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}}) \times (-W\hat{\boldsymbol{k}}) = -\frac{Wa}{2}\hat{\boldsymbol{\imath}} + \frac{Wa}{2}\hat{\boldsymbol{\jmath}} \\
\vec{r}_{\mathrm{D}} \times \vec{\boldsymbol{D}} &= b\hat{\boldsymbol{\jmath}} \times (D_x\hat{\boldsymbol{\imath}} + D_z\hat{\boldsymbol{k}}) = D_z b\hat{\imath} - D_x b\hat{\boldsymbol{k}}.
\end{aligned}$$

Substituting these products in eqn. (4.33) and dotting with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$, we get

$$T_2\sin\theta + D_z = \frac{W}{2} \tag{4.34}$$

$$T_2\cos\theta - D_x = 0 \tag{4.35}$$

$$(T_1 + T_2)\sin\theta = \frac{W}{2}. \tag{4.36}$$

The force equilibrium, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, gives

$$\vec{\boldsymbol{A}} + \vec{\boldsymbol{D}} + \vec{\boldsymbol{T}}_1 + \vec{\boldsymbol{T}}_2 + \vec{\boldsymbol{W}} = \vec{\boldsymbol{0}}.$$

Again, substituting the forces in their component form and dotting with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{k}}$ (there are no $\hat{\boldsymbol{\jmath}}$ components), we get

$$A_x + D_x - (T_1 + T_2)\cos\theta = 0$$

$$\Rightarrow \quad A_x - T_1\cos\theta = 0 \tag{4.37}$$

$$A_z + D_z + (T_1 + T_2)\sin\theta = 0$$

$$\Rightarrow \quad A_z + T_1\sin\theta = \frac{W}{2}. \tag{4.38}$$

These are all the equations that we can get. Now, note that we have five independent equations (eqns. (4.34) to (4.38)) but six unknowns. Thus we cannot solve for the unknowns uniquely. This is an indeterminate structure! No matter which point we use for our moment equilibrium equation, we will always have one more unknown than the number of independent equations. We can, however, solve the problem with an extra assumption (see comments below) — the structure is symmetric about the axis passing through G and parallel to x-axis. From this symmetry we conclude that $T_1 = T_2$. Then, from eqn. (4.37) we have

$$2T\sin\theta = \frac{W}{2} \quad \Rightarrow \quad T = \frac{W}{4\sin\theta}.$$

We can now find the maximum load that the plate can take subject to the maximum allowable tension in the strings.

$$\begin{aligned} W &= 4T\sin\theta \\ \Rightarrow \quad W_{\max} &= 4T_{\max}\sin\theta \\ &= 4(100\,\mathrm{N})\cdot\frac{1}{2} = 200\,\mathrm{N}. \end{aligned}$$

The total load as given is $(6+4)\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2} = 98.1\,\mathrm{N} \approx 100\,\mathrm{N}$. Thus we can double the load before the strings reach their break-points. Now the reactions at D and A follow from eqns. (4.34), (4.35), (4.37), and (4.38).

$$\begin{aligned} D_z = A_z = \frac{W}{2} - T\sin\theta &= \frac{W}{2} \\ D_x = A_x = T\cos\theta &= \frac{W}{4}\cot\theta. \end{aligned}$$

$$\boxed{W_{\max} = 200\,\mathrm{N}}$$

Comments:

1. We got only five independent equations (instead of the usual 6) because the force equilibrium in the y-direction gives a zero identity (0 = 0). There are no forces in the y-direction. The structure seems to be unstable in the y-direction — if you push a little, it will move. Remember, however, that it is so because we chose to model the hinges at A and D that way keeping in mind the only vertical loading. The actual hinges used on a bookshelf will not allow movement in the y-direction either. If we model the hinges as ball and socket joints, we introduce two more unknowns, one at each joint, and get just one more scalar equation. Thus we are back to square one. There is no way to determine A_y and D_y from equilibrium equations alone.
2. The assumption of symmetry and the consequent assumption of equality of the two string tensions is, mathematically, an extra independent equation based on deformations (strength of materials). At this point, you may not know any strength of material calculations or deformation theory, but your intuition is likely to lead you to make the same assumption. Note, however, that this assumption is sensitive to accuracy in fabrication of the structure. If the strings were slightly different in length, the angles were slightly off, or the wall was not perfectly vertical, the symmetry argument would not hold and the two tensions would not be the same.

Most real problems are like this — indeterminate. Our modelling, which requires insight, makes them determinate and solvable.

Problems for Chapter 4

Statics of one object

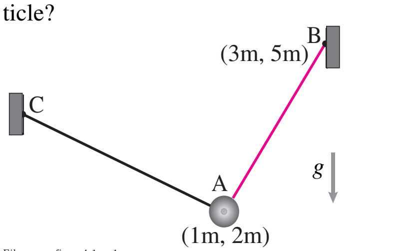
4.1 Static equilibrium of a particle

Preparatory Problems

4.1.1 What is a particle?

4.1.2 What are the equations of equilibrium for a particle (also called "*equilibrium conditions*", "*force balance*", or "*linear momentum balance for statics*"?

4.1.3 A string connects a particle A at (1m, 2m) to a support B at (3m, 5m). The tension in the string is 10N. There are other strings also holding the particle in place. What is the force of string AB on the particle?



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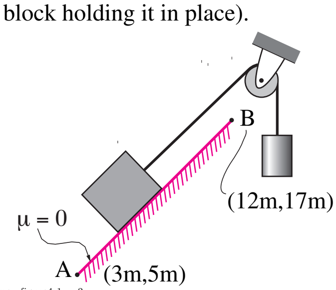
Problem 4.1.3

4.1.4 A frictionless ramp connects A at (3m, 5m) to B at (12m, 17m). The ramp pushes a block with a force of-50N. Express the force from the ramp as a vector $\vec{F}$ (ignore the other forces that also act on the block holding it in place).



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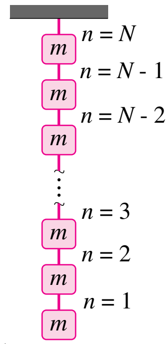
Problem 4.1.4

4.1.5 N small blocks each of mass m hang vertically as shown, connected by N inextensible strings. Find the tension T_n in string n. *



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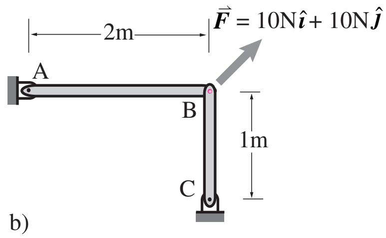
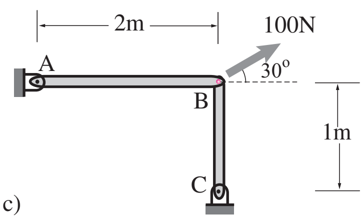
Problem 4.1.5

4.1.6 For each situation below, find the tensions in the two rods.

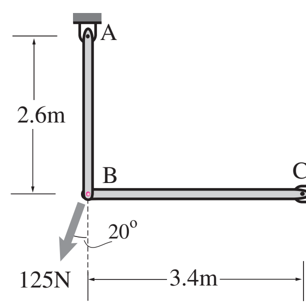
a)



b)

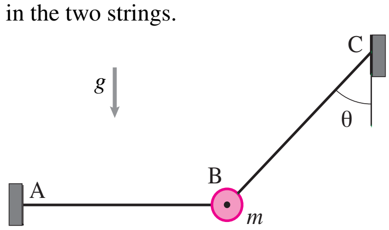


c)



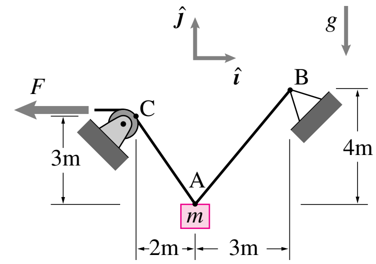
Filename:pfigure-particle-1
Problem 4.1.6

4.1.7 A particle of mass $m = 2\,\text{kg}$ hangs from strings AB and AC as shown. AB is horizontal and $\theta = 45^{\circ}$. Find the tension in the two strings.



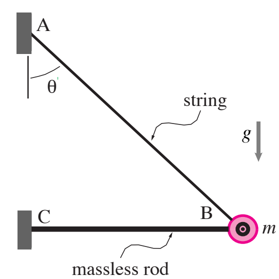
Filename:pfigure4-1-rp1
Problem 4.1.7

4.1.8 What force should be applied to the end of the string over the pulley at C so that the mass at A is at rest?



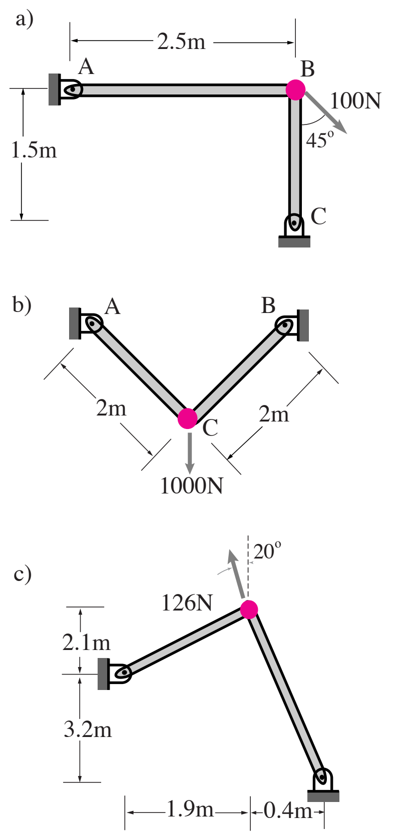
Filename:f92h1p1-a
Problem 4.1.8

4.1.9 A particle of mass $\text{m} = 5\,\text{kg}$ at the end of a horizontal massless rod CB of length 1.2 m is held in place with the help of a string AB that makes an angle $\theta = 45^{\circ}$ with the vertical in the equilibrium position. Find the tension in the bar CB (it is ok to have negative tension).



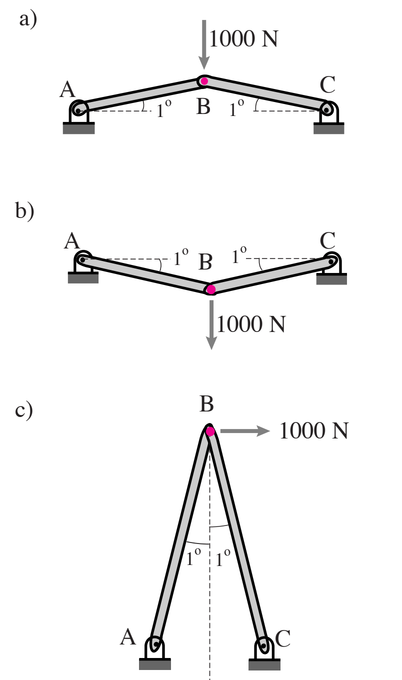
Filename:pfigure4-1-rp2
Problem 4.1.9

4.1.10 For each structure shown below, find the tension in each rod. (Note the tension can be less than zero.)



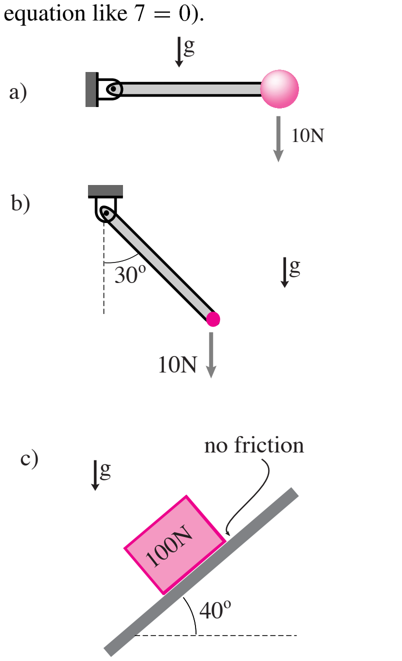
Filename:pfigure-particle-2a
Problem 4.1.10

4.1.11 In the following structures, a pin connects two thin bars that are very nearly either horizontal or vertical. Find the tensions in each rod under the applied loads. (Note the tension is less than zero for some of the rods.)



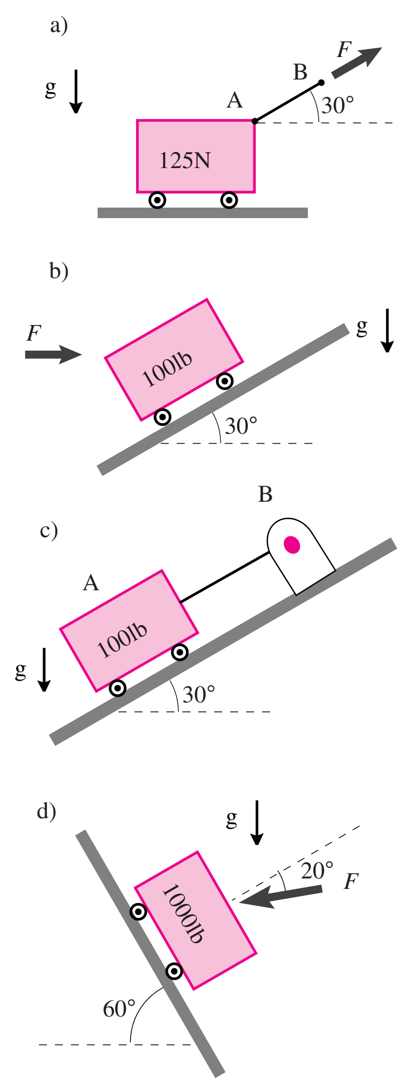
Filename:pfigure-particle-2b
Problem 4.1.11

4.1.12 For each situation shown below, equilibrium is not possible. Write the vector equation for force balance and show that it has no solutions (*i.e.,* leads to an equation like $7 = 0$).



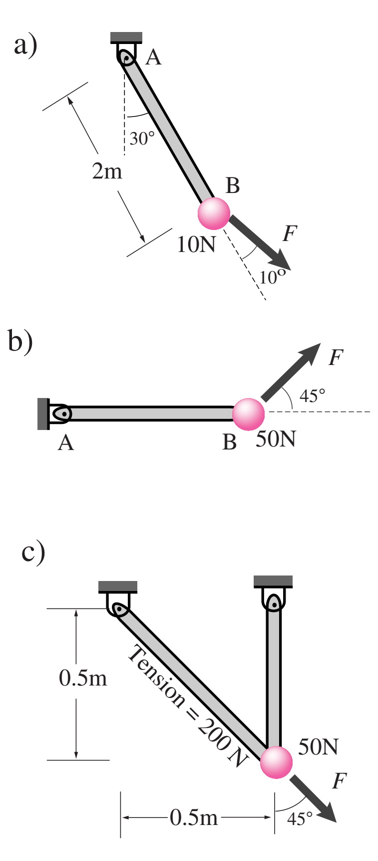
Filename:pfigure-particle-3
Problem 4.1.12

4.1.13 Assume no sliding friction ($\mu = 0$). Assume equilibrium. Find all reactions, tensions, and forces. *



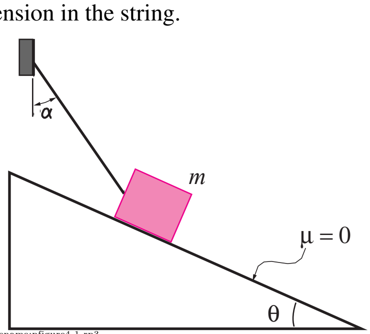
Filename:pfigure-particle-4
Problem 4.1.13

4.1.14 Find the unknown forces and tensions in each structure shown below.



Filename:pfigure-particle-5
Problem 4.1.14

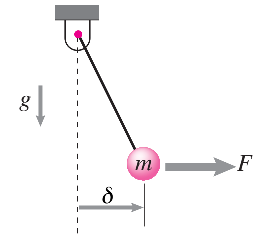
4.1.15 A block of mass $m = 5\,\mathrm{kg}$ rests on a frictionless inclined plane as shown in the figure. Let $\theta = \alpha = 30^\circ$. Find the tension in the string.



Filename:pfigure4-1-rp3
Problem 4.1.15

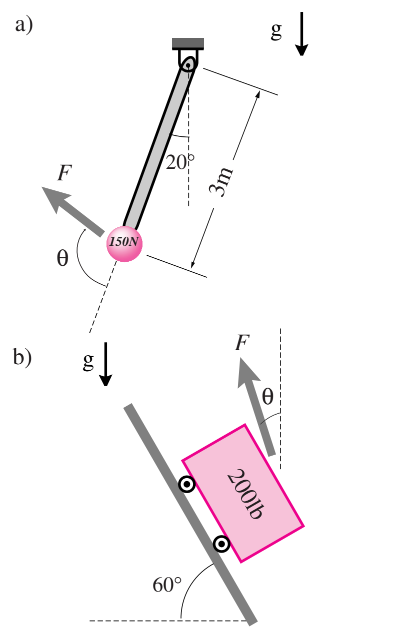
More-Involved Problems

4.1.16 For small δ what is the relation between F and δ (and g and ℓ) for a static pendulum?



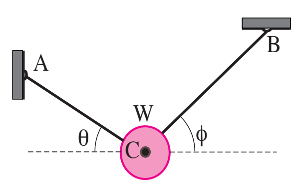
Filename:pfigure-spring3
Problem 4.1.16

4.1.17 In the situations shown in the figures, find the value of θ that minimizes F. What is the corresponding value of F in each case?



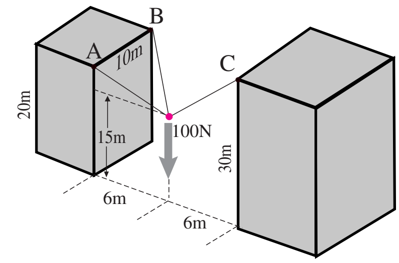
Filename:pfigure-particle-6
Problem 4.1.17

4.1.18 An object of weight $W = 10\,\mathrm{N}$ is held in equilibrium in the vertical plane by two strings AC and BC. Let $\theta = 30^\circ$ and $0 \le \phi \le 90^\circ$. Find and plot the tension in the two strings against ϕ and comment on the variation of the tension.



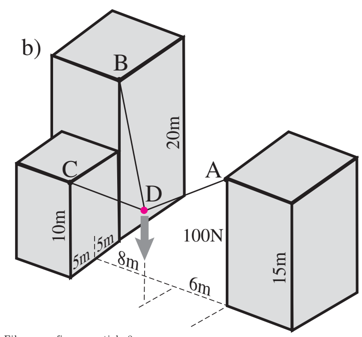
Filename:pfigure4-1-rp5
Problem 4.1.18

4.1.19 Find the tensions in the three strings shown in the figure.



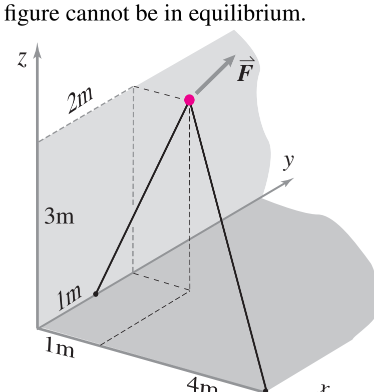
Filename:pfigure-particle-7
Problem 4.1.19

4.1.20 Find the tensions in the three strings shown in the figure. String CD is horizontal and the force at D is 100 N straight down. [Hint: this problem has a trick to it.]
*



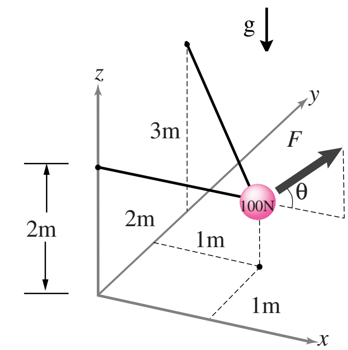
Filename:pfigure-particle-8
Problem 4.1.20

4.1.21 Show that the particle acted upon by the given force $\vec{F} = (3\hat{\imath} + 4\hat{\jmath} + 5\hat{k})\,\mathrm{N}$, and held by the two bars as shown in the figure cannot be in equilibrium.



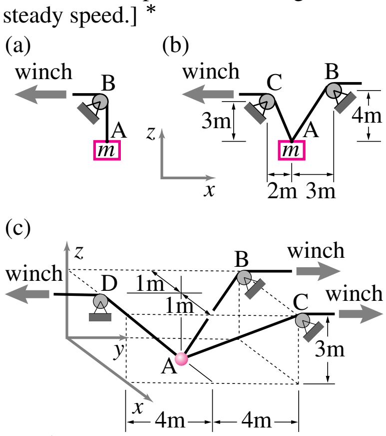
Filename:pfigure-particle-3D3
Problem 4.1.21

4.1.22 In the figure shown, the force $\vec{F}$ acts on the particle (weighing 100 N) in the x-z plane. Find F as a function of θ for equilibrium of the particle. For what value of θ, the required force is minimum?



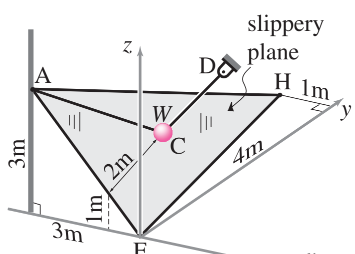
Filename:pfigure-particle-3D4
Problem 4.1.22

4.1.23 For the three cases (a), (b), and (c), below, find the tension in the string AB. In all cases the strings hold up the mass $m = 3\,\mathrm{kg}$. You may assume the local gravitational constant is $g = 10\,\mathrm{m/s^2}$. In all cases the winches are pulling in the string so that the velocity of the mass is a constant $4\,\mathrm{m/s}$ upwards (in the $\hat{k}$ direction). [Note that in problems (b) and (c), in order to pull the mass up at constant rate the winches must pull in the strings at an unsteady speed.] *



Filename:f92h1p1
Problem 4.1.23

4.1.24 A block of weight W , held by two strings AC and DC, rests on a slippery plane AEH. String CD is parallel to EH. Find the tensions in the two strings and the reaction of the plane. You may approximate AC to lie in the plane AEH.



Filename:pfigure-particle-9
Problem 4.1.24

4.2 Static equilibrium of one body

Preparatory Problems

4.2.1 For problems below, assume a 2D free-body diagram has been drawn where forces $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_5$ are applied at positions $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_5$ relative to the origin. Use this information in the answers below.

a) What is the force balance equation?

b) What is the moment balance equation about the origin?

c) What are equilibrium conditions?

d) Write equilibrium conditions as many different ways as you can.

e) How many independent scalar equations can one write using various force and moment balance equations?

f) If force $\vec{F}_4$ is moved to a new position along the its direction, which equilibrium equations are changed and which are not?

g) If force $\vec{F}_4$ is displaced sideways relative to its direction, which equilibrium equations are changed and which are not?

4.2.2 What is the meaning of *the line of action of a force*?

4.2.3 If only two forces, $\vec{F}_1$ and $\vec{F}_2$, act on a body at $\vec{r}_1$ and $\vec{r}_2$, what do the equilibrium conditions tell you about the two forces?

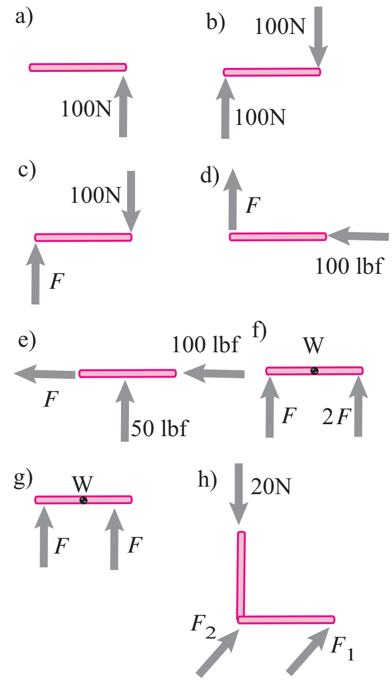
4.2.4 If only three forces, $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$, act on a body at $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$, what do the equilibrium conditions tell you about the three forces?

4.2.5 Which of the bars below cannot possibly be in equilibrium and which ones can? (Where the center of mass is indicated, assume non-zero weight acting vertically downwards. Assume dimensions as needed.)

i) Explain in words.

ii) Explain using equations.

Note that scalars (e.g., F, F_1, etc.) can be positive or negative.



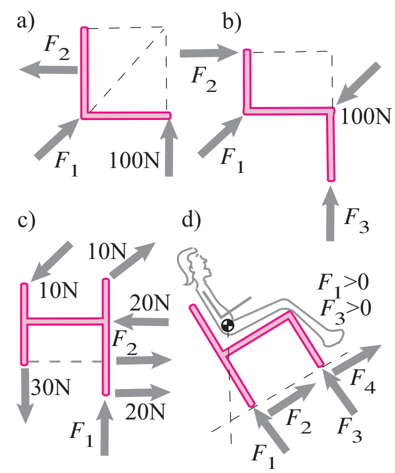
Filename:pfigure-4-2-manybars1
Problem 4.2.5

4.2.6 Which of the objects below cannot possibly be in equilibrium and which ones can? (Where the center of mass is indicated, assume non zero weight acting vertically downwards. Assume dimensions as needed.)

a) Explain in words.

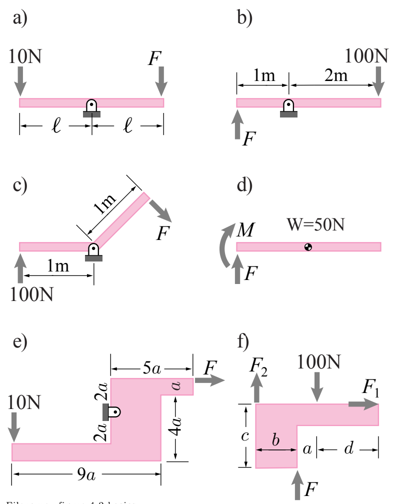
b) Explain using equations.

Note that scalars (e.g., F, F_1, etc.) can be positive or negative unless mentioned otherwise.



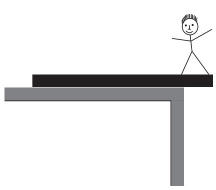
Filename:pfigure-4-2-manybars2
Problem 4.2.6

4.2.7 In the problems shown below, find F for equilibrium.



Filename:pfigure-4-2-basics
Problem 4.2.7

4.2.8 A straight uniform 2000 N beam is 6 m long. It rests on a flat roof with a 2 m overhang. How far out the overhang can an 800 N person walk without the beam tipping over?

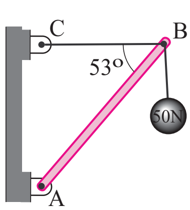


Filename:pfigureSoodak4-50
Problem 4.2.8

4.2.9 The uniform bar AB is 5 m long and weighs 100 N. It is pinned at A and supported by the horizontal cord BC attached at end B. A 50 N weight hangs from end B.

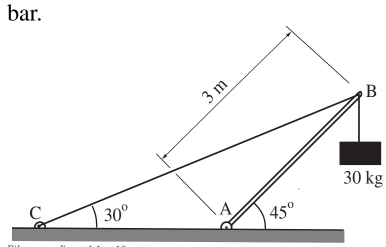
a) Find the tension in cord C.

b) Find the magnitude and direction of the force exerted on the pin at A by the bar.



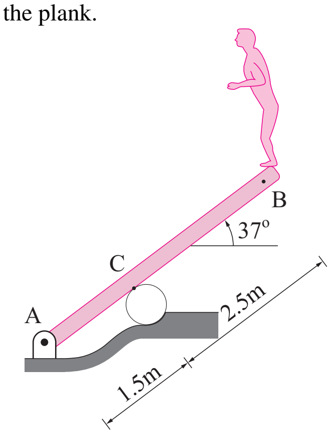
Filename:pfigureSoodak4-11
Problem 4.2.9

4.2.10 For static equilibrium of the system and the configuration shown in the figure, find the support reaction at end A of the bar.



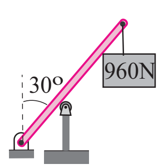
Filename:pfigure4-1-rp10
Problem 4.2.10

4.2.11 A 400 N child stands on the end of a uniform 800 N diving plank which is pinned on one end and which also rests on a log (idealized as frictionless). Find the force of the log on the plank and of the pin on the plank.



Filename:pfigureSoodak4-46
Problem 4.2.11

4.2.12 A negligible weight 6 m rod is pinned at one end and leans over a frictionless wall a third of the way up from the bottom. Find the forces of the wall and the pin on the rod.



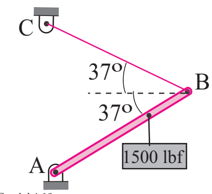
Filename:pfigureSoodak4-47
Problem 4.2.12

More-Involved Problems

4.2.13 The uniform boom AB is 20 ft long and weighs 150 lbf. A 1500 lbf weight is suspended from a point 5 ft from end B. The boom is pinned at A and supported by the cable BC attached at end B.

a) Find the tension in the cable.

b) Find the force exerted on the boom by the pin at A.

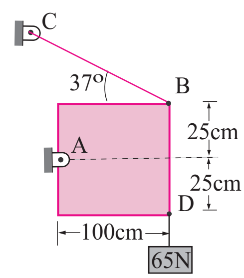


Filename:pfigureSoodak4-12
Problem 4.2.13

4.2.14 The 30 N uniform rectangular plate is supported by a pin at A and cable BC attached at corner B. A 65 N weight hangs from corner D.

a) Find the tension in the cable.

b) Find the force exerted on the plate by the pin at A.



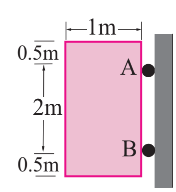
Filename:pfigureSoodak4-13
Problem 4.2.14

4.2.15 A uniform door of width 1 m and weight 200 N is supported by two hinges a distance 2 m apart.

a) Find the horizontal component of the force by the door on the upper hinge.

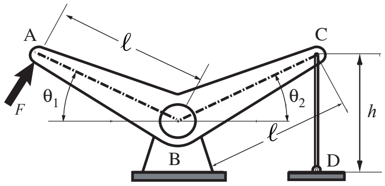
b) Find the horizontal component of the force by the door on the lower hinge.

c) Can you find the vertical force of the door on the upper or lower hinge? If not, what do you know about these forces?



Filename:pfigureSoodak4-15
Problem 4.2.15

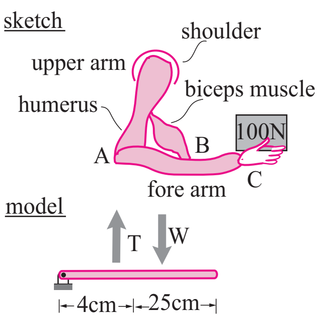
4.2.16 In the mechanism shown, find the maximum force F that can be applied at A normal to the link AB such that the magnitude of the force in rod CD does not exceed 10 kN.



Filename:pfigure4-1-rp13
Problem 4.2.16

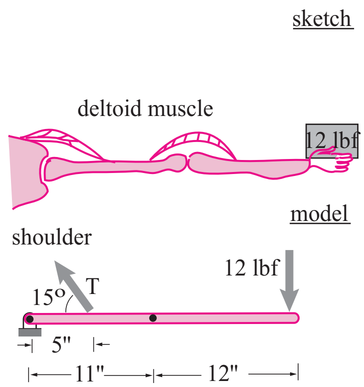
4.2.17 For biomechanics purposes muscles are commonly modeled as massless cables and joints (elbow, shoulder, hip, ankle, etc) as frictionless hinges connecting rigid bones. You will find that the muscle tension and joint reaction forces are large compared to the loads being carried. This is a general feature in biomechanics because muscles usually have short lever-arms relative to the bone lengths.

A human forearm weights 14 N and supports a 100 N weight. Find the muscle tension and the force of the upper arm on the forearm at the elbow.



Filename:pfigureSoodak4-23
Problem 4.2.17

4.2.18 See Problem 4.2.17. An arm weighs 7 pounds and supports a 12 pound weight. Find the tension in the deltoid muscle and the force of the body on the arm at the shoulder joint.

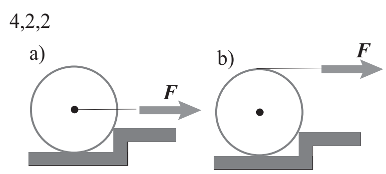


Filename:pfigureSoodak4-24
Problem 4.2.18

4.2.19 A 240 N roller is 1 m in diameter. It is being pulled over a 0.1 m curb with a horizontal rope. The roller does not slide on the curb.

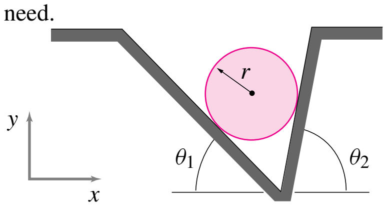
a) What is the force required to lift the roller over the curb with the rope attached at the middle?

b) What is the force required if instead the rope is instead wrapped around the roller as shown?



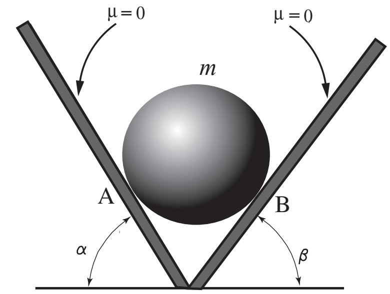
Filename:pfigureSoodak4-22
Problem 4.2.19

4.2.20 What are the forces on the disk due to the groove? Define any variables you need.



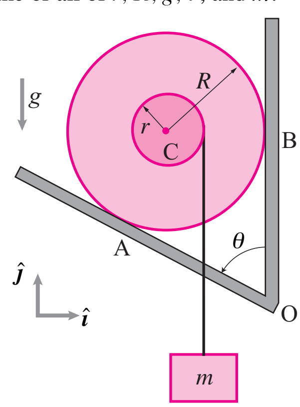
Filename:ch2-5-b
Problem 4.2.20

4.2.21 A solid sphere of mass $m = 5\,\mathrm{kg}$ and radius $R = 250\,\mathrm{mm}$ rests between two frictionless inclined planes. Let $\alpha = 60^\circ$. Find the magnitudes of normal reactions of the plane as functions of β and plot normalized reactions (N_1/mg and N_2/mg for $0 < \beta \leq 90^\circ$). Comment on the plot.



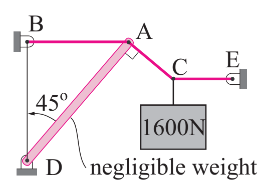
Filename:pfigure4-1-rp6
Problem 4.2.21

4.2.22 Assuming the spool is massless and that there is no friction at point A, find the force on the spool at point B in order to maintain equilibrium. Answer in terms of some or all of r, R, g, θ, and m.



Filename:pfigure-spoolinwedge
Problem 4.2.22

4.2.23 Find the tension in cord AB.



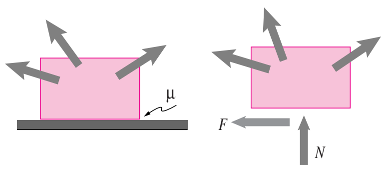
Filename:pfigureSoodak4-48
Problem 4.2.23

4.3 Friction and equilibrium

Preparatory Problems

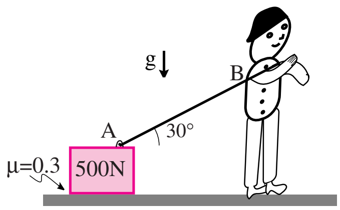
4.3.1 For the block shown in the figure, what do you know about F if

a) the block is sliding to the right
b) the block is sliding to the left
c) the block is not sliding.



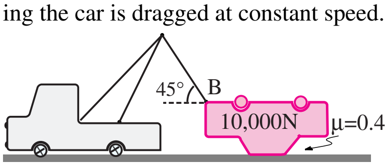
Filename:pfigure-particle-block1
Problem 4.3.1

4.3.2 A block weighing 500 N is dragged slowly on the ground as shown in the figure. Find the tension in the string?



Filename:pfigure-particle-10a
Problem 4.3.2

4.3.3 Find the tension in the cable assuming the car is dragged at constant speed.



Filename:pfigure-particle-10b
Problem 4.3.3

4.3.4 Consider the tow truck dragging the car in Problem 4.3.3 again. In order to ensure safety, you would like to minimize the tension in the rope attached to the car. Assume that the angle shown at point B is θ.

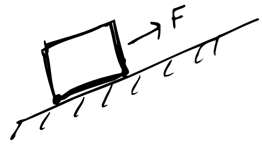
a) What value of θ minimizes the tension in the rope?
b) What is the corresponding value of T?
c) What is the force of the ground on the car?

.

More-Involved Problems

4.3.5 A 30,000 N stone cube one meter on a side was dragged up a 20° ramp by 100 of a Pharaoh's slaves by a rope parallel to the slope. The coefficient of friction was $\mu = 0.2$. Assume all the ground contact is at the front and back edges of the cube.

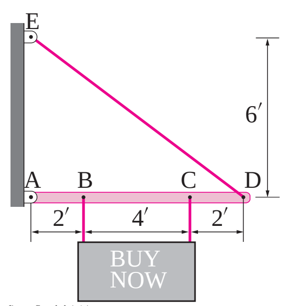
a) Find the dragging force.
b) Find the force on the front and back edges of the cube.



Filename:pfigureSoodak4-52
Problem 4.3.5

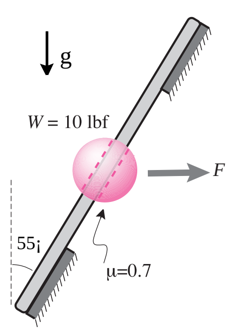
4.3.6 The 20 lbf uniform rectangular sign is suspended from the strut ABCD by two wires. The strut is supported by cable DE and a pin at A.

a) Find tension DE.
b) Suppose the workers who hung the sign forgot to pin the strut to the wall at point A. What is the least value of μ between the strut and wall for the system to maintain equilibrium.



Filename:pfigureSoodak4-14
Problem 4.3.6

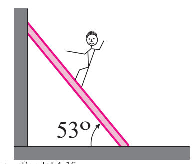
4.3.7 A horizontal force F is applied to slide the bead on the rod shown in the figure. Find the value of F that is required to initiate sliding. Why is F so big or small?



Filename:pfigure-particle-12a
Problem 4.3.7

4.3.8 A 130 pound person climbs a 120 pound ladder that is 30 ft long. The ladder leans against a frictionless wall and makes an angle of 53° with the ground.

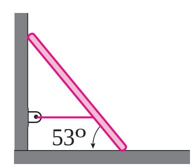
a) Find the force of the ground on the ladder when the person is one third of the way up the ladder.
b) When the person gets two thirds of the way up the bottom of the ladder starts to slip. What is μ between the ladder and ground?



Filename:pfigureSoodak4-16
Problem 4.3.8

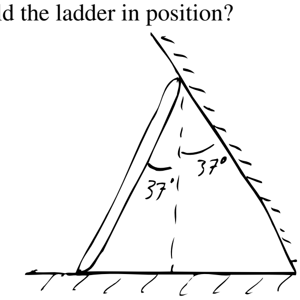
4.3.9 A uniform 200 N, 10 m ladder leans between a frictionless ground and a wall. It is kept from sliding away from the wall by a horizontal cable 2 m above the ground. Find

a) The tension in the cable.
b) The force of the ground on the ladder.
c) The force of the wall on the ladder.



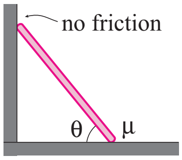
Filename:pfigureSoodak4-17
Problem 4.3.9

4.3.10 A uniform ladder of length ℓ and weight W rests against a frictionless slanted wall. What is the minimum μ between ladder and ground that is needed to hold the ladder in position?



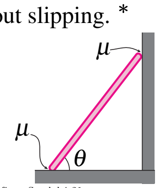
Filename:pfigureSoodak4-19
Problem 4.3.10

4.3.11 A uniform ladder with weight W and length ℓ leans against a frictionless vertical wall and makes an angle θ with the ground. In terms of the given quantities, find the values of μ at the ground for which the ladder will not slip.



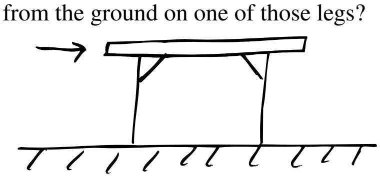
Filename:pfigureSoodak4-20
Problem 4.3.11

4.3.12 A uniform ladder with weight W and length ℓ leans against a frictional vertical wall and is supported by the frictional ground. The same coefficient of friction μ applies to the wall and to the ground. In terms of the given quantities, find the values of θ between the ladder and ground for which the ladder can be in equilibrium without slipping. *



Filename:pfigureSoodak4-21
Problem 4.3.12

4.3.13 A 2 m square 500 N 4-leg table is pushed across a floor by a horizontal force at its top surface and normal to one edge. Assume the table is 0.8 m high, that its center of mass is 0.6 m high and that all four legs slide on the floor with friction coefficient $\mu = 0.3$. Which legs carry the most load and what is the magnitude of the force from the ground on one of those legs?

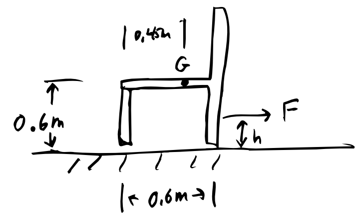


Filename:pfigureSoodak4-53
Problem 4.3.13

4.3.14 An 80 N chair is pulled steadily to the right by a rope. The coefficient of friction between the ground and floor is $\mu = 0.25$.

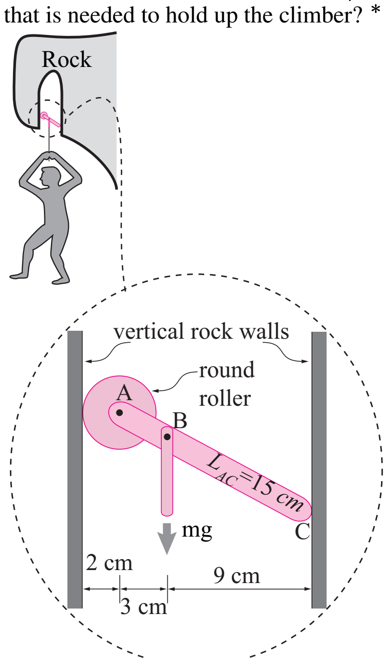
a) What is the force needed to pull the chair?

b) What is the highest point on the chair that the rope can be tied without the chair tipping over?



Filename:pfigureSoodak4-54
Problem 4.3.14

4.3.15 A candidate rock-climbing device consists of a roller (radius 2 cm) frictionlessly pinned at A to diagonal-member AC. The length of AC from point A to the wall-contact point at C is $L_{AC} = 15cm$. The climber ($m = 60$ kg) hangs from a rope connected to AC by a pin at B. B is on the line AC and located as shown in the figure. If needed, assume $g = 10$N/kg. What is the minimum coefficient of friction μ at C that is needed to hold up the climber? *



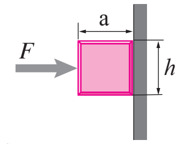
Filename:F02final1-Climber
Problem 4.3.15

4.3.16 A uniform $W = 50$ N block with width a and height h is held against a wall with a horizontal force of F acting on the left side half way up the block. The block is prevented from sliding down the wall by friction. There is no glue (no tension between wall and block).

a) Assuming friction is high enough to prevent slip, what is the minimum F to keep the block from tipping away from the wall?

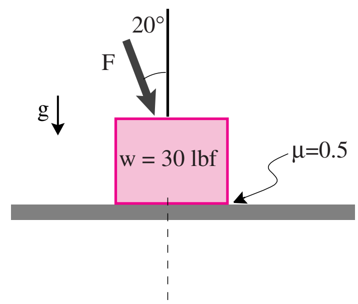
b) For twice that F what is the minimum friction to keep the block from sliding down the wall?

c) For $a = h$ and $F = 3W$ the resultant of all the wall normal and contact forces is a single force that acts on the right side of the block at what position y above the bottom of the block?



Filename:pfigureSoodak4-51
Problem 4.3.16

4.3.17 In the figure shown, what is force F required to push the block along the floor? This problem has no solution. Explain why (using free-body diagrams and mechanics equations).



Filename:pfigure-particle-13
Problem 4.3.17

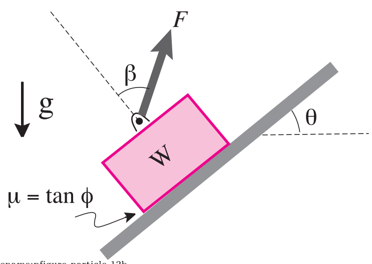
4.3.18 Consider the situation shown shown in the figure. Give your answers to the following questions in terms of some or all of W, θ, β, g, and ϕ or μ. Assume all values of β and $0 \le \theta \le \pi/2$, $0 \le \mu$, $g > 0$, $W > 0$.

a) Assume the block slides seadily uphill. Find F. For what values of θ, β, and μ does no such F exist (allow $F < 0$)?

b) Assume the block slides downhill. What is F? For what values of θ, β, and μ does no such solution exist?

c) assume the block is not sliding. What are the possible values of F? For what values of θ, β, and μ does such a solution exist?

d) For what values of θ, β, and μ can you have the block slide up, slide down, or lock (that is, no incipient slip) depending on the value of F?

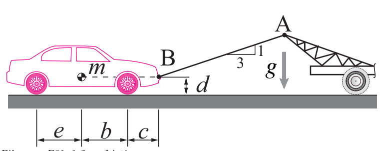


Filename:pfigure-particle-12b
Problem 4.3.18

4.3.19 A car is being towed. Unfortunately all the wheels are locked and skidding with friction coefficient μ. The tow cable AB has a slope of $1/3$.

a) In terms of some or all of e, b, c, d, m, g & μ, find the tension in the tow cable AB. *

b) Instead of an angle with slope $1/3$, what should the cable angle be to minimize the tension. *



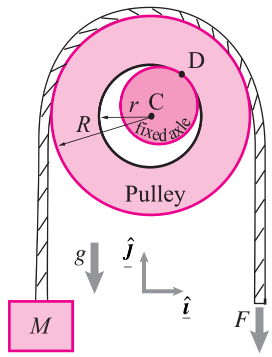
Filename:F01p1-2-carfriction
Problem 4.3.19

4.3.20 A weight M is steadily raised by pulling with a force F on a rope going over a negligible-mass pulley on an unlubricated journal bearing (no ball bearings). For an ideal frictionless pulley $F = Mg$. Here we have a friction coefficient between the bearing and its axle which is is $\mu = \tan\phi$. [Hint: Finding the location of the contact point D is probably part of your solution.]

a) Find F in terms of M, g, R, r and μ (or ϕ or $\sin\phi$ or $\cos\phi$ — whichever is most convenient. For example $\cos(\tan^{-1}(\mu))$ is more simply expressed as $\cos\phi$), *and*

b) Evaluate F in the special case that $M = 100\,\text{kg}, g = 10\,\text{m/s}^2, r = 1\,\text{cm}, R = 2\,\text{cm}$, and $\mu = \sqrt{3}/3$ (so $\phi = \pi/6, \sin\phi = 1/2, \cos\phi = \sqrt{3}/2$).

c) Referring back to the general case, for fixed r, R, M, and g what happens to F as $\mu \to \infty$ (does it go to ∞)?



Filename:pfigure-s01-p2-3
Problem 4.3.20

4.3.21 A reel of mass M and outer radius R is connected by a horizontal string from point P across a pulley to a hanging object of mass m. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle θ. There is no slip between the reel and the slope. There is gravity.

a) Find the ratio of the masses so that the system is at rest. *

b) Find the corresponding tension in the string, in terms of M, g, R, and θ. *

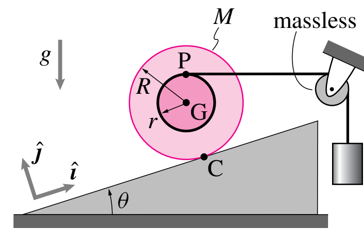
c) Find the corresponding force on the reel at its point of contact with the slope, point C, in terms of M, g, R, and θ. *

Harder Draw a careful sketch and find a point where the lines of action of the gravity force and string tension intersect. For the reel to be in static equilibrium, the line of action of the reaction force at C must pass through this point. Using this information, what must the tangent of the angle ϕ of the reaction force at C be, measured with respect to the normal to the slope? Does this answer agree with that you would obtain from your answer in part(c)? *

d) What is the relationship between the angle ϕ of the reaction at C, measured with respect to the normal to the ground, and the mass ratio required for static equilibrium of the reel? *

e) What is the minimum coefficient of friction μ at C needed to prevent slip.

Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{F}_C = Mg\hat{\jmath}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = 2$ and $\vec{F}_C = Mg(\hat{\imath} + 2\hat{\jmath})$.



Filename:pfigure2-blue-47-3-a
Problem 4.3.21

4.3.22 This problem is similar to problem 4.3.21. A reel of mass M and outer radius R is connected by an inextensible string from point P across a pulley to a hanging object of mass m. The inner cylinder of the reel has radius $r = \frac{1}{2}R$. The slope has angle θ. There is no slip between the reel and the slope. There is gravity. In terms of M, g, R, and θ, find:

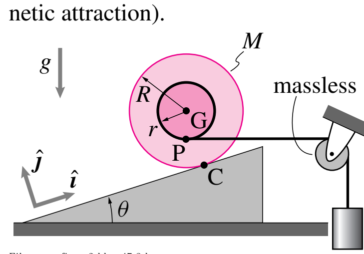
a) the ratio of the masses so that the system is at rest, *

b) the corresponding tension in the string, and *

c) the corresponding force on the reel at its point of contact with the slope, point C. *

d) What is the minimum coefficient of friction μ at C needed to prevent slip.

Check that for $\theta = 0$, your solution gives $\frac{m}{M} = 0$ and $\vec{F}_C = Mg\hat{\jmath}$ and for $\theta = \frac{\pi}{2}$, it gives $\frac{m}{M} = -2$ and $\vec{F}_C = Mg(\hat{\imath} - 2\hat{\jmath})$.The negative mass ratio is impossible since mass cannot be negative and the negative normal force is impossible unless the wall or the reel or both can 'suck' or they can 'stick' to each other (that is, provide some sort of suction, adhesion, or magnetic attraction).

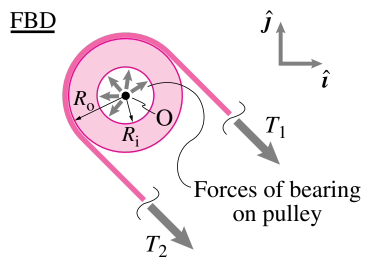


Filename:pfigure2-blue-47-3-b
Problem 4.3.22

4.3.23 Assume a massless pulley is round and has outer radius R_2. It slides on a shaft that has radius R_i. Assume there is friction between the shaft and the pulley with coefficient of friction μ, and friction angle ϕ defined by $\mu = \tan(\phi)$. Assume the two ends of the line that are wrapped around the pulley are parallel.

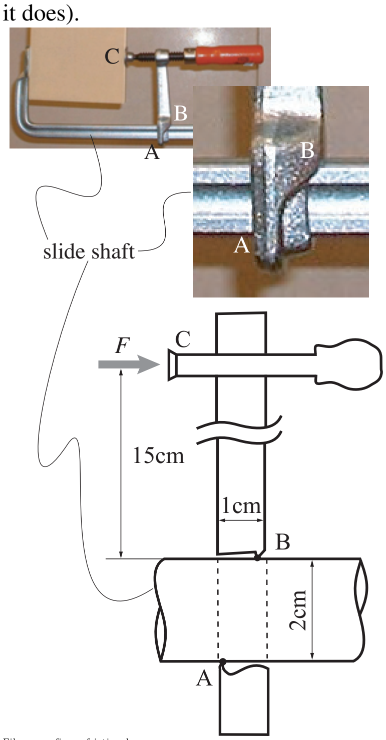
a) What is the relation between the two tensions when the pulley is turning? You may assume that the bearing shaft touches the hole in the pulley at only one point. *.

b) Plug in some reasonable numbers for R_i, R_o and μ (or ϕ) to see one reason why wheels (say pulleys) are such a good idea even when the bearings are not all that well lubricated. *



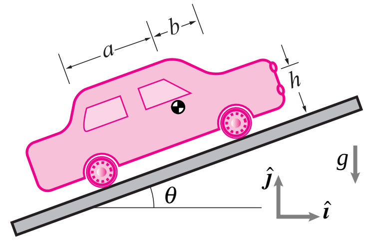
Filename:pfigure-blue-20-2
Problem 4.3.23

4.3.24 The so-called *pipe-clamp* has a bracket ABC which loosely fits around the slide-shaft (the'pipe'). When not clamped there is no big force at C and the bracket freely slides on the shaft. However the bracket frictionally locks once the load F at C gets large. Neglecting gravity, find the minimum coefficient of friction μ at A and B for which this clamp holds well (which it does).



Filename:pfigure-frictionclamp
Problem 4.3.24

4.3.25 Find the minimum coefficient of friction μ needed for a front wheel drive car to go up hill. Answer in terms of some or all of a, b, h, m, g and θ.



Filename:pfigure-frontwheelup
Problem 4.3.25

4.3.26 Solve Problem 4.3.25 for a rear wheel drive car.

4.3.27 Solve Problem 4.3.25 for a four wheel drive car.

4.4 Internal forces

Preparatory Problems

4.4.1 For the bar shown which ones of the following statements are true? *

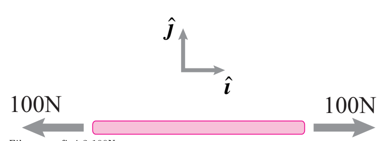
a) The two forces cancel so the tension is zero.

b) The two forces add so the tension is 200 N.

c) The tension is $100\,\mathrm{N}\hat{\imath}$.

d) The tension is $-100\,\mathrm{N}\hat{\imath}$.

e) The tension is $100\,\mathrm{N}\hat{\imath}$ on the right end and $-100\,\mathrm{N}\hat{\imath}$ on the left end.



Filename:pfig4-3-100N
Problem 4.4.1

4.4.2 What letters and case (upper or lower) are used in this book for tension, shear force, and bending moment?

4.4.3 Mechanics depends on free body diagrams. And free body diagrams only show the external forces on an object. So how can mechanical sense be made of the concept of "internal" force?

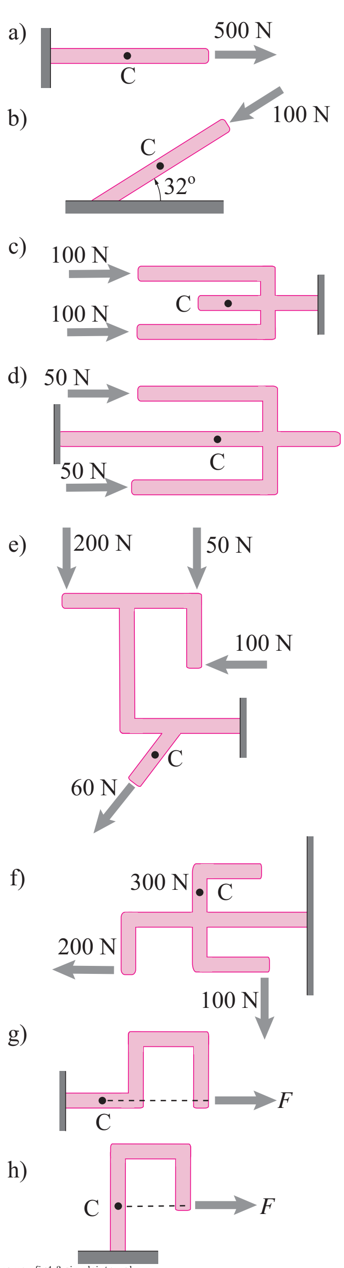
4.4.4 A string is conceptually cut in half by making free body diagrams of the left and right halves of the string. At the cut on the left half of the string acts the force $50\,\mathrm{N}\hat{\imath}$. At the cut on the right half of the string acts the force $-50\,\mathrm{N}\hat{\imath}$. With two different forces acting on the two halves how can one define a single 'tension'?

4.4.5 Define as precisely as you can:

a) Shear force

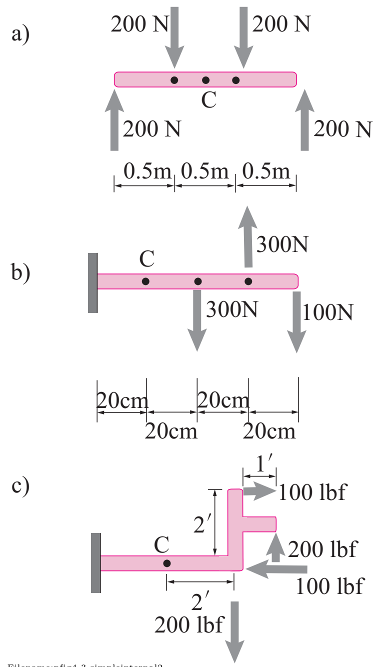
b) Bending moment

4.4.6 Find the tension, shear force and bending moment at C for each of the structures below. Neglect gravity. Assume dimensions as needed.



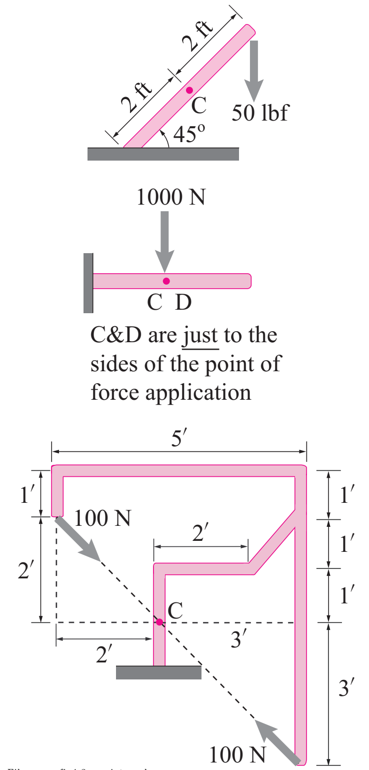
Filename:pfig4-3-simpleinternal
Problem 4.4.6

4.4.7 Find the tension, shear force and bending moment at C for each of the structures below. There is no gravity. Assume dimensions if needed.



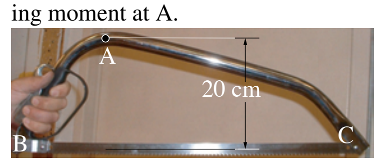
Filename:pfig4-3-simpleinternal2
Problem 4.4.7

4.4.8 Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Assume reasonable dimensions as needed.



Filename:pfig4-3-easyinternal
Problem 4.4.8

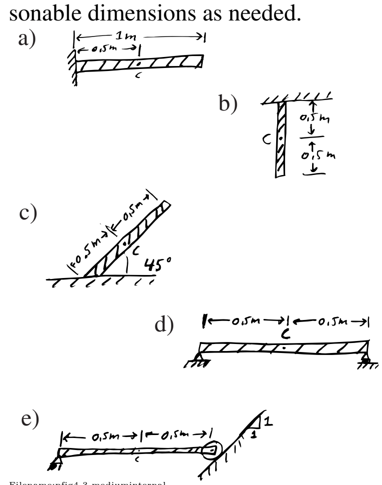
4.4.9 The tension in the bow-saw blade BC is 250 N. Find the tension, shear, and bending moment at A.



Filename:pfigure-bowsaw
Problem 4.4.9

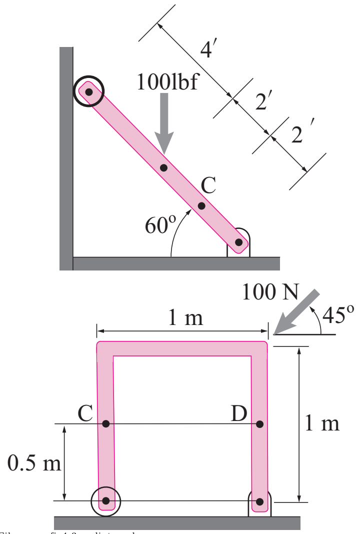
More-Involved Problems

4.4.10 Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Include gravity, assume all bars are uniform with density of (100 N/m). Assume reasonable dimensions as needed.



Filename:pfig4-3-mediuminternal
Problem 4.4.10

4.4.11 Find the tension, shear and bending moment at section C for each of the structures below. And also at D, if marked. Neglect gravity. Assume reasonable dimensions as needed.



Filename:pfig4-3-realinternal
Problem 4.4.11

4.5 Advanced statics

Preparatory Problems

4.5.1 In 2D, the force balance and moment balance equations for equilibrium of a body give three independent scalar equations that can be used to solve for three unknowns. How many independent scalar equations can you get from force and moment balance in 3D? Write down a set of such equations.

4.5.2 In 3D, how many independent scalar equations can you write for equilibrium of a particle?

4.5.3 How is moment balance equation about an axis different from moment balance about a point? Illustrate your answer with an example.

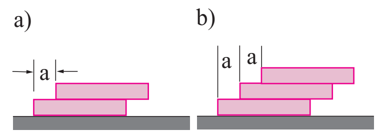
4.5.4 How many independent scalar equations of equilibrium can you get by writing moment balance equations about different lines or axes in 3D?

More-Involved Problems

4.5.5 Assume identical uniform rigid blocks with weight $W = 1\,\text{N}$, height $h = 1\,\text{cm}$, and length $\ell = 10\,\text{cm}$ are put one on top of the other. Assume there is no glue so blocks can only push against each other.

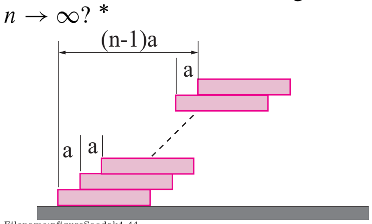
a) For two blocks what is the biggest overhang a so that the top block does not tip over?

b) For three blocks what is the biggest total overhang $= 2a$ (the same overhang a at each layer) so that the top block doesn't tip, nor does the middle block?



Filename:pfigureSoodak4-43
Problem 4.5.5

4.5.6 See Problem 4.5.5. For n stacked blocks what is the biggest possible overhang ($= na$) so that there is no tipping of any part of the pile relative to the rest? What is the maximum overhang in the limit $n \to \infty$? *



Filename:pfigureSoodak4-44
Problem 4.5.6

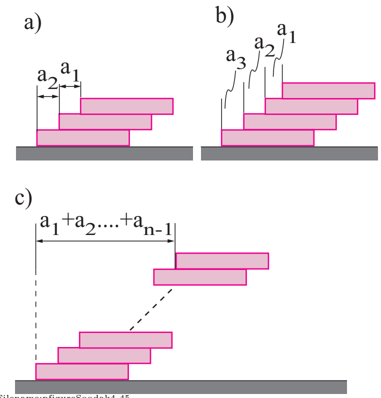
4.5.7 See simpler problems 4.5.5 and 4.5.6. Stacking identical rigid blocks one on top of each other one wants to get the biggest overhang possible without the tower toppling. Each block has, say, $W = 1\,\text{N}$, height $h = 1\,\text{cm}$, and length $\ell = 10\,\text{cm}$.

a) For three blocks find the biggest a_1 and a_2 so there is no toppling. [First put the top block as far to the right as you can, a_1, for no toppling. Then put that pair as far to the right as possible for no toppling over the bottom block.] The total overhang is $a_1 + a_2$.

b) For 4 blocks find the largest possible overhang $a_1 + a_2 + a_3$ by placing the tower of three above as far to the right as possible relative to the bottom block. [Note that you place the center of mass of the top 3 blocks over the right edge of the fourth bottom block].

c) For n blocks what is the biggest possible overhang ($= a_1 + a_2 + a_3 + \cdots + a_{n-1}$)?

d) Using blocks with length $\ell = 10\,\text{cm}$ how many blocks n are needed to get an overhang of 1 m? 2 m?



Filename:pfigureSoodak4-45
Problem 4.5.7

4.5.8 Uniform plate ADEH with mass m is connected to the ground with a ball and socket joint at A. It is also held by three massless bars (IE, CH and BH) that have ball and socket joints at each end, one end at the rigid ground (at I, C and B) and one end on the plate (at E and H).

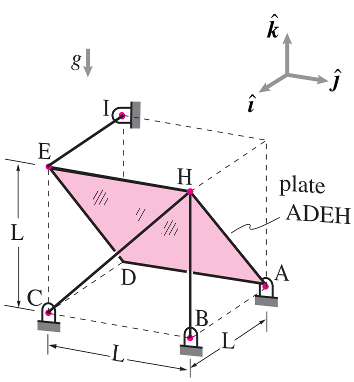
In terms of some or all of m, g, and L find

a) the reaction at A (the force of the ground on the plate),

b) T_{IE},

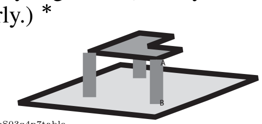
c) T_{CH},

d) T_{BH}.



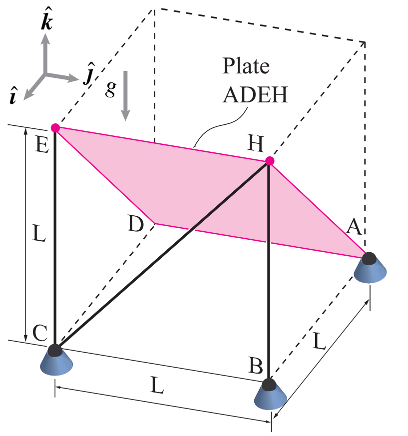
Filename:pfigure-01-f1
Problem 4.5.8

4.5.9 An 80 kg square table has one quarter cut away. The remaining 60 kg are supported on 3 massless legs on a level floor. Use $g = 10\text{N/kg}$. What is the load carried by leg AB? (State your assumptions clearly.) *



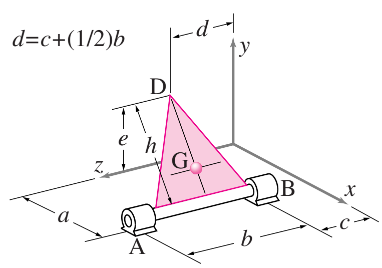
Filename:S03q4p7table
Problem 4.5.9

4.5.10 Uniform plate ADEH with mass m is connected to the ground with a ball and socket joint at A. It is also held by three massless bars (IE, CH and BH) that have ball and socket joints at each end, one end at the rigid ground (at I, C and B) and one end on the plate (at E and H). In terms of some or all of m, g, and L find the reaction at A (the force of the ground on the plate) and the three bar tensions T_{IE}, T_{CH} and T_{BH}. *



Filename:F01final1-plate
Problem 4.5.10

4.5.11 A massless triangular plate rests against a frictionless wall at point D and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor. A ball of mass m is fixed to the centroid of the plate. There is gravity and the system is at rest What is the reaction at point D on the plate?

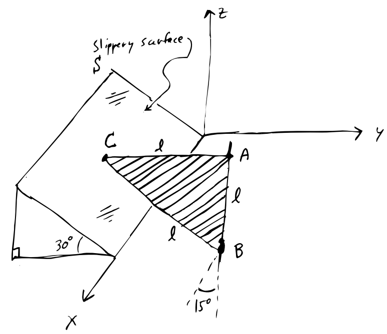


Filename:ch3-1b
Problem 4.5.11

4.5.12 A uniform equilateral triangular plate with weight $W = 1000N$ and sides $\ell = 2\,\mathrm{m}$ rests against a slippery plane S. Point C is 0.5 m above the xy plane. The bottom edge of the triangle has ball-and-socket joints at A and B, with the line AB on the xy plane making an angle of 15° with the x direction.

a) Find the reaction at C

b) Find all you can about the reactions at A and B.



Filename:pfigure-restingtriangle
Problem 4.5.12

4.5.13 A uniform 5 kg shelf is supported at one corner with a ball and socket joint and the other three corners with strings. At the moment of interest the shelf is at rest. Gravity acts in the$-\hat{k}$ direction. The shelf is in the xy plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try to find this force. If you can't, then come back to this question after you have done all the calculations.

c) Write down the equation of force equilibrium.

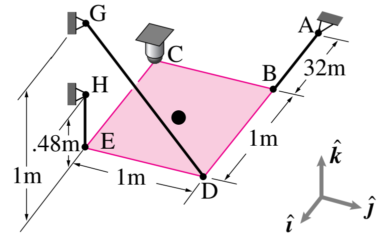
d) Write down the moment balance equation using the center of mass as a reference point.

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns.

f) Solve these equations by hand or on the computer.

g) Instead of using a system of equations try to find a single equation which can be solved for T_{EH}. Solve it and compare to your result from before. *

h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try moment balance about an appropriate axis as well as force balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (d)? Do they?



Filename:pfigure-s94h2p10-a
Problem 4.5.13

4.5.14 The sign is held up by 6 rods. Find the tension in bars

a) BH *

b) EB *

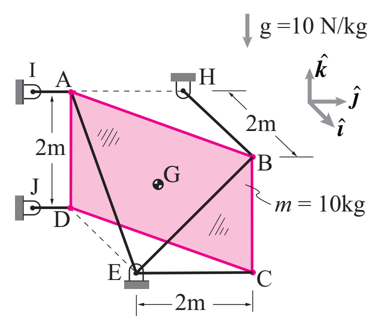
c) AE *

d) IA *

e) JD *

f) EC *

[One game you can play is to see how many of the tensions you can find *without* knowing any of the others. Another approach is to set up and solve 6 equations in 6 unknowns.]



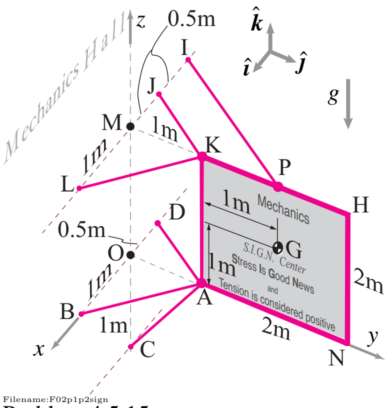
Filename:pfigure-crookedsign
Problem 4.5.14

4.5.15 The 100 kg, 2 m square, uniform sign KHNA is held up by 6 bars.
Structure and geometry clarifications: The sign is held vertically, 1m in front of, and orthogonal-to a vertical wall. Each bar holding the sign has a ball-and-socket joint both where it attaches to the sign and where it attaches to the wall. The points L, M, J, I, K, P and H lie in the same horizontal plane that includes the top edge of the sign. The points M, O, and C lie on a vertical line that is coplanar with the sign. Points B, O, D, A, and N lie in a horizontal plane shared with the bottom ege of the sign. The center of mass of the sign is at G. $g = 10\mathrm{N/kg}$.

a) Find the "bar force" in bar AC.
[hint: $\Sigma F_z = \left\{\Sigma \vec{F}\right\} \cdot \hat{k} = 0$]. *

b) Find the "bar force" in bar IP. [hint: $\Sigma M_{AK} = \left\{\Sigma \vec{M}_{/A}\right\} \cdot \hat{k} = 0$]. *

c) Find the "bar force" in bar KL. *



Filename:F02p1p2sign
Problem 4.5.15

4.5.16 Below is a highly schematic picture of a tricycle. The wheels are at C, B and A. The person-trike system has center of mass at G directly over the rear axle. The wheels at C and A are good free-turning, high friction wheels. The wheel at B is in a small ditch and can't move. Assume no slip and that F, m, g, w, ℓ, and h are given.

a) Of the 9 possible reaction components at A, B, and C, which do you know are zero *a priori*.

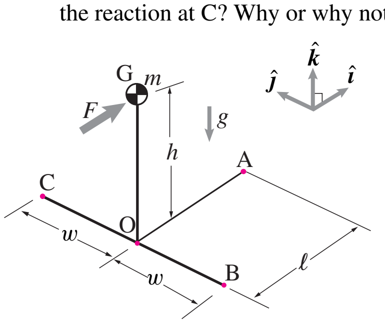
b) Find all the reaction components (the full reaction force) at A.

c) Find the vertical component of the reaction at C.

d) Find the x and z reaction components at B.

e) Find the sum of the y components of the reactions at B and C.

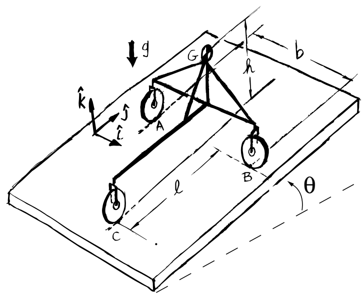
f) Can you find the y component of the reaction at C? Why or why not?



Filename:pfigure-trikerock
Problem 4.5.16

4.5.17 A 3-wheeled robot with mass m is parked on a hill with slope θ. The ideal massless robot wheels are free to roll but not to slip sideways. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\jmath}$ direction and the wheel at B is free to roll in the $\hat{\imath}$ direction. The center of mass of the robot at G is h above (normal to the slope) the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance b apart. The length of the robot is ℓ.

Find the force vector $\vec{F}_A$ of the ground on the robot at A in terms of some or all of $m, g, \ell, \theta, b, h, \hat{\imath}, \hat{\jmath}$, and $\hat{k}$. *



Filename:pfigure-threewheelparked
Problem 4.5.17

CHAPTER 5

Trusses and frames

Here we consider collections of parts assembled so as to hold something up or hold something in place. Emphasis is on trusses, assemblies of bars connected by pins at their ends. Trusses are analyzed by drawing free body diagrams of the pins or of bigger parts of the truss (method of sections). Frameworks built with other than two-force bodies are also analyzed by drawing free body diagrams of parts. Structures can be rigid or not and redundant or not, as can be determined by the collection of equilibrium equations.

Contents

Many structures are built from two or more parts. If the parts are well-modeled as rigid and the connections between them are also well-modeled as rigid① then the separateness of the parts is not visible to the laws of mechanics. The collection is then, effectively, a single object. And the best we can do with statics is to treat the group as one object. And this has been the approach of the previous chapter.

① Any connection between two parts that allows no relative motion (neither displacement nor rotation) is called a *rigid connection*. Examples of connections usually modeled as rigid are welds, bolted connections (with multiple bolts) and good glue joints.

Either by accident or design, however, the connections between solid parts often are not well-modeled as rigid. Rather, the connections are sometimes reasonably approximated as freely allowing some relative motion.

This chapter concerns the analysis of arrays of parts connected by these means.

- with **pin joints**. A pin joint allows relative rotation of the parts and does not transmit moments. Forces are transmitted in all directions. The pin connection is, by far, the most common model for connections in structures.
- a **round pin in a slot**. A pin in slot allows relative rotation of the two parts and relative motion in one direction. The only force transmitted is orthogonal to the slot.
- **square pin in a slot** (or shaft around a rod). This connection allows sliding in the slot but does not allow rotation. Force orthogonal to the slot is transmitted as is a moment.

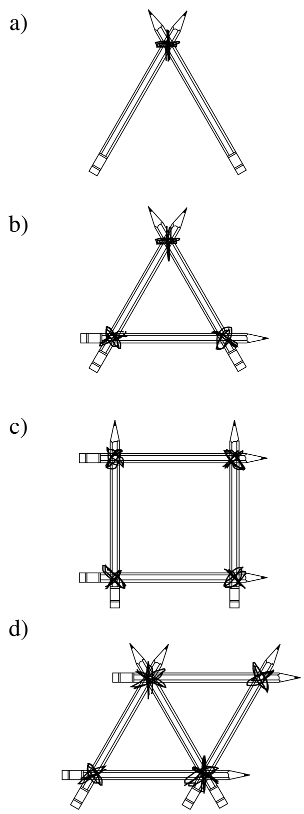
These are the standard non-rigid models for motion-allowing connections between parts. In 3D the array of standard connections is more complex, as discussed in Chapter 3.

In the previous chapter we only considered one object, and thus one free body diagram, at a time. Here we need to consider, all at once, a collection of objects and the associated collection of free body diagrams. The new skills that are thus needed are

- Use of the principal of action and reaction in the representation of forces on the free body diagrams of pairs of interacting objects, and
- The solution of a larger number of simultaneous equilibrium equations.

We start with the analysis of trusses, structures built out of straight bars connected to each other by pins at their ends.

5.1 Introduction to trusses and the method of joints



Filename:tfigure-pencil

Figure 5.1: a) Two pencils strapped together with a rubber band are not sturdy. b) A triangle made of pencils feels sturdy. c) A square made of 4 pencils easily distorts into a parallelogram. d) A structure made of two triangles feels sturdy (assuming it is kept flat).

①Some useable supplies are surely on hand. For sticks you can use pencils, pens, paper tightly rolled and taped into tubes, chopsticks, popsicle sticks, big wooden matches, knitting needles, barbeque skewers, plastic drinking straws, straw from a broom, Tinkertoy rods, table knives, or strips cut from cardboard. Similarly rubberbands can be replaced with masking, scotch or duct tape, used chewing gum, lashings using thread or string, hot-melt glue, or (for some of the sticks above) with a paperclip punched through the sticks and bent to make a home-made rivet.

Trusses are good.

Trusses are useful in engineering practice, they are easy to analyze, and they provide a good example of more general structural concepts. Your main goal here is to learn 'truss analysis', how to find the tensions in the bars of a given truss. But first, what is a truss and why are trusses so common?

You can quickly get a tactile sense of the truss concept. Get 9 short sticks and 9 rubber bands①. Put them together a few different ways and feel the resulting rigidity or lack thereof.

a) **A 'V' deforms**. First join two sticks tightly together with a rubber band so that they cannot easily slide along the connection, as in fig. 5.1a. Despite the tight joint connection you can feel that the sticks rotate relative to each other relatively easily; it is easy to open and close the upside-down V.

b) **A triangle is sturdy**. Add a third stick to complete the triangle (fig. 5.1b). The relative rotation of the first two pencils is now almost totally prohibited. Even though each joint on its own made a relatively flexible V, together the 3 joints make a very stiff triangle.

c) **A square deforms**. Now tightly strap four sticks into a square as in fig. 5.1c, making 4 rubber band joints at the corners. Put the square down on a table. The sticks don't stretch or bend visibly, nor do they slide much along each-other's lengths, but the connections allow the sticks to rotate relative to each other so the square easily distorts into a parallelogram.

d) **Two triangles are sturdy**. Now add two more sticks to your triangle to make two triangles (fig. 5.1d). So long as you keep this structure flat on the table, it is also sturdy.

Because a triangle is fully determined by the lengths of its sides and the V and quadrilateral are not, the structures made of triangles are *much* harder to distort. A triangle is sturdy even without rigid joints. And a V and a square (and a pentagon, etc) are not. You have just observed the essential inspiration of a truss:

Triangles make sturdy structures.

Swiss cheese

A different way to discover a truss is by means of subtraction. Imagine your first initial design for a bridge is to make it from one huge chunk of solid steel. This would be wildly heavy and expensive. So you could cut holes out of the chunk here and there, greatly diminishing the weight and amount of material used, but not much reducing the strength. Between these holes you would see other heavy regions of metal from which you might cut more holes leading to a more savings of weight at not much cost in strength. In fact, the reduced weight in the middle decreases the load on the outer parts of the structure possibly making the whole structure stronger rather than weaker②. Eventually you would find yourself with very holy swiss cheese, a structure that looks something like a collection of bars attached from end to end in vaguely triangular patterns; like a microscopic picture of spongy bone (fig. 5.2): As opposed to a solid block, a truss

- Uses less material;
- Puts less gravity load on other parts of the structure;
- Leaves space for other things of interest (*e.g.,* cars, cables, wires, people).

Real trusses are usually not made by removing material from a solid but by joining bars of steel, wood, or bamboo with welds, bolts, rivets, nails, screws, glue, or lashings. Once you are aware you will notice trusses in bridges, radio towers, and large-scale construction equipment. Early airplanes were flying trusses (fig. 5.3). Bamboo trusses have been used as scaffoldings for millennia. Birds have had bones whose internal structure is truss-like since they were dinosaurs.

Trusses are practical sturdy light structures.

But trusses are also prominent near the front of elementary mechanics books because

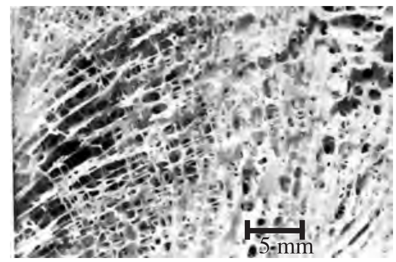
- They are perhaps the easiest example of a complex mechanical system that a student can analyze;
- They illustrate a variety of more general structural mechanics issues;
- They help build intuition about structures that are not really trusses (The engineering mind can see an underlying conceptual truss where no physical truss exists.).

What is a truss?

A *truss* is a structure made from long narrow bars connected at their ends.

The sturdiness of most trusses comes from the inextensibility of the bars, not the resistance to rotation at the joints (as in the sticks and rubberband examples at the start of this section). To make the analysis simpler the (generally small) resistance to rotation in the joints is totally neglected in truss analysis. Thus the interaction of the bars with their neighbors is by forces, with no couples; each bar has one net force acting on each end. So:

② Advanced aside: There are three ways that having more material in a structure can make it weaker: 1) the extra material adds to the gravitational load, as for the imagined bridge here, 2) The added material can be wedged in, causing the structure to fight itself (so called *locked-in* or *internal* stresses), and 3) material in the wrong place can cause *stress concentrations* and thus weak spots.



Filename:tfigure-BoneLakes

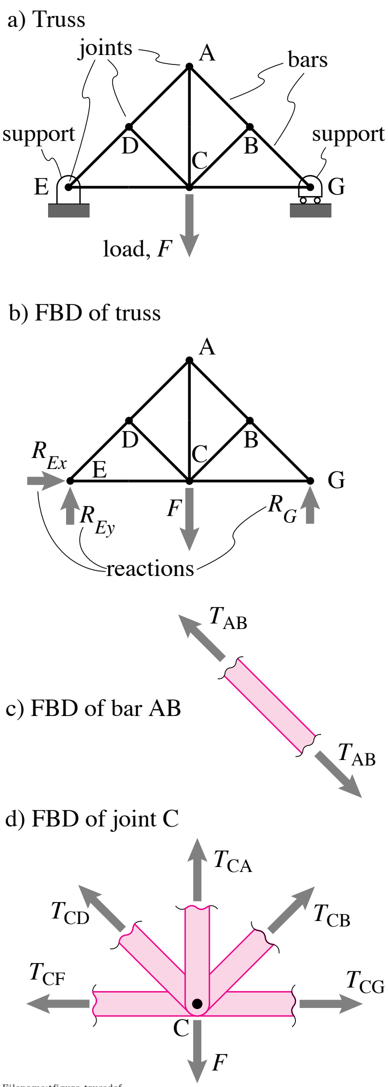
Figure 5.2: The truss inside you. The structure of spongy bone is vaguely truss-like. Shown here is human cancellous bone from the proximal femur, with the marrow removed. (courtesy Rod Lakes).

Filename:tfigureChanute33

Figure 5.3: Chanute Glider, 1897. The Wright brothers first planes were near copies of the 4-wing gliders built a few years earlier by Octave Chanute. Chanute was a retired bridge designer. Structurally these early biplanes were essentially flying bridges. The Wright bothers were preoccupied with trusses. Their first inspiration and main airplane patent was about how to control the distortion of a truss (‘wing warping’). The Wrights even thought of the threads in their wing fabric as bars in a truss. Take away the outer skin from many small modern planes and you will also find trusses. (National Air and Space Museum Negative 1A-2036084-10697)

Filename:tfigure-truss

Figure 5.4: Many bridges, especially old train bridges, are essentially trusses. Here's one that is partially obscured by the truss on a nearby car bridge.



Filename:tfigure-trussdef

Figure 5.5: a) a truss, b) FBD of truss (It is OK to show R_{Ex} even if it turns out to be zero), c) FBD of a bar, each bar is a two-force body, d) FBD of joint C, a joint is acted on by bar tensions and by applied loads. See Sample 5.1 on page 265 for an analysis of this truss.

An *ideal truss* is an assembly of two force members.

Or, if you like, an ideal truss is a collection of bars connected at their ends with frictionless pins. Loads are only applied at the pins. In engineering analysis, the word 'truss' refers to an ideal truss even though the object of interest might have, say, welded joint connections. Had we assumed the presence of welding equipment in your study room, the opening paragraph of this section would have described the welding of metal bars instead of the attachment of pencils with rubber bands. Even with welded-together steel you would have found that the triangles would be much more rigid than the V or square.

Bars, joints, loads, and supports

An ideal truss is a collection of *bars* connected at frictionless *joints* at which are applied *loads* as shown in fig. 5.5b (the load at a joint can be $\vec{\mathbf{0}}$ and thus not show on either the sketch of the truss or the free body diagram of the truss). A truss is held in place with *supports* which are idealized in 2D as either being fixed pins (as for joint E in fig. 5.5a) or as a pin on a roller (as for joint G in fig. 5.5a). *Reaction forces*, the forces on the truss at the supports, show on a FBD of the whole truss (fig. 5.5b) and also on a FBD of any joint at a support. Each bar is a two-force body (fig. 5.5c), with the same magnitude of tension pulling away from each end. A joint can be cut free with a conceptual chain saw, fooling each bar stub with the bar tension, as in the free body diagram of a joint in fig. 5.5d.

The bar tensions can be negative. A bar with a tension of, say, $T = -5000\,\text{N}$ is said to be in *compression*. A tension of $-5000\,\text{N}$ is a compression of $5000\,\text{N}$.

Elementary truss analysis

In elementary truss analysis you are given a truss design to which given loads are applied. Your goal is to 'solve the truss' which means you are to find the reaction forces and the tensions in the bars (sometimes called the 'bar forces'). As an engineer, this allows you to determine the needed strengths for the bars.

The 'method of free body diagrams'

Trusses are always analyzed by the same basic method used in all of mechancis, the 'method of free body diagrams'.

- Free body diagrams are drawn of the whole truss and of various parts of the truss.
- The equilibrium equations are applied to each free body diagram, and
- The resulting equations are solved for the unknown bar forces and reactions.

The ‘method of free body diagrams’ is classically subdivided into two sub-methods.

- In the *method of joints* you draw free body diagrams of every joint and apply the force balance equations to each free body diagram.
- In the *method of sections* you draw a free body diagrams of one or more parts of the structure each of which includes 2 or more joints and apply force and moment balance to the part or parts.

These two methods can be used separately or in conjunction. In the rest of this chapter we cover the method of joints, the method of sections, computer solution using the method of joints, and miscellaneous advanced truss topics.

The elementary truss analysis you are about to learn is straightforward and fun. You will learn it without difficulty. However, the analysis of trusses at a more advanced level is mysteriously deep and has occupied great minds from the mid-nineteenth century (*e.g.,* Maxwell and Cauchy) to the present (see, e.g., box 5.2 on page 306).

Filename:tfigureHowrah

Figure 5.6: The Howrah bridge. Trusses are such a good idea that they are used recursively. The bars in some large trusses, like this one or the Eiffel Tower, are themselves made of trusses (courtesy Bipin Chandra, www.bipinchandra.co.uk).

Method of joints

Let’s start with an example.

Example: Derrick arm.
Consider this planar model to the arm of a construction derrick (see fig.5.7). Assume F and d are known. This truss has joints A-S (skipping ‘F’ to avoid confusion with the load). As is common in truss analysis, we totally neglect the force of gravity on the truss elements ③. The goal is to find the tensions in the bars (the so-called ‘bar forces’).

③ **How to include gravity on the truss elements:** replace the single gravity force at the center of each bar with a pair of equivalent forces at the ends. The gravity loads are then only at the joints, and the truss can still be analyzed in the standard way as a collection of two-force members. Trusses are so efficient, however, that the load they carry is often much greater than their weight. So weights of the truss parts are often neglected when calculating bar forces. If weight is significant, the method in this sidenote will accurately calculate the tensions in the bars, but not the tendency of the bars to bend under their own weight.

The method of joints is a subset of the more general method of free body diagrams. Free body diagrams are drawn of the joints. Here is the method-of-joints recipe:

- Draw a free body diagram of the whole structure and write 3 independent equilibrium equations (6 in 3D) and solve for unknown reactions if you can. This step is technically superfluous, but is so-often a time-saver that its best to just do it.
- Draw free body diagrams of all n joints, 18 such in the example above.
- For each joint free body diagram you write the force balance equations, each of which can be broken down into 2 scalar equations (3 in 3D).
- Solve the $2n$ joint equations ($3n$ in 3D) for the unknown bar forces and reactions. In the example above this is $18 * 2 = 36$ equations for 33 unknown tensions and 3 unknown reactions (which you may have found from the FBD of the whole structure, but need not have).

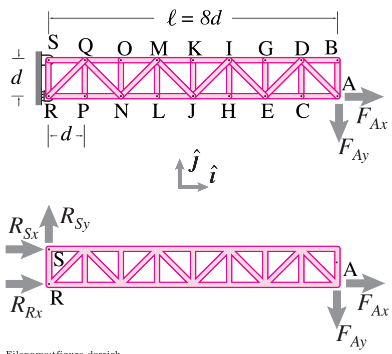


Figure 5.7: A truss with 33 bars, 18 joints and 3 unknown reaction forces. The reactions are sometimes written as, say, R_{Sx} instead of F_{Sx}.

Solving 36 simultaneous equations is generally only feasible with a computer, which is one way to go about things. However, for simple triangulated structures, like the one in fig. 5.7, you can find a sequence of joints for which hand solution is easy. If you solve the equilibrium equations as you go there are at most two unknown bar forces at each joint. By this means, the joint force-balance equations can be solved, even for some complex structures, without computers.

Example: Using the FBD of the whole structure
From the free body diagram of the whole structure (fig. 5.7) we find that

$$\begin{array}{lclcl}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} & \Rightarrow & R_{Sy} & = & F_{Ay} \\
\left\{\sum \vec{M}_{/S} = \vec{0}\right\} \cdot \hat{k} & \Rightarrow & R_{Rx} & = & 8F_{Ay} - F_{Ax}. \\
\left\{\sum \vec{M}_{/R} = \vec{0}\right\} \cdot \hat{k} & \Rightarrow & R_{Sx} & = & -8F_{Ay}.
\end{array}$$

Note, we picked a sign convention for the graphical representation of forces on the Free Body Diagram (see pages 46 and 154) and let the algebra possibly generate negative numbers: at S the support pushes on the arm with a force of $-8F_{Ay}$ which is pulling (if $F_{Ay} > 0$)

Note that for tension the order of subscripts is not meaningful. The tension T_{BC} is the same scalar as the tension T_{CB}. $T_{BC} = T_{CB}$ is the amount of pulling on joint B and also the amount of pulling on joint C. That the two force vectors are negatives of each other is accounted for by the definition of tension as pulling. This unimportance of the order of subscripts is in contrast with the case of position vectors where $\vec{r}_{BC}$ is the position vector from B to C (also called $\vec{r}_{C/B}$). For position vectors $\vec{r}_{BC} = \vec{r}_{C/B} = -\vec{r}_{B/C} = -\vec{r}_{CB}$. Summarizing, the subscript order has meaning for $\vec{r}_{AB}$ but not for T_{AB}.

FBDs of the joints

In the solve-by-hand method of joints we first find a joint with at most 2 bars connected. Then we work our way through the structure, one joint at a time, picking joints with at most 2 unknown bar tensions. For each joint we will use

$$\sum \vec{F}_i = \vec{0}.$$

Typically many bars in a truss are parallel to the x or y axis so we often fall into the routine of immediately reducing the above vector equilibrium equation to the component equations

$$\sum F_x = 0 \quad \text{and} \quad \sum F_y = 0.$$

For the truss in fig. 5.7

- Joint B has only two bars connected (see fig. 5.8). Force balance using FBD 5.8 tells us at a glance that

$$\sum F_x = 0 \Rightarrow T_{DB} = 0 \qquad \text{and} \qquad \sum F_y = 0 \Rightarrow T_{AB} = 0$$

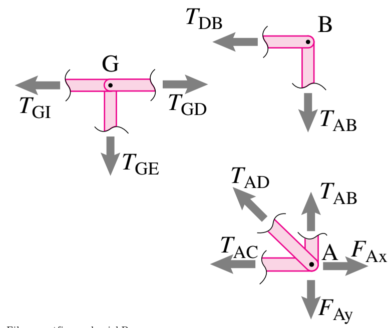
- Now you can draw a free body diagram of joint A where there are only two unknown tensions (since we just found T_{AB}), namely T_{AD} and T_{AC}. Force balance gives two scalar equations

$$\sum F_x = 0 \quad \Rightarrow \quad F_{Ax} - T_{AC} - \sqrt{2}T_{AD}/2 = 0$$
$$\sum F_y = 0 \quad \Rightarrow \quad -F_{Ay} + \underbrace{T_{AB}}_{0} + \sqrt{2}T_{AD}/2 = 0$$

which you can solve to find $T_{AD} = \sqrt{2}F_{Ay}$ and $T_{AC} = F_{Ax} - \sqrt{2}F_{Ay}$.

- Next is joint C. Force balance for joint C will tell you T_{CD} and T_{CE}.
- Then you can work your way through the alphabet of joints. Using the bar tensions you have already found you can find, one at a time, joints with only two unknown tensions.

That's it for the method of joints for simple structures.



Filename:tfigure-derrickB

Figure 5.8: Free body diagram of some joints from the truss in fig. 5.7. Note that bars DB, AB and GE are zero-force members.

Zero force members

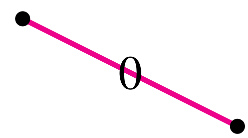
Just by looking at joint B and thinking about the free body diagram you could probably pick out that bars DB and AB must be *zero force members*. Here we explain the unnecessary but useful trick of recognizing such zero-force members even before systematically using the method of joints. *Zero-force members* are bars with $T = 0$, like bars AB, BD and CD in the truss of Fig. 5.7. The basic idea is this:

> If there is any direction for which only one bar contributes a force on a joint, then that bar is a zero-force member.

In particular:

- At any joint where
  - there are no loads, and
  - where there are only two unknown non-parallel bar forces, and
  - where all known bar-tensions are zero,

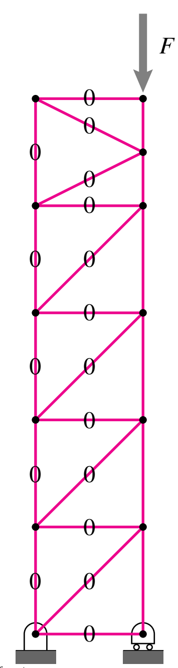
  then the two new bar tensions are both zero (e.g., joint B in fig. 5.8).
- At any joint where all bars but one are in the same direction as the applied load (if any), the one bar is a zero-force member (see joints C, G, H, K, L, O, and P in fig. 5.7).



Filename:tfigure-zeroforce

Figure 5.9: A zero force member is sometimes indicated by writing a zero on top of the bar.

In the truss of fig. 5.7 bars AB, BD, CD, EG, IH, JK, ML, NO, and PQ are all zero force members. Sometimes it is useful to keep track of the zero force members by marking them with a zero (see fig. 5.9).

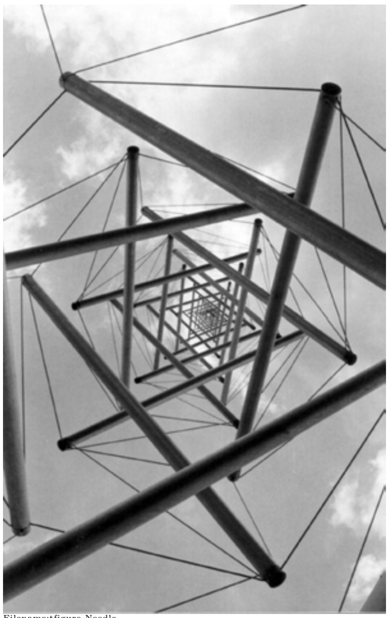


Filename:tfigure-zeroforcetower

Figure 5.10: A tower with many zero force members. Although they carry no load they prevent structural collapse. A common use of zero-force members is to brace long bars that are in compression and which would otherwise buckle (pop out to one side).

Zero-force members often have a non-zero purpose

Although with the given loading zero-force members have no tension, they are often needed because there are small loads not considered in the basic analysis. These could be from imperfections, or load induced asymmetries in a structure. This gives the 'zero-force' bars a small job to do, a job not noticed by the equilibrium equations in elementary truss analysis, but one that can prevent total structural collapse. Imagine, for example, the tower of fig. 5.10. In a real tower of that design the zero-force members might carry very small loads, say 100 or 1000 times smaller than the tensions (or compressions) calculated for the other bars. But if the zero-force members were removed the tower would collapse. Thus, in practice, you may observe large heavy structures with some very thin bars. Bars which in simple analyses carry no loads. But bars which prevent structural collapse



Filename:tfigure-Needle

Figure 5.11: Sometimes trusses are used *only* because they look nice. The tensegrity structure 'Needle Tower' was designed by artist Kenneth Snelson and is on display in the Hirshhorn Museum in Washington, DC. It doesn't hold up anything but itself. Here you are looking straight up the middle. (courtesy Christopher Rywalt)

Simple and not-simple trusses

Most elementary texts, like this one, start with structures that yield easily to the method of joints. These are structures where you can totally solve the equilibrium equations for the joints one at a time; each new joint only introduces two new unknown bar-tensions.

For more complex trusses this straightforward approach can fail a few ways:

- Some structures are not designed in a straightforward triangulated manner and cannot be solved 2 equations at a time. Although the method of joints may still yield a solution, it may require simultaneous solution of all of the equilibrium equations④.
- Many structures cannot be solved (the bar tensions can't be found) by using the laws of statics alone. Such are called 'statically indeterminate' structures.

For this first truss section we only consider structures that are statically determinate and easily solved. See sec. 5.5 for a detailed discussion of static determinacy.

Why aren't trusses everywhere?

Trusses can carry big loads with little use of material and can look nice (See fig. 5.11).They are used in many structures. Why don't engineers use trusses for all structural designs? Here are some reasons to consider not using a truss:

- Trusses are relatively difficult to build, involving many small parts⑤ and thus requiring much time and effort to assemble.
- Trusses can be sensitive to damage when forces are not applied at the anticipated joints. They are especially sensitive to loads on the middle of the bars.
- Trusses inevitably depend on the tension strength in some bars. Some common building materials (*e.g.,* concrete, stone, and clay) crack easily when pulled.
- Trusses often have little or no redundancy, so failure in one part can lead to total structural failure.

④ In the language of linear algebra: simple structures yield equilibrium equations that are naturally in upper triangular form, more complicated structures do not yield an upper-triangular form.

⑤If an Indian says to you "Go and count the rivets in the Howrah bridge." she means go away and do something that will take a very long time. The bridge has many rivets (and bars and joints).

- The triangulation that trusses require can use space that is needed for other purposes (*e.g.,* doorways, rooms).
- Trusses tend to be stiff, and sometimes more flexibility is desirable (*e.g.,* diving boards, car suspensions).
- In some places some people consider trusses unaesthetic. (e.g., the Washington Monument is not supposed to look like the Eiffel Tower).

Nonetheless, for situations where you want a stiff, light structure that can carry known loads at pre-defined points, a truss is often the best design choice.

Three-dimensional trusses

After you have mastered the elementary 2D truss analysis of the previous section you might wonder

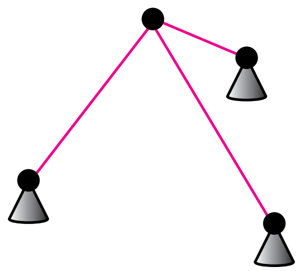
- **Do the ideas generalize to 3D?** Yes, with a only minor elaboration.
- **Does at least one of the methods presented always work?** Yes, if you just look at the homework problems for elementary truss analysis. And yes again for many practical structures. But some trusses cannot be analyzed by the simple methods. In this section we classify trusses into types. One type, statically determinate trusses, can be analyzed by simple statics methods, other trusses require study of deformations as well as statics.

The concepts for 3D trusses are basically the same as for 2D trusses with these differences;

- In the method of joints each joint is associated with 3 scalar force balance equations instead of 2;
- In the method of sections, and in the free body diagram of the whole structure one has 6 scalar equations instead of 3;
- To hold the structure in place takes at least 6 reaction components instead of 3;
- The rule-of-thumb check for static determinacy of a grounded structure in 3D is $b + r = 3j$ instead of the 2D relation $b + r = 2j$ (See sec. 5.5 for discussion of these formulas);
- The rule of thumb for rigidity for a floating truss in 3D is $b + 6 = 3j$ instead of the 2D relation $b + 3 = 2j$.

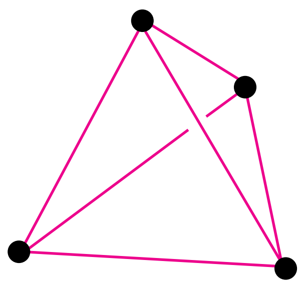
There are various ways to think about the number six in the counts above. Assuming the structure is more than a point, six is the number of ways a rigid structure can move in three dimensional space (three translations and three rotations), six is the number of equilibrium equations for the whole structure (one 3D vector moment, and one 3D vector force), and six is the number of constraints needed to hold a structure in place.

Example: A tripod



Filename:tfigure-tripod

Figure 5.12: A tripod is the simplest rigid 3D truss.



Filename:tfigure-tetrahedron

Figure 5.13: A tetrahedron is the simplest rigid truss in 3D that does not depend on grounding.

A tripod is the simplest rigid 3D structure. With four joints ($j = 4$), three bars ($b = 3$), and nine unknown reaction components ($r = 3\times3 = 9$), it exactly satisfies the equation $3j = b + r$, a check for determinacy of rigidity of 3D structures.

A tripod is the 3D equivalent of the two-bar truss shown in fig. 5.72a on page 304.

Example: A tetrahedron

The simplest 3D rigid floating structure is a tetrahedron. With four joints ($j = 4$) and six bars ($b = 6$) it exactly satisfies the equation $3j = b + 6$ which is a check for determinacy of rigidity of floating 3D structures.

A tetrahedron is thus, in some sense the 3D equivalent of a triangle in 2D.

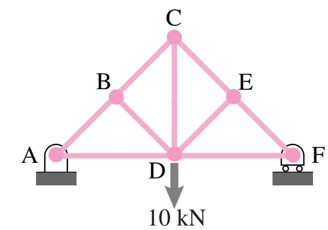
Example: Geodesic domes

Any closed polyhedron, with each face a triangle of rods, is a rigid structure. This includes a tetrahedron (above), an octahedron, a cube with a diagonal on each face, an icosohedron, and Buckminster Fuller's geodesic domes.

Well, so Cauchy thought. It turns out that there are some strange non-convex polyhedra that are not rigid. But, for practical purposes, if you see triangles all around the outside of a structure you can assume its rigid.

SAMPLE 5.1 The truss shown in the figure carries a load $F = 10\,\text{kN}$ at joint D. The truss is designed with nine rods, six of which (the inclined ones) have the same length $d = 2\,\text{m}$. Rods BC, EC, DE and BD form a square.

1. Find the support reactions at joints A and F.
2. Find the tensions in rods BD and BC.



Filename:sfig5-trussjoints-1

Figure 5.14:

Solution

1. **Support reactions:** To find the support reactions at A and F, we draw the free-body diagram of the entire truss (see fig. 5.15). We are given that $d = 2\,\text{m}$ and that $\angle\text{ABD} = \angle\text{DEF} = \pi/2$. Therefore, $\ell = \sqrt{2}d = 2\sqrt{2}\,\text{m}$.

   The scalar force balance equation in x-direction readily gives $R_{A_x} = 0$. The scalar moment balance equation about point A gives

   $$2\ell R_F - \ell F = 0 \quad \Rightarrow \quad R_F = \frac{F}{2} = 5\,\text{kN}.$$

   Now, from the scalar force balance in the y-direction, we have

   $$R_{A_y} + R_F - F = 0 \quad \Rightarrow \quad R_{A_y} = F - R_F = 5\,\text{kN}.$$

   $$\boxed{R_{A_x} = 0, \quad R_{A_y} = 5\,\text{kN}, \quad R_F = 5\,\text{kN}}$$

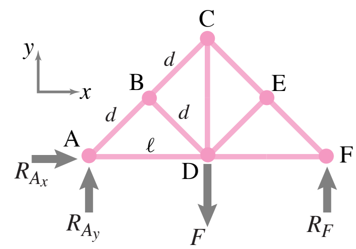


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Figure 5.15:

2. **Tensions in BD and BC:** We can find the tensions in rods BC and BD by analysing the equilibrium of joint B. As you can see, joint B has three unknown forces acting on it, namely the tensions of rods AB, BC and BD. Since the joint equilibrium equations (only two scalar equations) can only solve for two unknowns, we need to start at joint A, determine T_{AB} first and then move on to joint B.

   The free-body diagrams of the joints A and B are shown in fig. 5.16. Let us first consider the equilibrium of joint A. From the scalar force balance equations, we have

   $$\begin{aligned}
   \sum F_y = 0 \quad &\Rightarrow \quad R_{A_y} + T_{AB}\sin\theta = 0\\
   &\Rightarrow \quad T_{AB} = -R_{A_y}/\sin\theta = -5\,\text{kN}/(1/\sqrt{2}) = -7\,\text{kN}.\\
   \sum F_x = 0 \quad &\Rightarrow \quad T_{AB}\cos\theta + T_{AD} = 0\\
   &\Rightarrow \quad T_{AD} = -T_{AB}\cos\theta = -7\,\text{kN}(1/\sqrt{2}) = 5\,\text{kN}.
   \end{aligned}$$

   Now, we analyze joint B. From the geometry of forces, it is clear that writing scalar force balance equations in the x' and y' directions will be advantageous. For example, the force balance in the x' direction immediately gives $T_{BD} = 0$. The force balance in the y' direction gives

   $$-T_{AB} + T_{BC} = 0 \quad \Rightarrow \quad T_{BC} = T_{AB} = -7\,\text{kN}.$$

   $$\boxed{T_{BC} = -7\,\text{kN}, \quad T_{BD} = 0}$$

   Note that it is easy to spot bar BD as a zero force member since it is perpendicular to rods AB and BC.

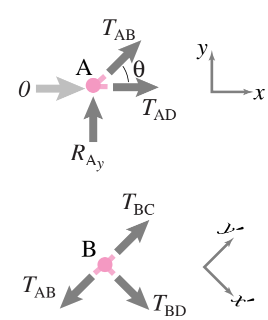

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Figure 5.16: Free-body diagrams of joints A and B. From the given geometry, $\theta = pi/4$.

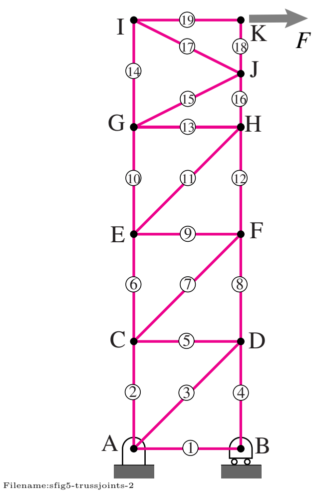

Figure 5.17:

SAMPLE 5.2 For the truss tower shown in the figure, assume all horizontal and vertical rods to be 1 mlong and rods numbered 16 and 18 to be 0.5 mlong. Given that the horizontal load on the truss $F = 500\,\text{N}$, find the tension in rod 15.

Solution To find the tension in rod 15, we can use the equilibrium of either joint G or joint K. In either case, the free-body diagram will have four unknown bar tensions (for four bars connected to each of these joints) at the joint. Therefore, we will not be able to solve for them. So, let us start at joint K and work through joint I to joint J. This sequence gets us only two unknown forces at each joint.

The free-body diagrams of the three joints are shown in fig. 5.18. Let us first consider the equilibrium of joint K. A simple inspection (or force balance in the y-direction) shows that bar 18 is a zero force member. The force balance in the horizontal direction then immediately gives $T_{19} = F = 500\,\text{N}$. Thus,

$$T_{19} = 500\,\text{N} \quad \text{and} \quad T_{18} = 0.$$

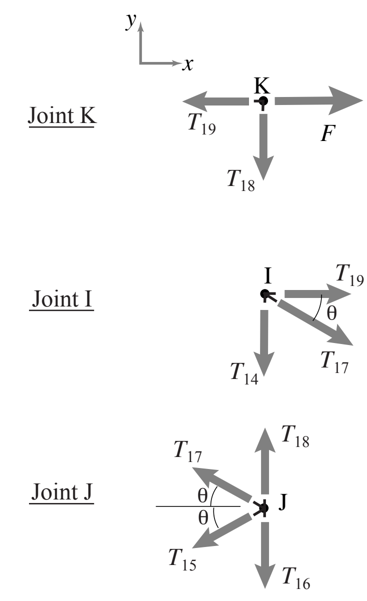

Figure 5.18: Free-body diagrams of joints K, I, and J. From the given geometry, $\theta = \tan^{-1}(\text{KJ}/\text{IK}) = 26.56^\circ$.

Next, we consider the equilibrium of joint I. Since T_{19} is already known, there are only two unknown forces, T_{14} and T_{17} at this joint. The force balance in the horizontal direction gives

$$\begin{aligned} T_{19} + T_{17}\cos\theta &= 0 \\ \Rightarrow \quad T_{17} &= -\frac{T_{19}}{\cos\theta} \\ &= -\frac{500\,\text{N}}{\cos(\tan^{-1}(0.5))} \\ &= -559\,\text{N}. \end{aligned}$$

Now we proceed to joint J. Note that we used only one scalar equation (force balance in the x-direction) at joint I, since we do not need T_{14}. Similarly, to find $T_1 5$, we only need the force balance in the horizontal direction at joint J:

$$\begin{aligned} -T_{17}\cos\theta - T_{15}\cos\theta &= 0 \\ \Rightarrow \quad T_{15} &= -T_{17} = 559\,\text{N}. \end{aligned}$$

$$\boxed{T_{15} = 559\,\text{N}}$$

Note: We did not have to find support reactions first in order to proceed to other joints as in the previous sample. As long as you can find a sequence of joints with just two unknown forces at each joint, up to the force that you need to determine, you can easily find the force with hand calculations.

SAMPLE 5.3 The truss shown in the figure is made up of five horizontal and six inclined rods. All inclined rods are 1 m long and at right angles to each other. The truss carries two vertical loads, $F_1 = 4\,\mathrm{kN}$ and $F_2 = 1\,\mathrm{kN}$ as shown. Find the tensions in rods CE, DE, and DF.

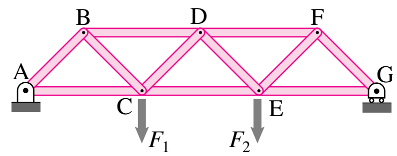

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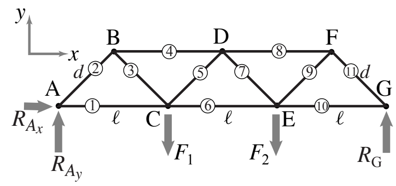
Figure 5.19:

Solution To find tensions in rods CE, DE and DF, we can either use joints C and D, or joints E and F. However, for either set we need to start from other joints since there are more than two unknown forces at each joint. Let us start from joint G and work our way through joints F and E. To start at joint G, however, we first need to determine the support reaction G.

The free-body diagram of the entire truss is shown in fig. 5.20 where we have numbered the rods for convenience. The scalar moment balance equation about point A in the z-direction gives



Filename:sfig5-trussjoints-3a

Figure 5.20:

$$3\ell R_G - \ell F_1 - 2\ell F_2 = 0 \quad \Rightarrow \quad R_G = \frac{F_1 + 2F_2}{3} = 2\,\mathrm{kN}.$$

The force balance equations give

$$\sum F_x = 0 \quad \Rightarrow \quad R_{A_x} = 0$$
$$\sum F_y = 0 \quad \Rightarrow \quad R_{A_y} = F_1 + F_2 - R_G = 3\,\mathrm{kN}.$$

Now, we are ready to proceed from joint G. The free-body diagrams of joints G, F, and E are shown in fig. 5.21.

At joint G:

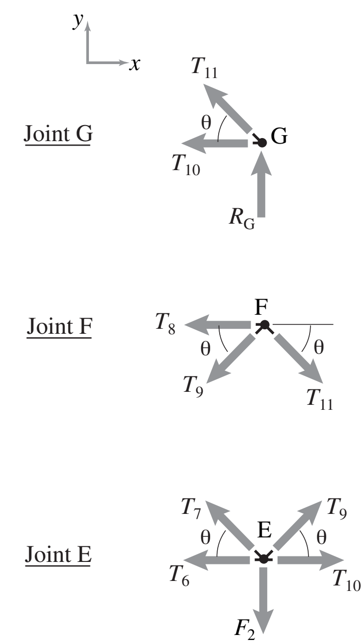
$$\sum F_y = 0 \quad \Rightarrow \quad T_{11}\sin\theta + R_G = 0$$
$$\Rightarrow \quad T_{11} = -\frac{R_G}{\sin\theta} = -\sqrt{2}R_G = -2.83\,\mathrm{kN}.$$
$$\sum F_x = 0 \quad \Rightarrow \quad -T_{11}\cos\theta - T_{10} = 0$$
$$\Rightarrow \quad T_{10} = -T_{11}\cos\theta = 2\,\mathrm{kN}.$$

At joint F:

$$\sum F_y = 0 \quad \Rightarrow \quad -T_{11}\sin\theta - T_9\sin\theta = 0$$
$$\Rightarrow \quad T_9 = -T_{11} = 2.83\,\mathrm{kN}$$
$$\sum F_x = 0 \quad \Rightarrow \quad (T_{11} - T_9)\cos\theta - T_8 = 0$$
$$\Rightarrow \quad T_8 = (T_{11} - T_9)\cos\theta = -4\,\mathrm{kN}.$$

At joint E:

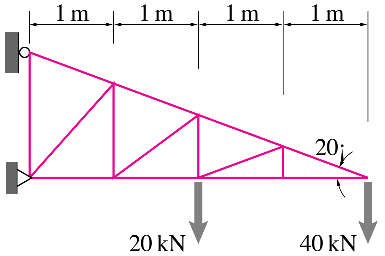
$$\sum F_y = 0 \quad \Rightarrow \quad (T_7 + T_9)\sin\theta - F_2 = 0$$
$$\Rightarrow \quad T_7 = \frac{F_2}{\sin\theta} - T_9 = -1.41\,\mathrm{kN}.$$
$$\sum F_x = 0 \quad \Rightarrow \quad (T_9 - T_7)\cos\theta + T_{10} - T_6 = 0$$
$$\Rightarrow \quad T_6 = (T_9 - T_7\cos\theta + T_{10} = 5\,\mathrm{kN}.$$



Filename:sfig5-trussjoints-3b

Figure 5.21: Free-body diagrams of joints G, F, and E. From the given geometry, $\theta = 45^\circ$.

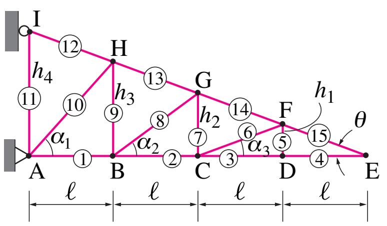
$$\boxed{T_{CE} = 5\,\mathrm{kN},\ T_{DE} = -1.41\,\mathrm{kN},\ T_{DF} = -4\,\mathrm{kN},}$$



Filename:sfig4-truss-comp

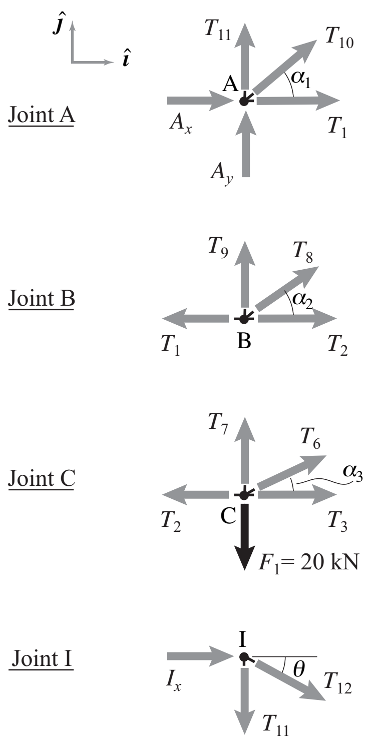
Figure 5.22:

SAMPLE 5.4 The truss shown in the figure has four horizontal bays, each of length 1 m. The top bars make 20° angle with the horizontal. The truss carries two loads of 40 kN and 20 kN as shown. Find the forces in each bar. In particular, find the bars that carry the maximum tensile and compressive forces.



Filename:sfig4-truss-comp-a

Figure 5.23: To keep track of joints and rods, we first label them. All joints are labeled with letters and rods with numbers.



Filename:sfig4-truss-comp-b

Figure 5.24: Free body diagrams of some representative joints.

Solution Since we need to find the forces in all the 15 bars, we need to find enough equations to solve for these 15 forces in addition to 3 unknown reactions A_x, A_y, and I_x. Thus we have a total of 18 unknowns. Note that there are 9 joints and therefore, we can generate 18 scalar equations by writing force equilibrium equations (one vector equation per joint) for each joint.

Number of unknowns:	15 bar forces + 3 reactions	= 18
Number of joints:	(A, B, C, . . ., and I)	= 9
Number of equations:	9 joint × 2 per joint	= 18.

So, we go joint by joint, draw the free-body diagram of each joint and write the equilibrium equations. After we get all the equations, we can solve them on a computer. All joint equations are just force equilibrium equations, *i.e.*, $\sum \vec{F} = \vec{\mathbf{0}}$.

- Joint **A**:
$$(A_x + T_1 + T_{10}\cos\alpha_1)\hat{\imath} + (A_y + T_{11} + T_{10}\sin\alpha_1)\hat{\jmath} = \vec{\mathbf{0}}. \tag{5.1}$$
- Joint **B**:
$$(-T_1 + T_2 + T_8\cos\alpha_2)\hat{\imath} + (T_9 + T_8\sin\alpha_2)\hat{\jmath} = \vec{\mathbf{0}}. \tag{5.2}$$
- Joint **C**:
$$(-T_2 + T_3 + T_6\cos\alpha_3)\hat{\imath} + (T_7 + T_6\sin\alpha_3)\hat{\jmath} = F_1\hat{\jmath}. \tag{5.3}$$
- Joint **D**:
$$(T_4 - T_3)\hat{\imath} + T_5\hat{\jmath} = \vec{\mathbf{0}}. \tag{5.4}$$
- Joint **E**:
$$(-T_4 - T_{15}\cos\theta)\hat{\imath} + T_{15}\sin\theta\,\hat{\jmath} = F_2\hat{\jmath}. \tag{5.5}$$
- Joint **F**:
$$(-T_6\cos\alpha_3+(T_{15}-T_{14})\cos\theta)\hat{\imath}+(-T_6\sin\alpha_3+(T_{14}-T_{15})\sin\theta-T_5)\hat{\jmath} = \vec{\mathbf{0}}. \tag{5.6}$$
- Joint **G**:
$$(-T_8\cos\alpha_2+(T_{14}-T_{13})\cos\theta)\hat{\imath}+((T_{13}-T_{14})\sin\theta-T_8\sin\alpha_2-T_7)\hat{\jmath} = \vec{\mathbf{0}}. \tag{5.7}$$
- Joint **H**:
$$(-T_{10}\cos\alpha_1+(T_{13}-T_{12})\cos\theta)\hat{\imath}+((T_{12}-T_{13})\sin\theta-T_{10}\sin\alpha_1-T_9)\hat{\jmath} = \vec{\mathbf{0}}. \tag{5.8}$$
- Joint **I**:
$$(I_x + T_{12}\cos\theta)\hat{\imath} + (-T_{11} - T_{12}\sin\theta)\hat{\jmath} = \vec{\mathbf{0}}. \tag{5.9}$$

Dotting each equation from (5.1) to (5.9) with $\hat{\imath}$ and $\hat{\jmath}$, we get the required 18 equations. We need to define all the angles that appear in these equations ($\alpha_1, \alpha_2, \alpha_3$, and θ) before we are ready to solve the equations on a computer.

Let ℓ be the length of each horizontal bar and let DF $= h_1$, CG $= h_2$, and BH $= h_3$. Then, $h_1/\ell = h_2/2\ell = h_3/3\ell = \tan\theta$. Therefore,

$$\begin{aligned}
\tan\alpha_1 &= \frac{h_3}{\ell} = \frac{3\ell\tan\theta}{\ell} &\Rightarrow\quad& \alpha_1 = \tan^{-1}(3\tan\theta)\\
\tan\alpha_2 &= \frac{h_2}{\ell} = 2\tan\theta &\Rightarrow\quad& \alpha_2 = \tan^{-1}(2\tan\theta)\\
\tan\alpha_3 &= \frac{h_1}{\ell} = \tan\theta &\Rightarrow\quad& \alpha_3 = \tan^{-1}(\tan\theta) = \theta.
\end{aligned}$$

Now, we are ready for a computer solution. You can enter the 18 equations in matrix form or as your favorite software package requires and get the solution by solving for the unknowns. Here is a pseudocode to set up and solve the matrix equation. Let us order the unknown forces in the form

$$x = [T_1 \quad T_2 \quad \ldots \quad T_{15} \quad A_x \quad A_y \quad I_x]^T$$

so that x_1 to x_{15} = T_1 to T_{15}; $x_{16} = A_x$, $x_{17} = A_y$, and $x_{18} = I_x$.

Entering and solving full matrix equation:

```
theta = pi/9                       % specify theta in radians
alpha1 = atan(3*tan(theta)         % calculate alpha1
alpha2 = atan(2*tan(theta))        % calculate alpha2 from arctan
alpha3 = theta                     % calculate alpha3 from arctan
C = cos(theta), S = sin(theta)     % compute all sines and cosines
C1 = cos(alpha1), S1 = sin(alpha1)
C2 = .. ..

F1 = 20;                            % input given external loads
F2 = 40;

A = [1 0 0 0 0 0 0 0 0 C1 0 0 0 0 0 1 0 0      % enter matrix A row-wise
     0 0 0 0 0 0 0 0 0 S1 1 0 0 0 0 0 1 0
     .
     .
     0 0 0 0 0 0 0 0 0 0 -1 -S 0 0 0 0 0 0]
b = [0 0 0 0 0 F1 0 0 0 F2 0 0 0 0 0 0 0 0]'  % enter column vector b

solve A*x = b for x
```

The solution obtained from the computer is

T_1	=	−128.22 kN,	T_2	=	−109.9 kN,	T_3	=	−109.9 kN,
T_4	=	−109.9 kN,	T_5	=	0,	T_6	=	0,
T_7	=	20 kN,	T_8	=	−22.66 kN,	T_9	=	13.33 kN,
T_{10}	=	−13.33 kN,	T_{11}	=	−50 kN,	T_{12}	=	146.19 kN,
T_{13}	=	136.44 kN,	T_{14}	=	116.95 kN,	T_{15}	=	116.95 kN,
A_x	=	137.37 kN,	A_y	=	60 kN,	I_x	=	−137.37 kN.

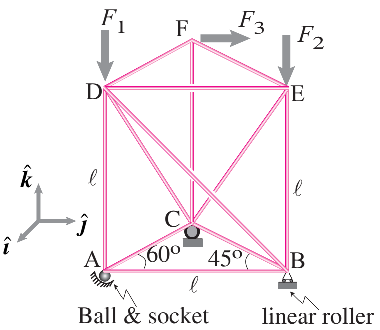


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Figure 5.25: This 3-D truss has a triangular base ABC with support points A, B, and C on the ground, and a triangular top DEF. The structure is thus like a prism.

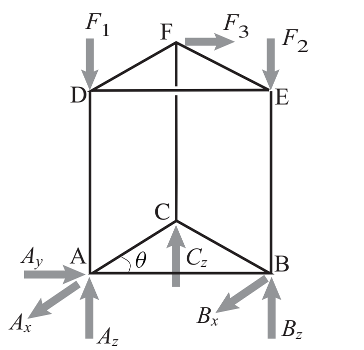

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Figure 5.26: The free body diagram of the structure as whole. Note that we do not need to show the internal braces DB, DC, or EC here.

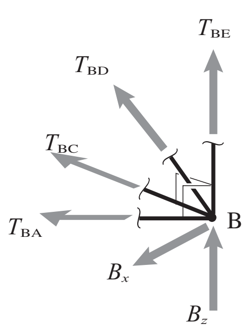

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Figure 5.27: The free body diagram of joint B.

SAMPLE 5.5 **A simple 3-D truss:** The 3-D truss shown in the figure has 12 bars and 6 joints. Nine of the 12 bars that are either horizontal or vertical have length $\ell = 1\,\mathrm{m}$. The truss is supported at A on a ball and socket joint, at B on a linear roller, and at C on a planar roller (all three supports are on the ground). The loads on the truss are $\vec{F}_1 = -50\,\mathrm{N}\hat{k}$, $\vec{F}_2 = -60\,\mathrm{N}\hat{k}$, and $\vec{F}_3 = 30\,\mathrm{N}\hat{j}$. Find all support reactions and the tension in bar BC.

Solution The free-body diagram of the entire structure is shown in fig. 5.26. Let the support reactions at A, B, and C be $\vec{R}_A = R_{A_x}\hat{\imath} + R_{A_y}\hat{j} + R_{A_z}\hat{k}$, $\vec{R}_B = R_{B_x}\hat{\imath} + R_{B_z}\hat{k}$, and $\vec{R}_C = R_C\hat{k}$. Then the moment balance about point A, $\sum \vec{M}_A = \vec{0}$, gives

$$\vec{r}_{B/A} \times \vec{R}_B + \vec{r}_{C/A} \times \vec{R}_C + \vec{r}_{E/A} \times \vec{F}_2 + \vec{r}_{F/A} \times \vec{F}_3 = \vec{0}. \tag{5.10}$$

Note that $\vec{F}_1$ passes through A and, therefore, produces no moment about A. Now we compute each term in the equation above.

$$\begin{aligned}
\vec{r}_{B/A} \times \vec{R}_B &= \ell\hat{j} \times (R_{B_x}\hat{\imath} + R_{B_z}\hat{k}) &&= -R_{B_x}\ell\hat{k} + R_{B_z}\ell\hat{\imath},\\
\vec{r}_{C/A} \times \vec{R}_C &= \ell(\cos\theta\,\hat{j} - \sin\theta\,\hat{\imath}) \times R_C\hat{k} &&= R_C\tfrac{\ell}{2}\hat{\imath} + R_C\tfrac{\sqrt{3}\ell}{2}\hat{j}\\
\vec{r}_{E/A} \times \vec{F}_2 &= (\ell\hat{j} + \ell\hat{k}) \times (-F_2\hat{k}) &&= -F_2\ell\hat{\imath},\\
\vec{r}_{F/A} \times \vec{F}_3 &= [\ell(\cos\theta\,\hat{j} - \sin\theta\,\hat{\imath}) + \ell\hat{k}] \times F_3\hat{j} &&= -F_3\ell\hat{\imath} - F_3\tfrac{\sqrt{3}\ell}{2}\hat{k}.
\end{aligned}$$

Substituting these products in eqn. (5.10), and dotting the resulting equation with $\hat{j}, \hat{k}$, and $\hat{\imath}$, respectively, we get

$$\begin{aligned}
R_C &= 0\\
R_{B_x} &= -\frac{\sqrt{3}}{2}F_3 = -15\sqrt{3}\,\mathrm{N}\\
R_{B_z} &= -\frac{1}{2}R_C + F_2 + F_3 = 90\,\mathrm{N}.
\end{aligned}$$

Thus, $\vec{B} = R_{B_x}\hat{\imath} + R_{B_z}\hat{k} = -15\sqrt{3}\,\mathrm{N}\hat{\imath} + 30\,\mathrm{N}\hat{k}$ and $\vec{R}_C = \vec{0}$. Now from the force balance, $\sum \vec{F} = \vec{0}$, we find $\vec{R}_A$ as

$$\begin{aligned}
\vec{R}_A &= -\vec{R}_B - \vec{R}_C - \vec{F}_1 - \vec{F}_2 - \vec{F}_3\\
&= -(-15\sqrt{3}\,\mathrm{N}\hat{\imath} + 90\,\mathrm{N}\hat{k}) - (-50\,\mathrm{N}\hat{k}) - (-60\,\mathrm{N}\hat{k}) - (30\,\mathrm{N}\hat{j})\\
&= 15\sqrt{3}\,\mathrm{N}\hat{\imath} - 30\,\mathrm{N}\hat{j} + 20\,\mathrm{N}\hat{k}.
\end{aligned}$$

To find the force in bar BC, we draw a free-body diagram of joint B (which connects BC) as shown in fig. 5.27. Now, writing the force balance for the joint in the x-direction, *i.e.*, $[\sum \vec{F} = \vec{0}] \cdot \hat{\imath}$, gives

$$\begin{aligned}
R_{B_x} + \vec{T}_{BC} \cdot \hat{\imath} &= 0\\
\text{or} \qquad R_{B_x} + T_{BC}\sin\theta &= 0\\
\Rightarrow \qquad T_{BC} &= -\frac{R_{B_x}}{\sin\theta}\\
&= -\frac{-15\sqrt{3}\,\mathrm{N}}{\sqrt{3}/2} = 30\,\mathrm{N}.
\end{aligned}$$

Thus, the force in bar BC is $T_{BC} = 30\,\mathrm{N}$ (tensile force).

$$\vec{R}_A = 15\sqrt{3}\,\mathrm{N}\hat{\imath} - 30\,\mathrm{N}\hat{j} + 20\,\mathrm{N}\hat{k},\ \vec{R}_B = -15\sqrt{3}\,\mathrm{N}\hat{\imath} + 90\,\mathrm{N}\hat{k},\ \vec{R}_C = \vec{0},\ T_{BC} = 30\,\mathrm{N}$$

SAMPLE 5.6 A 3-D truss solved on the computer: The 3-D truss shown in the figure is fabricated with 12 bars. Bars 1–5 are of length $\ell = 1$ m, bars 6–9 have length $\ell/\sqrt{2}(\approx 0.71$ m), and bars 10–12 are cut to size to fit between the joints they connect. The truss is supported at A on a ball and socket, at B on a linear roller, and at C on a planar roller. A load $F = 2$ kN is applied at D as shown. Write all equations required to solve for all bar forces and support reactions and solve the equations using a computer.

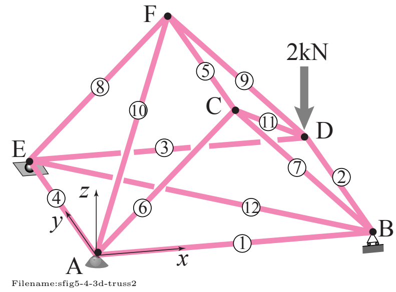

Figure 5.28:

Solution There are 12 bars and 6 joints in the given truss. The unknowns are 12 bar forces and six support reactions (3 at A (R_{A_x}, R_{A_y}, R_{A_z}), 2 at B (R_{B_y}, R_{B_z}), and 1 at E (R_{E_z})). Therefore, we need 18 independent equations to solve for all the unknowns. Since the force equilibrium at each joint gives one vector equation in 3-D, *i.e.*, three scalar equations, the 6 joints in the truss can generate the required number ($6 \times 3 = 18$) of equations. Therefore, we go joint by joint, draw the free-body diagram of the joint, write the force equilibrium equation, and extract the 3 scalar equations from each vector equation. We switch from the letters to denote the bars in the force vectors to numbers in its scalar representation (T_1, T_2, *etc.*) to facilitate computer solution.

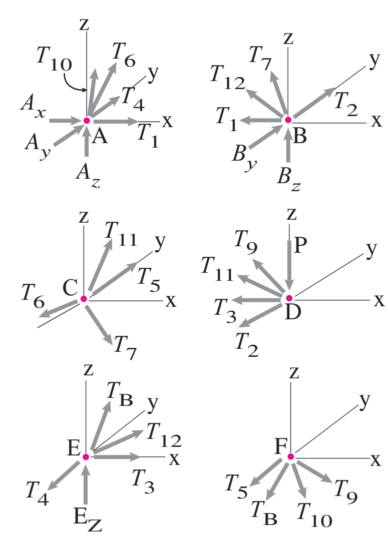

Figure 5.29:

- **Joint A:**

$$T_1\hat{\imath} + \frac{T_6}{\sqrt{2}}(\hat{\imath}+\hat{k}) + \frac{T_{10}}{\sqrt{6}}(\hat{\imath}+2\hat{\jmath}+\hat{k}) + T_4\hat{\jmath} + R_{A_x}\hat{\imath} + R_{A_y}\hat{\jmath} + R_{A_z}\hat{k} = \vec{\mathbf{0}}.$$

- **Joint B:**

$$-T_1\hat{\imath} + \frac{T_7}{\sqrt{2}}(-\hat{\imath}+\hat{k}) + T_2\hat{\jmath} + \frac{T_{12}}{\sqrt{2}}(-\hat{\imath}+\hat{\jmath}) + R_{B_y}\hat{\jmath} + R_{B_z}\hat{k} = \vec{\mathbf{0}}.$$

- **Joint C:**

$$-\frac{T_6}{\sqrt{2}}(\hat{\imath}+\hat{k}) - \frac{T_7}{\sqrt{2}}(-\hat{\imath}+\hat{k}) + T_5\hat{\jmath} + \frac{T_{11}}{\sqrt{6}}(\hat{\imath}+2\hat{\jmath}-\hat{k}) = \vec{\mathbf{0}}.$$

- **Joint D:**

$$-T_2\hat{\jmath} - \frac{T_{11}}{\sqrt{6}}(\hat{\imath}+2\hat{\jmath}-\hat{k}) - T_3\hat{\imath} + \frac{T_9}{\sqrt{2}}(-\hat{\imath}+\hat{k}) - F\hat{k} = \vec{\mathbf{0}}.$$

- **Joint E:**

$$-T_4\hat{\jmath} + \frac{T_{12}}{\sqrt{2}}(\hat{\imath}-\hat{\jmath}) + T_3\hat{\imath} + \frac{T_8}{\sqrt{2}}(\hat{\imath}+\hat{k}) + R_{E_z}\hat{k} = \vec{\mathbf{0}}.$$

- **Joint F:**

$$-T_5\hat{\jmath} - \frac{T_8}{\sqrt{2}}(\hat{\imath}+\hat{k}) - \frac{T_{10}}{\sqrt{6}}(\hat{\imath}+2\hat{\jmath}+\hat{k}) - \frac{T_9}{\sqrt{2}}(-\hat{\imath}+\hat{k}) = \vec{\mathbf{0}}.$$

Now we can separate out 3 scalar equations from each of the joint vector equations by dotting them with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$.

Joint	[Eqn.] $\cdot \hat{\boldsymbol{\imath}}$	[Eqn.] $\cdot \hat{\boldsymbol{\jmath}}$	[Eqn.] $\cdot \hat{\boldsymbol{k}}$
A	$T_1 + \frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + R_{A_x} = 0,$	$\frac{2}{\sqrt{6}}T_{10} + T_4 + R_{A_y} = 0,$	$\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{6}}T_{10} + R_{A_z} = 0$
B	$-T_1 - \frac{1}{\sqrt{2}}T_7 - \frac{1}{\sqrt{2}}T_{12} = 0,$	$T_2 + \frac{1}{\sqrt{2}}T_{12} + R_{B_y} = 0,$	$\frac{1}{\sqrt{2}}T_7 + R_{B_z} = 0$
C	$-\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0,$	$T_5 + \frac{2}{\sqrt{6}}T_{11} = 0,$	$\frac{1}{\sqrt{2}}T_6 + \frac{1}{\sqrt{2}}T_7 + \frac{1}{\sqrt{6}}T_{11} = 0$
D	$-\frac{1}{\sqrt{6}}T_{11} - T_3 - \frac{1}{\sqrt{2}}T_9 = 0,$	$-T_2 - \frac{2}{\sqrt{6}}T_{11} = 0,$	$\frac{1}{\sqrt{6}}T_{11} + \frac{1}{\sqrt{2}}T_9 = F$
E	$\frac{1}{\sqrt{2}}T_{12} + T_3 + \frac{1}{\sqrt{2}}T_8 = 0,$	$-T_4 - \frac{1}{\sqrt{2}}T_{12} = 0,$	$\frac{1}{\sqrt{2}}T_8 + R_{E_z} = 0$
F	$-\frac{1}{\sqrt{2}}T_8 - \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0,$	$-T_5 - \frac{2}{\sqrt{6}}T_{10} = 0,$	$\frac{1}{\sqrt{2}}T_8 + \frac{1}{\sqrt{6}}T_{10} + \frac{1}{\sqrt{2}}T_9 = 0.$

Thus, we have 18 required equations for the 18 unknowns. Before we go to the computer, we need to do just one more little thing. We need to order the unknowns in some way in a one-dimensional array. So, let

$$x = [R_{A_x} \quad R_{A_y} \quad R_{A_z} \quad R_{B_x} \quad R_{B_y} \quad R_{E_z} \quad T_1 \ldots T_{12}].$$

Thus $x_1 = R_{A_x},\ x_2 = R_{A_y},\ \ldots, x_7 = T_1,\ x_8 = T_2,\ \ldots, x_{18} = T_{12}$. Now we are ready to go to the computer, feed these equations, and get the solution. We enter each equation as part of a matrix [A] and a vector {b} such that `[A]{x} = {b}`. Here is the pseudocode:

```
sq2i = 1/sqrt(2)             % define a constant
sq6i = 1/sqrt(6)             % define another constant
F = 2                        % specify given load
A(1,[1 7 12 16]) = [1 1 sq2i sq6i]
A(2,[2 10 16]) = [1 1 2*sq6i]
.
.
A(18,[14 15 16]) = [sq2i sq2i sq6i]
b(12,1) = F
form A and b setting all other entries to zero
solve A*x = b for x
```

The solution obtained from the computer is the one-dimensional array `x` which after decoding according to our numbering scheme gives the following answer.

$$R_{A_x} = R_{A_y} = 0,\ R_{A_z} = -2\,\text{kN},\ R_{B_y} = 0,\ R_{B_z} = 2\,\text{kN},\ R_{E_z} = 2\,\text{kN},$$
$$T_1 = T_3 = -2\,\text{kN},\ T_2 = T_4 = T_5 = -4\,\text{kN},\ T_6 = 0,$$
$$T_7 = T_8 = -2.83\,\text{kN},\ T_9 = 0,\ T_{10} = T_{11} = 4.9\,\text{kN},\ T_{12} = 5.66\,\text{kN},$$

5.2 The method of sections

The central concept for mechanics, and thus for truss analysis, is of a free body diagram. For truss analysis we have already found it fruitful to draw free body diagrams of the whole structure, of the bars (to see that they are two-force bodies), and of the individual joints. But you can draw a free body diagram of any part of a system you are studying. Assuming static equilibrium, force and moment balance apply to that subsystem.

In *the method of sections* you find bar tensions by drawing a free body diagram of a part of the truss that includes more than one joint and less than the whole structure.

The place where the truss is cut is called the *section*.

What's wrong with the method of joints?

The method of joints can solve any solvable truss. So why learn a different method? There are two basic reasons.

1. Sometimes one only wants to know a little and the method of joints is cumbersome.

   Example: Difficulty in finding just one bar tension.
   Say you are interested only in T_{KM} in the truss of fig. 5.7 on page 260. With the method of joints we could find T_{KM} using the method of joints or by working through the joints one at time. To get to joint K we would have to draw free body diagrams of at least 8 other joints first. And for each we would have to solve two simultaneous equations.

2. Sometimes the method of joints doesn't best reveal basic structural ideas.

   Example: Difficulty in understanding trends.
   Again look at the truss of fig. 5.7. With the method of joints we would find, after all the algebra, that all the bars on the bottom (AC, CE, EH, HJ, JL, LN, NP, PR) have compression (negative tension) and that each bar has more compression than the one to its right. Similarly the top is all tension with the tension increasing with the bars more to the left. Are these trends just a consequence of lots of algebra?

The method of sections provides a shortcut, particularly for elementary textbook-like problems. And the method of sections can explain some structural trends.

The basic method of sections recipe

Say you are just trying to find one bar tension, for example T_{KM} in the truss of fig. 5.7. For simplicity we limit our attention to 2D structures.

- Find a way to cut the structure into two parts, using a *section cut* that
  - cuts the bar of interest and
  - cuts at most 3 bars in total and
  - where one of the two parts of the truss have all loads known because
    - ∗ all loads are given applied loads, or
    - ∗ the loads are reactions that have been found using a free body diagram of the whole structure.
- Write and solve the equations of moment balance for one side of the structure. This should be 3 equations in 3 unknowns.
  - Either use 3 random equations (say force balance and moment balance), or
  - Look for a shortcut. Try to find one equation that contains the unknown of interest and no other unknowns using
    - ∗ moment balance about the point of intersection of the lines of action of the two unknown forces that are not of interest, or
    - ∗ if the two uninteresting unknown forces are parallel, use force balance in a direction orthogonal to them.

For a given truss and given bar tension of interest there is no guarantee that the recipe applies. You can always find a section cut through the bar of interest, but there may be too-many unknowns in the free body digrams of both of the resulting sub-structures.

Because 2D statics of finite objects gives three scalar equations we can generally find all three unknown bar tensions from a section cut that goes through 3 bars.

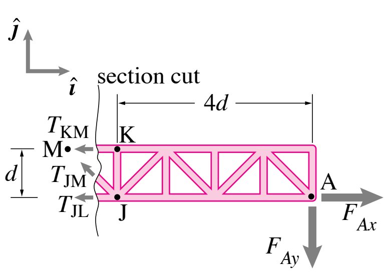

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Figure 5.30: Free body diagram of a 'section' of the structure.

Example: Three bar-forces from one FBD.

Look at the free body diagram from a section cut in fig. 5.30. Moment balance about point J (about an axis through J in the z direction) gives:

$$\left\{\sum \vec{M}_{\mathrm{J}} = \vec{0}\right\} \cdot \hat{k} \qquad \Rightarrow \qquad T_{\mathrm{KM}} = 4F_{Ay}.$$

Using the FBD with this same section cut we can also find:

$$\left\{\sum \vec{M}_{\mathrm{M}} = \vec{0}\right\} \cdot \hat{k} \qquad \Rightarrow \qquad T_{\mathrm{JL}} = -4F_{Ay} + F_{Ax}, \text{ and}$$

$$\left\{\sum \vec{F}_{\mathrm{i}} = \vec{0}\right\} \cdot \hat{\jmath} \qquad \Rightarrow \qquad F_{\mathrm{JM}} = \sqrt{2}F_{Ay}.$$

Note that in the free body diagram of fig. 5.30 moment balance about point J eliminates T_{JM} and T_{JL} and gives one equation for T_{KM}. And force balance in the $\hat{\jmath}$ direction eliminates T_{KM} and T_{JL} giving one equation for T_{JM}.

Using sections to gain insight

In the method of joints, as you worked your way along the structure fig. 5.7 from right to left you would have found the tensions getting bigger and bigger on the top bars and the compressions (negative tensions) getting bigger and bigger on the bottom bars. With the method of sections you can see that this comes from the lever arm of the load F being bigger and bigger for longer and longer sections of truss. The moment caused by the vertical load F_{Ay} is carried by the tension in the top bars and compression in the bottom bars.

A warning

Because of positive experiences with the method of sections for textbook-like problems and very simple structures, many people are left with the impression that the method of sections is more powerful than the method of joints. It isn't. The method of sections is of less general utility than the method of joints. And, unlike for the method of joints, there is no simple systematic way to find all of the bar tensions in all statically-determinate trusses (See fig. 5.31).

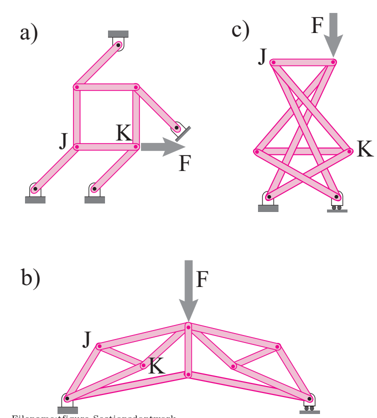

Figure 5.31: Some statically determinate trusses that do not yield easily to the method of sections. No section cut reveals a free body diagram from which you can find T_{JK}. In the truss (c) there are joints only where dots are marked (this mildly unusual truss has no closed triangles). To find T_{JK} in these trusses one has to write the equilibrium equations for many joints and solve these simultaneously.

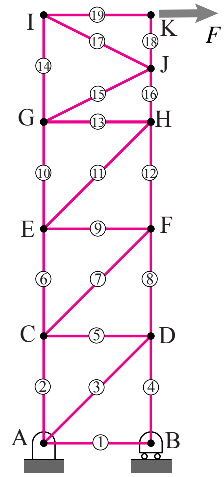

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Figure 5.32:

SAMPLE 5.7 The tower truss shown in the figure is fabricated with 19 rods. All the horizontal and vertical rods are one meter long. Joint J is halfway between joints K and H. The horizontal force applied at joint K is 1 kN. Find the tensions in

1. rod GJ, and
2. rod CE.

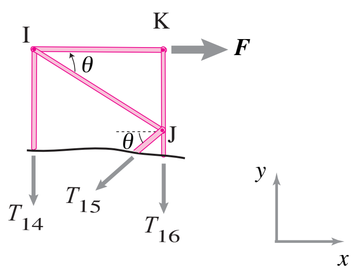

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Figure 5.33:

Solution To find the tension in rod GJ, numbered 15, let us make a cut through the truss as shown in fig. 5.33. The section taken here cuts rods 14, 15, and 16. The free-body diagram has only three unknown tensions acting on the part of the truss under consideration.

From the force balance in the x direction, we see at once,

$$\begin{aligned} F - T_{15}\cos\theta &= 0 \\ \Rightarrow \quad T_{15} &= \frac{F}{\cos\theta} \\ &= \frac{1\,\text{kN}}{\cos 26.56^\circ} \\ &= 1.12\,\text{kN}. \end{aligned}$$

$$T_{GJ} \equiv T_{15} = 1.12\,\text{kN}$$

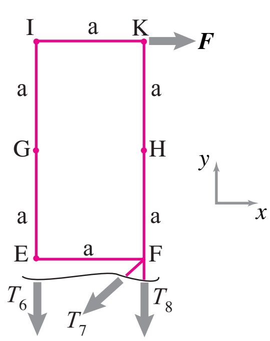

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Figure 5.34:

To determine the tension in rod CE, we consider a section that cuts rods CE, CF, and DF. The free-body diagram of the truss above this section is shown in fig. 5.34. Once again, we have only three unknown forces on the body under consideration (note that we will have six unknown forces that include three support reactions if we considered the lower part of the truss, below the selected section).

To find T_6, we write the scalar moment balance equation in the z-direction about point F:

$$\begin{aligned} aT_6 - 2aF &= 0 \\ \Rightarrow \quad T_6 &= 2F \\ &= 2\,\text{kN}. \end{aligned}$$

$$T_{CE} \equiv T_6 = 2\,\text{kN}$$

SAMPLE 5.8 A 2-D truss: The box truss shown in the figure is loaded by three vertical forces acting at joints A, B, and E. All horizontal and vertical bars in the truss are of length 2 m. Find the forces in members AB, AC, and DC.

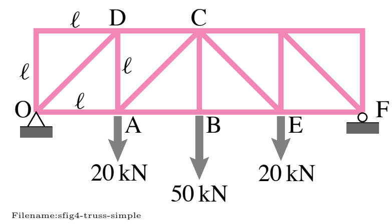


Figure 5.35:

Solution First, we need to find the support reactions at points O and F. We do this by drawing the free-body diagram of the whole truss and writing the equilibrium equations for it. Referring to Fig. 5.36, the force equilibrium, $\sum \vec{F} = \vec{0}$ implies,

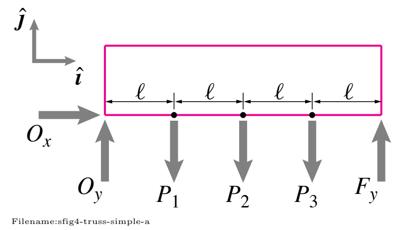


Figure 5.36:

$$O_x\hat{\imath} + (O_y + F_y - P_1 - P_2 - P_3)\hat{\jmath} = \vec{0}. \tag{5.11}$$

Dotting eqn. (5.11) with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} O_x &= 0 \\ O_y + F_y &= P_1 + P_2 + P_3. \end{aligned} \tag{5.12}$$

The moment equilibrium about point O, $\vec{M}_O = \vec{0}$, gives

$$(-P_1\ell - P_2\,2\ell - P_3\,3\ell + F_y\,4\ell)\hat{k} = \vec{0} \tag{5.13}$$

$$\text{or} \qquad F_y = \frac{1}{4}(P_1 + 2P_2 + 3P_3). \tag{5.14}$$

Solving eqns. (5.12) and (5.14), we get

$$F_y = 45\,\text{kN}, \qquad \text{and} \qquad O_y = 45kN.$$

In fact, from the symmetry of the structure and the loads, we could have guessed that the two vertical reactions must be equal, *i.e.,* $O_y = F_y$. Then, from eqn. (5.12) it follows that $O_y = F_y = (P_1 + P_2 + P_3)/2 = 45\,\text{kN}$.

Now, we proceed to find the forces in the members AB, AC, and DC. For this purpose, we make a cut in the truss such that it cuts members AD, AC, and DC, just to the right of joints A and D. Next, we draw the free-body diagram of the left (or right) portion of the truss and use the equilibrium equations to find the required forces. Referring to Fig. 5.37, the force equilibrium requires that

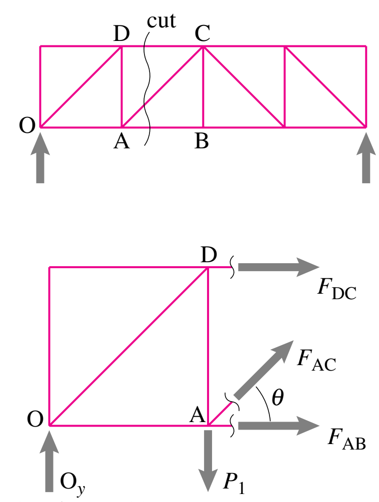


Figure 5.37:

$$(F_{AB} + F_{DC} + F_{AC}\cos\theta)\hat{\imath} + (O_y - P_1 + F_{AC}\sin\theta)\hat{\jmath} = \vec{0}. \tag{5.15}$$

Dotting eqn. (5.15) with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$F_{AB} + F_{DC} + F_{AC}\cos\theta = 0 \tag{5.16}$$
$$O_y - P_1 + F_{AC}\sin\theta = 0. \tag{5.17}$$

So far, we have two equations in three unknowns (F_{AB}, F_{DC}, F_{AC}). We need one more independent equation to be able to solve for the unknown forces. We now write moment equilibrium equation about point A, *i.e.,* $\sum \vec{M}_A = \vec{0}$,

$$\begin{aligned} (-O_y\ell - F_{DC}\ell)\hat{k} &= \vec{0} \\ \Rightarrow \qquad O_y + F_{DC} &= 0. \end{aligned} \tag{5.18}$$

We can now solve eqns. (5.16–5.18) any way we like, *e.g.,* using elimination or a computer. The solution we get (see next page for details) is:

$$F_{AC} = -25\sqrt{2}\,\text{kN}, \quad F_{DC} = -45\,\text{kN}, \qquad \text{and} \qquad F_{AB} = 70\,\text{kN}.$$

$$\boxed{F_{AC} = -25\sqrt{2}\,\text{kN}, \quad F_{DC} = -45\,\text{kN}, \quad F_{AB} = 70\,\text{kN}.}$$

Comments:

- Note that the values of F_{AC} and F_{DC} are negative which means that bars AC and DC are in compression, not tension, as we initially assumed. Thus the solution takes care of our incorrect assumptions about the directionality of the forces.
- **Short-cuts:** In the solution above, we have not used any tricks or any special points for moment equilibrium. However, with just a little bit of mechanics intuition we can solve for the required forces in five short steps as shown below.
  (i) No external force in $\hat{\imath}$ direction implies $O_x = 0$.
  (ii) Symmetry about the middle point B implies $O_y = F_y$. But,
  $$O_y + F_y = \sum P_i = 90\,\text{kN} \quad \Rightarrow \quad O_y = F_y = 45\,\text{kN}.$$
  (iii) $(\sum \vec{M}_A = \vec{0}) \cdot \hat{k}$ gives
  $$O_y \ell + F_{DC} \ell = 0 \quad \Rightarrow \quad F_{DC} = -O_y = -45\,\text{kN}.$$
  (iv) $(\sum \vec{M}_C = \vec{0}) \cdot \hat{k}$ gives
  $$-O_y\, 2\ell + P_1 \ell + F_{AB} \ell = 0 \quad \Rightarrow \quad F_{AB} = 2O_y - P_1 = 70\,\text{kN}.$$
  (v) $(\sum \vec{F} = \vec{0}) \cdot \hat{\jmath}$ gives
  $$O_y - P_1 + F_{AC} \sin\theta = 0 \quad \Rightarrow \quad F_{AC} = (P_1 - O_y)/\sin\theta = -25\sqrt{2}\,\text{kN}.$$
- **Solving equations:** On the previous page, we found F_{AB}, F_{DC}, and F_{AC} by solving eqns. (5.15–5.17) simultaneously. Here, we show you two ways to solve those equations.
  1. *By elimination:* From eqn. (5.17), we have
  $$F_{AC} = \frac{O_y - P_1}{\sin\theta} = \frac{20\,\text{kN} - 45\,\text{kN}}{1/\sqrt{2}} = -25\sqrt{2}\,\text{kN}.$$
  From eqn. (5.18), we get
  $$F_{DC} = -O_y = -45\,\text{kN},$$
  and finally, substituting the values found in eqn. (5.15), we get
  $$F_{AB} = -F_{DC} - F_{AC}\cos\theta = 45\,\text{kN} + 25\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 70\,\text{kN}.$$
  2. *On a computer:* We can write the three equations in the matrix form:
  $$\underbrace{\begin{bmatrix} 1 & 1 & \cos\theta \\ 0 & 0 & \sin\theta \\ 0 & 1 & 0 \end{bmatrix}}_{\texttt{A}} \underbrace{\begin{Bmatrix} F_{AB} \\ F_{DC} \\ F_{AC} \end{Bmatrix}}_{\texttt{x}} = \begin{Bmatrix} 0 \\ P_1 - O_y \\ -O_y \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ -25 \\ -45 \end{Bmatrix}}_{\texttt{b}} \text{kN}.$$
  We can now solve this matrix equation on a computer by keying in matrix `A` (with θ specified as $\pi/4$) and vector `b` as input and solving for `x`.①

① Pseudocode:
```
A = [1 1 cos(pi/4)
     0 0 sin(pi/4)
     0 1 0]
b = [0 -25 -45]
```
solve A*x = b for x

SAMPLE 5.9 Consider the truss shown in the figure. Rods AB, BC, EC, EF, BD, and DE are each 2 m long, and $\angle$ ABD $= \angle$DEF $= \pi/2$. Find the tensions in rods DE and CD.

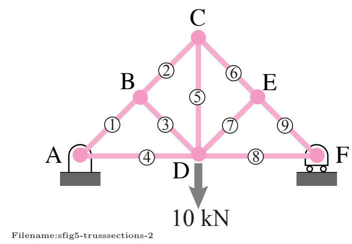


Figure 5.39:

Solution We do not need any analysis to find the tension in rod DE. Since DE is normal to CF, DE has to be a zero force member for equilibrium of joint E.② However, let us find out the same result using the method of sections. Let us take a section just to the left of joint E that cuts through rods CE, DE and DF. The free-body diagram of the truss to the right of the section is shown in fig. 5.38(a).

② In fact, that both rods BD and DE are zero force members. Since they carry no tension, rods AD and DF must also be zero force members for equilibrium of joint D

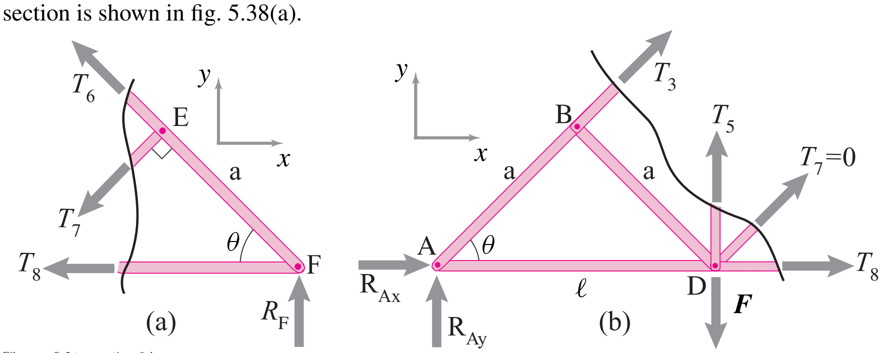


Figure 5.38: Sectional cuts of the truss, (a) A sectional cut that cuts through rods CE, DE and DF, (b) A sectional cut that cuts through rods BC, CD, DE, and DF.

The scalar moment balance equation, $\sum M_z = 0$, about point F gives at once,

$$aT_7 = 0 \quad \Rightarrow \quad T_7 = 0.$$

Thus rod CE is tension free. Now, we make another cut, taking the section shown in fig. 5.38(b) to determine the tension in rod CD. Since $T_7 = 0$, we can write the scalar moment balance equation in the z-direction about point A to give

$$\begin{aligned} \ell T_5 - \ell F &= 0 \\ \Rightarrow \quad T_5 &= F = 10\,\text{kN}. \end{aligned}$$

$$\boxed{T_7 = 0, \quad T_5 = 10\,\text{kN}}$$

5.3 Solving trusses on a computer

The method of joints is routine and is easily implemented on a computer.

- First, some software packages will accept a collection of algebraic equations, say the joint equilibrium equations, and solve them as a set for the unknowns.
- Second, one can take the set of algebraic equations as written by hand, and organize them into matrix form and solve that form on a computer as described in Section 2.5 (see page 104).
- Finally, one can treat the whole truss problem as one for which you want to do all the algebra and solution on the computer.

The first two approaches are general purpose, using the linearity of the equations and nothing special about trusses. They are as useful for trusses as for any other situation in which you have several simultaneous equations to solve.

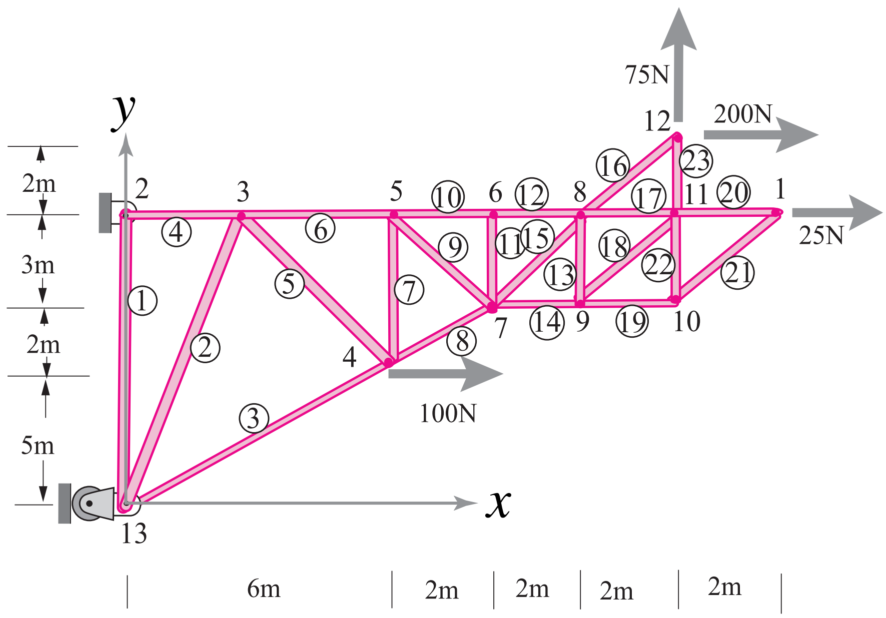
Here we present a method for both setting up and solving the equations for a truss using no hand-calculations whatsoever. That is, we present a program which you can write in whatever your preferred computer package. The advantages of having a general purpose computer program available include:

- it is quicker to then solve any given truss
- you are less likely to make an error
- if you find an error in data entry, you can quickly correct it without having to redo all other data entry and calculation
- you can change the truss geometry easily to see the effect on the bar tensions and reactions
- you can just as easily solve non-simple trusses where neither the method of joints nor the method of sections allows solution of only 2 or 3 simultaneous equations at a time.

The rest of this section is a description of the recipe, a presentation of the final program (on page 288), and some samples using that program. This is all just a systematic use of the method of joints.

The data that defines a truss problem

We first show how to define the truss, how it is supported, and the loads on it, with an organized collection of numbers rather than a picture. For definiteness, refer to the picture in fig. 5.40 which we want to communicate to a computer.



Filename:tfigure-trussnum

Figure 5.40: A truss with 23 bars (numbered with circles around them), 13 joints, 4 loads and 3 reaction components.

First pick an origin, coordinate directions, units to use for length and units to use for force. First a few numbers that say how many other numbers are needed.

n_{joints} is the number of joints, often called j. In the example $n_{joints} = 13$.

n_{bars} is the number of bars (rods), often called b. In the example $n_{bars} = 23$.

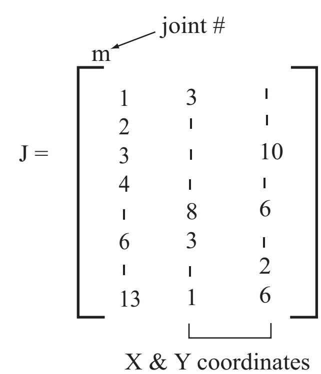
n_{bcs} is the number of reaction components (or boundary conditions), often called r. n_{bcs} is commonly 3: an x and y component at one joint and just an x or y component at another, as in the example.

The descriptions of the joints, the bars, the reactions and the loads are held in 4 matrices①.

[J] is a matrix defining the joints. Each joint is identified by a number (1 or 2 or ...) with each number from 1 to n_{joints} associated with one joint. It doesn't matter which joint has which number. Each row of [J] is the information for one joint. The first entry of a row is the joint number, and the next two numbers are the coordinates of the joint. If joint 6 is at $x = 8$ m, $y = 10$ m then the row 6 of [J] would be [6 8 10]. [J] has n_{joints} rows and 3 columns (fig. 5.41).

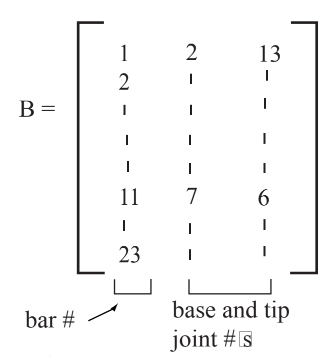
[B] is a matrix defining the bars. It has one row for each bar (n_{bars} of them) and three columns. The bars are identified by numbers 1, 2, ...(sometimes circled, to distinguish them from the joint numbering).

① If you use and are comfortable with object-oriented programming some of the data structures below can be written in a more transparent form using suggestive naming rather than array locations for the various bits of data.



Filename:tfigure-trussnumJ

Figure 5.41: The matrix [J] gives the location of the joints.



Filename:tfigure-trussnumB

Figure 5.42: The matrix [B] gives the base and tip joint # of each bar.

It doesn't matter which bar has which number so long as every integer from 1 to n_{bars} is associated with a bar (see fig. 5.42).

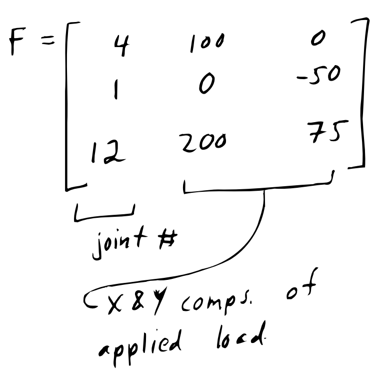
The first row of $[B]$ describes bar 1, the second describes bar 2, etc. The first element of each row is the bar number. This is also the number of the row, but it makes your data easier to read. The second two numbers are the numbers of the joints at the two ends of the bar. So if bar 11 connects base joint 7 with tip joint 6 the 11th row of $[B]$ is [11 7 6]. It is equivalent and ok to have instead the 11th row of $[B]$ be [11 6 7]; neither end of a bar is more special than the other. But once you have set the base and tip they are used to define angles in the calculations below.

$[R]$ is a matrix of reactions. It has as many rows as there are unknown reaction components, typically 3. $[R]$ has 4 columns. For easier reading, the first element of each row is the number of the row. The second element is the node at which the reaction applies. The next two numbers indicate the direction of the force acting on the truss (x and y components of a unit vector in the direction of the reaction):

- for a roller at a joint the last two numbers in the row are in the direction normal to the rollers. For normal support rollers they would be [0 1], for rollers against a vertical wall to the right of the structure they would be $[-1\ 0]$. For a roller on a 45° slope the two components could be $[0.707\ 0.707]$
- for a pin joint there are two rows in $[R]$: one for the x direction and one for the y.

Often $[R]$ will have exactly 3 rows. For the example matrix $[R]$ would be

$$[R] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \\ 3 & 13 & 1 & 0 \end{bmatrix}$$



Filename:tfigure-trussnumF

Figure 5.43: The matrix $[F]$ defines the applied loads, one row for each joint at which a load is applied.

$[F]$ is a matrix of applied loads. It has a row for each joint at which there is a non-zero load. It has three columns. The first entry of each row is the joint to which the load is applied. The next two numbers are the x and y components of the load applied to that joint. Any units can be used, they just have to be the same units for all loads. And the numerical answer for the tensions will be in these same units. If there is a rightwards load of 100 N at joint 4 one line of $[F]$ will read [4 100 0] (see fig. 5.43).

All the information about a truss that we usually communicate with a sketch is in the 4 matrices $[J]$, $[B]$, $[R]$, and $[F]$.

These specify the locations of the joints, which joints the bars are connected to, the directions and locations of reaction forces and the applied loads. Given these matrices and nothing else one could draw the truss, supports, and loading.

The unknowns

Solving the truss, finding the tensions in the bars and reaction components, is just a matter of manipulating the numbers in the four data matrices. We will hold that answer in the list $[T]$:

$[T]$ is a column vector holding the unknowns. It has as many elements as there are unknowns ($n_u \equiv n_{bars} + n_{bcs}$). The first n_{bars} elements are the unknown tensions, the last n_{bcs} elements are the unknown reaction components.

The problem

Our goal now is to use the data matrices $[J]$, $[B]$, $[R]$, and $[F]$ to find the unknowns $[T]$. We know it can be done by hand and, because the equations are linear, computer solutions should be straightforward.

Setting up the joint equations in matrix form

We now apply the method of joints.

For each joint we draw a free body diagram (in our mind). And we apply force balance in the x and y directions. Thus we will have $2n_{joints}$ equations in terms of our $n_u \equiv n_{bars} + n_{bcs}$ unknowns. The strategy is to write all these equations long hand (in our mind) and then assemble those into matrix form.

If joint 1 has emanating from it bars 3 and 7 and also has a 25 N horizontal load to the right the first of these $2n_{joints}$ equations is (see fig. 5.44):

$$\cos\theta_{20}T_{20} + \cos\theta_{21}T_{21} + 25\,\mathrm{N} = 0,$$

where θ_{20} and θ_{21} are the angles of bars 20& 21, measured CCW from the plus x direction. We can write this again as

$$0 * T_1 + 0 * T_2 + \cdots + A_{1,20} * T_{20} + A_{1,21} * T_{21} = -25\,\mathrm{N}$$

where the cosines have been rewritten as elements of a matrix. If we assume that lots of these matrix elements are zero we can rewrite the first equation once again as

$$A_{1,1} * T_1 + A_{1,2} * T_2 + A_{1,3} * T_3 + \cdots + A_{1,n_u} * T_{n_u} = -F_{11}.$$

using $[A]$ as a matrix with lots of zeros, but sines and cosines of bar angles where appropriate. Recall that n_u is the number of unknown bar tensions and reaction components and $F_{11} = 25\,\mathrm{N}$ is the x component of the load applied to joint 1.

For the second equation we similarly write the equation for force balance in the y direction for joint 1.

$$\sin\theta_{20}T_{20} + \sin\theta_{21}T_{21} = 0,$$

which can also be written out with the terms of $[A]$ (see fig. 5.45) as

$$A_{21} * T_1 + A_{22} * T_2 + A_{23} * T_3 + \cdots + A_{2n_u} * T_{n_u} = -F_{12}.$$

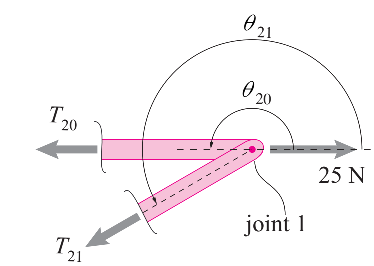


Figure 5.44: Free body diagram of joint 1 on the truss of fig. 5.40 on page 281.

The next two equations describe joint 2, etc. Thus the assembly of $2n_{joints}$ equations looks like this

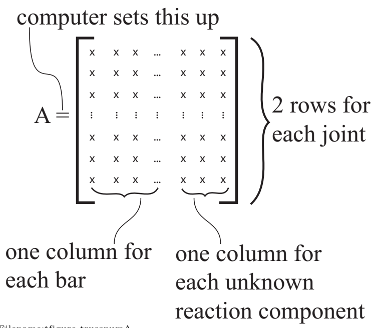
$$\begin{array}{ccccc}
A_{11} * T_1 & A_{12} * T_2 & \dots & = & -F_{11} \\
A_{21} * T_1 & A_{22} * T_2 & \dots & = & -F_{12} \\
\dots & & & & \\
\dots & & & & \\
A_{2n_{joints}\ 2} T_1 & A_{2n_{joints}\ 2} T_2 & \dots & = & -F_{n_{joints}\ 2}
\end{array}$$

which we can write more compactly as

$$[A][T] = [L] \qquad (5.19)$$

where $[A]$ is a matrix with cosines and sines of the bar angles and lots of zeros (because most bars don't touch a given joints) and $[L]$ is a list of negative of the loads applied in the x and y directions at the joints.

The point is, that all the information needed to calculate all the terms in $[A]$ and $[L]$ are in our four truss-definition matrices $[J]$, $[B]$, $[R]$ and $[F]$. And eqn. (5.19) for the unknown $[T]$ is exactly of the type that computers are great at solving.



Filename:tfigure-trussnumA

Figure 5.45: The matrix $[A]$ contains all the coefficients in the linear joint equations. It is made of sines and cosines of bar angles and the direction cosines of reaction-force directions.

Some preliminary geometry

The matrix $[A]$ is made up of sines and cosines of bar angles and we have specified the truss by the x and y positions of the ends of the bars. We first tell the computer to do some simple trig to find the sines and cosines.

$[X]$ is a list of x coordinates of each bar tip relative to its base. $[X]$ is a single column with n_{bars} entries. To find the entries of $[X]$ subtract the base-joint x coordinate from the tip-joint x coordinate. For bar 13 this would be

```
X(13) = J( B(13,3), 2 ) - J( B(13,2), 2 )
```

because `B(13,3)` is the joint at the tip of bar 13 and `B(13,2)` is the joint at the base. Thus `J(B(13,3),2)` and and `J(B(13,2),2)` are the x coordinates of the joints at the tip and base of bar 13. To find all of the elements of $[X]$ you may need to loop through all the bars or, depending on your package, you may be able to do the subtraction in one step.

$[Y]$ is a list of base-to-tip y coordinates for the bars defined analogously to $[X]$ above. Thus

```
Y(13) = J( B(13,3), 3 ) - J( B(13,2), 3 )
```

$[D]$ is a list of bar lengths (distances), so

```
D(13) =  ( X(13)^2 + y(13)^2 )^.5
```

$[C]$ is a list of n_{bars} cosines for the bars, one cosine for each bar. It is defined as the counter-clockwise angle of the base-to-tip bar relative to the positive x axis. Thus

```
C(13) = X(13)/D(13) % cosine
```

$[S]$ is a similar list of sines so

```
S(13) = Y(13)/D(13) % sine
```

All we need from the above are the $[C]$ and $[S]$ column vectors②.

② The calculation of $[X]$, $[Y]$ and $[D]$ are just intermediate steps to simplify the presentation. If you can tolerate dense coding and use a package that deals well with matrices, $[C]$ and $[S]$ can be generated with as few as 2 dense lines of code.

Building up $[A]$ from $[J]$, $[B]$ and $[R]$

The only difficult work in setting up a statically-determinate truss for computer solution is making up the matrix $[A]$. First lets set $[A]$ to be a matrix with $2n_{joints}$ rows and n_u columns and with every entry zero.

```
A = [0]
```

We now need to put a bunch of cosines and sines into the right places.

Cycling through the bars. If we look at the whole $[A]$ matrix we see that the information about bar 7, say, only occurs in column 7 of $[A]$; column 7 of $[A]$ consists of the terms that multiply T_7. Furthermore, information about bar 7 only shows up in the rows corresponding to the x and y force balance for the the joints at its two ends; that's 4 places in total.

- Bar 7 pulls on its base joint `B(7, 2)` in the x direction. Because we write 2 equations for each joint this equation corresponds to row `2*B(7, 2)-1`. Thus we can make the assignment

  ```
  A( (2*B(7, 2)-1), 7 ) = C(7)
  ```

- Bar 7 pulls on its base joint in the y direction. This equation corresponds to the next row `2 * B(7, 2)` Thus we can make the assignment

  ```
  A( (2*B(7, 2) + 1), 7 ) = S(7)
  ```

- Bar 7 pulls in the opposite direction on its tip joint `B(7, 3)` so

  ```
  A( 2*B(7, 3),        7 ) = -C(7)
  ```

- and

  ```
  A( (2*B(7, 3) + 1), 7 ) = -S(7)
  ```

One needs to cycle through all the bars③ and make these 4 assignments, 7 was just used as an example. In a package that deals well with matrices all four assignments associated with one bar could be in a single line of code.

③ **Naive approaches.** One could imagine working one row at a time, corresponding to working one joint equation at a time rather than one bar at a time. For each joint we then would need to hunt through the list of bars and see which are connected to that joint. One could write a program to do this, its just more complex than the approach we present. Alternately, you might imagine that in our original data set we would have associated each joint with the bars that connect to it (rather than the other way around as we did). This is also legitimate. But, because the number of connected bars varies from joint to joint the data structure would be more complex. Finally, because the key information is the location of the bar ends, we could have used those coordinates in our data array for the bars. But this would have required our entering the coordinates of each joint over and over, once for each bar-end connected to that joint.

Cycling through the reactions to fill in the right-most columns of $[A]$. The unknown reaction components have much the same role as do the bar tensions. But they act on only one joint. Thus each reaction component only affects 2 rows of $[A]$, the x and y components of that joint equation.

For reaction 3, say, the relevant joint is `R(3, 2)` and thus the relevant rows are `2*R(3, 2)-1` and `2*R(3, 2)`. The relevant column is $n_{bars}+3$.

- for the x component of reaction 3

```
A( (2*R(3, 2)-1), (nbars+3) )  =  R(3,3)
```

- for the y component of reaction 3

```
A((2*R(3, 2)   ), (nbars+3) )  =  R(3,4)
```

Most often, for trusses that are rigid even when floating, one only has three such reaction components to cycle through.

The load vector $[L]$ The load vector is just made up of the forces applied to the joints. For load 2, for example, applied at joint `F(2,1)` , the two relevant rows of $[L]$ are `2*F(2,1)` and `2*F(2,1)+1` at which act the x, and y components of the force `F(6,2)` and `F(6,3)`, respectively. Thus, for load 6, we have

```
L( 2*F(2,1)   )  =   -F(2,2)
L( 2*F(2,1)+1 )  =   -F(2,3)
```

Recall that the minus sign follows from moving the applied load to the right side of the equation. This pair of commands needs to be applied to each line of the $[F]$ matrix.

Solution

We have now constructed all the unknowns in eqn. (5.19)

$$[A][T] = [L] \tag{5.20}$$

and can thus hand the problem to the computer for solution

```
Solve {A T = L} for T
```

The resulting column vector $[T]$ is a list of bar tensions and reaction components.

The complete truss program

The complete truss program, in pseudo-code that you need to convert to your preferred computer language/package, is shown in fig. 5.40 on page 288. Some of the loops can be 'vectorized' if your package supports such. The output $[T]$ is a column with the tensions followed by the reaction components.

What can go wrong?

Besides the various careless errors you will discover the first 10 or so times you try to run your code, there are possible deeper problems.

Because we are not trying to write general purpose super-robust software we assume the simple check for determinacy (number of unknowns = number of equations):

$$n_{rods} + n_{bcs} = 2n_{joints} \quad \text{or} \quad b + r = 2j$$

has been satisfied. Thus $[A]$ will be square. If the truss is determinate the computer will give you a nice solution. If the truss is not determinate, with $[A]$ square or not, the result of the computer calculation will depend on the software package, ranging from an error message (e.g., "`Matrix singular!`" or "`Divide by zero!`") to the computer's making its best guess at what you want (even though the equations may have no solution, or may be many solutions to select from). Some computer packages don't tell you when they are guessing.

How the pros solve trusses

The algorithm here is one way to set up and solve statically-determinate mechanics problems on a computer. In detail, however, this recipe is simpler than that commonly used in the *finite-element* method. Finite-element programs can also solve statically-indeterminate problems. A statically-indeterminate truss has tensions which can't be found from statics alone, but which can be found if the bar stiffnesses are known. Finite-element programs don't assume the bars are rigid. Rather, they take account of the small deformations of the bars.

A simple finite-element program for statically-indeterminate trusses would not use the tensions in the bars as unknowns, but rather the displacements of the joints. Such a program would be a little longer than the one presented here, and also requires introduction of a 'stiffness matrix'④, a topic a shade too advanced to cover in detail here.

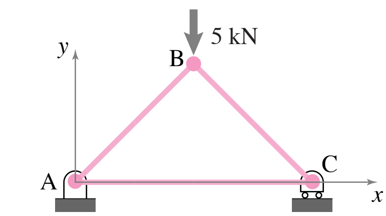
④**Stiffness matrix.** The stiffness matrix $[K]$ for a truss has $2n_{joints}$ rows and columns. It satisfies the equation $[\Delta] = [K][L]$ where $[L]$ is a list of x and y components of the loads applied to all the joints and $[\Delta]$ are the x and y displacements of the joints for those loads. The matrix $[K]$ can be *assembled* if the properties of all the bars are given.

```
%PSEUDO-CODE TO SOLVE ANY 2D STATICALLY DETERMINATE TRUSS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Assign values to the matrices which define the truss and loading
  J = [ 1 . . . ]   % specify the joint locations
  B = [ 1 . . . ]   % specify the joints that the bars connect
  R = [ 1 . .   ]   % specify which nodes connect to the ground and how
  F = [ . . . . ]   % specify which nodes have what applied loads
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
Program TRUSS, input is (J,B,R,F)  output is (T)
% Set up
 A     =  a square matrix  of zeros with twice as many rows as J
 L     =     a column      of zeros with twice as many rows as J
 nbars =  the number of rows of B
% Fill in the columns of the matrix A associated with bar tensions
 Loop for every bar  (each row i of B)
   base = B(i,2)                    % joint at one end of a bar
   tip  = B(i,3)                    % joint at the other end
   X = J( tip, 2 ) - J( base, 2 )   % base to tip x shadow of bar
   Y = J( tip, 3 ) - J( base, 3 )   % base to tip y shadow of bar
   D = ( X^2 + y^2 )^.5             % length of bar
   C = X/D                          % cosine of bar angle
   S = Y/D                          % sine   of bar angle
   A( (2*base-1), i ) =  C          % x comp of pull direction on base
   A( (2*base  ), i ) =  S          % y comp of pull direction on base
   A( (2*tip -1), i ) = -C          % x comp of pull direction on tip
   A( (2*tip   ), i ) = -S          % y comp of pull direction on tip
 End Loop
% Fill in  rightmost columns of  A, associated with reaction forces
 Loop for every reaction component (each row j of R)
   joint = R(j,2)                              % joint at ground connection
   A( (2*joint-1),  (nbars+j) )  = R(j,3)    % x comp of reaction direction
   A( (2*joint  ),  (nbars+j) )  = R(j,4)    % y comp of reaction direction
 End Loop
 Loop for all joints with loads (each row k of F)
   joint  =  F(k,1)                   % joint at which load is applied
   L( 2*joint -1 )  = - F(k,2)        % x component of load
   L( 2*joint    )  = - F(k,3)        % y component of load
 End Loop


% Solve the truss (solve the  set of simultaneous joint-equilibrium equations)
 Solve  {AT = L} for T     % The whole calculation is done in this one line.
                           % T is a list of bar tensions
                           %   followed  by reaction components
End Program
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

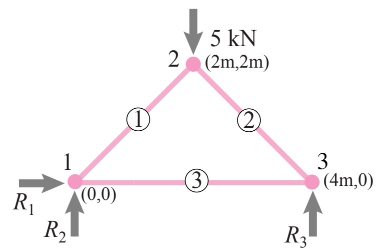
Figure 5.46: **Pseudo-code for solving any statically determinate truss.** This ‘program’ calculates the bar tensions and reactions in a statically determinate truss. The algorithm is described in detail starting on page 280. This program could be reduced to 10 lines of code in some common computer languages (with some loss of clarity).

SAMPLE 5.10 For the truss shown in the figure, the coordinates of the three joints are: A(0,0), B(2m,2m), and C(4m,0). Find all reactions and bar forces using computer analysis. Show the input data to the program used and the matrices [A] and [L] generated by the program.



Filename:sfig5-3-simpletruss

Figure 5.47:



Filename:sfig5-3-simpletruss-a

Figure 5.48:

Solution The free-body diagram of the truss with the unknown reactions serially numbered is shown in fig. 5.48. We have also numbered the bars and joints for preparing the input data file as described in the text. Here, we have three bars and three joints, three unknown reactions, and one externally applied load. Therefore, the input matrices [B] for bar data, [J] for joint data, [R] for support reaction data, and [F] for applied load data are as follows (see page 281 for row and column descriptions).

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 3 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0.0 & 0.0 \\ 2 & 2.0 & 2.0 \\ 3 & 4.0 & 0.0 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 1 & 1.0 & 0.0 \\ 2 & 1 & 0.0 & 1.0 \\ 3 & 3 & 0.0 & 1.0 \end{bmatrix}, \quad F = \begin{bmatrix} 2 & 0.0 & -5.0 \end{bmatrix}$$

The computer program based on the pseudocode described in the text generates the following matrices [A] and [L], before solving for the tensions and reactions:

```
A = [ 0.7071         0    1.0000    1.0000         0         0
      0.7071         0         0         0    1.0000         0
     -0.7071    0.7071         0         0         0         0
     -0.7071   -0.7071         0         0         0         0
           0   -0.7071   -1.0000         0         0         0
           0    0.7071         0         0         0    1.0000 ]
L = [ 0
      0
      0
      5
      0
      0 ]
```

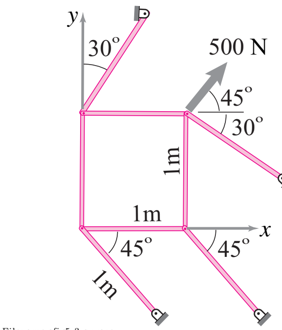
The final step, `Solve {A T = F} for T,` gives the following output

```
T = [ -3.5355
      -3.5355
       2.5000
            0
       2.5000
       2.5000 ]
```

which means, $T_1 = T_2 = -3.5355\,\text{N}$, $T_3 = 2.5\,\text{N}$, $R_1 = 0$, and $R_2 = R_3 = 2.5\,\text{N}$.

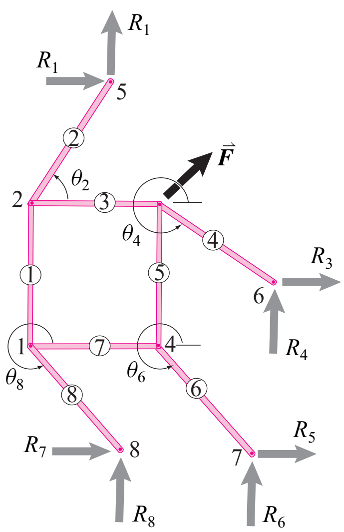
$$\boxed{T_1 = T_2 = -3.5355\,\text{N},\ T_3 = 2.5\,\text{N},\ R_1 = 0,\ R_2 = R_3 = 2.5\,\text{N}}$$

Note: If you write a truss code, you can use this sample to check your code.



Filename:sfig5-3-square

Figure 5.49:



Filename:sfig5-3-square-a

Figure 5.50: Free-body diagram of the truss along with bar and joint numbers. From the given geometry, $\theta_1 = \pi/3, \theta_2 = 11\pi/6 = -\pi/6$, and $\theta_3 = \theta_4 = 7\pi/4$. Assuming the bar length to be ℓ, we can write the coordinates of all joints in terms of ℓ and θ's. For example, $x_5 = \ell\cos\theta_1$, $y_5 = \ell + \ell\sin\theta_1$.

SAMPLE 5.11 The truss shown in the figure has no triangles, yet it is rigid in the configuration shown as discussed in the text. It is also an example of a truss where you cannot find a sequence of joints that will let you solve for the bar forces 'locally', that is, without solving all joint equations simultaneously. Assume all bars to be 1 m long. Find all reactions and bar forces. Show the input data to the program used.

Solution The free-body diagram of the truss with the unknown reactions serially numbered is shown in fig. 5.50. Note that support reactions have been taken as unknown x and y components of the reaction at each support point. We could have, alternatively, taken the reaction components to be along and normal to the bars at each support point.

The bars and joints are numbered as shown. Here, we have eight bars and eight joints, eight unknown reactions, and one externally applied load. Let the length of each bar be $\ell = 1$ m. The angle of outer bars with the x-axis are $\theta_2 = \pi/3$, $\theta_4 = -\pi/6$, $\theta_6 = \theta_8 = -\pi/4$. Therefore, the input matrices [B] (bar data), [J] (joint data), [R] (support reaction data), and [F] (applied load data) are as follows (see page 281 for row and column descriptions).

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 5 \\ 3 & 2 & 3 \\ 4 & 3 & 6 \\ 5 & 3 & 4 \\ 6 & 4 & 7 \\ 7 & 1 & 4 \\ 8 & 1 & 8 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 0.0 & 0.0 \\ 2 & 0.0 & \ell \\ 3 & \ell & \ell \\ 4 & \ell & 0.0 \\ 5 & \ell\cos\theta_2 & \ell + \ell\sin\theta_2 \\ 6 & \ell + \ell\cos\theta_4 & \ell + \ell\sin\theta_4 \\ 7 & \ell + \ell\cos\theta_6 & \ell\sin\theta_6 \\ 8 & \ell\cos\theta_8 & \ell\sin\theta_8 \end{bmatrix},$$

$$R = \begin{bmatrix} 1 & 5 & 1.0 & 0.0 \\ 2 & 5 & 0.0 & 1.0 \\ 3 & 6 & 1.0 & 0.0 \\ 4 & 6 & 0.0 & 1.0 \\ 5 & 7 & 1.0 & 0.0 \\ 6 & 7 & 0.0 & 1.0 \\ 7 & 8 & 1.0 & 0.0 \\ 8 & 8 & 0.0 & 1.0 \end{bmatrix}, \quad F = \begin{bmatrix} 3 & 500/\sqrt{2} & 500/\sqrt{2} \end{bmatrix}$$

The computer program based on the pseudocode described in the text generates the following output, [T], for the tensions and reactions: The final step, `Solve {A T = F} for T,` gives the following result for bar tensions and reactions.

T_1	=	−418.26 N,	R_1	=	−241.48 N,
T_2	=	−482.96 N,	R_2	=	−418.26 N,
T_3	=	241.48 N,	R_3	=	−112.07 N,
T_4	=	−129.41 N,	R_4	=	64.70 N,
T_5	=	418.26 N,	R_5	=	418.26 N,
T_6	=	591.51 N,	R_6	=	−418.26 N,
T_7	=	418.26 N,	R_7	=	−418.26 N,
T_8	=	−591.51 N,	R_8	=	418.26 N.

Note: It is easy to check that $R_1 + R_3 + R_5 + R_7 = -F\cos(45^\circ)$ and $R_2 + R_4 + R_6 + R_8 = -F\sin(45^\circ)$ where $F = 500$ N. That is, for the free-body diagram of the truss, $\sum F_x = 0$ and $\sum F_y = 0$.

5.4 Frames and structures

Although trusses are good, they are not good enough for all purposes, nor necessarily good-enough models of very truss-looking structures. *Frames* are structures that are more general than trusses. In a truss every bar is a two-force body. In general structures one or more components is not a two-force body. The analysis of non-truss frames is generally less formulaic, and thus more subtle, than analysis of trusses using the method of joints.

Example: A-frame ladder.
The two-diagonal parts of an A-frame ladder are not two-force bodies and thus the ladder, triangular as it looks, is not a truss. And truss analysis is not appropriate.

The overall mechanics recipe applies to frames, of course: a) draw free body diagrams, b) apply the laws of mechanics to each free body diagram, and c) solve the mechanics equations for unknowns of interest. For trusses, the free body diagrams of each bar, with the 3 equilibrium equations (6 in 3D) just yield the "two-force" body result that the bar has equal tensions at the two ends. Because there was no more to learn from the bar free body diagrams we didn't even draw them. Instead we used the bar tensions as forces on free body diagrams of the joints. Its as if the bars were just a means to mediate action-reaction pairs between joints.

For more general frameworks we we have to pay full respect to the free body diagrams of all of the parts, not just the pins. At least for all of the parts that are not two-force bodies. Here is the analysis of frameworks recipe:

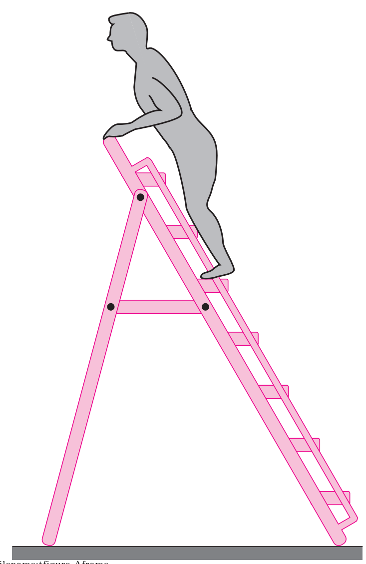
- Draw free body diagrams of
  - the whole structure; and
  - the separate parts of the structure; and
  - collections of parts of the structure if such seems likely to be fruitful;
  - Use the principal of action and reaction in the free body diagrams so that one action-reaction pair has only one unknown;
- For each free body diagram write equilibrium conditions (force and moment balance).
- Solve the equilibrium equations for desired unknowns.

For the parts that are not two-force bodies, we will not know the directions of the interaction forces *a priori*, that's why the method of joints is not used for frames ①. Naturally one can be on the look out for shortcuts:

- for any two force bodies assign an equal valued tension to each end (thus eliminating any need or use for equilibrium equations for that object)
- consider each pin as part of one of the bodies to which it is connected (ie, there is no need to draw a separate FBD of the pin).

Filename:tfigure-ladderphoto

Figure 5.51: A ladder is well modeled as a collection of rigid pieces connected by pin joints . . .



Filename:tfigure-Aframe

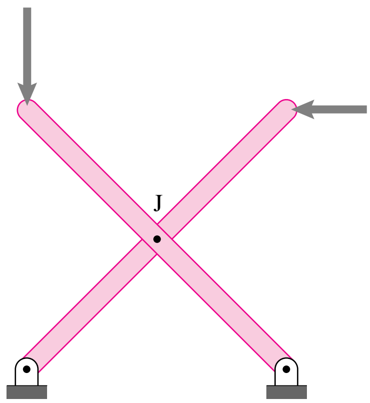
Figure 5.52: ... but A-frame ladder is not a truss because the bars are not all two-force members. The piece the person is standing on has forces at more than two places: the hands, the top pin, the middle brace, the feet and the ground. The left diagonal leg also has three ($3 > 2$) forces acting on it.

① Conversely, we could have analyzed trusses the way we are now going to analyze frames. This seldom-used approach to trusses, the 'method of bars and pins', is discussed in box 5.1 on page 294.

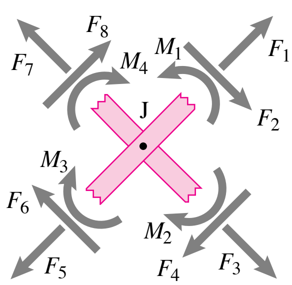
- To minimize calculation, look for a subset of the equilibrium equations that
  - contains your unknowns of interest, and
  - has as many unknowns as scalar equations, and
  - contains as few equations as possible.

Our general goal here is to find the reaction forces, the interaction forces and the 'internal' forces in the components of a statically determinate structure.

Example: An X structure
Two bars are joined in an 'X' by a pin at J. Neither of the bars is a two-force body so a free body diagram of the 'joint' at J, made by cutting and leaving stubs as we did with trusses, has 12 unknown force and moment components.



FBD of 'joint' J



Filename:tfigure-xstructure

Figure 5.53: A simple frame and a free body diagram of a joint. Unlike the case for trusses, free body diagrams of joints (pins) are nearly useless in the study of more general frames. There are too many unknown forces and moments acting at the pin.

Instead of drawing free body diagrams of the connections, our approach here is to draw free body diagrams of each of the structure or machine's parts. Sometimes, as was the case with trusses, it is also useful to draw a free body diagram of a whole structure or of some multi-piece part of the structure

Static determinacy

A statically determinate structure has

- a solution for all possible applied loads, and
- only one solution, and
- this solution can be found by using equilibrium equations applied to each of the pieces.

Not all practical structures are statically determinate. Some structures are rigid but redundant, thus precluding finding all unknowns from statics. Some structures cannot carry all loads, but can carry the loads of interest (e.g., a vertical cable that can usefully carry a weight but cannot carry a side load). None-the-less, for starters here we emphasize determinate structures. The basic counting formula

$$\text{number of equations} = \text{number of unknowns}$$

is necessary for determinacy but does not guarantee determinacy. For frameworks in 2D there are three equilibrium equations for each (non-point) object. There are two unknown force components for every pin connection, whether to the ground or to another piece. And there is one unknown force component for every roller connection whether to the ground or between objects. Applied forces do not count in this determinacy check, even if they are unknown.

Example: 'X' structure counting

In the 'X' structure above we can count as follows.

$$\begin{aligned}\text{number of equations} &\stackrel{?}{=} \text{number of unknowns}\\ (3 \text{ eqs per bar}) \cdot (2 \text{ bars}) &\stackrel{?}{=} (2 \text{ unknown force comps per pin}) \cdot (3 \text{ pins})\\ 6 \text{ eqs} &\stackrel{\surd}{=} 6 \text{ unknown force components}\end{aligned}$$

So the 'X' structure passes the counting test for static determinacy.

Redundant structures

A redundant structure can carry whatever loads it can carry in more than one way. If not also indeterminate, a redundant structure has fewer equilibrium equations than unknown reaction or interaction force components. Finding all the reaction components is only possible if one models the deformation, a topic for more advanced structural mechanics. **Example: Overbraced 'X'**

The structure is evidently redundant because it has a bar added to a structure which was already statically determinate. By counting we get

$$\begin{aligned}\text{number of equations} &\stackrel{?}{=} \text{number of unknowns}\\ 3 \cdot (\underbrace{\text{number of bars}}_{3}) &\stackrel{?}{=} 2 \cdot (\underbrace{\text{number of joints}}_{5})\\ 9 \text{ eqs} &< 10 \text{ unknown force components}\end{aligned}$$

thus demonstrating redundancy.

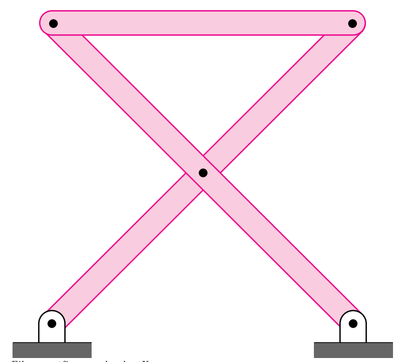


Figure 5.54: Overbraced X. A rigid frame that is not statically determinate.

5.1 The 'method of bars and pins' for trusses

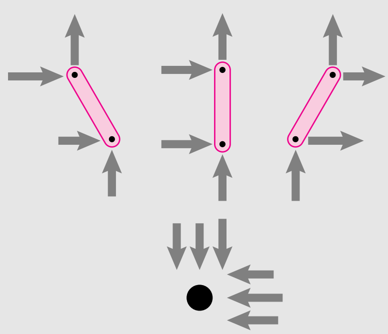
This is an aside for those who wonder why truss analysis seems so different than frame analysis.

Trusses are simple frameworks. So the methods used for more general frameworks should work for trusses. They do. The resulting method, which is essentially never used in such detail, we will call 'the method of bars and pins'.

In the method of bars and pins you treat a truss like any other structure. You draw a free body diagram of each part.

One approach: treat the pins as parts. One approach is to draw free body diagrams of each pin also. You use the principle of action and reaction to relate the forces on the different bars and pins. Then you solve the collection of equilibrium equations.

Consider one joint of a truss where three bars meet at a hinge (pin). Below are free body diagrams of the three bars and of the pin. Assuming a frictionless round pin at the hinge, all the bar forces on the pin pass through its center.

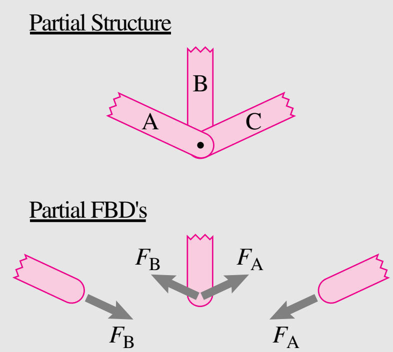


Filename:tfigure-barsandpins

Thus, in 2D, you get two equilibrium equations for each pin and three for each bar. If you apply the three bar equations to a given bar you find that it obeys the two-force body relations. Namely, the reactions on the two bar ends are equal and opposite and along the connecting points. Now application of the pin equilibrium equations is identical to the joint equations we had previously. Thus, the 'method of bars and pins' reduces to the method of joints in the end.

Approach two: draw FBDs of just the bars. Another approach is to associate each pin with one of the bars to which it is attached. Then just think of a truss as bars that are connected with forces and no moments. Draw free body diagrams of each piece, use the principle of action and reaction, and write the equilibrium equations for each bar. This is the approach that is used in this section for other structures.

If three bars A, B, and C are connected to a pin, consider the pin as part of, say, A. Then consider action-reaction pairs between A and B, and between A and C, but not between B and C. Similarly if there are four or more bars, consider interactions between each bar and the one-bar that has the pin.



Filename:tfigure-ABC

Determinate equations. In all cases, if the truss is statically determinate the equilibrium equations generated from the free body diagrams above will produce a solvable set of linear algebraic equations. But, in all cases above, these will not be the more minimal set of equations we generated in the method of joints in the truss analysis. The methods of this box work, they are just harder to implement.

SAMPLE 5.12 The braced X-frame shown in the figure carries two vertical loads $F_1 = 2\,\text{kN}$ and $F_2 = 3\,\text{kN}$. Points G and H are directly above points A and B respectively. If $d = h = 2\,\text{m}$, find the tension in the brace CD.

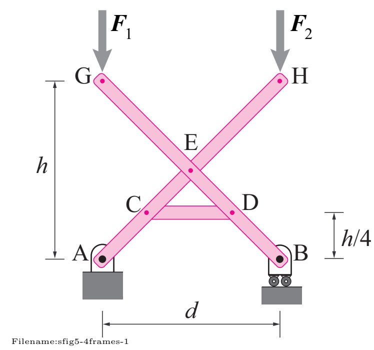


Figure 5.55:

Solution The brace CD is pinned to the X-frame at C and D. The only loads acting on the brace are at its ends C and D. Therefore, it is a two-force body. Let us assume that the tension in brace is C_x. We need to find C_x under the given loads.

The free-body diagram of the whole frame is shown in fig. 5.56. Since the frame is supported by a hinge at A and a roller at B, there are three scalar support reactions acting on the frame. We can now determine all the three reactions from the static analysis of the frame:

$$
\begin{aligned}
\sum F_x = 0 \quad &\Rightarrow \quad A_x = 0 \\
\sum M_A = 0 \quad &\Rightarrow \quad B_y d - F_2 d = 0 \\
&\Rightarrow \quad B_y = F_2 \\
\sum F_y = 0 \quad &\Rightarrow \quad A_y = F_1 + F_2 - B_y = F_1.
\end{aligned}
$$

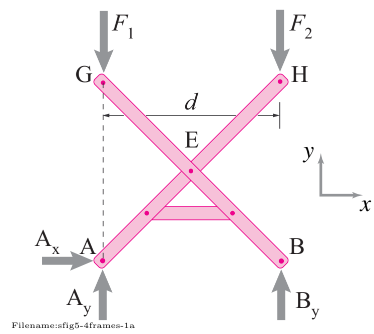


Figure 5.56:

Thus all the reactions are known. Now we can analyze either bar AH or bar BG (the analysis is identical) to determine the tension C_x in the brace. The free-body diagram of bar AH is shown in fig. 5.57. Since we are only interested in C_x, we can carry out moment balance about point E ($\sum M_E = 0$) to give

$$
\begin{aligned}
C_x \frac{d}{4} - F_2 \frac{d}{2} - A_y \frac{d}{2} &= 0 \\
\Rightarrow \quad C_x &= 2(F_2 + A_y) \\
&= 2(F_2 + F_1) \\
&= 2(3k\,\text{N} + 2\,\text{kN}) \\
&= 10\,\text{kN}.
\end{aligned}
$$

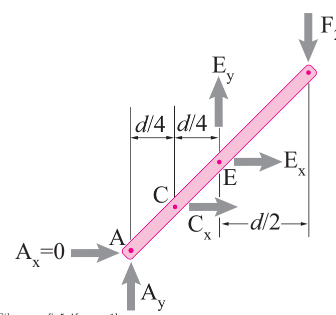


Figure 5.57:

Thus the tension in the brace is twice the total load on the structure.

Tension in brace CD $= 10\,\text{kN}$

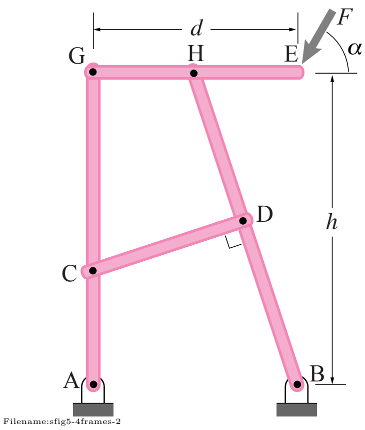


Figure 5.58:

SAMPLE 5.13 The frame shown in the figure is supported by hinges at both A and B. Bar GE is as long as the base AB and bar BH is pinned to GE at the mid point H. Brace CD is pinned at D, the mid-point of bar BH, and is orthogonal to bar BH. The load on the structure, $F = 1\,\text{kN}$, is applied at E, at an angle $\alpha = 60°$. Given that $d = 2\,\text{m}$, $h = 3\,\text{m}$, find the forces on the inclined bar BH and the support reactions at A and B.
[Note: Usually, determinate framed structures are made up of overhangs and extensions on a rigid triangle. This structure is an example of a frame that does not contain any rigid traingle.]

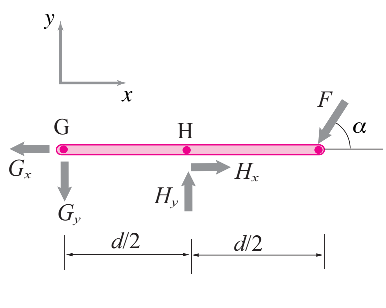


Figure 5.59:

Solution The given structure has hinges at both A and B. Therefore, there are four scalar support reactions, two each at A and B. So, from the free-body diagram of the whole structure, we cannot determine all support reactions. In fact, the free-body diagram of each rod will have more than three unknown forces (you can check this mentally). Thus, we are not likely to find all unknown forces on a bar without analyzing other bars. Since bar CD is a two-force member bar, it only contributes one scalar force, the tension in this rod. Now, there are two unknown scalar forces at each pin joint, A, B, G, and H, and one force at C and D (the same force). Thus we have nine unknown scalar forces. We have three bars AG, GE, and BH, each with three independent scalar equations of static equilibrium. Thus we have nine independent equations in nine unknowns. Therefore, we can solve for all the unknown forces.

Consider the free-body diagram of bar GE. The static equilibrium of this bar requires

$$\begin{aligned}
\sum M_H = 0 \quad &\Rightarrow \quad G_y(d/2) - F\sin\alpha(d/2) = 0\\
&\Rightarrow \quad G_y = F\sin\alpha\\
\sum F_y = 0 \quad &\Rightarrow \quad -G_y + H_y - F\sin\alpha = 0\\
&\Rightarrow \quad H_y = F\sin\alpha + G_y = 2F\sin\alpha\\
\sum F_x = 0 \quad &\Rightarrow \quad H_x - G_x - F\cos\alpha = 0. \qquad (5.21)
\end{aligned}$$

Thus we have found G_y and H_y but only a relationship between G_x and H_x. Since G_x and H_x are colinear, we cannot solve for them from the static analysis of bar GE alone. Now, let us consider bar AG (or bar BH; does not make a difference). The equilibrium analysis of this bar gives

$$\sum F_y = 0 \quad \Rightarrow \quad A_y + G_y + R_{CD}\sin\theta = 0 \qquad (5.22)$$

$$\sum M_C = 0 \quad \Rightarrow \quad A_x h_1 - G_x h_2 = 0 \qquad (5.23)$$

$$\sum F_x = 0 \quad \Rightarrow \quad A_x + G_x + R_{CD}\cos\theta = 0. \qquad (5.24)$$

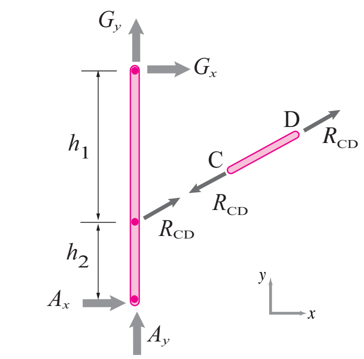


Figure 5.60:

Since none of these equations contains only one unknown, we cannot solve for these forces from the equilibrium equations of bar AG alone. Note that we have written these equations in terms of h_1, h_2, and θ, thus far, undetermined geometric variables. However, we can easily find them from the given geometry. Now let us analyze bar BH.

$$\sum F_y = 0 \quad \Rightarrow \quad B_y - H_y - R_{CD}\sin\theta = 0 \qquad (5.25)$$

$$\sum F_x = 0 \quad \Rightarrow \quad B_x - H_x - R_{CD}\cos\theta = 0 \qquad (5.26)$$

$$\sum M_D = 0 \quad \Rightarrow \quad (H_x + B_x)\frac{h}{2} + (H_y + B_y)\frac{d}{2} = 0 \qquad (5.27)$$

So, now we have seven independent equations, eqns. (5.21)–(5.27), in seven unknowns — A_x, A_y, B_x, B_y, R_{CD}, G_x, and H_x (we have already solved for G_y and H_y). We can solve these seven equations on a computer.

Before we go to the computer, let us find the undetermined geometric quantities h_1 and h_2. From fig. 5.62, we see that

$$\begin{aligned} h_1 &= \frac{h}{2} - \Delta \\ h_2 &= \frac{h}{2} + \Delta \end{aligned}$$

where $\Delta = d' \sin\theta$, $d' = 3d/4$, and $\theta = \tan^{-1}(d/2h)$. Now, we are ready to solve the seven equations on a computer.

```
% input given quantities
d = 2; h = 3; F = 1; alpha = pi/3;
% Define other used quantities in the equations
Delta = 3*d^2/(8*h);
h1 = h/2 - Delta;  h2 = h/2 + Delta;
theta = arctan(0.5*d/h);
% Input equations
eqset = { Hx - Gx = F*cos(alpha)
          Ay + RCD*sin(theta) = -F*sin(alpha)
          Ax*h1 - Gx*h2 = 0
          Ax + Gx + RCD*cos(theta) = 0
          By - RCD*sin(theta) = 2*F*sin(alpha)
          Bx - Hx - RCD*cos(theta) = 0
          (Hx+Bx)*h/2 + By*d/2 = -F*d*sin(alpha) }

solve eqset for Ax, Ay, Bx, By, Gx, Hx, and RCD
```

Including the values of G_y and H_y obtained from the first two equations of equilibrium of bar GE, we get the following values for all unknown forces from the computer solution.

$$\begin{array}{ll} A_x = -9.93\,\text{kN} & A_y = -10.79\,\text{kN} \\ B_x = 10.43\,\text{kN} & B_y = 11.66\,\text{kN} \\ R_{CD} = 31.40\,\text{kN} & \\ G_x = -19.86\,\text{kN} & G_y = 0.87\,\text{kN} \\ H_x = -19.36\,\text{kN} & H_y = 1.73\,\text{kN} \end{array}$$

$R_{CD} = 31.4\,\text{kN}$

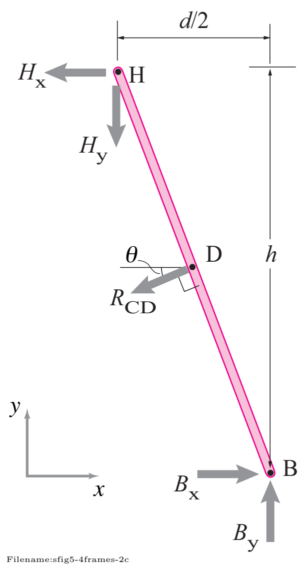


Figure 5.61:

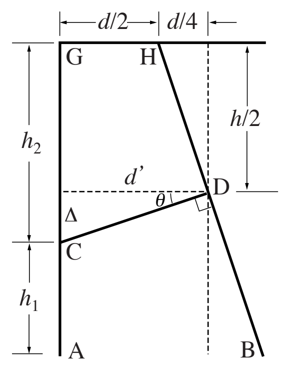
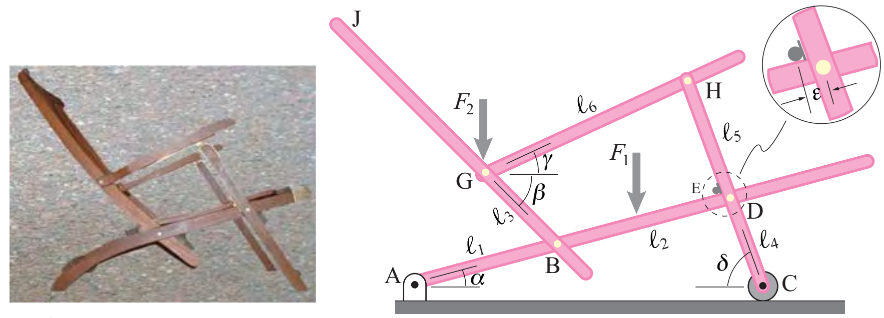


Figure 5.62: The geometry of h_1 and h_2. Since $d' = 3d/4$, $\Delta = d' \tan\theta$ and $\tan\theta = \frac{d/2}{h}$, we get $\Delta = \frac{3d}{4}\frac{d}{2h} = \frac{3d^2}{8h}$ and $h_1 = h/2 - \Delta$, $h_2 = h/2 + \Delta$.

SAMPLE 5.14 An easy-chair uses a curved frame as shown in the small picture in fig. 5.64. To simplify geometry, we can model the chair with straight bars as shown in the figure. Of special significance is the small pin at E that is rigidly attached to bar CDH and slides with negligible friction on bar ABD (see inset). This pin keeps the chair from collapsing and bears a large load. Assume the pin is $\epsilon = 2.5$ cm away from joint D towards B along bar ABD.



Filename:sfig5-4-chair-col

Figure 5.64: Dimensions: $\ell_1 = 45$ cm, $\ell_2 = 60$ cm, $\ell_3 = 30$ cm, $\ell_4 = 30$ cm, $\ell_5 = 70$ cm, $\epsilon = 2.5$ cm, $\alpha = 15°$, $\beta = 45°$, $\gamma = 25°$, and $\delta = 70°$. Loads: $F_1 = 500$ N and $F_2 = 200$ N. F_1 acts in the middle of bar segment BD and F_2 acts at G.

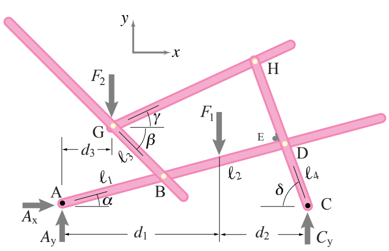


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Figure 5.63: Free-body diagram of the whole chair. Note that $d_1 = (\ell_1 + \ell_2/2)\cos\alpha$, $d_2 = (\ell_2/2)\cos\alpha + \ell_3\cos\delta$, and $d_3 = \ell_1\cos\alpha - \ell_4\cos\beta$.

Solution Since the chair is supported by a hinge at A and a roller at B, there are three scalar support reactions. So, we can determine them from the static analysis of the whole chair frame. The free-body diagram of the chair is shown in fig. 5.63. The moment and force equilibrium equations give

$$\sum F_x = 0 \quad \Rightarrow \quad A_x = 0$$

$$\sum M_A = 0 \quad \Rightarrow \quad C_y(d_1 + d_2) - F_1(d_1) - F_2(d_3) = 0$$

$$\Rightarrow \quad C_y = \frac{F_1 d_1 + F_2 d_3}{d_1 + d_2}$$

$$\sum F_y = 0 \quad \Rightarrow \quad A_y = F_1 + F_2 - C_y.$$

From the given geometry,

$$\begin{aligned} d_1 &= (\ell_1 + \ell_2/2)\cos\alpha = 72.44\,\text{cm} \\ d_2 &= (\ell_2/2)\cos\alpha + \ell_3\cos\delta = 39.24\,\text{cm} \\ d_3 &= \ell_1\cos\alpha - \ell_4\cos\beta = 22.25\,\text{cm}. \end{aligned}$$

Substituting these dimensions above with their numerical values, we get

$$C_y = 364\,\text{N}, \quad \text{and} \quad A_y = 336\,\text{N}.$$

The support reactions are thus determined. To find the force on the pin E, we can use either bar ABD or bar CDH. In either case however, we have more unknown force on the bars that we can determine from the equilibrium equations of that bar alone. So, we will have to use equilibrium of some other bar as well. Note that bar GH is a two-force body. Therefore, the tension in this rod can be shown as a single scalar force R_{GH}. Let us now analyze the equilibrium of bar BGJ since it has only three unknown forces on it (see fig. 5.65). The moment and force equilibrium equations give

$$\sum M_G = 0 \quad\Rightarrow\quad F_2(\ell_4\cos\beta) - R_{GH}(\ell_4\sin(\gamma+\beta)) = 0$$
$$\Rightarrow\quad R_{GH} = \frac{F_2(\ell_4\cos\beta)}{\ell_4\sin(\gamma+\beta)} = 183\,\text{N}.$$
$$\sum F_x = 0 \quad\Rightarrow\quad B_x = R_{GH}\cos\gamma = 166\,\text{N}$$
$$\sum F_y = 0 \quad\Rightarrow\quad B_y = R_{GH}\sin\gamma - F_2 = -123\,\text{N}.$$

Now that we know A_x, A_y, B_x and B_y, we can analyze bar ABD and determine the rest of the unknown forces on it including the force in the pin E, R_E (see the free-body diagram in fig. 5.66):

$$\sum M_D = 0 \quad\Rightarrow\quad -A_y(d_4+d_5) - B_y d_5 + B_x h + F_1 d_6 + R_E\epsilon = 0$$
$$\Rightarrow\quad R_E = \frac{A_y(d_4+d_5) + B_y d_5 - B_x h - F_1 d_6}{\epsilon}$$
$$\sum F_x = 0 \quad\Rightarrow\quad B_x + D_x + R_E\sin\alpha = 0$$
$$\Rightarrow\quad D_x = -B_x - R_E\sin\alpha$$
$$\sum F_y = 0 \quad\Rightarrow\quad D_y = R_E\cos\alpha - A_y - B_y.$$

From geometry,

$$\begin{aligned} d_4 &= \ell_1\cos\alpha \\ d_5 &= \ell_2\cos\alpha \\ d_6 &= d_5/2 = (\ell_2/2)\cos\alpha \\ h &= \ell_2\sin\alpha. \end{aligned}$$

Substituting these variables with their numerical values above, we get

$$R_E = 3953\,\text{N}, \qquad D_x = -1189\,\text{N}, \qquad \text{and} \qquad D_y = 4106\,\text{N}.$$

$$\boxed{A_x = 0,\ A_y = 336\,\text{N},\ C_y = 364\,\text{N},\ R_E = 3953\,\text{N}}$$

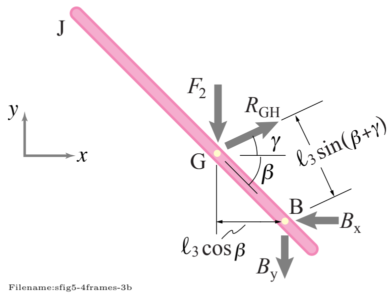


Figure 5.65:

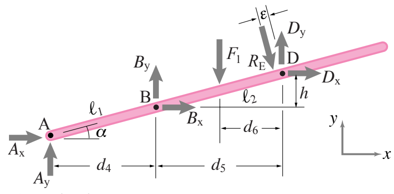
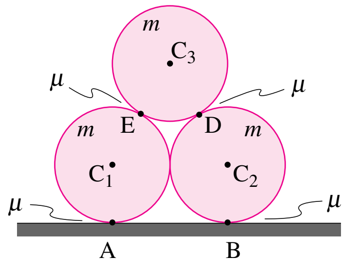


Figure 5.66: Free-body diagram of bar ABD. Note that $d_4 = \ell_1\cos\alpha$, $d_5 = \ell_2\cos\alpha$, $d_6 = d_5/2 = (\ell_2/2)\cos\alpha$, and $h = \ell_2\sin\alpha$.



Filename:sfig4-single-3balls

Figure 5.67:

SAMPLE 5.15 Can a stack of three cylinders be in static equilibrium? Three identical cylinders, each of mass m and radius R, are stacked such that the top cylinder rests on the lower two cylinders. The two cylinders at the bottom do not touch each other. Let the coefficient of friction at each contact surface be μ. Find the minimum value of μ so that the three cylinders are in static equilibrium.

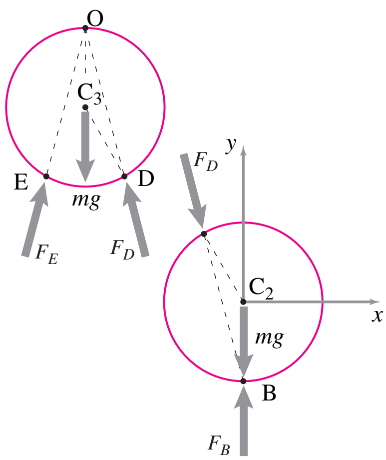


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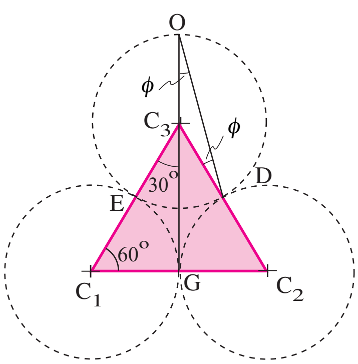
Figure 5.68:

Solution Let us assume that the three cylinders are in equilibrium. We can then find the forces required on each cylinder to maintain the equilibrium. If we can find a plausible value of the friction coefficient μ from the required friction force on any of the cylinders, then we are done, otherwise our initial assumption of static equilibrium is wrong.

The free body diagrams of the upper cylinder and the lower right cylinder (why the right cylinder? No particular reason) are shown in fig. 5.68. The contact forces, $\vec{F}_E$ and $\vec{F}_D$, act on the upper cylinder at points E and D, respectively. Each contact force is the resultant of a tangential friction force and a normal force acting at the point of contact. From the free body diagrams, we see that each cylinder is a three-force-body. Therefore, all the three forces — the two contact forces and the force of gravity — must be concurrent. This requires that the two contact forces must intersect on the vertical line passing through the center of the cylinder (the line of action of the force of gravity). Now, if we consider the free body diagram of the lower right cylinder, we find that force $\vec{F}_D$ has to pass through point B since the other two forces intersect at point B. Thus, we know the direction of force $\vec{F}_D$.

Let ϕ be the angle between the contact force $\vec{F}_D$ and the normal to the cylinder surface at D. Now, from geometry, $\angle C_3DO + \angle C_3OD + OC_3D = 180°$. But, $\phi = \angle C_3DO = \angle C_3OD$. Therefore,

$$\begin{aligned}\phi &= \frac{1}{2}(180° - \angle OC_3D) = \frac{1}{2}(\angle GC_3D)\\ &= \frac{1}{2}30° = 15°\end{aligned}$$



Filename:sfig4-single-3balls-b

Figure 5.69:

where $\angle GC_3D = 30°$ follows from the fact that $C_1C_2C_3$ is an equilateral triangle and C_3G bisects $\angle C_1C_3C_2$.

Now, from fig. 5.69, we see that

$$\tan\phi = \frac{F_s}{N}.$$

But, the force of friction $F_s \leq \mu N$. Therefore, it follows that

$$\mu \geq \tan\phi = \tan 15° = 0.27.$$

Thus, the friction coefficient must be at least 0.27 if the three cylinders have to be in static equilibrium.

$\boxed{\mu \geq 0.27}$

5.5 Advanced truss concepts: determinacy

Your first concern when studying trusses is to develop the ability to solve a truss using free body diagrams and equilibrium equations. You can do this with the method of joints. For some trusses you can use the method of sections as a short cut. However, not all trusses give a unique solution. In algebra there are equations with non-unique solutions (*e.g.,* $x + y = 4$) and sets of equations with no solutions ($x + y = 4$ and $2x + 2y = 9$). We have seen this issue before in the context of static equilibrium of a particle (see box 4.1 on page 190). With trusses the issues of existence and uniqueness remain.

Determinate, rigid, and redundant trusses

A truss that yields a solution, and only one solution, to such an analysis for all possible loadings is called *statically determinate* or just *determinate*. The braced box supported with one pin joint and one pin on rollers (see fig. 5.70a) is a classic statically determinate truss. A statically determinate truss is *rigid* and does not have *redundant* bars.

You should beware, however, that there are a few other possibilities.

Some trusses are *non-rigid*, like the one shown in fig. 5.70b, and can not carry arbitrary loads at the joints.

Example: Joint equations and non-rigid structures
Free body diagrams of joints A and B of fig. 5.70b are shown in fig. 5.71.

$$\begin{aligned}
\text{jointB}: &\quad \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\imath} &\Rightarrow&\quad T_{AB} &=&\quad F\\
\text{jointA}: &\quad \left\{\sum \vec{F}_i = \vec{0}\right\}\cdot\hat{\imath} &\Rightarrow&\quad T_{AB} &=&\quad 0
\end{aligned}$$

The contradiction that T_{AB} is both F and 0 implies that the equations of statics have no solution for a horizontal load at joint B.

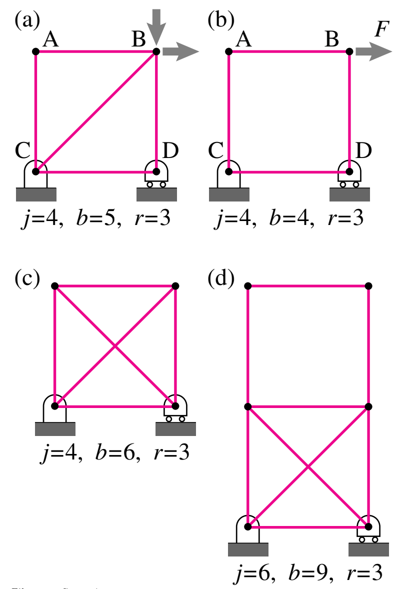


Figure 5.70: a) a statically determinate truss, b) a non-rigid truss, c) a redundant truss, and d) a non-rigid and redundant truss.

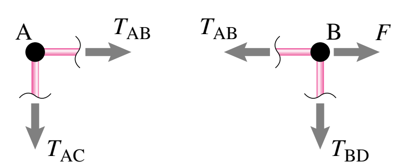


Figure 5.71: Free body diagrams of joints A and B from 5.70b. The solutions to the equilibrium equations are in conflict. That the equilibrium equations can't be satisfied means the structure cannot carry the given load. It is not rigid.

A non-rigid truss can carry some loads, and you can find the bar tensions using the joint equilibrium equations when these loads are applied. For example, the structure of fig. 5.70b can carry a vertical load at joint B. Engineers sometimes choose to design trusses that are not rigid, the simplest example being a single piece of cable hanging a weight. A more elaborate example is a suspension bridge which, when analyzed as a truss, is not rigid.

A *redundant* truss has more bars than needed for rigidity. As you can tell from inspection or analysis, the braced square of fig. 5.70a is rigid. None the less engineers will often choose to add extra redundant bracing as in fig. 5.70c for a variety of reasons.

- Redundancy is a safety feature. If one member brakes the whole structure holds up.
- Redundancy can increase a structure's strength.
- Redundancy can allow tensile bracing. In the structure of fig. 5.70a top load to the left puts bar BC in compression. Thus bar BC can't be, say,

a cable. But in structure fig. 5.70c both diagonals can be cables and neither need carry compression for any load①.

A property of redundant structures is that you can find more than one set of bar forces that satisfy the equilibrium equations. Even when the loads are all zero these structures can have non-zero *locked in* forces (sometimes called ('locked in stress', or 'self stress'). In the structure of fig. 5.70c, for example, if one of the diagonals got cool and contracted both it and the opposite diagonal would be put in tension while the outside was in compression. For structures whose parts are likely to expand or contract, or for which the foundation may shift, this locked in stress can be a contributor to structural failure. So redundancy is not all good.

Finally, a structure can be both non-rigid and redundant as shown in fig. 5.70d. This structure can't carry all loads, but the loads it can carry it can carry with various locked in bar forces.

More examples of statically determinate, non-rigid, and redundant truss are given on pages 309 and 310.

Note, one of the basic assumptions in elementary truss analysis which we have thus far used without comment is that

> motions and deformations of the structure are not taken into account when applying the equilibrium equations.

If a bar is vertical in the drawing then it is taken as vertical for all joint equilibrium equations.

Example: Hanging rope
For elementary truss analysis, a hanging rope would be taken as hanging vertically even if side loads are applied to its end. This obviously ridiculous assumption manifests itself in truss analysis by the discovery that a hanging rope cannot carry any sideways loads (if it must stay vertical this is true).

Determining determinacy: counting equations and unknowns

How can you tell if a truss is statically determinate? The only sure test is to write all the joint force balance equations and see if they have a unique solution for all possible joint loads. Because this is an involved linear algebra calculation (which we skip in this book), it is nice to have shortcuts, even if not totally reliable. Here are three:

- See, using your intuition, if the structure can deform without any of the bars changing length. You can see that the structures of fig. 5.70b and d can distort. If a structure can distort it is not rigid and thus is not statically determinate.
- See, using your intuition, if there are any redundant bars. A redundant bar is one that prevents a structural deformation that already is prevented. It is easy to see that the second diagonal in structures of

①**Tensegrity structures.** Notice that you could make the diagonals in fig. 5.70c both sticks and all of the outside square from cables and the truss would still carry all loads. This is the simplest 'tensegrity' structure. In a tensegrity structure no more than one bar in compression is connected to any one joint. (See fig. 5.11 for a more elegant example.). The label 'Tensegrity structure' was coined by the truss-pre-occupied designer Buckminster Fuller. Fuller is also responsible for re-inventing the "geodesic dome" a type of structure studied previously by Cauchy.

fig. 5.70c and d is clearly redundant so these structures are not statically determinate.

- Count the total number of joint equations, two for each joint. See if this is equal to the number of unknown bar forces and reactions. If not, the structure is not statically determinate.

The counting formula in the third criterion above is:

$$2j = b + r \tag{5.28}$$

where j is the number of joints, including joints at reaction points, b is the number of bars, and r is the number of reaction components that shows on a free body diagram of the whole structure (2 from pin joints, 1 from a pin on a roller).

If $2j > b + r$ the structure is necessarily not rigid because then there are more equations than unknowns②. For such a structure there are some loads for which there is no set of bar forces and reactions that can satisfy the joint equilibrium equations. A structure that is non-redundant and non-rigid always has $2j > b + r$ (see fig. 5.70b).

② A non-rigid truss is sometimes called 'over-determinate' because there are more equations than unknowns. However, the term 'over-determinate' may incorrectly conjure up the image of there being too many bars (which we call *redundant*) rather than too many joints. So we avoid use of this phrase.

If $2j < b + r$ the structure is redundant because there are not as many equations as unknowns; if the equations can be solved there is more than one combination of forces that solve them. A structure that is rigid and redundant always has $2j < b + r$ (see fig. 5.70b).

But the possibility of structures that are both non-rigid and redundant makes the counting formulas an imperfect way to classify structures③.Non-rigid redundant structures can have $2j < b + r$, $2j = b + r$, or $2j > b + r$. The redundant non-rigid structure in fig. 5.70d has $2j = b + r$.

③ In the language of mathematics we would say that satisfaction of the counting equation $2j = b + r$ is a *necessary* condition for static determinacy but it is not *sufficient*.

The discussion above can be roughly summarized by this table (refer to fig. 5.70 for a simple example of each entry and to pages 309 and 310 for several more examples).

Truss Type	Rigid	Non-rigid
Non-redundant	a) $2j = b + r$ (Statically determinate)	b) $2j > b + r$
Redundant	c) $2j < b + r$	d) $2j < b + r$, $2j = b + r$, or $2j > b + r$

A basic summary is this:

If

- $2j = b + r$ *and*
- you cannot see any ways the structure can distort, *and*

- you cannot see any redundant bars

then the truss is likely statically determinate. But the only way you can know for sure is through either a detailed study of the joint equilibrium equations, or familiarity with similar structures.

On the other hand if

- $2j > b + r$, *or*
- $2j < b + r$, *or*
- you can see a way the structure can distort, *or*
- you can see one or more redundant bars,

then the truss is *not* statically determinate.

Example: The classic statically determinate structure
A *triangulated truss* can be drawn as follows:

1. draw one triangle,
2. then another by adding two bars to an edge,
3. then another by adding two bars to an existent edge
4. and so on, but never adding a triangle by adding just one bar, and
5. you hold this structure in place with a pin at one joint and one pin on roller at another joint

then the structure is statically determinate. Many elementary trusses are of exactly this type. (*Note*: if you violate the 'but' in the 4th rule you can make a truss that looks 'triangulated' but is redundant, and therefore not statically determinate.)

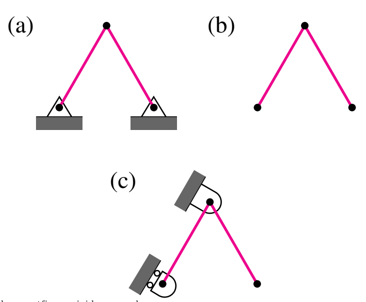


Figure 5.72: a) a determinate two bar truss connected to the ground, b) the same truss is not rigid when floating, which you can tell by seeing that c) it is not rigid when one bar is fixed to the ground.

Floating trusses

Sometimes one wants to know if a structure is rigid and non-redundant when it is floating unconnected to the ground (but still in 2D, say). For example, a triangle is rigid when floating and a square is not. The truss of fig. 5.72a is rigid as connected but not when floating (fig. 5.72b).A way to find out if a floating structure is rigid is to connect one bar of the truss to the ground by connecting one end of the bar with a pin and the other with a pin on a roller, as in fig. 5.72c. All determinations of rigidity for the floating truss are the same as for a truss grounded this way. The counting formula eqn. 5.28, is reduced to

$$2j = b + 3$$

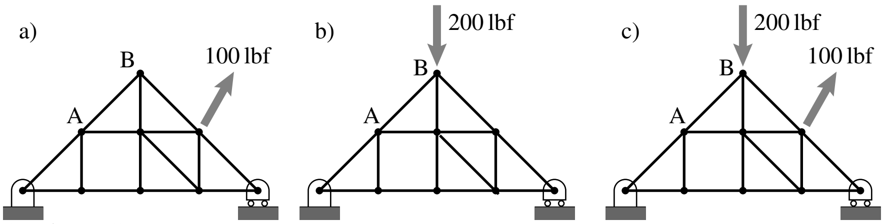
because this minimal way of holding the structure down uses $r = 3$ reaction force components.

The principle of superposition for trusses

Say you have solved a truss with a certain load and have also solved it with a different load. Then if both loads were applied the reactions would be the sums of the previously found reactions and the bar forces would be the sums of the previously found bar forces.

This useful fact follows from the linearity of the equilibrium equations④.

④A careful derivation would also show that the linearity depends on the nature of the foundation. Linearity holds for pins and pins on rollers, but not for frictional contact.



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Example: Superposition and a truss
If for the loading (a) you found $T_{AB} = 50\,\text{lbf}$ and for loading (b) you found $T_{AB} = -140\,\text{lbf}$ then for loading (c) $T_{AB} = 50\,\text{lbf} - 140\,\text{lbf} = -90\,\text{lbf}$

The principle of superposition can only hold if the solution for zero load is zero tension in all the bars. Any truss that only has bars in tension when there is no load does not satisfy the principle of superposition.

Example: Spider web
A network of taut strings is a kind of a truss. So a spider web is a kind of a truss. But a spider web is only a coherent structure if it is kept taut. So there is tension in the strands even when there is no load (from, say the weight of a spider). Thus the principle of superposition does not apply. The tension in a given strand is not the tension due to the spider added to the tension due to an insect.

5.2 Stuctural rigidity and geometric congruence

This box is only for the curious. It will not help you solve truss homework problems.

In high school geometry one learns to prove that two shapes are congruent (the same shape and size) if they have enough in common. High school geometry proofs are based on triangles. For example on proof, called "side-side-side" (SSS), says that if two triangles have three sides with corresponding lengths then the corresponding angles are also equal.

Now, here, we claim that structures made of triangles tend to be rigid. Is there a relation between the central role of triangles in geometry proofs and their role in structural rigidity? The answer is yes, but more subtly than you may expect.

Consider one triangle. If the lengths are specified it is like three sticks connected with rubber bands (page 256). That two different triangles each with the same 3 side lengths are congruent means that one triangle whose side-lengths are given has no choice about its shape. So for one triangle the SSS proof corresponds exactly to structural rigidity.

More generally, imagine looking at a structure and thinking of certain aspects of it as fixed and others as not fixed. For example, think of of a collection of bars with the lengths fixed (each bar cannot stretch or shrink) and the angles between them as not-fixed (the angles are flexible). This would be a model, say, of bars connected with pin joints. If one could find a geometry proof that these two structures had identical shapes it would mean that each one of them had no choice about its shape. So a geometry proof of congruency, based on the aspects of a structure that are approximately fixed, is a proof of structural rigidity. This shows that there *is* a connection between congruence proofs and structural rigidity.

Here's the subtlety. Neither congruence proofs nor rigidity actually depend essentially on triangles. There are congruence proofs for shapes that do not have any closed triangles, and the related structures are rigid.

In fact, there is a whole arcane mathematics of rigidity. And the things mathematicians have learned about rigidity are incredible.

The case of K_{33}

Take 3 points on a plane and mark them with dots. Take 3 more points on the plane and mark them with little x's. Connect each dot with each x. Thats 9 connection lines. In topology-speak they call this set of dots and lines "K three three" (*K*onnections between *three* dots and *three* x's).

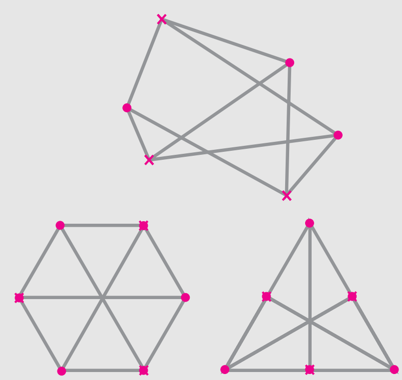
Now think of that criss-crossed K_{33} drawing as a structure made of sticks connected with hinges at the dots and x's. Note that, neglecting where the sticks cross but are not connected, there are no closed triangles. Yet, incredibly, this structure is always rigid. Well, almost always. If all 6 points happen to lie on one circle, ellipse, parabola or hyperbola then the structure is not rigid.

Example: Regular hexagon

If you take a regular hexagon made of sticks (length ℓ and hinges and brace it with three cross bars (each with length 2ℓ) you will see that you have K33; every-other corner is a dot and the alternate ones are x's. But the points on a hexagon are on a circle, so that structure is not rigid.

Example: Triangle with two bars per side

On the other hand, take an equilateral triangle and cut each side in half so you have six bars around the outside (each with length $\ell/2$). Now brace that hexagon (that is shaped like a triangle) with the three triangle altitudes (each with length $\sqrt{3}\ell$) and you again have K33. But this time it's rigid.



Filename:pfigure-boxtrussgeometry

The examples above are used in the text and homework to illustrate structures that don't lend themselves to the simple joint-by-joint method-of-joints, nor the method of sections. For these trusses the method of joints leads to a set of equations that need to be solved simultaneously.

K_{nn}

The mathematical magic goes on. If you take any n dots and any n x's and connect each dot to each x with a rigid rod (K_{nn}) you get a rigid structure. Unless all $2n$ dots happen to lie on a conic section.

The proofs of such rigidity theorems are way over our heads. But you can simply check such structures for rigidity with the computer program developed in section 5.3.

Rigidity and congruence?

So yes, geometric congruence and structural rigidity are the same subject. But that subject does not totally depend on triangles. Triangles just provide the simple examples and what we vaguely think of as the essence of both subjects.

5.3 Rigidity, redundancy, linear algebra and maps

This mathematical aside is only for people who have had a course in linear algebra. For definiteness this discussion is limited to 2D trusses, but the ideas also apply to 3D trusses.

For beginners trusses fall into two types, those that are uniquely solvable (statically determinate) and those that are not. Statically determinate trusses are rigid and non-redundant. However, a truss could be non-rigid and non-redundant, rigid and redundant, or non-rigid and redundant. These four possibilities are shown with a simple example each in fig. 5.70 on page 301, as a simple table on page 303, and as a big table of examples on pages 309 and 310.

Another approach is the table in fig. 5.73 which we now proceed to discuss in detail. It is a more abstract mathematical representation of this same set of possibilities.

• • •

To start with we use the matrix form of the truss joint equations from page 284. To make contact with linear algebra here we take the unknowns as $[v]$ with $[v] \equiv [T]$ being the n unknown tensions and reaction components. The set of lists of all conceivable tensions and reaction forces we call the "vector space" V (it is also R^n).

The m possible loads at the joints are written in the column vector $[w]$ (called $[L]$, for loads, in the numeric set up). The set of all possible loads we call the vector space W.

If we use the method of joints we can write two scalar equilibrium equations for each joint. These are linear algebraic equations. Thus we can write them in matrix form as (see eqn. (5.19) on page 284),

$$[A][v] = [w] \tag{5.29}$$

The classification of trusses is really a statement about the solutions of eqn. 5.29. This classification follows, in turn, from the properties of the matrix $[A]$.

Another point of view is to think of eqn. 5.29 as a function that maps one vector space onto another. For any $[v]$ eqn. 5.29 maps that $[v]$ to some $[w]$. That is, if one were given all the bar tensions and reactions one could uniquely determine the applied loads from eqn. 5.29. This map, from V to W we call T.

• • •

We can now discuss each of the truss categorizations in turn, with reference to the table at the end of this box.

The first column of the table corresponds to rigid trusses. These trusses have at least one set of bar forces that can equilibrate any particular load. This means that for every $[w]$ there is some $[v]$ that maps to (whose image is) $[w]$. In these cases the map T is onto. And the column space of $[A]$ is W. Thus $[A]$ needs to have at least as many columns as the dimension of W which is the number of rows of $[A]$.

On the other hand if the structure is not rigid there are some loads that cannot be equilibrated by any bar forces. This is the second column of the table. There is at least some $[w]$ with no pre-image $[v]$. Thus the map T is not onto and the column space of $[A]$ is less than all of W.

The first row of the table describes trusses which are not-redundant. Thus, any loads which can be equilibrated can be equilibrated with a unique set of bar tensions and reactions. Thus the columns of $[A]$ are linearly independent and the map T is one-to-one. The matrix $[A]$ must have at least as many rows as columns.

If a truss is redundant, as in the second row of the table, then there are various ways to equilibrate loads which can be carried. Points in W in the image of one, and the columns of A are linearly dependent.

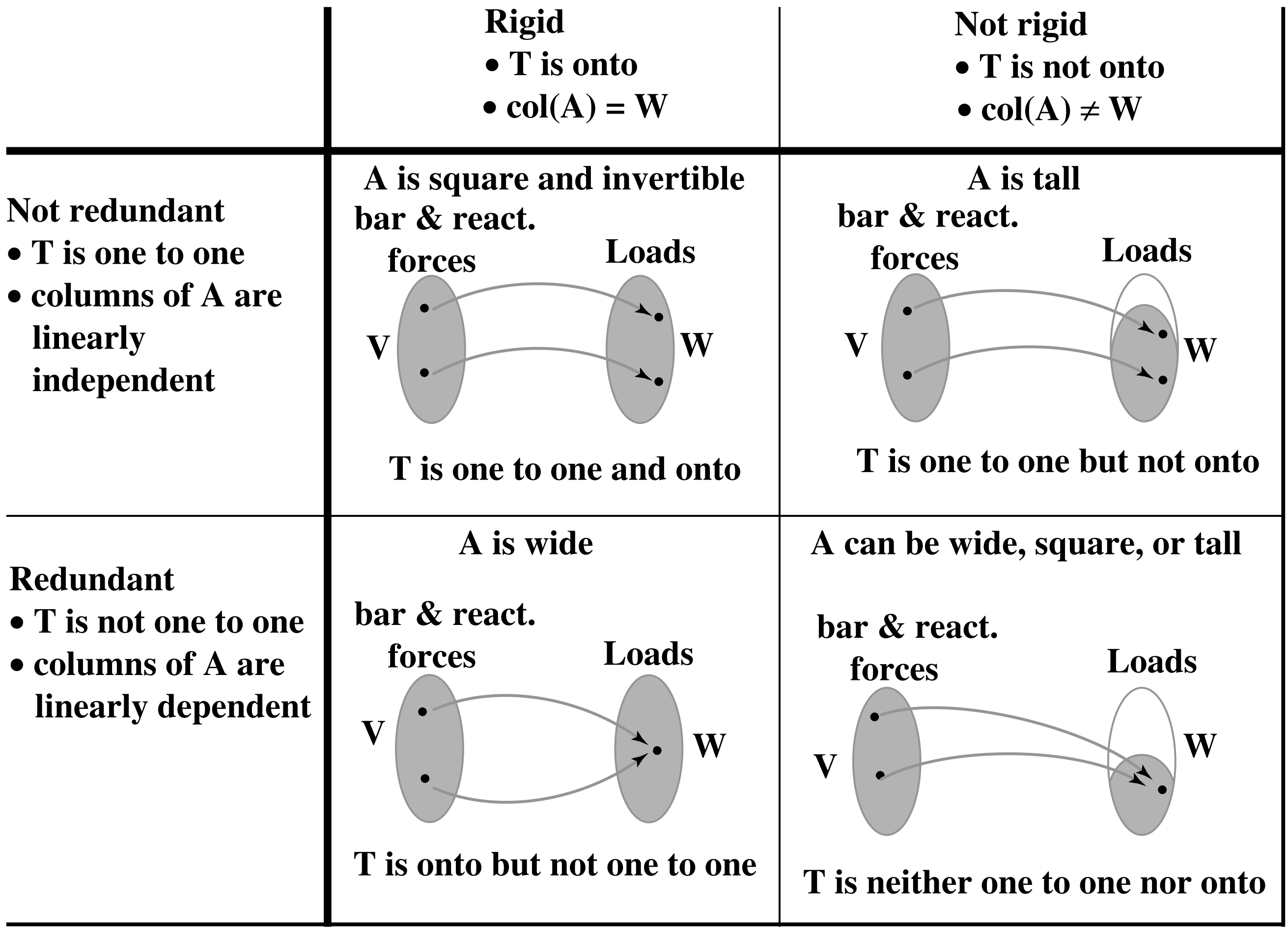
• • •

We can now look at the four entries in the table. The top left case is the statically determinate case where the structure is rigid and non-redundant. The map T is one to one and onto, $V = W$, and the matrix $[A]$ is square and non-singular.

The bottom left case corresponds to a truss that is rigid and redundant. The map to is onto but not one to one. The columns of $[A]$ are linearly dependent and it has more columns than rows (it is wide).

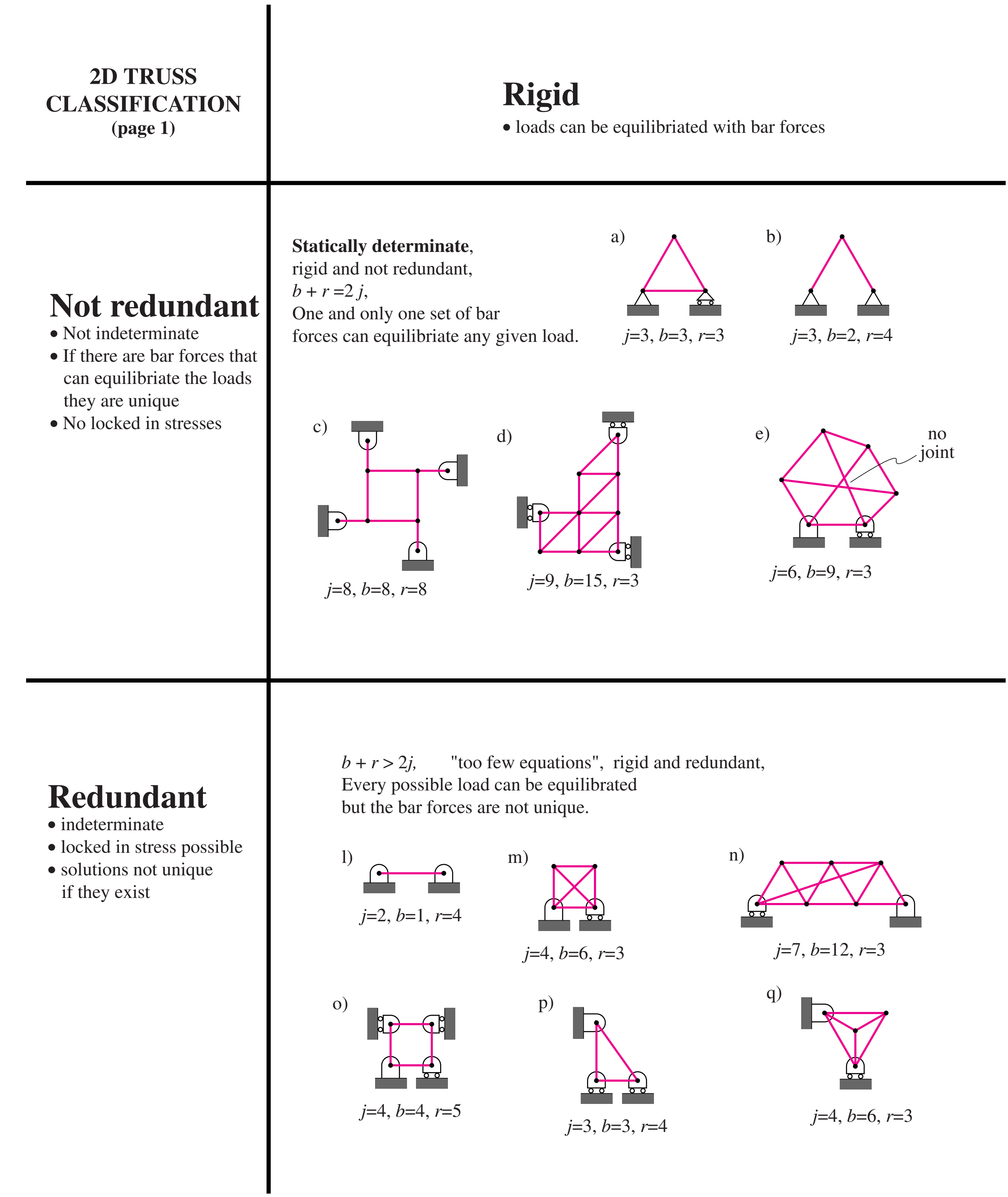
The top right case is not rigid and not redundant. Some loads cannot be equilibrated and those that can be are equilibrated uniquely. T is one to one but not onto. The columns of $[A]$ are linearly independent but they do not span W. The matrix $[A]$ has more rows than columns and is thus tall.

The bottom right case is the most perverse. The structure is not rigid but is redundant. Not all loads can be equilibrated but those that can be equilibrated are equilibrated non-uniquely. The matrix $[A]$ could have any shape but its columns are linearly dependent and do not span W. The map T is neither one to one nor onto.



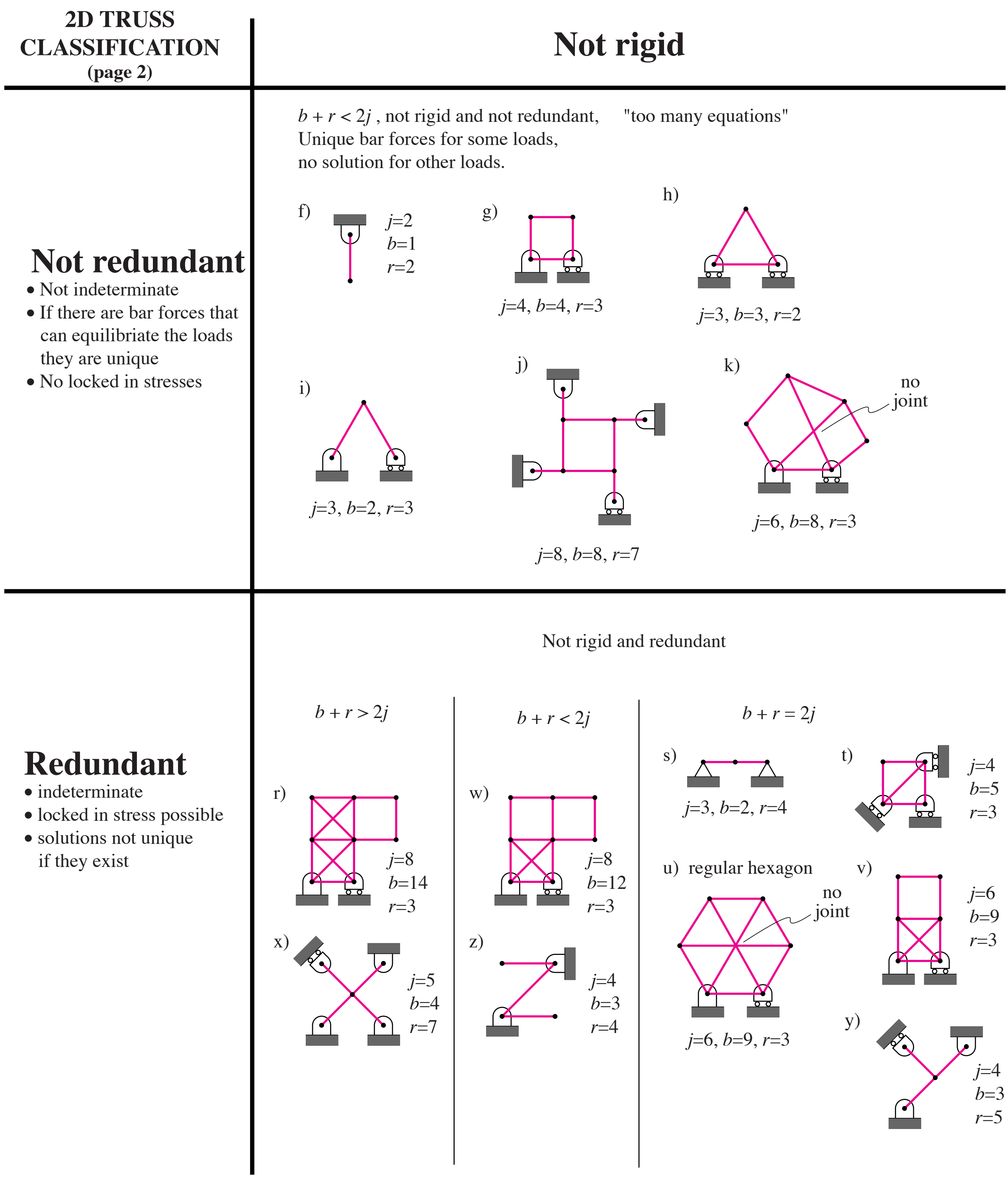
Filename:tfigure-trussmap

Figure 5.73: **Rigidity, redundancy and the structural matrices**. T is the linear transformation from the bar and reaction forces to the applied loads, it is represented with the matrix $[A]$. See box 5.3 on page 307 for a more detailed description.



Filename:tfigure-trussclass1

Figure 5.74: Examples of 2D trusses. These two pages concern the 2-fold system for identifying trusses. Trusses can be rigid or not rigid (the two columns) and they can be redundant or not redundant (the two rows). Elementary truss analysis is only concerned with rigid and not redundant trusses (*statically determinate* trusses). Note that the only difference between trusses (b) and (s) is a change of shape (likewise for the far more subtle examples (e) and (u)). Truss (e) is interesting as a rare example of a determinate truss with no triangles. Continued on page 310



Filename:tfigure-trussclass2

Figure 5.75: (Second page of a two page table.)

SAMPLE 5.16 **An indeterminate truss:** For the truss shown in the figure, find all support reactions.

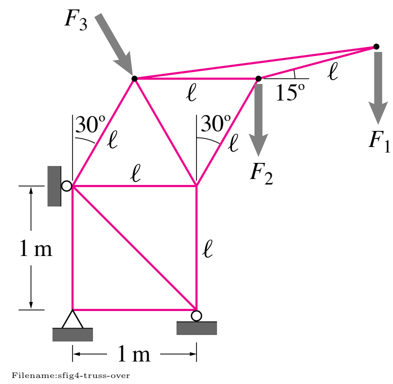


Figure 5.76:

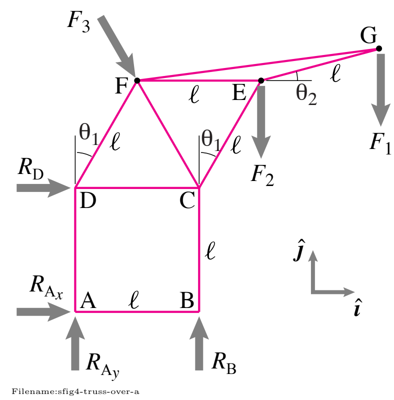


Figure 5.77: The free body diagram of the structure.

Solution The free-body diagram of the truss is shown in Fig. 5.77. We need to find the support reactions R_{A_x}, R_{A_y}, R_B, and R_D.

The x and y components of the force equilibrium, $\sum \vec{F} = \vec{0}$, give

$$\sum F_x = 0 \quad \Rightarrow \quad R_{A_x} + R_D = -F_3 \cos\theta_1 \tag{5.30}$$

$$\sum F_y = 0 \quad \Rightarrow \quad R_{A_y} + R_B = F_1 + F_2 + F_3 \sin\theta_1. \tag{5.31}$$

Now we apply moment balance about point A, $\sum \vec{M}_A = \vec{0}$. Let A be the origin of our xy-coordinate system (so that we can write $\vec{r}_{D/A} = \vec{r}_D$, etc.).

$$\vec{r}_D \times \vec{R}_D + \vec{r}_F \times \vec{F}_3 + \vec{r}_G \times \vec{F}_1 + \vec{r}_E \times \vec{F}_2 + \vec{r}_B \times \vec{R}_B = 0$$

where,

$$\begin{aligned}
\vec{r}_D \times \vec{R}_D &= \ell\hat{\jmath} \times R_D\hat{\imath} = -R_D\ell\hat{k} \\
\vec{r}_F \times \vec{F}_3 &= (\vec{r}_D + \vec{r}_{F/D}) \times \vec{F}_3 \\
&= [\ell\hat{\jmath} + \ell(\sin\theta_1\hat{\imath} + \cos\theta_1\hat{\jmath})] \times F_3(\cos\theta_1\hat{\imath} - \sin\theta_1\hat{\jmath}) \\
&= F_3\ell\cos\theta_1\hat{k} - F_3\ell\hat{k} = -F_3\ell(1+\cos\theta_1)\hat{k} \\
\vec{r}_G \times \vec{F}_1 &= (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) \times (-F_1\hat{\jmath}) = -r_{G_x}F_1\hat{k} \\
&= -F_1\ell(1+\sin\theta_1+\cos\theta_2)\hat{k} \\
\vec{r}_E \times \vec{F}_2 &= -F_2(\ell + \ell\sin\theta_1\hat{k} = -F_2\ell(1+\sin\theta_1)\hat{k} \\
\vec{r}_B \times \vec{R}_B &= \ell\hat{\imath} \times R_B\hat{\jmath} = R_B\ell\hat{k}.
\end{aligned}$$

Adding them together and dotting with $\hat{k}$ we get

$$-R_D\ell - F_3\ell(1+\cos\theta_1) - F_1\ell(1+\sin\theta_1+\cos\theta_2) - F_2\ell(1+\sin\theta_1) + R_B\ell = 0$$

$$\begin{aligned}
\Rightarrow \quad R_B - R_D &= F_1(1+\sin\theta_1+\cos\theta_2) \\
&\quad + F_2(1+\sin\theta_1) + F_3(1+\cos\theta_1).
\end{aligned} \tag{5.32}$$

We have three equations (5.30–5.32) containing four unknowns R_{A_x}, R_{A_y}, R_B, and R_D. So, we cannot solve for the unknowns uniquely. This was expected as the truss is indeterminate. However, if we assume a value for one of the unknowns, we can solve for the rest in terms of the assumed one. For example, let $R_D = \alpha$. For simplicity let the right hand sides of eqns. (5.30, 5.31, and 5.32) be C_1, C_2, and C_3 (computed values), respectively. Then, we get $R_{A_x} = C_1 - \alpha$, $R_{A_y} = C_2 - C_3 - \alpha$, and $R_B = C_3 + \alpha$. The equilibrium is satisfied for any value of α. Thus there are infinite number of solutions! This is true for all indeterminate systems. However, when deformations of structures are taken into account (extra constraint equations), then solutions do turn out to be unique. You will learn about such things in courses dealing with strength of materials.

Problems for Chapter 5

Trusses

5.1 Method of joints

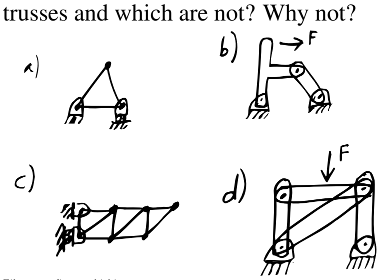
Preparatory Problems

5.1.1 Define these terms: a) truss, b) ideal truss, c) bar, d) joint, e) load, f) “bar force”, g) bar tension, h) bar compression, i) reaction, j) roller support and k) pin support.

5.1.2 Name as many positive attributes of trusses as you can.

5.1.3 Name as many negative attributes of trusses as you can.

5.1.4 Which of the structures below are trusses and which are not? Why not?



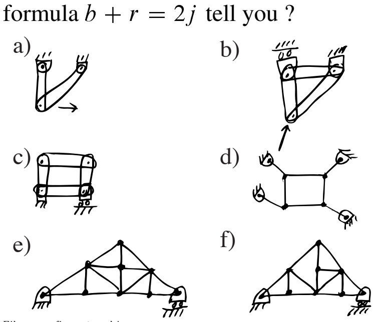
Filename:pfigure-whichistruss
Problem 5.1.4

5.1.5 Consider this formula

$$b + r = 2j$$

a) What do b, r, and j stand for?
b) What is the use of this formula?
c) What is the source of this formula?

5.1.6 For each of the trusses below: **i)** What are b, j, and r? **ii)** What does the formula $b + r = 2j$ tell you ?

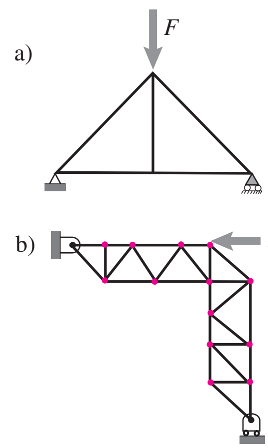


Filename:pfigure-trussbjr
Problem 5.1.6

5.1.7 For a given truss you are told values for b, j, and r.

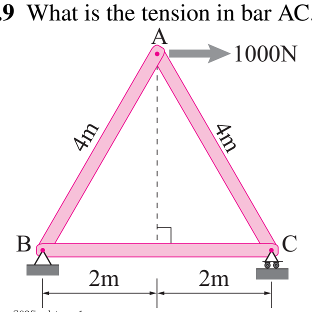
a) When solving the truss how many unknowns are you trying to solve for?
b) How many independent scalar equations do you have from using the method of joints on the whole structure?

5.1.8 Find the zero-force members in the trusses below.



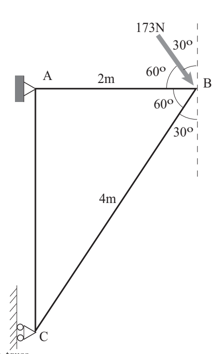
Filename:pfigure-zeroforcetruss
Problem 5.1.8

5.1.9 What is the tension in bar AC. *



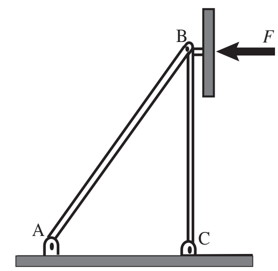
Filename:S03final-truss1
Problem 5.1.9

5.1.10 The only force acting on the negligible-weight truss ABC is the 173 N force shown. Find the tension in the bar AB. *



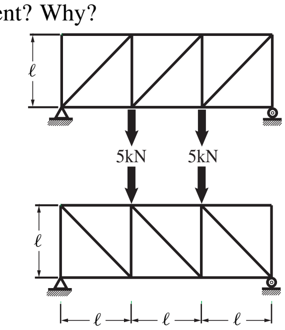
Filename:S03q3a-truss
Problem 5.1.10

5.1.11 A billboard is supported by a two bar truss as shown in the figure. The two bars have pin joints at A, B, and C. If the total wind load on the board is estimated to be 300 N, find the forces in bars AB and BC.



Filename:pfigure4-1-rp11
Problem 5.1.11

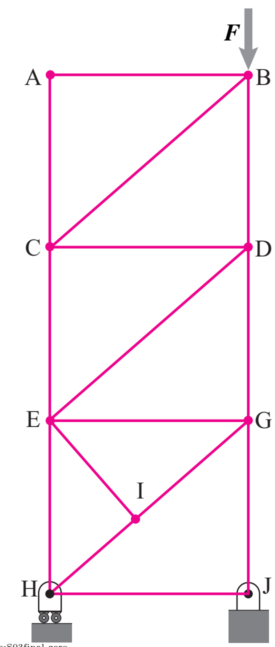
5.1.12 Find the support reactions for the two trusses without any (written) calculations. Should the support reactions be different? Why?



Filename:pfigure4-2-truss5
Problem 5.1.12

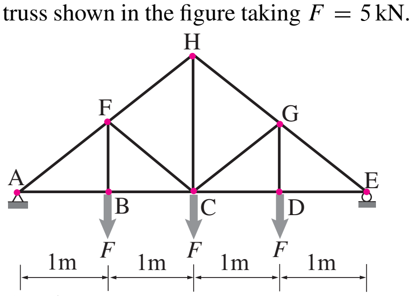
More-Involved Problems

5.1.13 Sketch the truss below. Write a big clear zero on top of each of the zero-force members. *



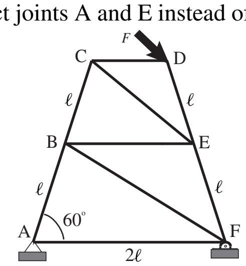
Filename:S03final-zero
Problem 5.1.13

5.1.14 Find the support reactions on the truss shown in the figure taking $F = 5\,\text{kN}$.



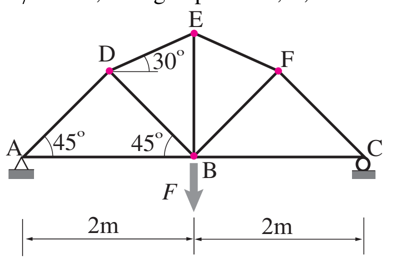
Filename:pfigure4-2-truss1
Problem 5.1.14

5.1.15 Find the support reactions at A and F for a load $F = 3\,\text{kN}$ acting at D at 45° with respect to CD, if $\ell = 1\,\text{m}$ and $\theta = 60^\circ$. How will the support reactions change if bar BF was removed and used to connect joints A and E instead of B and F?



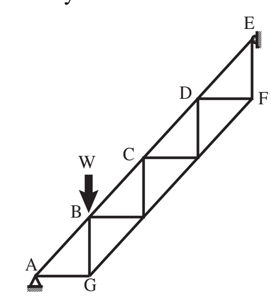
Filename:pfigure4-2-truss4
Problem 5.1.15

5.1.16 How do the support reactions on the truss shown in the figure change if the load at point B is replaced by three equal loads, $F/3$ each, acting at points D, E, and F?



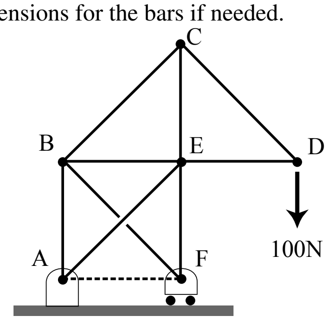
Filename:pfigure4-2-truss2
Problem 5.1.16

5.1.17 The stairstep truss shown in the figure has 500 mm long horizontal and vertical bars. Find the support reactions at A and E when a load $W = 1\,\text{kN}$ is applied at (a) point B, (b) Point C, and (c) point D, respectively.



Filename:pfigure4-2-truss3
Problem 5.1.17

5.1.18 In the truss shown in the figure, how does the force in bar EF change if the diagonal bar BF is removed and another bar AF (shown by dotted line) is introduced instead? You can assume any reasonable dimensions for the bars if needed.



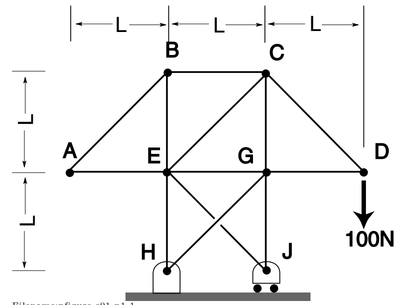
Filename:pfigure-zeroforce2
Problem 5.1.18

5.1.19 For the truss shown, find:

a) The reaction at J.

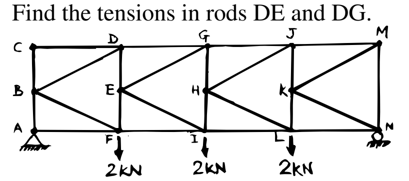
b) The bar force in BC (tension or compression).

c) The force in bar CG (tension or compression).



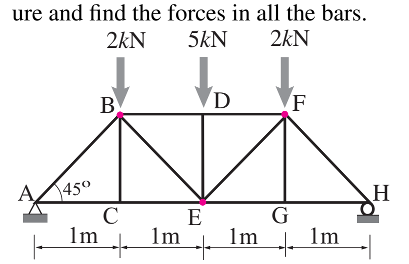
Filename:pfigure-s01-p1-1
Problem 5.1.19

5.1.20 The truss shown in the figure consists of-4 bays of 'K' structure. Each bay has 2 m long horizontal and vertical bars. Find the tensions in rods DE and DG.



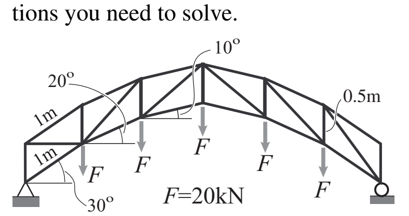
Filename:pfigure5-1-truss7
Problem 5.1.20

5.1.21 Analyze the truss shown in the figure and find the forces in all the bars.



Filename:pfigure4-2-truss8
Problem 5.1.21

5.1.22 Analyze the truss shown in the figure and find out forces in all bars. Use symmetry to reduce the number of equations you need to solve.



Filename:pfigure4-2-truss10
Problem 5.1.22

5.2 Method of sections

Preparatory Problems

5.2.1 What is the method of sections?

5.2.2 When is the method of sections most useful?

5.2.3 With the free body diagram associated with one section cut how many bar tensions can you hope to find?

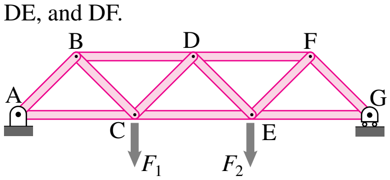
5.2.4 Given a truss and a particular bar in that truss

a) Can you always find one section cut with which you can find the desired bar tension?

b) If so, how do you find that cut? If not, why not?

c) Whichever your answer above, give an example of a bar in a truss that illustrates your point.

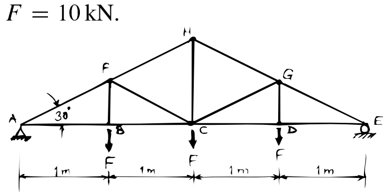
5.2.5 This problem is exactly the same as Sample 5.2 where it was solved using method of joints. The truss is made up of five horizontal and six inclined rods. All inclined rods are 1 m long and at right angles to each other. The truss carries two vertical loads, $F_1 = 4\,\text{kN}$ and $F_2 = 1\,\text{kN}$ as shown. Find the tensions in rods CE, DE, and DF.



Filename:pfigure5-2-sample3truss

Problem 5.2.5

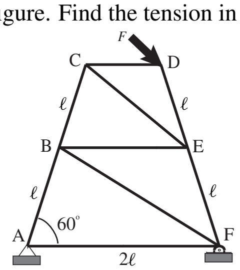
5.2.6 For the truss shown in the figure, find the tensions in rods BC and FH, assuming $F = 10\,\text{kN}$.



Filename:pfigure5-2-truss1

Problem 5.2.6

5.2.7 A force $F = 3\,\text{kN}$ acts at 45° with the horizontal at joint D of the truss shown in the figure. Find the tension in rod BE.

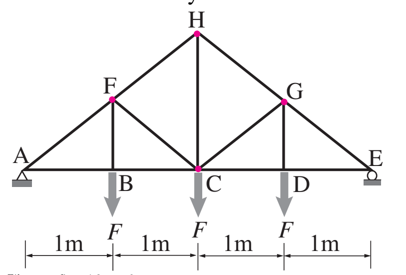


Filename:pfigure5-2-truss4

Problem 5.2.7

More-Involved Problems

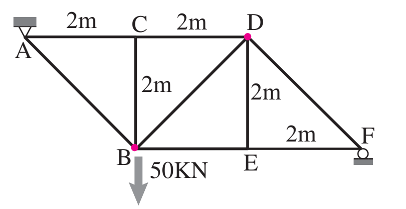
5.2.8 Find the forces in bars FH, FB, and BC of the truss shown in the figure taking $F = 10\,\text{kN}$. Now pretend that bars FC and CG are removed and two new bars BH and HD are put in place. Find the forces in bars FH, FB, and BC again. Are the forces different now? Why?



Filename:pfigure4-2-truss6

Problem 5.2.8

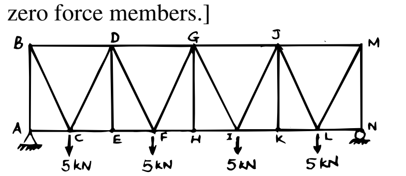
5.2.9 Find the forces in bars BC and BD in the truss shown in the figure. How does the force change in each of these bars if the load is moved to joint B from joint E?



Filename:pfigure4-2-truss7

Problem 5.2.9

5.2.10 For the truss shown in the figure, assume that AC=CE=1 m, and AB=BD=2 m. The rest of the bays are identical to bay ABDE. For the given loads, find the tensions in rods GH, GI, and GJ. [Hint: you can use information about zero force members.]

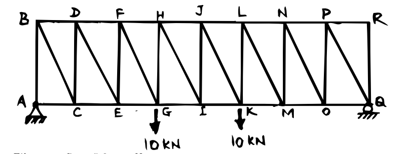


Filename:pfigure5-2-trussM

Problem 5.2.10

5.2.11 Consider the truss shown in Problem 5.2.6. Find the tension in rod CH. [Hint: you may have to use multiple sections or solve Problem 5.2.6 first.]

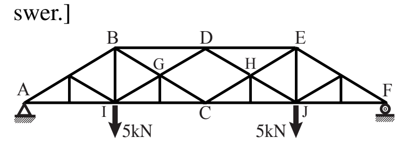
5.2.12 The truss shown in the figure consists of 8 'N' bays. In each bay, the vertical rod is 2 m long and the horizontal rod is 1 m long. For the given loads, find the tensions in rods HJ, HI, and GH.



Filename:pfigure5-2-trussN

Problem 5.2.12

5.2.13 A complex symmetric truss spanning a length of 16 m is shown in the figure. The outermost inclined rods make an angle of 30° with the horizontal. Find the tension in rod BD. [Note: you may have to use more than one section to get the answer.]



Filename:pfigure5-2-truss6

Problem 5.2.13

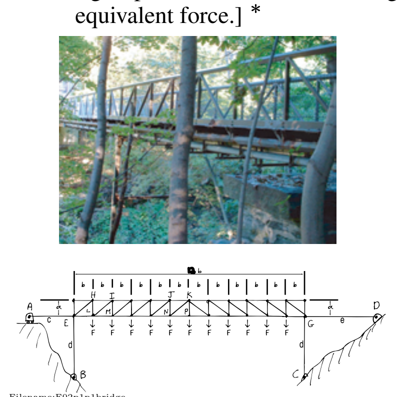
5.2.14 The 2D truss shown consists of 12 diagonally braced rectangles (each a high and b wide). Thus the slope of the diagonal elements is a/b. The whole structure is supported by 4 bars (with lengths c, d, and e as marked). The loading is idealized as 11 identical loads F shown. Give your answers in terms of some or all of a, b, c, d, e and F.

a) On a sketch of the figure below clearly mark all the zero-force members (put a '0' on the middle of each bar that has a 'bar force' of zero).

b) Find the 'bar-force' in bar EB. *

c) Find the 'bar-force' in bar HI. *

d) Find the 'bar-force' in bar JK. [Hint: Use the method of sections and, to reduce calculations, replace a group of the F forces with a single equivalent force.] *



Filename:F02p1p1bridge

Problem 5.2.14

5.3 Solving trusses on computers

Preparatory Problems

You should check your mastery of the method of joints problems before working on this section. **5.3.1** Define these matrices and column vectors used to define a truss, the loading on it, the bar tensions , the reactions, and the coefficients in the matrix form of the joint equilibrium equations:

a) $[J]$

b) $[B]$

c) $[R]$

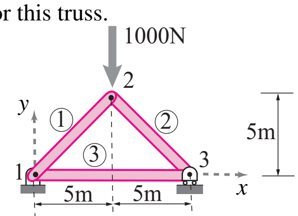
d) $[F]$

e) $[T]$

f) $[L]$

g) $[A]$

5.3.2 By hand, with no use of a computer, find all of the matrices and column vectors above for this truss.



Filename:pfigure-trussnumericsimple
Problem 5.3.2

5.3.3 When does the numerical recipe presented here succeed and when does it fail? When it fails, how does it fail?

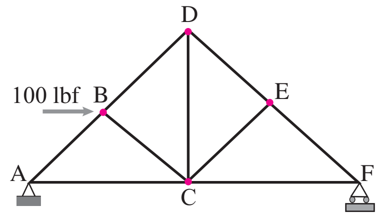
5.3.4

a) Write a computer program, using your preferred language or package, that takes as input the matrices $[J]$, $[B]$, $[R]$, and $[F]$ and calculates $[T]$.

b) Test this program on the truss of problem 5.3.2.

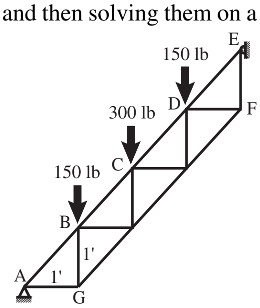
More-Involved Problems

5.3.5 All of the bars in the symmetric truss below are either level or at 30° from the horizontal. Find all the bar forces and reactions.



Filename:pfigure-simpletruss
Problem 5.3.5

5.3.6 Find the force in each bar of the staircase truss shown in the figure by writing the required number of equilibrium equations and then solving them on a computer.



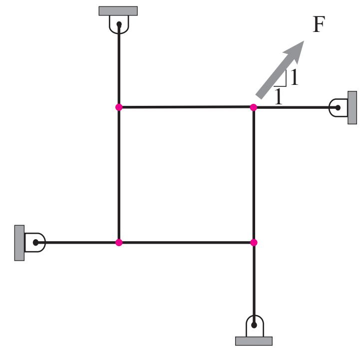
Filename:pfigure4-2-truss9
Problem 5.3.6

5.3.7 Find the tensions in all the bars, and all the reactions for these structures.

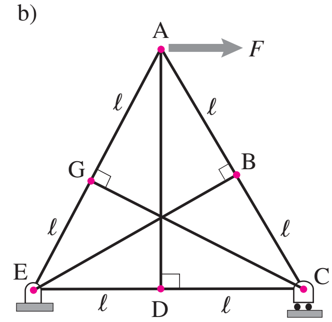
a) A square supported by four bars. This is perhaps the simplest rigid structure that has no triangles.

b) The 9-bar structure shown. This structure also has no triangles in that there is no closed circuit that involves only three bars (for example, from D to A to B to C and back to D involves 4 bars).

a)



b)



no pin at the center, total 9 bars

Filename:pfigure-advtruss1
Problem 5.3.7

5.3.8 Analyse the truss given in Problem 5.1.20 and solve for all bar tensions and support reactions.

5.3.9 Solve Problem 5.1.21.

5.3.10 Solve Problem 5.1.22.

5.3.11 Using your program from problem 5.3.4 solve each of the following problems:

a) Problem 5.1.10

b) Problem 5.1.11

c) Problem 5.1.13

d) Problem 5.1.20

e) Problem 5.1.21

f) Problem 5.1.22

g) Problem 5.2.5

h) Problem 5.2.7

i) Problem 5.2.12

j) Problem 5.3.6

5.4 Frames

Preparatory Problems

5.4.1 In what way(s) is/are trusses different from more general frames?

5.4.2 Consider a frame made of-3 pieces connected together. Assume that no free body diagram cut is within a part.

a) How many different free body diagrams can you draw?

b) For each free body diagram how many independent scalar equations can be extracted from the equilibrium relations?

c) In total, from all the free body diagrams, how many independent scalar equations can be extracted from the various equilibrium conditions.

5.4.3 Consider the two-bar frame shown. Choose appropriate coordinate axes. Find

a) The reaction at D.

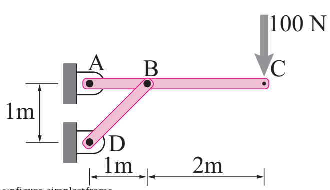
b) The tension in bar DB.

c) The reaction at A.

d) The force of DB on ABC.

e) The moment in ABC

- just (an infinitesimal distance) to the right of A
- just to the left of B
- just to the right of B
- just to the left of C



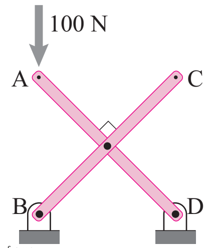
Filename:pfigure-simplestframe
Problem 5.4.3

5.4.4 Two equal length bars are pinned together at right angles as shown. Find

a) the reactions at B and D

b) the force of BC on AD

c) the moment in BD just above the hinge



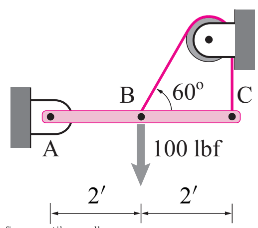
Filename:pfigure-xframe
Problem 5.4.4

5.4.5 For the structure shown find

a) the tension in the string

b) the reaction at A

c) the moment in ABC just to the right of B



Filename:pfigure-cantileverpulley
Problem 5.4.5

More-Involved Problems

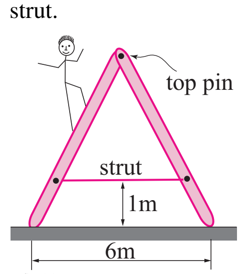
5.4.6 An A-frame aluminum ladder consists of two uniform 5 m, 150 N sections that are pinned at the top and held from splitting by a massless strut $1m$ above the slippery floor. An 800 N person has climbed halfway up the left side.

a) Find the reactions (the forces of the ground on the two ladder sections).

b) Find the force of the left section on the right at the top pin.

c) Find the tension in the connection strut.

d) Find the moment in the right leg of the ladder just above the tension strut.



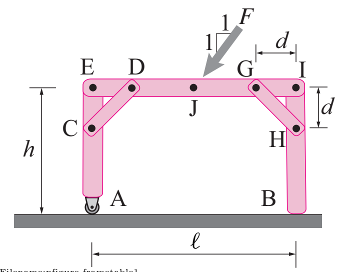
Filename:pfigureSoodak4-49
Problem 5.4.6

5.4.7 To make a model of a table statically determinate we assume that one leg slides easily on the floor. Assume the other leg does not slip. Use $F = 200\,\mathrm{N}$, $h = 1\,\mathrm{m}$, $\ell = 2\,\mathrm{m}$, and $d = .25\,\mathrm{m}$. Find

a) the reactions at A and B,

b) the tension in GH,

c) the moment in IHB just below H.



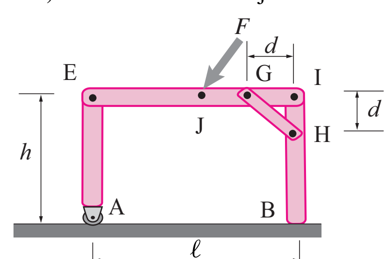
Filename:pfigure-frametable1
Problem 5.4.7

5.4.8 To make a model of a table statically determinate we assume that one leg is not braced. Assume the other leg does not slip. Use $F = 200\,\mathrm{N}$, $h = 1\,\mathrm{m}$, $\ell = 2\,\mathrm{m}$, and $d = .25\,\mathrm{m}$. Find

a) the reactions at A and B

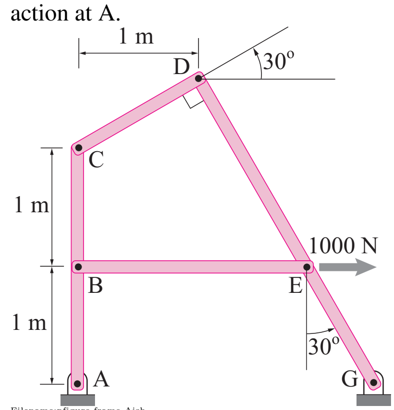
b) the tension in GH

c) the moment in IHB just below H



Filename:pfigure-frametable2
Problem 5.4.8

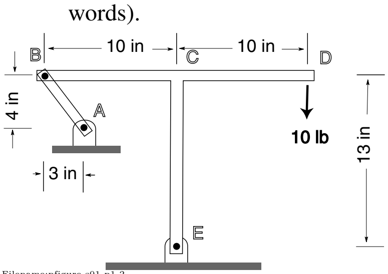
5.4.9 For the structure shown find the reaction at A.



Filename:pfigure-frame-Aish
Problem 5.4.9

5.4.10 The structure consists of two pieces: bar AB and 'T' EBCD. They are connected to each other with a hinge at B. They are connected to the ground with hinges at A and E. The force of gravity is negligible. Find

a) The reaction at A.
b) The reaction at E.
c) The moment in BCED just to the left of C.
d) Why are these forces so big or small? (Your answer should be in words).



Filename:pfigure-s01-p1-2
Problem 5.4.10

5.5 Advanced truss analysis: determinacy, rigidity, and redundancy

Preparatory Problems

5.5.1 Define these terms

a) statically determinate
b) rigid and non-rigid
c) redundant and non-redundant

5.5.2 For each set of conditions below, find 2 trusses both of which fit the description

a) rigid and non-redundant
b) rigid and redundant
c) not rigid and not redundant
d) not rigid and redundant

5.5.3 In 2D trusses we used the formula $b + r = 2j$. With what formula do we replace this for 3D trusses? Explain why.

5.5.4 For the 3D method of joints, for a whole truss how many independent scalar equilibrium equations can one write?

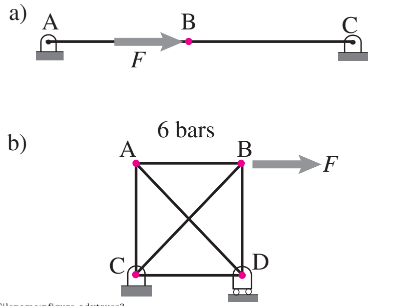
5.5.5 For one section cut in 3D how many bar tensions can you hope to find?

5.5.6 For a 3D truss that is rigid when not grounded, how many independent reaction components do you need to make it a statically determinate structure for any loading.

More-Involved Problems

5.5.7 For the following structures find at least 2 different sets of bar forces that can equilibrate the applied load shown.

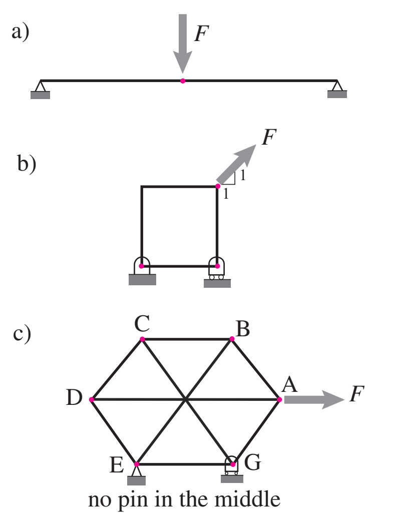
a) Two bars in a line with a force in the same line.
b) A square with two diagonal braces.



Filename:pfigure-advtruss3
Problem 5.5.7

5.5.8 For the structures and loading shown show that there is no set of bar forces for which equilibrium is possible (at least with the geometry shown). All of these structures are not rigid, they require either infinite bar forces or some (or a lot of) deformation to withstand the load applied

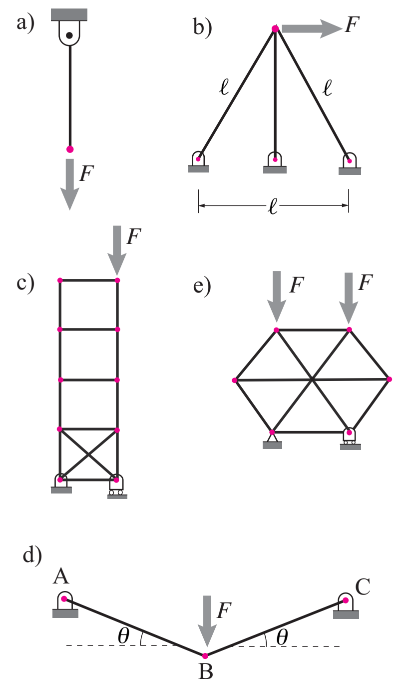
a) Two bars in a straight line.
b) A square without a diagonal.
c) A regular hexagon with three diameters. [This problem is hard and might best be answered using linear algebra methods on the matrix form of the system of equilibrium equations.]



Filename:pfigure-advtruss2
Problem 5.5.8

5.5.9 For each of the structures below and the shown loading answer these questions: i) Does a set of equilibrium bar forces and ground reactions exist? ii) If so, find one such set. iii) Are the solutions, if they exist, unique? iv) If not find at least two solutions. v) Is the structure rigid? vi) If not, how can it deform?

a) One hanging rod
b) A braced pole
c) A tower
d) Two bars holding a vertical load. Comment in your answers how they change in the limit $\theta \to 0$.
e) A regular hexagon with three diagonals (this is a hard problem).



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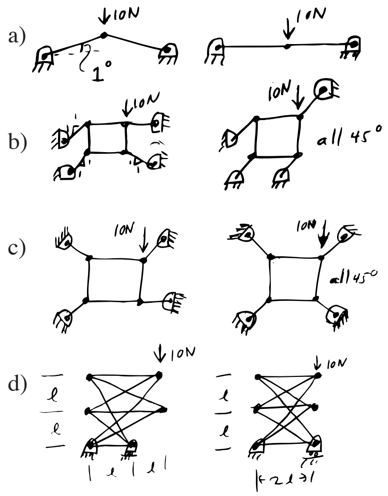
Problem 5.5.9

5.5.10 Use your program from problem 5.3.4 to analyze each pair of structures shown. In each case the output of your program should be radically different for the right structure than for the superficially similar structure on the left. **i)** Describe the difference in your computer program behavior, **ii)** As well as you can, explain what it is about the structures that causes this difference in computer behavior.



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Problem 5.5.10

CHAPTER 6

Transmissions and mechanisms

Some collections of solid parts are assembled so as to cause force or torque in one place given a different force or torque in another. These include levers, gear boxes, presses, pliers, clippers, chain drives, and crank-drives. Besides solid parts connected by pins, a few special-purpose parts are commonly used, including springs and gears. Tricks for amplifying force are usually based on principals idealized by pulleys, levers, wedges and toggles. Force-analysis of transmissions and mechanisms is done by drawing free body diagrams of the parts, writing equilibrium equations for these, and solving the equations for desired unknowns.

Contents

Here we consider collections of parts assembled to transmit motion or force. We are not going to address the conversion of thermal, chemical (or biological) or electrical source to a useful force. Rather we discuss the transmission of that force. We are concerned with the passive parts of machines, or with passive machines that have no energy source within them. Most often there is an input force or torque and a desired output which does the machine's job.

The categorization of an assembly of parts as a structure or as a machine is mostly a matter of intent. Is the main job to hold or support something still (a structure) or to move something. There is no useful intrinsic aspect of an assembly of parts that well-defines the difference between a structure and a machine①. So the statics analysis of mechanisms and transmissions is the same as for frames. Our concern is as in the rest of statics:

① A candidate delineation between machine and structure might be whether the assembly allows any motion (mechanism) or not (structure). Even that concept is ambiguous. Common machines, like presses, wrenches and clamps, are as motion-restricted in their end-use as many structures. And, conversely, many structures are adjustable and thus are designed to move.

> Given some information about the forces on or in a mechanism find out more about the forces.

The practice of mechanism design is often dominated by kinematic analysis, the study of the geometry of the interacting motions of the parts as the mechanism configuration changes. Such is not our concern here. Rather we focus on the relations between the various forces in a given configuration of the mechanism②.

② The dynamics portion of this book is largely an introduction to the kinematics of mechanisms.

Building blocks

In the same way that machines and buildings are built from bricks, gears, beams, bolts and other standard pieces, elementary mechanics models of the world are made from a few elementary building blocks. Conspicuous so far, roughly categorized, are:

- Special objects:
  - Point masses.
  - Rigid bodies:
    - Two force bodies,
    - Three force bodies,
    - Pulleys, and
    - Wheels.
- Special connections:
  - Hinges,
  - Welds,
  - Sliding contact, and
  - Rolling contact.

Products and models Some of these things have dual lives, as products and as models. On the one hand a mechanical hinge corresponds to a product you can buy in a hardware store called a hinge. On the other hand a hinge in mechanics represents a constraint that restricts certain motions and freely allows others. A hinge in a mechanics model may or may not correspond to hardware called a hinge. For example, when considering a box balanced on an edge, we may *model* the contact as a hinge meaning we would use the same equations for the forces of contact as we would use for a hinge. Although you can buy a pulley, you might model a rope sliding around a post as a rope on a pulley even though there was no literal pulley in sight.

The connection between product and model can even sound contradictory. Although 'like a rock' means 'solid' in English, one may model a rock as a spring (which is done for foundation engineering, understanding waves in rocks, and understanding the energy of earthquakes). A coil spring may be modeled as a rigid rod for a simple structure-like study of a machine. And a hinge might be modeled as a spring if its deformation is important. The appropriate mechanics model for a thing and its common name don't always correspond.

What's new in this chapter The new content in this chapter is

- Detailed discussion of a few components used in mechanisms and transmissions that are not used commonly in simple 'structures'. These include springs, pulleys, wheels, and gears.
- Introduction to a variety of design tricks to, say, cause a big force when only a small force is available.

We start the chapter by discussing a few special parts and assemblies of those parts. Then we consider more general assemblies.

6.1 Springs

A *spring* is a deformable solid that regains its original shape after being compressed, extended or otherwise deformed. The word spring has a dual personality.


Filename:tfigure-SpringVikingProduct

Figure 6.1: Some commercially available springs (Viking Products).

1) **Spring as product.** Springs, in various forms, most characteristically as helices made of steel wire, can be purchased from hardware stores and mechanical parts suppliers(fig. 6.1). Springs are used to hold things in place (in a clothes pin), to store energy (in a clock or wind-up toy), to reduce contact forces (bumpers), to isolate something from vibrations (a car suspension), and to modulate the feel for human interaction (under keyboard keys). You will find springs in most any complicated machine. Take apart a disposable camera, a laser printer, a gas lawn mower, a bicycle, a cruise missile, or a washing machine and you will find springs.
2) **Spring as model.** On the other hand, springs are used in mechanical 'models' of many things that are not, by name, springs (see page 33 for discussion of 'models'). For much of this book we approximate solids as rigid. But sometimes the flexibility or *elasticity* of an object is an

important part of its mechanics. The simplest accounting for this is to think of the object as a spring. So a tire may be modeled as a spring as might be the near-contact-point material of a bouncing ball, a strut in a truss, the snapping-back part of the earth's crust in an earthquake, your achilles tendon, or the give of soil under a concrete slab. Engineer Tom McMahon idealized the give of a running track as that of a spring when he designed the record breaking track used in the Harvard stadium.

For simplicity we only concern ourselves with *tension and compression*

6.1 'Zero-length' springs

Zero rest-length springs

A special case of linear springs that has remarkable mechanical consequences is a *zero-rest-length* spring (also called a 'zero-length' spring for short) with $\ell_0 = 0$. These ideas are useful for design, but not essential for basic understanding of statics.

The defining equations for a zero-rest-length spring, in scalar and vector form, are

$$T = k\ell \quad \text{and} \quad \vec{F}_B = k \cdot \vec{r}_{AB}.$$

The tension versus length curve for a zero-length spring is shown in fig. 6.4b.

At first blush such a spring seems *non-physical*, meaning that it seems to represent something which you can't build. If you take a coil spring all the metal gets in the way of the spring collapsing to zero length, when the ends would coincide. In fact, however, there are many ways to make zero-rest-length springs springs. For example, the tension versus length curve of a rubber band (or piece of surgical tubing) looks something like that shown in fig. 6.4c. Over some portion of the curve the zero-length spring approximation is reasonable (a sign of this is that the vibration frequency is almost independent of stretch for some range of stretch). For other physical implementations of zero-length springs see box 6.1 on page 323.

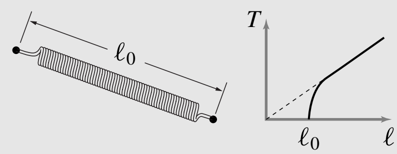
The mathematics in many mechanics problems is simpler for $\ell_0 = 0$ springs than for $\ell_0 \neq 0$ springs.

Rubber bands. As shown in fig. 6.4c straps of rubber behave like zero-length springs over some of their length. If this is the working length of your mechanism then the zero-length spring approximation may be good.

A stretchy conventional spring. Some springs are stretched way beyond their rest lengths. Thus the approximation that $k(\ell - \ell_0) = k\ell\left(1 - \frac{\ell_0}{\ell}\right) \approx k\ell$ may be reasonable.

A pre-stressed coil spring. Some door springs and many springs used in desk lamps are made close wound so that each coil of wire is pressed against the next one. It takes some tension just to start to stretch such a spring. The tension versus length curve for such springs can look very much like a zero-length spring once stretch has started. In fact, in the original elegant 1930's patent, which commonly seen present-day parallelogram-mechanism lamps imitate, specifies that the spring should behave as a zero-length spring.

Such a pre-stressed zero-length coil spring was a central part of the design of the long period seismometer featured on a 1959 Scientific American cover.



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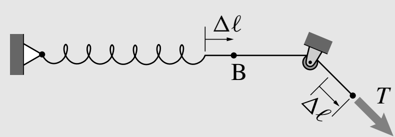
A spring, string, and pulley. If a spring is connected to a string that is wrapped around a pulley then the end of the string can feel like a zero force spring if the attachment point is at the pulley when the spring is relaxed.



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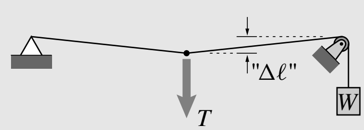
A string pulled from the side. If a taut string is pulled from the side it acts like a zero-length spring in the plane orthogonal to the string.



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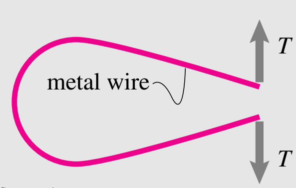
A 'U' clip. If a springy piece of metal is bent so that its unloaded shape is a pinched 'U' then it acts very much like a zero length spring. This is perhaps the best example in that it needs no anchor (unlike the pulley) and can be relaxed to almost zero length (unlike a pre-stressed coil).



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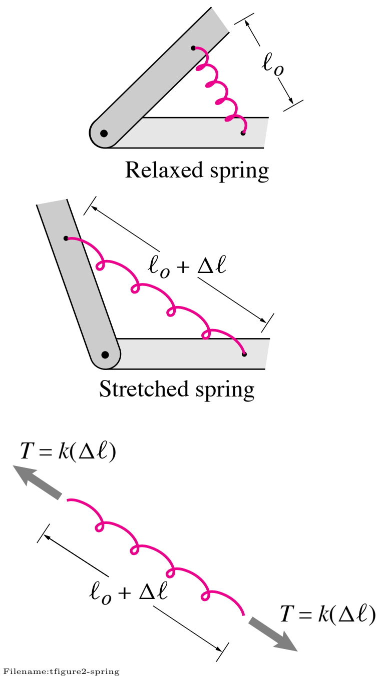

Figure 6.2: **Spring connection.** The tension in a spring is usually assumed to be proportional to its change in length, with proportionality constant k: $T = k\Delta\ell$.

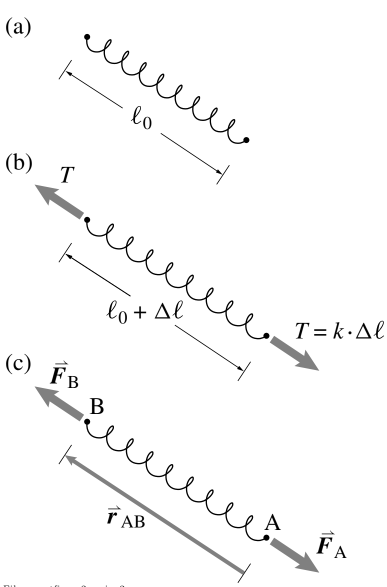

Figure 6.3: An ideal spring with rest length ℓ_0 and stretched length $\ell_o + \Delta\ell$. The tension in the spring is T and the vector forces at the ends are $\vec{F}_A$ and $\vec{F}_B$.

springs here. These are springs which only have axial loads applied and only at the ends.

If the tension in a spring is a function of its length alone, independent of its rate of lengthening, the spring is said to be 'elastic.' Many materials are well-modeled as elastic for small-enough deformation. If the tension in the spring is proportional to its stretch, the spring is said to be 'linear.' Most elastic materials are close to linear in their behavior. Thus the word *spring* is often short for *linear elastic spring*. The stretch of a spring is the amount by which the spring is longer than when it is relaxed. This relaxed length is also called the *unstretched* length, the *rest* length, or the *reference* length. If the relaxed length (the length at zero tension) is ℓ_0, and the present length ℓ, then the stretch of the spring is

$$\Delta\ell = \ell - \ell_0 = \text{Increase in length from rest length}$$

An *ideal spring* is a massless two-force body characterized by its *rest length* ℓ_0 (also called the *relaxed length*, or *reference length*), its *spring constant* k, and the defining equation (or constitutive law), *Hook's law*:

$$T = k \cdot (\ell - \ell_0) \quad \text{or} \quad T = k \cdot \Delta\ell$$

where ℓ is the present length and $\Delta\ell$ is the increase in length or *stretch* (see fig. 6.3).

The spring constant k is also sometimes called the spring *rate*, the spring *stiffness* or the spring *proportionality constant*.

The ideal spring is called *linear* because of the formula $k\Delta\ell$ and not, say, $k(\Delta\ell)^3$. The defining spring formula is sometimes, although we don't recommend this, memorized as '$F = kx$'

Note: the formula '$F = kx$' can lead to errors: the direction of the force is not evident, and some people are unclear about the meaning of x in this formula. The safest way to avoid sign errors when dealing with springs is to

- Draw a free body diagram of the spring;
- Write the *increase* in length $\Delta\ell$ in terms of geometry variables in your problem (even if you know that this increase is going to be a negative number);
- Use $T = k\Delta\ell$ to find the tension in the spring (even if you know the tension will turn out negative); and then
- Use the principle of action and reaction to find the forces on the objects to which the spring is connected.

The main idea is to pick a sign convention (tension and lengthening are positive) and stick with it, accepting the arithmetic of negative numbers if it arises. A plot of tension versus length for an ideal spring is shown in fig. 6.4a.

A comment on the notation $\Delta\ell$ Often in engineering we write $\Delta(something)$ to mean the change of '$something$.' Most often one also has in mind a small change. In the context of springs, however, $\Delta\ell$ is allowed to be a rather large change. A useful way to think about springs is that increments of force are proportional to increments of length change, whether the force or length is already large or small:

$$\Delta T = k\Delta\ell \quad \text{or} \quad \frac{dT}{d\ell} = k$$

Compliance. A spring with a large stiffness is called *stiff* or *hard*. The reciprocal of stiffness $\frac{1}{k}$ is called the *compliance*. A spring with a small stiffness and large compliance is called *compliant* or *soft* and has a lot of 'give'.

The force vector on one end of a spring. Because the spring force is along the spring, a known direction, we can write a vector formula for the force on the B (say) end of the spring as (see fig. 6.3)

$$\vec{F}_{\mathrm{B}} = k \cdot \underbrace{(|\vec{r}_{\mathrm{AB}}| - \ell_0)}_{\Delta\ell} \underbrace{\left(\frac{\vec{r}_{\mathrm{AB}}}{|\vec{r}_{\mathrm{AB}}|}\right)}_{\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}}. \tag{6.1}$$

where $\hat{\boldsymbol{\lambda}}_{\mathrm{AB}}$ is a unit vector along the spring. This explicit formula is useful for, say, numerical calculations. This formula becomes especially simple if the rest-length of the spring is zero ($\ell_0 = 0$) so

$$\vec{F}_{\mathrm{B}} = k\vec{r}_{\mathrm{B/A}}.$$

Absurd as this seems, how could a spring have zero rest length, the idea is useful both as a model and for engineering design (see box 6.1 on page 323.

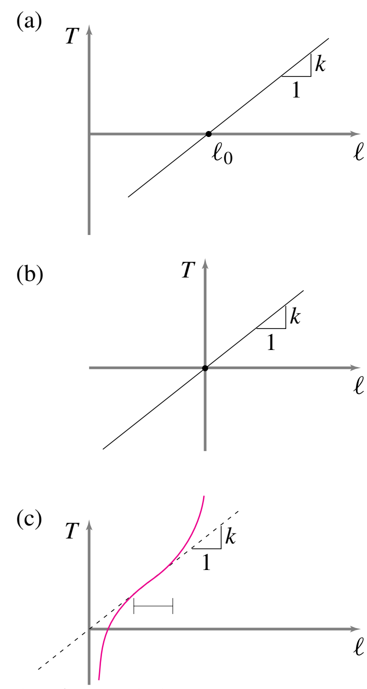

Figure 6.4: a) Tension versus length for an ideal spring, b) for a zero-length spring, and c) for a strip of rubber.

Assemblies of springs

Here we see how springs are put together with other springs *in parallel* and *in series*. For starters we'll put together just two springs with rest lengths ℓ_{01} and ℓ_{02}. The extensions and tensions of the two springs are $\Delta\ell_1$, $\Delta\ell_2$, T_1, and T_2.

The assembly of springs also acts like a single spring. The central issue is determination of the properties of the combined spring.

Much of what you need to know about the words 'in parallel' and 'in series' follows easily from these phrases:

In parallel,	forces	and	stiffnesses	add.
In series,	displacements	and	compliances	add.

which we discuss in detail below.

Springs in parallel

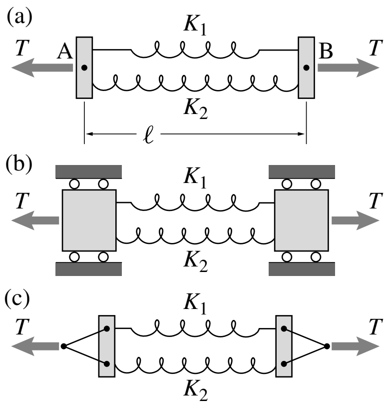

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Figure 6.5: a) Schematic of parallel springs, b) genuinely parallel springs, c) a simple implementation that gives a reasonable approximation of parallel springs. The triangle of cables limits the rotation of the end bars if the triangles are long.

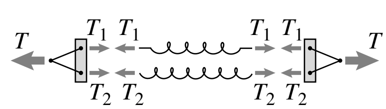

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Figure 6.6: Free body diagrams of the components of a parallel spring arrangement.

Two springs that share the burden of a load and stretch the same amount are said to be *in parallel*.

fig. 6.5a shows the standard schematic for springs in parallel. This schematic is a non-physical cartoon because the applied tension would likely cause the end-bars to rotate. What is meant by the schematic in fig. 6.5a is the somewhat clumsy constrained mechanism of fig. 6.5b. In engineering practice one rarely builds such a structure. For the purposes of discussion here, we assume that any of fig. 6.5abc represent a situation where the springs both stretch the same amount.

For each spring we have the defining constitutive relation:

$$T_1 = k_1 \Delta\ell_1 \quad \text{and} \quad T_2 = k_2 \Delta\ell_2. \tag{6.2}$$

Using the free body diagrams in fig. 6.6), force balance for one of the end supports shows that

$$T = T_1 + T_2. \tag{6.3}$$

This is what is meant by the two springs sharing the load. Springs in parallel stretch the same amount thus we have the kinematic relation:

$$\Delta\ell_1 = \Delta\ell_2 = \Delta\ell. \tag{6.4}$$

For simplicity we have assumed that the two springs have the same rest length. Put the two results above together and we have

$$\begin{aligned} T &= T_1 + T_2 \\ &= k_1\Delta\ell_1 + k_2\Delta\ell_2 \\ &= k_1\Delta\ell + k_2\Delta\ell \\ &= \underbrace{(k_1+k_2)}_{k}\Delta\ell\ . \end{aligned}$$

Thus the effective spring constant of the pair of springs in parallel is, naturally enough:

$$k = k_1 + k_2. \tag{6.5}$$

The loads carried by the springs are

$$T_1 = \frac{k_1}{k_1 + k_2} T \quad \text{and} \quad T_2 = \frac{k_2}{k_1 + k_2} T$$

which add up to T as they must.

Example: Two springs in parallel.
Take $k_1 = 99\,\mathrm{N/cm}$ and $k_2 = 1\,\mathrm{N/cm}$. The effective spring constant of the parallel combination is:

$$k = k_1 + k_2 = 99\,\mathrm{N/cm} + 1\,\mathrm{N/cm} = 100\,\mathrm{N/cm}.$$

Note that $T_1/T = .99$ so even though the two springs share the load, the stiffer one carries 99% of it. For practical purposes, or for the design of this system, it would be reasonable to remove the much less stiff spring.

The reasoning above with two springs in parallel is easy enough to reproduce with 3 or more springs. The result is:

$$k_{\text{tot}} = k_1 + k_2 + k_3 + \ldots \quad \text{and} \quad T_1 = T k_1/k_{\text{tot}}, \quad T_2 = T k_2/k_{\text{tot}} \ldots$$

That is,

- The net spring constant is the sum of the constants of the separate springs; and
- The load carried by springs is in proportion to their spring constants.

Some comments on parallel springs

Once you understand the basic ideas and calculations for two side-by-side springs connected to common ends, there are a few things to think about for context.

The simplest redundant truss For the purposes of drawing pictures (*e.g.,* fig. 6.5a) parallel springs are drawn side by side. But in the mechanics analysis we treated them as if they were on top of each other. A pair of parallel springs is like a two bar truss where the bars are on top of each other but connected at their ends. With 2 bars and 2 joints we have $2j < b + 3$, and a redundant truss. In fact this is the simplest redundant truss, as one spring (read bar) does exactly the same job as the other (carries the same loads, resists the same motions). With statics alone we can not find the tensions in the springs since the statics equation $T_1 + T_2 = T$ has non-unique solutions.

Statically indeterminate problems. Calculating the forces in a set of parallel springs is solving (using more than just statics, namely the spring constitutive law) the simplest statically-indeterminate problem.

Parallel springs and the three pillars of mechanics The laws of statics allow multiple solutions to redundant problems. But a bar in a real physical structure has, at one instant of time, some unique bar tension determined by the deformations and material properties. This is the first, and perhaps most conspicuous, occasion in this book that you see a problem where the three pillars of mechanics (see page 25) are assembled in such clear harmony, namely, material properties (eq. 6.2), the laws of mechanics (eq. 6.3), and the geometry of motion and deformation (eq. 6.4). In strength of materials calculations, where the distribution of stress is not determinable by statics alone, this threesome (geometry of deformation, material properties and statics) clearly come together in almost every calculation.

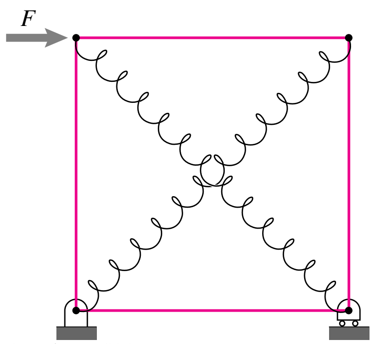


Figure 6.7: Parallel springs are not always geometrically parallel. The deformation of the structure above into a diamond is resisted by the two springs. They share the load and they have stretches that are linked by the kinematics. Thus these two perpendicular springs are 'in parallel.' (Detailed analysis of this structure is a little beyond the coverage here.)

Parallel springs are not necessarily geometrically parallel n the discussion above 'in parallel' corresponded to the springs being geometrically parallel. In common mechanics usage the words 'in parallel' are more general and mean that the net load is the sum of the loads carried by the two springs, and the stretches of the two springs are the same (or in a ratio restricted by kinematics). You will see cases where 'in parallel' springs are not the least bit parallel (*e.g.,* see fig. 6.7).

Springs in series

Two springs that share a displacement and carry the same load are *in series*.

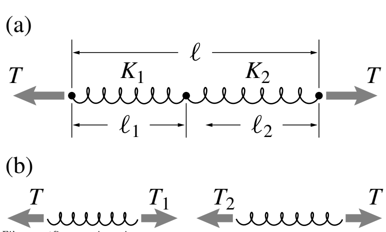


Figure 6.8: Schematic of springs in series.

A schematic of two springs in series is shown in fig. 6.8a where the springs are aligned serially, one after the other. To determine the net stiffness of this simple spring network we again assemble the three pillars of mechanics, using the free body diagram of fig. 6.8b.

$$
\begin{array}{llll}
\text{Constitutive law:} & T_1 = k_1(\ell_1 - \ell_{1_0}), & T_2 = k_2(\ell_2 - \ell_{2_0}), & \\
\text{Kinematics:} & \ell_0 = \ell_{1_0} + \ell_{2_0}, & \ell = \ell_1 + \ell_2, & \quad (6.6)\\
\text{Force Balance:} & T_1 = T, \quad \text{and} & T_2 = T. &
\end{array}
$$

(where,*e.g.,* ℓ_{1_0} is the rest length of spring 1). We can manipulate these equations much as we did for the similar equations for springs in parallel. The manipulation differs in structure the same way the equations do. For springs in parallel the tensions add and the displacements are equal. For springs in series the displacements add and the tensions are equal:

$$\begin{aligned}
\Delta\ell &= \ell - \ell_0 \\
&= (\ell_1 + \ell_2) - (\ell_{1_0} + \ell_{2_0}) \\
&= (\ell_1 - \ell_{1_0}) + (\ell_2 - \ell_{2_0}) \\
&= \Delta\ell_1 + \Delta\ell_2 \\
&= \frac{T_1}{k_1} + \frac{T_2}{k_2} \\
&= \frac{T}{k_1} + \frac{T}{k_2} \\
&= \underbrace{\left(\frac{1}{k_1} + \frac{1}{k_2}\right)}_{\frac{1}{k}} T \;.
\end{aligned}$$

Thus we get that the net compliance is the sum of the compliances:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{or} \quad k = \frac{1}{1/k_1 + 1/k_2} = \frac{k_1 k_2}{k_1 + k_2},$$

which you should compare with the case of springs in parallel (Eqn. 6.5). The sharing of the net stretch is in proportion to the compliances:

$$\Delta\ell_1 = \frac{1/k_1}{1/k_1 + 1/k_2}\Delta\ell \quad \text{and} \quad \Delta\ell_2 = \frac{1/k_2}{1/k_1 + 1/k_2}\Delta\ell$$

which add up to $\Delta\ell$ as they must.

Example: Two springs in series.
Take $1/k_1 = 99\,\text{cm/N}$ and $1/k_2 = 1\,\text{cm/N}$. The effective compliance of the parallel combination is:

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} = 99\,\text{cm/N} + 1\,\text{cm/N} = 100\,\text{cm/N}.$$

Note that $\Delta\ell_1/\Delta\ell = .99$ so even though the two springs share the displacement, the more compliant one has 99% of it. For design purposes, or for modeling this system, it would be fair to replace the much more stiff spring with a rigid link.

Consequences of series and parallel springs for modeling

As the previous two examples illustrate, springs can sometimes be replaced with 'air' (nothing) or with rigid links without changing the system or model behavior much. One way to think about this is that in the limit as $k \to \infty$ a spring becomes a rigid bar and in the limit $k \to 0$ a spring becomes air.

These ideas are used by engineers, often intuitively or even subconsciously and with no substantiating calculations, when making a model of a mechanical system.

6.2 How stiff a spring is a solid rod

Here we derive the formula for stiffness of a rod:

$$k = EA/\ell$$

This foreshadowing of Strength of Materials concepts is not central to the study of statics.

Let's take a reference bar with cross sectional area A_0 and rest length ℓ_0 and pull it with tension T and measure the elongation $\Delta\ell_0$ (figure below). The stiffness of this reference rod is $k_0 = T_0/\Delta\ell_0$. Now put two such rods side by side and you have parallel springs. You might imagine this sequence: two bars are near each other, then side by side, then touching each other, then glued together, then melted together into one rod with twice the cross section. The same tension in each causes the same elongation, or it takes twice the tension to cause the same elongation when you have twice the cross sectional area. Likewise with three side by side bars and so on, so for bars of equal length

$$k = \frac{A}{A_0} k_0.$$

On the other hand we could put the reference rods end to end in series. Then the same tension causes twice the elongation. We could be three or more rods together in series thus for bars with equal cross sections:

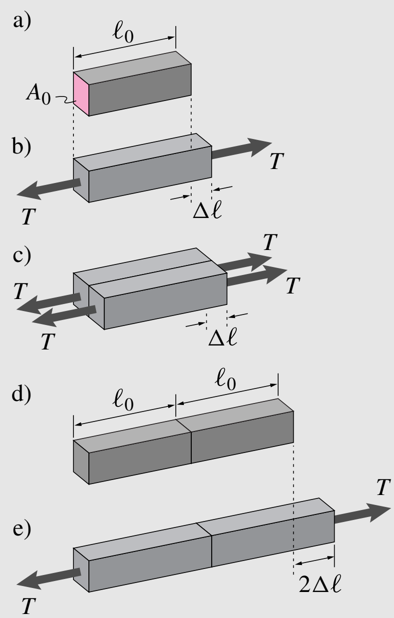
$$k = \frac{\ell_0}{\ell} k_0.$$

Putting these together we get:

$$k = \left(\frac{A}{A_0}\right)\left(\frac{\ell_0}{\ell}\right) k_0 = \left(\frac{k_0\ell_0}{A_0}\right)\frac{A}{\ell}.$$

Now presumably if we took a rod with a given material, length, and cross section the stiffness would be k, no matter what the dimensions of the reference rod. So $\left(\frac{k_0\ell_0}{A_0}\right)$ has to be a material constant. It is called E, the *modulus of elasticity* or Young's modulus. For all steels $E \approx 30 \cdot 10^6\,\mathrm{lbf/in^2} \approx 210 \cdot 10^9\,\mathrm{N/m^2}$ (consistent with fig. 6.10c). Aluminum has about a third this stiffness. So, a solid bar is a linear spring, obeying the spring equations:

$$k = \frac{EA}{\ell} \quad \text{or} \quad \Delta\ell = \frac{T\ell}{EA} \quad \text{or} \quad T = \frac{\Delta\ell\, EA}{\ell}.$$



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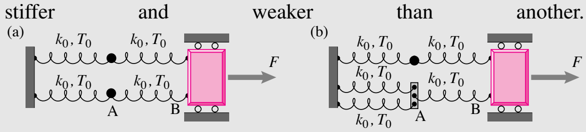
6.3 Stiffer but weaker

This is an aside for those who wonder how one thing can be both stiffer and weaker than another.



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The structure on the left is made with identical 4 springs. The structure on the right is made with 5 of the same springs. All 9 springs have stiffness k_0 and break when the tension in them reaches T_0. We now want to compare the stiffness and strength of the two structures. Because of the mixture of parallel and series springs, the net stiffness of the structure in (a) is

$$k_{\text{net}} = k_0 \qquad \text{and} \qquad \text{strength} = 2T_0$$

because none of the springs reaches its breaking tension until $F = T_0$.

By doubling up one of the springs in (a) to get (b) we get

$$k_{\text{net}} = 7k_0/6 \qquad \text{and} \qquad \text{strength} = 21T_0/12.$$

The structure is made 16% stiffer but spring AB now reaches its breaking point T_0 when the applied load is 12.5% smaller.

What's going on? The second structure is made stiffer by reducing the deflection of point A. But this causes spring AB to stretch more and thus carry more of the total load. In some sense, the load is concentrated in spring AB. This load concentration, where the total load is unevenly carried, is one reason that stiffness and strength need to be considered separately. Load concentration (or *stress concentration*) is a major cause of structural failure.

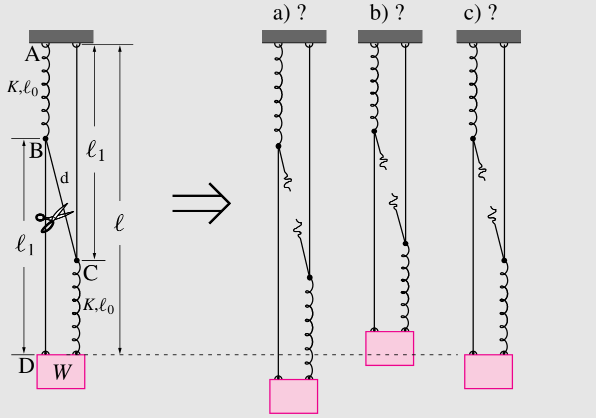
Consider the extreme case: put hundreds of springs in parallel where the pair of springs is now to the left of A in (b). Effectively this welds point A to the left wall. In (b) the load is concentrated in spring AB so (b) is about 50% stiffer than (a) and only has about 75% of (a)'s strength.

In common experience stiffness and strength *do* correlate. Something that feels rickety (is very compliant) also tends to fail with a small load. But this common experience can be misleading: 1) A stiff structure can be weak because of stress concentrations, and 2) Given two materials, one may be both stiffer and weaker than the other.

6.4 A puzzle with two springs and three ropes.

This tricky puzzle is an aside.

A weight hangs from 3 strings (BD,BC, and AC) and 2 springs (AB and CD). Point B is above point C and all ropes are taut.



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When rope BC is cut does the weight go (a) down?, (b) up?, or (c) stay put? (Three dots represents a time for you to stop and think.)

• • •

Do the experiment. In 15 minutes you can do this experiment with 3 pieces of string, 2 rubber bands and a soda bottle. Hang the partially filled soda bottle from a door knob (or the top corner of a door, or a ruler cantilevered over the top of a refrigerator). Adjust the string lengths and amount of weight so that no strings or rubber bands are slack and make sure point B is above point C. The two points A can coincide as can the two points D. You can separate the strings a little with, say, a small wad of paper so you can see which string is which. Look at your experimental setup, but don't pull and poke at it. Try to predict whether your bottle will go up down or not move when you cut the middle string.

• • •

Spoiler, Answer: *If you did the experiment you know the answer, the weight goes up a little when you cut. That's what really happens.*

This puzzle was published as one for which people have bad intuitions. And that's true, as you probably just found out. Why the experiment comes out the way it does? If you got it wrong (like most people do), can you find the error in your ways?

• • •

Clue: All simple explanations are based on the assumption that the lengths of the two strings AC and BD are constant at ℓ_1.

• • •

Explanation 1: To simplify the reasoning assume that springs AB and CD are identical and carry the same tension T_s and that the ropes AC and BD carry the same tension T_r. As usual, start with free body diagrams (below). With the symmetry we have assumed diagrams (a) and (c) provide identical information. The three free body diagrams can be considered before and after the middle string removal by having $T_d > 0$ or $T_d = 0$, respectively. Vertical force balance gives (approximating T_d as vertical):

$$T_s + T_r = W \quad \text{and} \quad 2T_r + T_d = W \quad \Rightarrow \quad T_s = (W + T_d)/2.$$

Because we approximate AC as rigid with length ℓ_1, the downwards position of the weight is the string length ℓ_1 plus the rest length of the spring ℓ_0 plus the stretch of the spring T_s/k:

$$\ell = \ell_1 + \ell_0 + T_s/k = \ell_1 + \ell_0 + (W + T_d)/(2k).$$

In the course of this experiment ℓ_1, ℓ_0, W and k are constants. So as the tension T_d goes from positive to zero (when the rope BC is cut) and ℓ decreases. So the weight goes up.

• • •

Explanation 2: More intuitively, start with the configuration with the rope already cut and apply a small upwards force at C. It has no effect on the tension in spring CD thus the weight does not move. Now apply a small downwards force at B. This does stretch spring AB and thus lower point B, thus lowering the weight since ℓ_1 is constant. Applying both simultaneously is like attaching the middle rope. Thus attaching the middle rope lowers the weight so cutting the middle rope raises the weight.

• • •

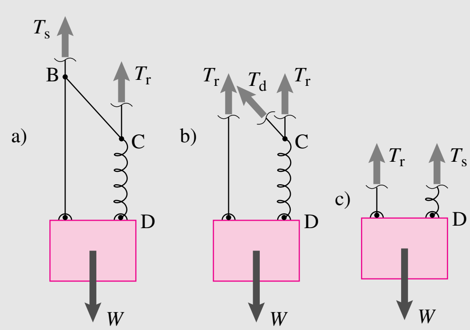
Explanation 3: Here is another intuitive approach. Point C can't move. Point B moves up and down just as much as the weight does. Point B is a distance d above point C. Since the rope BC is taut, releasing it will allow B and C to separate, thus increasing d and raising the weight.

• • •

A wrong explanation: What about springs in parallel and series? Here is a quick but wrong explanation for the experimental result, though it happens to predict the right direction of motion.

> "Before rope BC is cut the two springs are more or less in series because the load is carried from spring through BC to spring. Afterwards they are more or less in parallel because they have the same stretch and share the load. Two springs in parallel have 4 times the stiffness of the same two springs in series. So in the parallel arrangement the deflection is less. So the weight goes up when the springs switch from series to parallel."

What is the error in this thinking? The position of the weight comes from spring deflection added to the position when there is no weight. For the argument just presented to make sense, the rest-position of the mass (with gravity switched off) would have to be the same for the supposed 'series' and 'parallel' cases, which it is not ($\ell_1 + \ell_0 \neq \ell_0 + d + \ell_0$).



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Another way to see the fallacy of the 'parallel versus series' argument is that the incremental stiffness of the system is, assuming inextensible ropes, infinite. That is, if you add or subtract a small load to the bottle only moves because of a small stretch of the ropes (which is neglected in the correct simple explanations above). If the springs were in series or parallel we would expect an incremental stiffness that was related to spring stretch not rope stretch.

- If one of several pieces in series is much stiffer than the others it is often replaced with a rigid link.
- If one of several pieces in parallel is much more compliant than the others it is often replaced with air (nothing, sailboat fuel).

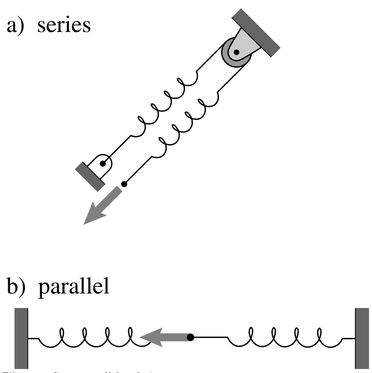


Figure 6.9: a) The two springs shown are in series because the carry the same load and their displacements add. b) These two springs are in parallel because the have a common displacement and their forces add.

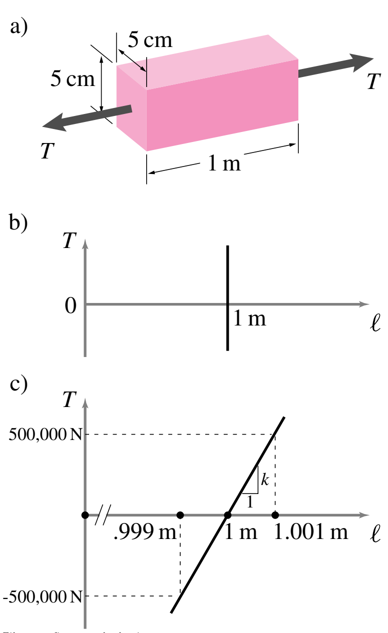


Figure 6.10: a) a steel rod in tension, b) tension versus length curve, c) zoom in on the tension versus length curve.

For example:

- When a coil spring is connected to a linkage, the other pieces in the linkage, though undoubtedly somewhat compliant, are typically modelled as rigid. They are stiffer than the spring and in series with it.
- A single hinge resists rotation about axes perpendicular to the hinge axis. But a door connected at two points along its edge is stiffly prevented against such rotations. Thus the hinge stiffness is in parallel with the greater rotational stiffness of the two connection points and is thus often neglected (see the discussion and figures in section 3.8 starting on page 157).
- Welded joints in a determinate truss are modeled as frictionless pins. The rotational stiffness of the welds is 'in parallel' with the axial stiffness of the bars. To see this look at two bars welded together at an angle. Imagine trying to break this weld by pulling the two far bar ends apart. Now imagine trying to break the weld if the two far ends are connected to each other with a third bar. The third bar is 'in parallel' with the weld material. See the first few sentences of section 5.1 for a do-it-yourself demonstration of the idea.
- Human bones are often modeled as rigid because, in part, when they interact with the world they are in series with more compliant flesh.

Note, again, that the mechanics usage of the words 'in parallel' and 'in series' don't always correspond to the geometric arrangement. For example the two springs in fig. 6.9a are in series and the two springs in fig. 6.9b are in parallel.

Strength and stiffness

Most often when you build a structure you want to make it stiff and strong. The ideas of stiffness and strength are so intimately related that it is sometimes hard to untangle them. For example, you might examine a product in discount store by putting your hand on it, applying small forces and observing the motion. Then you might say: "pretty shaky, I don't think it will hold up" meaning that the stiffness is low so you think the thing may break if the loads get high.

Although stiffness and strength are often correlated, they are distinct concepts. Something is stiff if the force to cause a given motion is high. Something is strong if the force to cause any part of it to break is high. In fact, it is possible for a structure to be made weaker by making it stiffer (see box 6.3 on page)

Why aren't springs in all mechanical models?

All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions which would be little effected by such deformation. In particular, if a problem is statically determinate then very small deformations only have a very small effect on the equilibrium equations and calculated forces.

Linear springs are just one way to model 'give'

If it is important to consider the deformability of an object, the linear spring model is just one simple model. It happens to be a good model for the small deformation of many solids. But the linear spring model is defined by the two words 'linear' and 'elastic'. For some purposes one might want to model the force due to deformation as being *non-linear*, like $T = k_1(\Delta\ell) + k_2(\Delta\ell)^3$. And one may want to take account of the dissipative or *in-elastic* nature of something. The most common example being a linear dashpot $T = c\dot{\ell}$. Various mixtures of non-linearity and inelasticity may be needed to model the large deformations of a yielding metal, for example.

Solid bars are linear springs

When a structure or machine is built with literal springs (*e.g.,* a wire helix) it is common to treat the other parts as rigid. But when a structure has no literal springs the small amount of deformation in rigid looking objects can be important, especially for determining how loads are shared in redundant structures.

Let's consider a 1 m (about a yard) steel rod with a 5 cm square (about $(2\,\text{in})^2$) cross section (fig. 6.10a). If we plot the tension versus length we get a curve like fig. 6.10b. The length just doesn't visibly change (unless the tension got so large as to damage the rod, not shown.) But, when you pull on anything, it does deform at least a little. If we zoom in on the tension versus length plot we get fig. 6.10c. To change the length by one part in a thousand (a millimeter, a twenty fifth of an inch) we have to apply a tension of about $500,000\,\text{N}$ (about 60 tons). Nonetheless the plot reveals that the solid steel rod behaves like a (very stiff) linear spring.

Surprisingly perhaps this little bit of compliance is important to structural engineers. Modeling solid metal rods as linear springs is essential for finding internal forces in statically indeterminate structures. Because it is hard to picture steel deforming, your intuition may be helped by exaggerating the deformation. Think of all solids as being rubber. Or, if you want to look inside the solid in your mind, think of every solid as if it was a piece deforming Jello. (Jello is colored sugar water held together, jelled, by long springy gela-

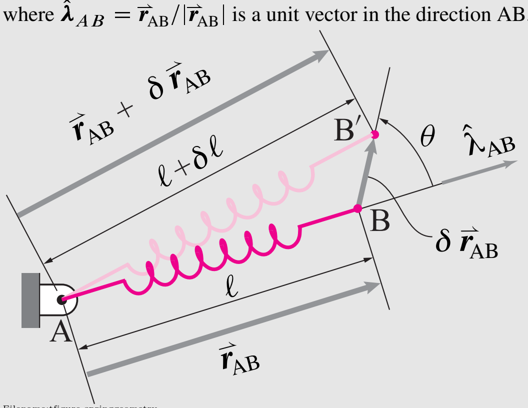
6.5 2D geometry of spring stretch

The material here is used in advanced sample 6.5 on page 341 and some of the later homework problems.

The key result concerns a spring with one end fixed at A and the other at moving point B. When point B moves from $\vec{r}_B$ to $\vec{r}_B + \Delta\vec{r}_B$ then the spring length changes from ℓ to $\ell + \Delta\ell$ with

$$\Delta\ell \approx \hat{\boldsymbol{\lambda}}_{AB} \cdot \Delta\vec{r}_B \tag{6.7}$$

where $\hat{\boldsymbol{\lambda}}_{AB} = \vec{r}_{AB}/|\vec{r}_{AB}|$ is a unit vector in the direction AB.



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We use $\Delta\ell$ and $\delta\ell$ interchangeably. Before we derive this result a few ways, lets discuss its relevance.

Fixed-configuration statics: the usual approach

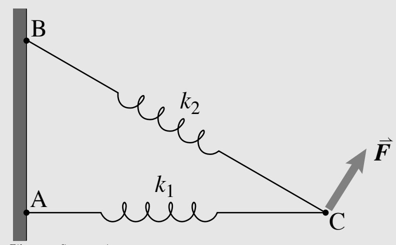
The forces and moments on a system in static equilibrium satisfy force and moment balance. In these equations the force magnitudes and directions, the moments and the locations of points of application of these are those *in the equilibrium configuration*. The equilibrium of the deformed state is expressed in terms of the geometry of that deformed state. Where the structure was before loading doesn't appear in the equilibrium equations.

However, often we know the geometry of a structure before the loads are applied, not after. To avoid calculation and confusion, we assume that the deformations cause negligible changes in positions. This is one reason people mistakenly think of statics as being limited to rigid bodies. Rather, for bodies that don't deform much, we can use the before-load geometry of a structure for reasonably accurate estimation of the deformed geometry.

Statics, taking account of deflection

In principal, the statics of deformable solids is the same as for rigid solids. You just need to use the deformed geometry in the statics calculations. Unfortunately, to find that geometry one needs the forces and their points of application. And one can't find all the locations without finding the deformation which depends on the forces, etc. This dizzying circle is escapable using the 'three pillars' (page 25).

Example: A structure made of springs.



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Assume all the lengths and geometry of the two-bar truss are known when there is no load at C. We can find all the tensions and deflections as follows (See page 16 for the general strategy):

1. Assume that the equilibrium loaded location of C is displaced from the rest location by $\delta\vec{r}_C = \delta x_C\,\hat{\imath} + \delta y_C\,\hat{\jmath}$ where δx_C and δy_C are *unknowns*;
2. Calculate the lengths of the springs in terms of δx_C and δy_C (this will be a complex expression with squares and square roots);
3. Find the tensions in the springs in terms of their new lengths and thus in terms of δx_C and δy_C;
4. Draw a free body diagram of C, using the spring orientations and tensions you have found (still in terms of unknowns δx_C and δy_C);
5. Write the force balance equations. These are two equations for two unknowns δx_C and δy_C.
6. Solve for δx_C and δy_C.
7. Use δx_C and δy_C to find the lengths and thus the tensions in the springs.

What's wrong with taking account of the deflection?

The trap — having to know the deflection to find the tensions but having to know the tensions to find the deflection — is avoided by setting up and solving simultaneous non-linear equations.

Although this non-linear-equation approach is correct, given our spring model, it is generally not used in structural mechanics because:

- **Confusing.** The equations are a mess.
- **Hard.** It is hard to solve non-linear equations, sometimes even hard on a computer.
- **Non-uniqueness.** There may be more than one solution. For example in the math problem above, if F is not too large, there will be two solutions. One solution with C deflected up and to the right, and another with C way to the left of the wall. To get rid of such off-the-wall solutions you need to either use judgment after you find them, or further specify your math problem to eliminate them.

(continued...)

6.5 2D geometry of spring stretch (continued)

- **Linear equations are good enough.** There are simpler methods that give approximate solutions that are accurate for small-enough loads. That is, if the deflection is small compared to the size of the structure then there are linear equations which reasonably approximate the non-linear equations above.

Small-deflection theory: the structural-mechanics approach.

So long as $\boldsymbol{F}$ is not too large, the motion of point C will be small compared to the lengths of the springs. Especially since, in practice, those springs are often solid metal rods. The usual *small deformation* assumption is that

- The deflection is small enough so that the spring angle changes have negligible effect on the equilibrium equations, and
- The deflection is small enough for the approximate formula for spring length change, eqn. (6.7), to be adequate.

The recipe for finding the deflection of C in the example above is greatly simplified with these approximations:

1. Assume that the equilibrium loaded location of C is displaced from the rest location by $\delta\vec{\boldsymbol{r}}_C = \delta x_C\hat{\boldsymbol{\imath}} + \delta y_C\hat{\boldsymbol{\jmath}}$ where δx_C and δy_C are *unknowns* (unchanged);
2. Calculate the lengths of the springs in terms of δx_C and δy_C using eqn. (6.7) (simplified);
3. Find the tensions in the springs in terms of their new lengths (unchanged) and thus in terms of δx_C and δy_C (much simpler expressions);
4. Draw a free body diagram of C, using the original undeflected geometry (much simplified);
5. Write the force balance equations. These are two equations for two unknowns δx_C and δy_C. (These will now be linear equations instead of a non-linear mess.)
6. Solve for δx_C and δy_C. (This is now the solution of linear instead of non-linear equations.)
7. (simplified) Use δx_C and δy_C to find the lengths and thus the tensions in the springs. (This now uses eqn. (6.7) instead of complicated relations with square roots, etc.)

This simplified recipe depends on the simplified formula for the spring length change eqn. (6.7), derived below four different ways.

Derivations of equation 6.7

Derivation 1 of eqn. (6.7). The law of cosines (page 95) says

$$(\ell + \delta\ell)^2 = \ell^2 + |\delta\vec{\boldsymbol{r}}_{AB}|^2 + 2\ell|\delta\vec{\boldsymbol{r}}_{AB}|\cos\theta$$

(θ here is negative of that used in the statement of the law of cosines). Expanding the left side and dropping terms in $\delta\ell^2$ and $|\delta\vec{\boldsymbol{r}}_{AB}|^2$ on both sides (assuming $\delta\ell/\ell \ll 1$ and $|\delta\vec{\boldsymbol{r}}_{AB}|/\ell \ll 1$), and dividing both sides by ℓ we get

$$\delta\ell \approx |\delta\vec{\boldsymbol{r}}_{AB}|\cos\theta = \hat{\boldsymbol{\lambda}}_{AB}\cdot\delta\vec{\boldsymbol{r}}_{AB}$$

where the last equality comes from the definition of the dot product (Section 2.2).

Derivation 2 of eqn. (6.7). Use the pythogorean theorem to determine the lengths of $\vec{\boldsymbol{r}}_{AB}$ and of $\vec{\boldsymbol{r}}_{AB} + \delta\vec{\boldsymbol{r}}_{AB}$:

$$\begin{aligned}\ell &= \sqrt{(x_{AB}^2 + y_{AB}^2)}\\ \ell + \delta\ell &= \sqrt{\left((x_{AB}+\delta x_{AB})^2 + (y_{AB}+\delta y_{AB})^2\right)}\end{aligned}$$

Subtracting the first from the second, dividing both sides by ℓ, and expanding the contents of the square root we get

$$\delta\ell/\ell = \sqrt{1 + 2\left(x_{AB}\delta x_{AB} + y_{AB}\delta y_{AB}\right)/\ell^2 + \delta x_{AB}^2\delta x_{AB}^2/\ell^2} - 1.$$

Neglecting δx_{AB}^2 and δy_{AB}^2 (assuming $\delta\ell \ll \ell$) and expanding the square root ($\sqrt{1+\epsilon} \approx 1 + \epsilon/2$), and multiplying through by ℓ, we get

$$\delta\ell \approx (x_{AB}/\ell)\delta x_{AB} + (y_{AB}/\ell)\delta y_{AB})$$

which is eqn. (6.7) because $\hat{\boldsymbol{\lambda}}_{AB} = (x_{AB}/\ell)\hat{\boldsymbol{\imath}} + (y_{AB}/\ell)\hat{\boldsymbol{\jmath}}$.

Derivation 3 of eqn. (6.7). Using vector notation throughout:

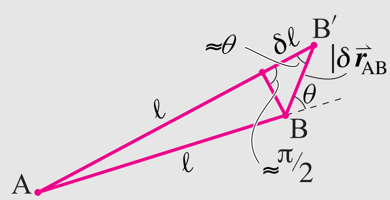
$$\begin{aligned}\ell^2 &= \vec{\boldsymbol{r}}_{AB}\cdot\vec{\boldsymbol{r}}_{AB}\\ (\ell+\delta\ell)^2 &= (\vec{\boldsymbol{r}}_{AB} + \delta\vec{\boldsymbol{r}}_{AB})\cdot(\vec{\boldsymbol{r}}_{AB} + \delta\vec{\boldsymbol{r}}_{AB})\end{aligned}$$

Expanding the second equation, neglecting second order terms and subtracting the first we get

$$\ell\delta\ell \approx \vec{\boldsymbol{r}}_{AB}\cdot\delta\vec{\boldsymbol{r}}_{AB}$$

dividing by ℓ and noting that $\hat{\boldsymbol{\lambda}}_{AB} = \vec{\boldsymbol{r}}_{AB}/\ell$ we again get eqn. (6.7).

Derivation 4 of eqn. (6.7). Finally, and most intuitively, look at this sketch.



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The line AB and its deflected self are nearly parallel. Thus the triangle at the end is nearly a right triangle. So, approximately, $\delta\ell = |\delta\vec{\boldsymbol{r}}_{AB}|\cos\theta$ which is $\delta\vec{\boldsymbol{r}}_{AB}\cdot\hat{\boldsymbol{\lambda}}_{AB}$, again giving eqn. (6.7).

What the pros do

To find the loads in metal structures the pros treat solid bars as springs, as per box 6.2 on page 330. Then they use eqn. (6.7) in the *small-deflection theory* described above. Generally this is all automated in computer code called a *finite-element* program. Such programs are standard commercial products used by hundreds of thousands of engineers round the world daily.

tine molecules extracted from animal hooves. Vegetarians can use sea-weed based Agar jell for their deformation fantasies.)

How does a solid bar's stiffness depend on its shape and composition?In box 6.2 on page 330 we show that the stiffness of a solid elastic bar is

$$k = \frac{EA}{\ell}$$

where E is a material property called the Young's modulus. It's that E is big that keeps most solids from deforming visibly①.

①**Why aren't springs in all mechanical models?** All things deform a little under load. Why don't we take this deformation into account in all mechanics calculations by, for example, modeling solids as elastic springs? Because many problems have solutions which would be little effected by such deformation. In particular, if a problem is statically determinate then very small deformations only have a very small effect on the equilibrium equations and calculated forces.

SAMPLE 6.1 Springs in series versus springs in parallel: Two springs with spring constants $k_1 = 100\,\mathrm{N/m}$ and $k_2 = 150\,\mathrm{N/m}$ are attached together as shown in Fig. 6.11. In case (a), a vertical force $F = 10\,\mathrm{N}$ is applied at point A, and in case (b), the same force is applied at the end point B. Find the force in each spring for static equilibrium. Also, find the equivalent stiffness for (a) and (b).

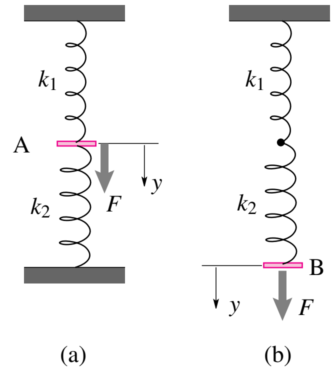

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Figure 6.11:

Solution In static equilibrium, let Δy be the displacement of the point of application of the force in each case. We can figure out the forces in the springs by writing force balance equations in each case.

- **Case (a):** The free body diagram of point A is shown in Fig. 6.12. As point A is displaced downwards by Δy, spring 1 gets stretched by Δy whereas spring 2 gets compressed by Δy. Therefore, the forces applied by the two springs, $k_1 \Delta y$ and $\Delta k_2 y$, are in the same direction. Then, the force balance in the vertical direction, $\hat{\jmath} \cdot (\sum \vec{F} = \vec{0})$, gives:

$$
\begin{aligned}
F &= F_1 + F_2 = (k_1 + k_2)\Delta y \\
\Rightarrow \quad \Delta y &= \frac{F}{k_1 + k_2} = \frac{10\,\mathrm{N}}{(100 + 150)\,\mathrm{N/m}} = 0.04\,\mathrm{m} \\
\Rightarrow \quad F_1 &= k_1 \Delta y = 100\,\mathrm{N/m} \cdot 0.04\,\mathrm{m} = 4\,\mathrm{N} \\
\Rightarrow \quad F_2 &= k_2 \Delta y = 150\,\mathrm{N/m} \cdot 0.04\,\mathrm{m} = 6\,\mathrm{N}.
\end{aligned}
$$

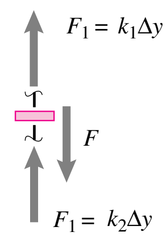

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Figure 6.12: Free body diagram of point A.

The equivalent stiffness of the system is the stiffness of a single spring that will undergo the same displacement Δy under F. From the equilibrium equation above, it is easy to see that,

$$k_e \equiv \frac{F}{\Delta y} = k_1 + k_2 = 250\,\mathrm{N/\,m}.$$

$$\boxed{F_1 = 4\,\mathrm{N}, \quad F_2 = 6\,\mathrm{N}, \quad k_e = 250\,\mathrm{N/\,m}}$$

- **Case (b):** The free body diagrams of the two springs is shown in Fig. 6.13 along with that of point B. In this case both springs stretch as point B is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = \Delta y.$$

Now, using the free body diagram of point B and writing the force balance equation in the vertical direction, we get $F = k_2 y_2$, and from the free-body diagram of spring 2, we get $k_2 y_2 = k_1 y_1$. Thus the force in each spring is the same and equals the applied force, *i.e.,*

$$F_1 = k_1 y_1 = F = 10\,\mathrm{N} \qquad \text{and} \qquad F_2 = k_2 y_2 = F = 10\,\mathrm{N}.$$

The springs in this case are in series. Therefore, their equivalent stiffness, k_e, is

$$k_e = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)^{-1} = \left(\frac{1}{100\,\mathrm{N/m}} + \frac{1}{150\,\mathrm{N/m}}\right)^{-1} = 60\,\mathrm{N/m}.$$

Note that the displacements y_1 and y_2 are different in this case. They can be easily found from $y_1 = F/k_1$ and $y_2 = F/k_2$.

$$\boxed{F_1 = F_2 = 10\,\mathrm{N}, \quad k_e = 60\,\mathrm{N/m}}$$

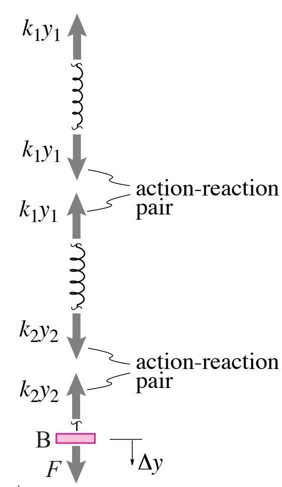

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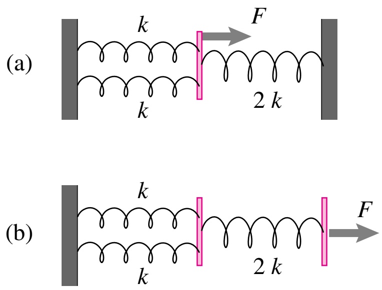
Figure 6.13: Free body diagrams

Comments: Although the springs attached to point A do not visually seem to be in parallel, from mechanics point of view they are parallel. Springs in *parallel* have the *same displacement* but *different forces*. Springs in *series* have *different displacements* but the *same force*.



Filename:sfig4-3springs

Figure 6.14:

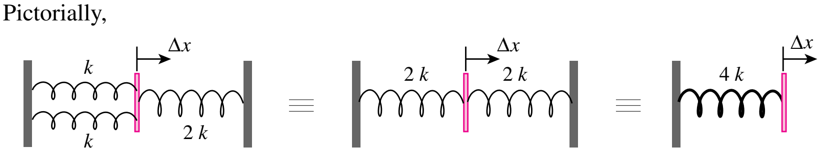
SAMPLE 6.2 Stiffness of three springs: For the spring networks shown in Fig. 6.14(a) and (b), find the equivalent stiffness of the springs in each case, given that each spring has a stiffness of $k = 20\,\mathrm{N/m}$.

Solution

1. In Fig. 6.14(a), all springs are in parallel since all of them undergo the same displacement Δx in order to balance the applied force F. Each of the two springs on the left stretches by Δx and the spring on the right compresses by Δx. Therefore, the equivalent stiffness of the three springs is

$$k_p = k + k + 2k = 4k = 80\,\mathrm{kN/\,m}.$$

Pictorially,



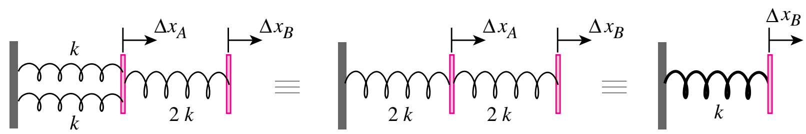
Filename:sfig4-3springs-a

Figure 6.15:

$$\boxed{k_{\text{equiv}} = 80\,\mathrm{kN/\,m}}$$

2. In Fig. 6.14(b), the first two springs (on the left) are in parallel but the third spring is in series with the first two. To see this, imagine that for equilibrium point A moves to the right by Δx_A and point B moves to the right by Δx_B. Then each of the first two springs has the same stretch Δx_A while the third spring has a net stretch $= \Delta x_B - \Delta x_A$. Therefore, to find the equivalent stiffness, we can first replace the two parallel springs by a single spring of equivalent stiffness $k_p = k + k = 2k$. Then the springs with stiffnesses k_p are $2k$ are in series and therefore their equivalent stiffness k_s is found as follows:

$$\begin{aligned} \frac{1}{k_s} &= \frac{1}{k_p} + \frac{1}{2k} = \frac{1}{2k} + \frac{1}{2k} = \frac{1}{k} \\ \Rightarrow \quad k_s &= k = 20\,\mathrm{kN/\,m}. \end{aligned}$$

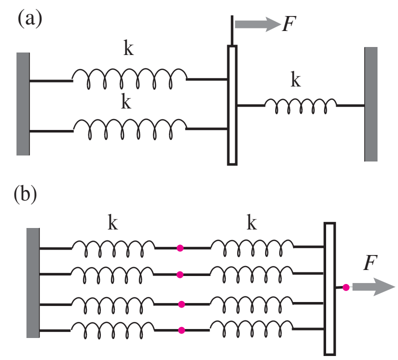


Filename:sfig4-3springs-b

Figure 6.16:

$$\boxed{k_{\text{equiv}} = 20\,\mathrm{kN/\,m}}$$

SAMPLE 6.3 Stiffness vs strength: Which of the two structures (network of springs) shown in the figure is stiffer and which one has more strength if each spring has stiffness $k = 10\,\mathrm{kN/m}$ and strength $T_0 = 10\,\mathrm{kN}$.



Filename:sfig4-manysprings

Figure 6.17:

Solution In structure (a), all the three springs are in parallel. Therefore, the equivalent stiffness of the three springs is

$$k_a = k + k + k = 3k = 30\,\mathrm{kN/\,m}.$$

For figuring out the strength of the structure, we need to find the force in each spring. From the free-body diagram in Fig. 6.18 we see that,

$$\begin{aligned} k\Delta x + k\Delta x + k\Delta x &= F \\ \Rightarrow \quad \Delta x &= \frac{F}{3k}. \end{aligned}$$

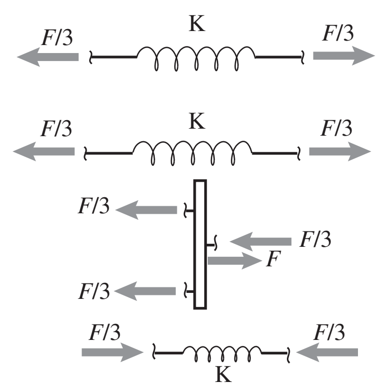
Therefore, the force in each spring is

$$F_s = k\Delta x = \frac{F}{3}.$$

But the maximum force that a spring can take is $(F_s)_{max} = T_0 = 10\,\mathrm{kN}$. Therefore, the maximum force that the structure can take (*i.e.,* the strength of the structure), is

$$F_{max} = 3T_0 = 30\,\mathrm{kN}.$$

$$\boxed{\text{Stiffness} = 30\,\mathrm{kN/m},\ \text{Strength} = 30\,\mathrm{kN}}$$



Filename:sfig4-manysprings-a

Figure 6.18:

Now we carry out a similar analysis for structure (b). There are four parallel chains in this structure, each chain containing two springs in series. The stiffness of each chain, k_c, is found from

$$\begin{aligned} \frac{1}{k_c} &= \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \\ \Rightarrow \quad k_c &= \frac{k}{2} = 5\,\mathrm{kN/\,m}. \end{aligned}$$

So, the stiffness of the entire structure is

$$k_b = k_c + k_c + k_c + k_c = 4k_c = 20\,\mathrm{kN/\,m}.$$

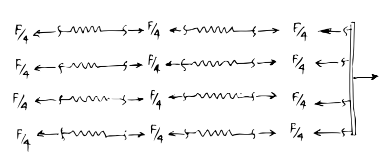


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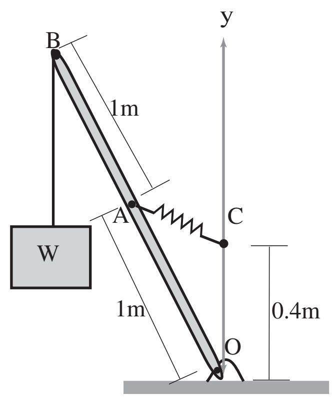
Figure 6.19:

From the free-body diagram shown in fig. 6.19, we find the force in each spring to be F/4 . Therefore, the maximum force that the structure can take is

$$F_{max} = 4T_0 = 40\,\mathrm{kN}.$$

$$\boxed{\text{Stiffness} = 20\,\mathrm{kN/m},\ \text{Strength} = 40\,\mathrm{kN}}$$

Thus, structure (a) is stiffer but structure (b) is stronger (higher strength).



SAMPLE 6.4 Zero length springs are special. A rigid and massless rod OAB of length 2 m supports a weight $W = 100\,\mathrm{kg}$ hung from point B. The rod is pinned at O and supported by a zero length (in relaxed state) spring attached at mid-point A and point C on the vertical wall. Find the equilibrium angle θ and the force in the spring.

Solution The free-body diagram of the rod is shown in Fig. 6.21 in an assumed equilibrium state. Let $\hat{\boldsymbol{\lambda}} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ be a unit vector along OB. The spring force can be written as $\vec{F}_s = k\vec{r}_{C/A}$ (since AC is a *zero-length* spring, the stretch in the spring is $|\vec{r}_{C/A}|$). We need to determine θ and F_s.

Let us write moment equilibrium equation about point O, *i.e.,* $\sum \vec{M}_O = \vec{0}$,

$$\vec{r}_{B/O} \times \vec{W} + \vec{r}_{A/O} \times \vec{F}_s = \vec{0}.$$

Noting that

$$\begin{aligned} \vec{r}_{B/O} &= \ell\hat{\boldsymbol{\lambda}}, \quad \vec{r}_{A/O} = \frac{\ell}{2}\hat{\boldsymbol{\lambda}}, \\ \vec{F}_s &= k\vec{r}_{C/A} = k(\vec{r}_C - \vec{r}_A) \\ &= k\left(h\hat{\jmath} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}\right), \end{aligned}$$

we get,

$$\begin{aligned} \ell\hat{\boldsymbol{\lambda}} \times (-W\hat{\jmath}) + \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times k\left(h\hat{\jmath} - \frac{\ell}{2}\hat{\boldsymbol{\lambda}}\right) &= \vec{0} \\ -W\ell(\hat{\boldsymbol{\lambda}} \times \hat{\jmath}) + kh\frac{\ell}{2}(\hat{\boldsymbol{\lambda}} \times \hat{\jmath}) &= \vec{0}. \end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\jmath})$, we get,

$$\begin{aligned} -W\ell + kh\frac{\ell}{2} &= 0 \\ \Rightarrow \quad kh &= 2W. \end{aligned}$$

Thus the result is independent of θ! As long as the spring stiffness k and the height h of point C are such that their product equals $2W$, the system will be in equilibrium at any angle. This, however, is in general not possible if AC is not a zero-length spring.

Equilibrium is satisfied at any angle if $kh = 2W$

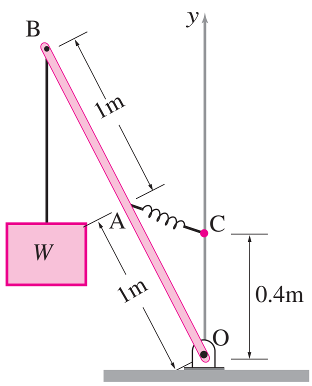


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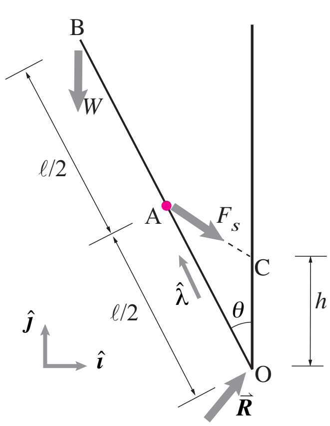
Figure 6.20:



Filename:sfig4-zerospring-a

Figure 6.21:

SAMPLE 6.5 Deflection of an elastic structure: For the two-spring structure shown in the figure, find the deflection of point C when

1. $\vec{F} = 1\,\mathrm{N}\hat{\imath}$,
2. $\vec{F} = 1\,\mathrm{N}\hat{\jmath}$,
3. $\vec{F} = 30\,\mathrm{N}\hat{\imath} + 20\,\mathrm{N}\hat{\jmath}$,

The spring stiffnesses are $k_1 = 10\,\mathrm{kN}/m$ and $k_2 = 20\,\mathrm{kN}/m$.

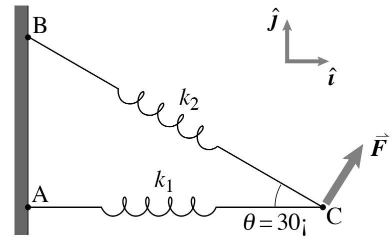


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Figure 6.22:

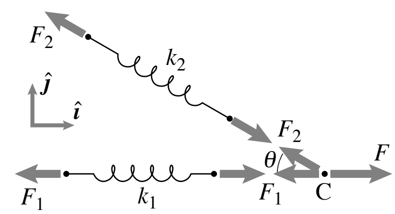

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Figure 6.23:

Solution Let $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$ be the displacement of point C of the structure due to the applied load. We can figure out the deflections in each spring as follows. Let $\hat{\lambda}_{AC}$ and $\hat{\lambda}_{BC}$ be the unit vectors along AC and BC, respectively (see fig. 6.24). Then, the change in the length of spring AC due to the (assumed small) displacement of point C is (see page 334 for a discussion)

$$
\begin{aligned}
\Delta_{AC} &= \hat{\lambda}_{AC} \cdot \Delta\vec{r} \quad \text{(this is the key equation)} \\
&= \hat{\imath} \cdot (\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x.
\end{aligned}
$$

Similarly, the change in the length of spring BC is

$$
\begin{aligned}
\Delta_{BC} &= \hat{\lambda}_{BC} \cdot \Delta\vec{r} \\
&= (\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \cdot (\Delta x\hat{\imath} + \Delta y\hat{\jmath}) = \Delta x\cos\theta - \Delta y\sin\theta.
\end{aligned}
$$

Now we can find the force in each spring since we know the deflection in each spring.

$$
\begin{aligned}
\text{Force in spring AC} = F_1 &= k_1\Delta x && (6.8) \\
\text{Force in spring BC} = F_2 &= k_2(\Delta x\cos\theta - \Delta y\sin\theta). && (6.9)
\end{aligned}
$$

The forces in the springs, however, depend on the applied force, since they must satisfy static equilibrium. Thus, we can determine the deflection by first finding F_1 and F_2 in terms of the applied load and substituting in the equations above to solve for the deflection components.

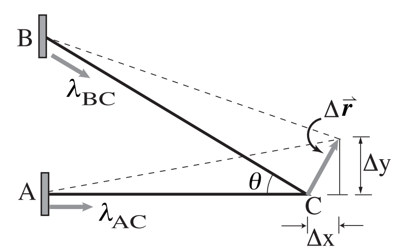

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Figure 6.24:

1. *Deflections with unit force in the x-direction:*

   Let $\vec{F} = f_x\hat{\imath} = 1\,\mathrm{N}\hat{\imath}$, (we have adopted a special symbol f_x for the unit load). Then, from the free-body diagram of the springs and the end pin shown in fig. 6.23 and the force equilibrium ($\sum\vec{F} = \vec{0}$), we have,

   $$f_x\hat{\imath} - F_1\hat{\imath} + F_2(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) = \vec{0}.$$

   Dotting this eqn. with $\hat{\jmath}$ and $\hat{\imath}$, respectively, we get,

   $$
   \begin{aligned}
   F_2 &= 0 \\
   F_1 &= f_x = 1\,\mathrm{N}.
   \end{aligned}
   $$

   Substituting these values of F_1 and F_2 in eqns. (6.8) and (6.9), and solving for Δx and Δy we get,

   $$\left(\begin{array}{c}\Delta x \\ \Delta y\end{array}\right)_{\vec{F}=f_x\hat{\imath}} = \left(\begin{array}{c}\frac{1}{k_1} \\ \frac{1}{k_1}\cot\theta\end{array}\right) f_x. \qquad (6.10)$$

   Substituting the given values of θ, k_1, and $f_x = 1\,\mathrm{N}$, we get

   $$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} = (100\hat{\imath} + 173\hat{\jmath}) \times 10^{-6}\,\mathrm{m}.$$

   $$\boxed{\Delta\vec{r} = (100\hat{\imath} + 173\hat{\jmath}) \times 10^{-6}\,\mathrm{m}}$$

2. *Deflections with unit force in the y-direction:* We carry out a similar analysis for this case. We again assume the displacement of point C to be $\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath}$. Since the geometry of deformation and the associated results are the same, eqns. (6.8) and

(6.9) remain valid. We only need to find the spring forces from the static equilibrium under the new load. From the free-body diagram in Fig. 6.25 we have,

$$(-F_1 - F_2\cos\theta)\hat{\imath} + (F_2\sin\theta + F)\hat{\jmath} = \vec{0} \tag{6.11}$$

$$\begin{aligned}
\{\text{eqn. (6.11)}\}\cdot\hat{\jmath} \quad &\Rightarrow \quad F_2 = -\frac{F}{\sin\theta} \\
\{\text{eqn. (6.11)}\}\cdot\hat{\imath} \quad &\Rightarrow \quad F_1 = -F_2\cos\theta = F\cot\theta.
\end{aligned}$$

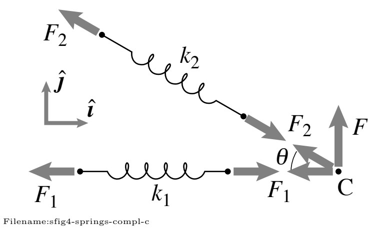


Figure 6.25:

Substituting these values of F_1 and F_2 in terms of $F = f_y$ in eqns. (6.8) and (6.9), we get

$$\begin{aligned}
f_y\cot\theta &= k_1\Delta x \quad \Rightarrow \quad \Delta x = \frac{f_y}{k_1}\cot\theta \\
-\frac{f_y}{\sin\theta} &= k_2(\Delta x\cos\theta - \Delta y\sin\theta) \\
\Rightarrow \quad \Delta y &= \frac{1}{\sin\theta}\left(\Delta x\cos\theta + \frac{f_y}{k_2\sin\theta}\right) \\
&= f_y\left(\frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta\right).
\end{aligned}$$

Thus,

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{F}=f_y\hat{\jmath}} = \begin{pmatrix}\frac{1}{k_1}\cot\theta \\ \frac{1}{k_1}\cot^2\theta + \frac{1}{k_2}\csc^2\theta\end{pmatrix} f_y. \tag{6.12}$$

Substituting the values of θ, k_1, k_2, and $f_y = 1\,\text{N}$, we get

$$\Delta\vec{r} = \Delta x\hat{\imath} + \Delta y\hat{\jmath} = (173\hat{\imath} + 500\hat{\jmath})\times 10^{-6}\,\text{m}.$$

$$\boxed{\Delta\vec{r} = (173\hat{\imath} + 500\hat{\jmath})\times 10^{-6}\,\text{m}}$$

3. *Deflection under general load:* Since we have already got expressions for deflections in the x and y-directions under unit loads in the x and y-directions, we can now combine the results (using superposition, see page 206) to find the deflection under any general load $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath}$ as follows.

$$\begin{aligned}
\Delta\vec{r} &= \begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = F_x\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{F}=1\hat{\imath}} + F_y\cdot\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix}_{\vec{F}=1\hat{\jmath}} \\
&= \begin{bmatrix} k_1^{-1} & k_1^{-1}\cot\theta \\ k_1^{-1}\cot\theta & k_1^{-1}\cot^2\theta + k_2^{-1}\csc^2\theta\end{bmatrix}\begin{pmatrix}F_x\\ F_y\end{pmatrix}.
\end{aligned}$$

Once again, substituting all given values and $F_x = 30\,\text{N}$ and $F_y = 20\,\text{N}$, we get

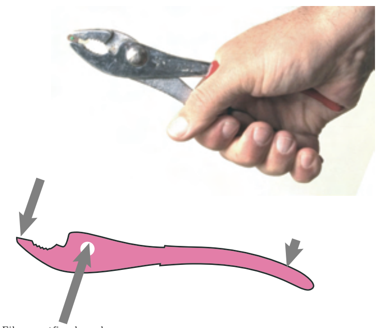
$$\Delta\vec{r} = (6.4\hat{\imath} + 15.2\hat{\jmath})\times 10^{-3}\,\text{m}.$$

$$\boxed{\Delta\vec{r} = (6.4\hat{\imath} + 15.3\hat{\jmath})\times 10^{-3}\,\text{m}}$$

Note: The matrix obtained above for finding the deflection under general load is called the *compliance matrix* of the structure. Its inverse is known as the *stiffness matrix* of the structure and is used to find forces given deflections.

6.2 Force amplification devices: Levers, wedges, toggles, gears, and pulleys

Simple objects can be connected in various arrangements for various purposes. Here we describe 5 machine fragments that can be used to amplify force. Most machines use these ideas in combination. It might help intuitive understanding of machines to recognize one of these methods in use, although precise categorization of every machine part as one or another of these devices is not possible.



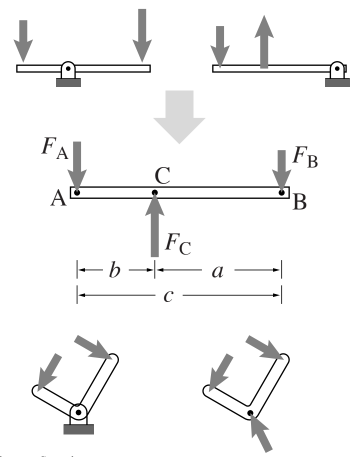
Filename:tfig-plyers-lever

Figure 6.26: One half of a pair of plyers is one of many classic lever examples.

A lever

One of the simplest machines, long understood longer used by humans, is a lever (fig. 6.27). Although now we think of statics as a special case of dynamics, the statics of a lever was well understood 50 generations before Newton and Euler.

> An ideal lever is a rigid body held in place with a frictionless hinge and with two other applied loads.



Filename:tfigure-lever

Figure 6.27: A lever can have the pivot in various places. The free body diagram looks the same in any case.

The free body diagram fig. 6.27 is the same whether the hinge is at point A, B or C①.

Lots of things can be viewed as levers including, for example, a wheelbarrow, a hammer pulling a nail, a boat oar, one half of a pair of tweezers, a break lever, a gear, and, most generally, any three-force body. Using the equilibrium relations on the free body diagram in fig. 6.27 you can find that

$$\frac{F_A}{a} = \frac{F_B}{b} = \frac{F_C}{c}$$

from which you could find the relation between any pair of the forces. In practice it is easier to use moment balance about an appropriate point than to memorize and recall this formula.

①People studied levers before mechanics was a quantitative subject. So they have a specialized antiquated vocabulary, classifying them according to which of A, B, or C is the pivot point and which of the other two forces you think of as input and which as the output: A 'class one' lever has the pivot in between; a 'class two' lever has the pivot at one end and the input force at the other; and a 'class three' lever has the pivot at one end and the input force in the middle.

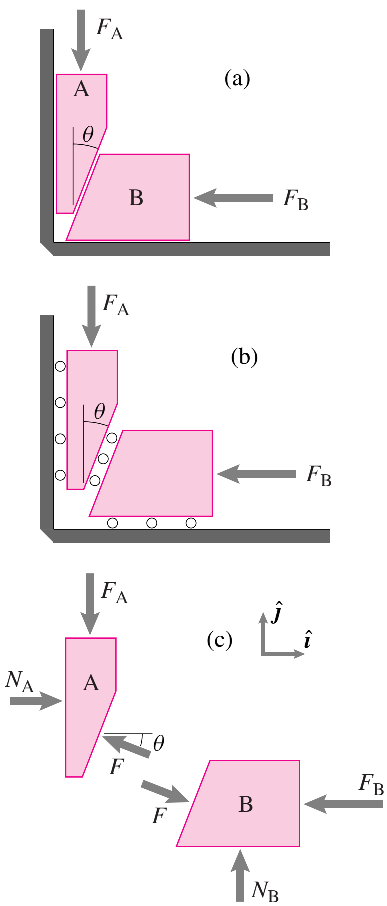
An ideal wedge

Wedges are a kind of machine. You see them used in exactly their wedge configuration, to split and separate things (see fig. 6.28). But screws are also, effectively wedges, with torque replacing the downwards force. For an ideal wedge one neglects friction, effectively replacing sliding contact with rolling contact (see fig. 6.29ab). Although this approximation may not be accurate, it is helpful for building intuition. Because the key idea depends on force balance and not moment balance, for the free body diagrams of fig. 6.29c we have not specified the exact location of the contact forces . Neglecting gravity,

$$\begin{aligned}
&\text{For block A,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} && \Rightarrow && -F_A + F \sin\theta && = 0 \\
&\text{For block B,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} && \Rightarrow && -F_B + F \cos\theta && = 0
\end{aligned}$$

Filename:tfigure-hatchetclip-1301

Figure 6.28: A hatchet splitting wood is a classic wedge. In this case it is used dynamically, with the force coming from the deceleration of mass. But other more complex wood splitting devices push the wedge in slowly.



Filename:tfigure-wedge

Figure 6.29: a) A wedge, b) treated as frictionless in the ideal case, c) free body diagrams (assuming no friction)

$$\text{eliminating } F \quad \Rightarrow \quad F_B = \frac{1}{\tan\theta} F_A.$$

Using the small-angle approximation that $\tan\theta \approx \theta$ we have that

The force amplification of a wedge is about the reciprocal of the wedge angle (in radians).

To multiply the force F_A by 10 takes a wedge with a taper of $\theta = \tan^{-1} 0.1 \approx 6^\circ$. With this taper, an ideal wedge could also be viewed as a device to attenuate the force F_B by a factor of 10, although wedges are never used for force attenuation in practice, because they easily bind because of friction. A wedge with friction is considered in 6.12 on page 353.

A toggle

The classic *toggle* mechanism for amplifying force tends to have a 'snap-through' or bi-stable aspect which is used in the design of some electrical switches. Hence, perhaps, the two dictionary meanings of the word toggle: 1) a force amplifying mechanism, 2) a switch between two states (see fig. 6.32).

The simplest version of the toggle mechanism is shown in fig. 6.30. The force amplification is $N/F = 1/\tan\theta$.

Usually the toggle concept is not used with a wall but with a pair of bars (fig. 6.31) Simple truss analysis shows the bar compressions are $-T = N/2\sin\theta$ and $N = F/2\tan\theta$. The toggle-like force amplification occurs for tension as well as compression. But, because of the oft-desirable snap-through and because the amplification increases as the applied force F moves down, the toggle is most often used in compression.

A toggle as lever and wedge. The distinction between toggles and wedges and levers is not precise. On the one hand the toggle is a lever where the lever arm of F is $\ell\cos\theta$ and the lever arm of N is $\ell\sin\theta$. On the other hand the toggle is sort of a rotary wedge with wedge angle θ.

Pulleys and Gears

Here we discuss a few more common machine components which are used to transmit and amplify or attenuate a force or moment.

Gears

One type of transmission is based on gears (fig. 6.36a). If we think of the input and output as the moments on the two gears, we find from the free body diagram in fig. 6.36b that

$$\begin{aligned} &\text{For gear A, } \left\{\sum \vec{M}_{\mathrm{i/A}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad -R_{\mathrm{A}} F + M_{\mathrm{A}} &= 0 \\ &\text{For gear B, } \left\{\sum \vec{M}_{\mathrm{i/B}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad -R_{\mathrm{B}} F + M_{\mathrm{B}} &= 0 \end{aligned}$$

$$\text{eliminating } F \quad \Rightarrow \quad M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B$$

depending on which you want to think of input and which as output. The force amplification or attenuation ratio is just the radius ratio, just like for a lever.

Because the spacing of gear teeth for both of a meshed pair of gears is the same, a gears circumference, and hence its radius is proportional to the number of teeth. And formulas involving radius ratios can just as well be expressed in terms of ratios of numbers of teeth. The tooth ratio is not just used as an approximation to the radius ratio. Averaged over the passage of several teeth, it is exactly the reciprocal ratio of the turning rates of the meshed gears.

Two gears pulled out of a bigger transmission are shown in fig. 6.36c. Gear A has an inner part with radius R_{A_i} welded to an outer part with radius R_{A_o}. Gear B also has an inner part welded to an outer part.

Moment balance about A in the first free body diagram in fig. 6.36d gives that $R_{A_i} F_A = R_{A_o} F$. You can think of the one gear as a lever (see fig. 6.33). Moment balance about B in the second free body diagram gives that $R_{B_i} F = R_{B_o} F_B$. Combining we get

$$F_B = \frac{R_{A_i}}{R_{A_o}} \frac{R_{B_i}}{R_{B_o}} F_A \quad \text{or} \quad F_A = \frac{R_{A_o}}{R_{A_i}} \frac{R_{B_o}}{R_{B_i}} F_B$$

depending on which force you want to find in terms of the other. The transmission attenuates the force if you think of F_A as the input and amplifies the force if you think of F_B as the input. If the inner gears have one tenth the radius of the outer gears than the multiplication or attenuation is a factor of 100.

Trains of gears can build up large net gear ratios. The ratio of the fastest to slowest gear in a common clock or mechanical watch is on the order of 10,000.

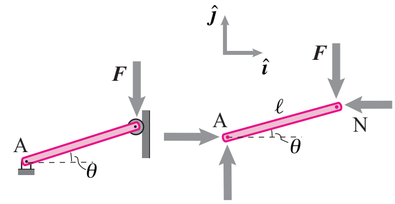
In some gear trains, like the example above, large torque amplification comes from a large ratio of concentrically welded gears. A large amplification can also come from differences rather than ratios. The designs based primarily on differences rather than ratios are called 'differentials', 'harmonic drives', or 'planetary gears'.

Example: Planetary gear with a large ratio

fig. 6.37 shows a gear design where the ratio of the input torque on the drive gear, to the output torque, on the spider can be huge. In particular, for the design shown the torque ratio is approximately:

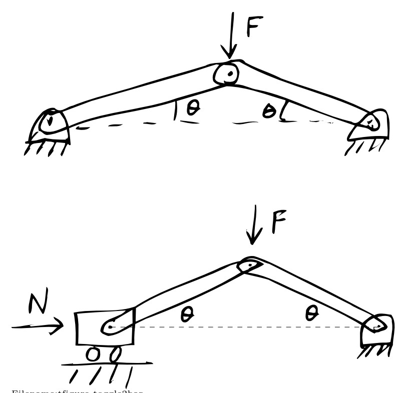
$$\frac{M_{out}}{M_{in}} \approx \frac{2}{R_D/R_R - 1}$$

where R_D is the ratio of the inner drive gear to outer drive gear radius and R_R is the ratio of the inner ring gear to outer ring gear radius. Thus if the inner and outer drive gears have 49 and 50 teeth, respectively, and the inside and outside of the ring gear have 50 and 51 teeth then the torque multiplication is nearly 5000. (See homework 6.2.18).



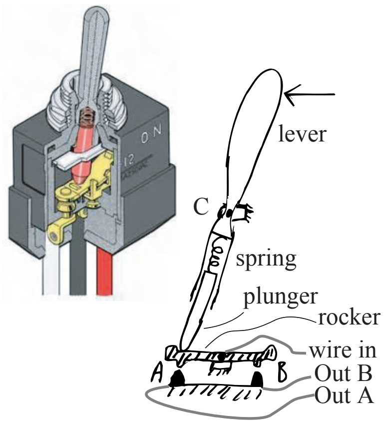
Filename:tfigure-simpletoggle

Figure 6.30: **A simple toggle mechanism.** Moment balance about point A gives that $N = F/\tan\theta$ so if θ is small N is large.



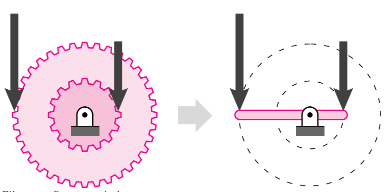
Filename:tfigure-toggle2bar

Figure 6.31: Two bars in a toggle mechanism.



Filename:tfigure-NKKtoggleswitch

Figure 6.32: Cutaway drawing of a toggle switch and a simplified schematic of the idea. The lever, spring and plunger rotate about C. The spring is prestressed so that the plunger presses hard against the rocker. The plunger slides on the rocker (input line) which is pressed against contact at outputs A or B. The relevant idea here is that the force of the plunger on the rocker can be much bigger than the input force on the lever (cutaway from NKK Switches online catalogue).



Filename:tfigure-gearisalever

Figure 6.33: One gear may be thought of as a lever.

Pulleys

We have already studied a pulley as a single object (see page 204. Now we show, as you probably have learned a few times before in school, how to use pulleys to amplify or attenuate force. We assume pulleys are round, massless, and have frictionless bearings.

The key fact for statics analysis is

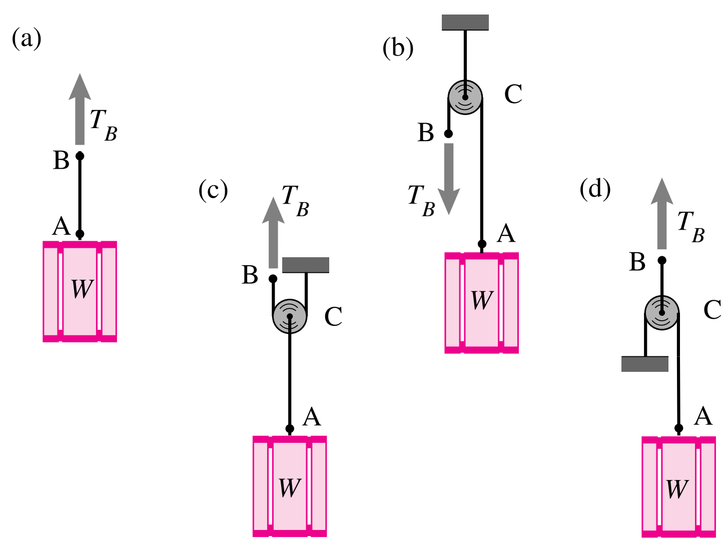
For an ideal round pulley with negligible mass (or negligible angular acceleration) the tension on the cable is the same on both sides of the pulley:

$$T_1 = T_2$$

The classic problem is shown in fig. 6.34a where you would like to use a pulley to make the task easier. Figures 6.34b-c show three possible uses of pulleys. If, at a glance, you can't see that these three designs are quite different in their effects you should puzzle them out slowly now.

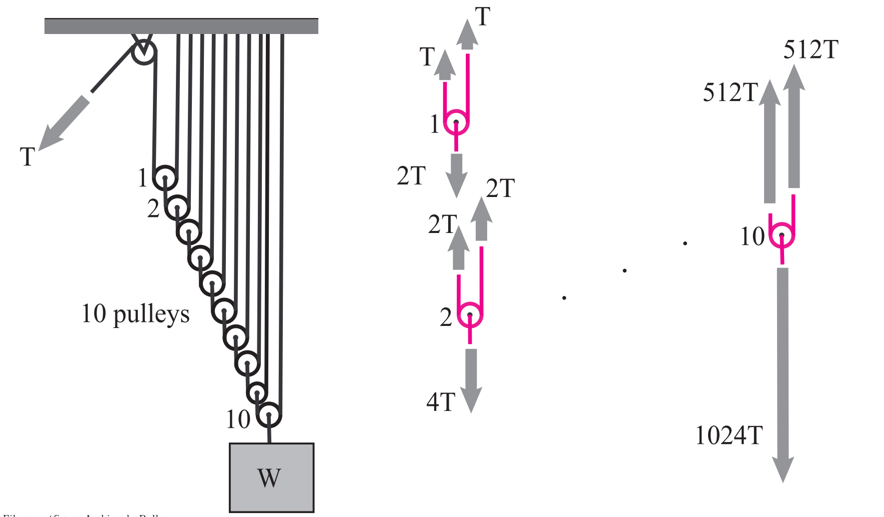
Because the two tension in the rope that wraps around the pulley is the same on both sides, the central rope has twice the tension. Design (b) gives no mechanical advantage but does allow one to pull down in order to lift the weight. Design (c) halves the effort. Design (d), which might look superficially similar to (c) doubles the required pulling force, requiring 4 times the force of (c).

By using pulleys in combination one can get various force attenuations and gains. The 10-pulley design in fig. 6.35 multiplies the force by about 1000.



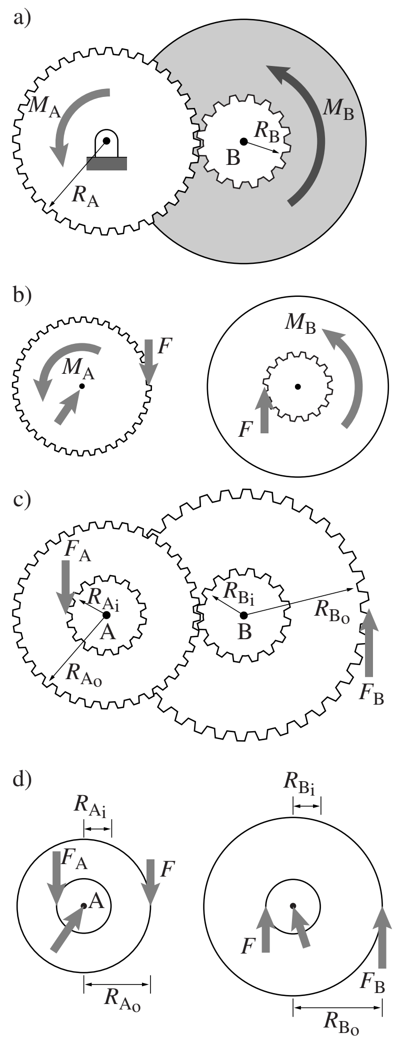
Filename:tfigure-pulleystatics

Figure 6.34: **Simple pulley arrangements.** **a)** Lifting a weight with no pulley, **b)** a pulley lets you pull down instead of up, **c)** a pulley halves the needed pull, **d)** a pulley doubles the needed pull.



Filename:tfigure-ArchimedesPulley

Figure 6.35: **Archimedes pulley.** A pulley arrangement sometimes commonly attributed to Archimedes. With 10 pulleys, he could lift a weight of W with a pull of about $T = 2^{-10}W \approx W/1000$. With 20 pulleys the force amplification is by more than a million. With 67 pulleys he, assuming he was as strong as the average person, could hoist a rock the size of the moon.



Filename:tfigure-gears

Figure 6.36: a) Two gear pairs pulled out of a transmission with forces on the teeth, b) Free body diagrams, c) The same gear pair, but loaded with tooth-forces from unseen gears, d) the consequent free body diagrams.

General force amplification concepts

If a mechanism generates a large force ratio (output/input) this usually corresponds to a large ratio in some geometric quantities. For a lever we have the ratio of two lever arms. For a wedge the small wedge angle, and for a toggle also a small angle.

More precisely

For a frictionless transmission the ratio of the input force to output force is the reciprocal of the ratio of input motion to output motion.

For a high-gain lever the handle moves much further than the load. For a narrow wedge the slip distance is much bigger than the spreading distance. For a toggle the motion of the compressed end is much smaller than that of the applied load. That the force amplification is identical to the motion attenuation follows from energy conservation: the work into the mechanism is equal to the work out.

So when Archimedes pulls in a kilometer of rope while lifting a rock with 67 pulleys. the moon, only lifts 2^{-67} km or about one hundred millionth of a nanometer. Similarly with levers. When Archimedes famously said:

> *Give me a lever long enough and a fulcrum on which to place it,*
> *and I shall move the world,*

he was careful not to say how far he would move it.

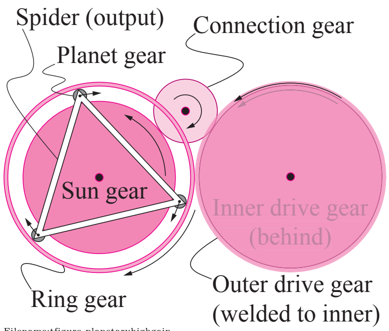
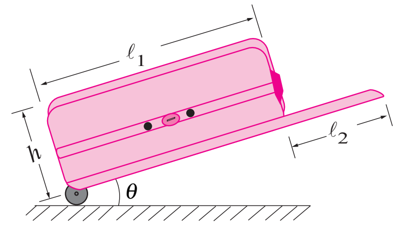


Figure 6.37: **A high gain planetary gear system,** so named because the 'planet' gears go around the 'sun' gear. Often planetary gears depend on the sun or the spider or the ring not rotating. In this design all three rotate. A CCW torque is applied to the Inner drive gear which is welded to the outer drive gear. The outer drive gear turns the ring gear CW. The inner drive gear turns the connection gear CW which turns the sun gear CCW. The three planet gears spin between the inside of the ring gear and the outside of the sun gear. If the inside of the ring gear moves a little faster down than the outside of the sun gear moves up, then the spider turns CW with the center of the planet gears.

SAMPLE 6.6 A wheeled suitcase of length 60 cm, height 30 cm and 'weighing' 20 kg on the airport check-in counter, has a telescopic handle of length 40 cm. The suitcase is dragged at an angle $\theta = 30°$. Assuming good wheels (negligible friction), find the force applied on the handle in order to wheel the suitcase steadily. (Take $g \approx 10\,\mathrm{m/s^2}$).



Filename:sfig6-2-suitcase

Figure 6.38:

Solution The free-body diagram of the suitcase is shown in fig. 6.37. The reaction force at the wheel is almost vertical because of negligible friction. So, we can also assume the force F applied at the handle to be almost vertical. We assume that the center of mass G is located at the geometric center of the rectangular suitcase. Now the moment balance equation about point A, $\sum M_A = 0$, gives

$$\vec{r}_{G/A} \times (-mg\,\hat{\jmath}) + \vec{r}_{B/A} \times F\,\hat{\jmath} = \vec{0}.$$

Substituting $\vec{r}_{G/A} = \frac{\ell_1}{2}\hat{\lambda} + \frac{h}{2}\hat{n}$, $\quad \vec{r}_{B/A} = (\ell_1 + \ell_2)\hat{\lambda}$, and noting that $\hat{\lambda} \times \hat{\jmath} = \cos\theta\hat{k}$ and $\hat{n} \times \hat{\jmath} = -\sin\theta\hat{k}$, we have

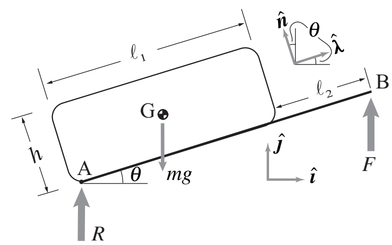
$$-\frac{1}{2}mg(\ell_1\cos\theta - h\sin\theta)\hat{k} + F(\ell_1 + \ell_2)\cos\theta\hat{k} = 0$$

$$\Rightarrow \quad F = \frac{1}{2}mg\left[\frac{\ell_1}{\ell_1 + \ell_2} - \frac{h}{\ell_1 + \ell_2}\tan\theta\right].$$

Substituting $\ell_1 = 60\,\mathrm{cm}$, $\ell_2 = 40\,\mathrm{cm}$, $h = 30\,\mathrm{cm}$, $\theta = 30°$ and $mg = 200\,\mathrm{N}$, we get

$$F = \frac{1}{2}\cdot 200\,\mathrm{N}\left[\frac{60\,\mathrm{cm}}{100\,\mathrm{cm}} - \frac{30\,\mathrm{cm}}{100\,\mathrm{cm}}\tan 30°\right] = 42.7\,\mathrm{N}.$$

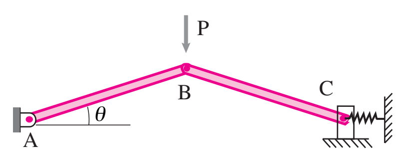
$F = 42.7\,\mathrm{N}$



Filename:sfig6-2-suitcase-a

Figure 6.39: Free body diagram of the suitcase. Unit vectors $\hat{\lambda}$ and $\hat{n}$ are along the length and height of the suitcase, respectively. Thus, $\hat{\lambda} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{n} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$.

SAMPLE 6.7 The figure shows a basic toggle mechanism. (a) If the applied force is $P = 20\,\mathrm{N}$ and the mechanism is in equilibrium at $\theta = 5°$, find the force applied by the spring. (b) If doubling the load P (to $P = 40\,\mathrm{N}$) causes a decrease of θ by 1° (to $\theta = 4°$), does the spring force at C double too?



Filename:sfig6-2-togglemech

Figure 6.40: Toggle mechanism.

Solution The free-body diagrams of the pin connecting the two rods and the BC are shown in fig. 6.39. From the static equilibrium of the pin B, we have

$$\sum F_x = 0 \quad \Rightarrow \quad T_2\cos\theta - T_1\cos\theta = 0 \quad \Rightarrow \quad T_1 = T_2$$

$$\sum F_y = 0 \quad \Rightarrow \quad -(T_1 + T_2)\sin\theta - P = 0 \quad \Rightarrow \quad T_2 = -\frac{P}{2\sin\theta}.$$

which follows from setting $T_1 + T_2 = 2T_2$ since $T_1 = T_2$. Now, we consider the free-body diagram of rod BC. The force balance equation in the x-direction ($\sum F_x = 0$) gives

$$-T_2\cos\theta - F = 0 \quad \Rightarrow \quad F = -T_2\cos\theta = \frac{P\cos\theta}{2\sin\theta}.$$

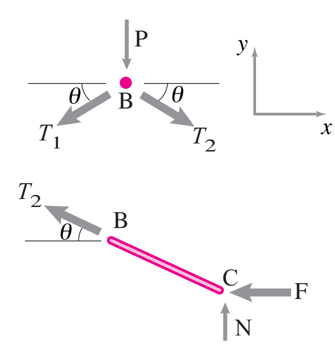
Since θ is small, we have $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Thus $F = P/2\theta$ where θ is in radians. Substituting $P = 20\,\mathrm{N}$ and $\theta = 5\pi/180$, we get

$$F = \frac{20\,\mathrm{N}}{2\pi/36} = 115\,\mathrm{N}$$

which is almost 6 times P.

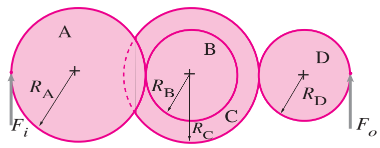
(b) If P is doubled, we might expect F to double because $F \approx P/2\theta$. But if θ also decreases to $\theta = 4°$, repeating the calculation above with $P = 40\,\mathrm{N}$, and $\theta = 40°$ we get $F = 286\,\mathrm{N}$ which is 2.5 times the previous spring force.

For $P = 20\,\mathrm{N}, \theta = 5°, F = 115\,\mathrm{N}$; and for $P = 40\,\mathrm{N}, \theta = 4°, F = 286\,\mathrm{N}$



Filename:sfig6-2-togglemech-a

Figure 6.41: Free body diagrams.



Filename:sfig4-mech-gear

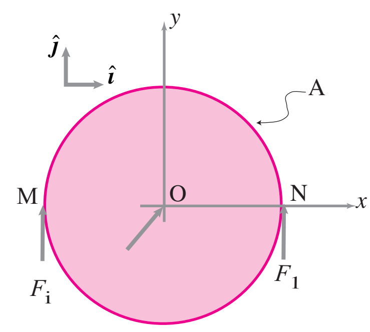
Figure 6.42:

SAMPLE 6.8 **A gear train:** In the compound gear train shown in the figure, the various gear radii are: $R_A = 10\,\text{cm}$, $R_B = 4\,\text{cm}$, $R_C = 8\,\text{cm}$ and $R_D = 5\,\text{cm}$. The input load $F_i = 50\,\text{N}$. Assuming the gears to be in static equilibrium find the machine load F_o.

Solution You may be tempted to think that a free-body diagram of the entire gear train will do since we only need to find F_o. However, it is not so because there are unknown reactions at the axle of each gear and, therefore, there are too many unknowns. On the other hand, we can find the load F_o easily if we go gear by gear from the left to the right.

The free-body diagram of gear A is shown in Fig. 6.41. Let F_1 be the force at the contact tooth of gear A that meshes with gear B. From the moment balance about the axle-center O, $\sum \vec{M}_O = \vec{0}$, we have

$$
\begin{aligned}
\vec{r}_M \times \vec{F}_i + \vec{r}_N \times \vec{F}_1 &= \vec{0} \\
-F_i R_A \hat{k} + F_1 R_A \hat{k} &= \vec{0} \\
\Rightarrow \quad F_1 &= F_i .
\end{aligned}
$$

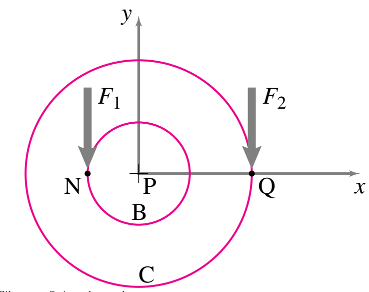


Filename:sfig4-mech-gear-a

Figure 6.43:

Similarly, from the free-body diagram of gear B and C (together) we can write the moment balance equation about the axle-center P as

$$
\begin{aligned}
F_1 R_B \hat{k} + F_2 R_C \hat{k} &= \vec{0} \\
\Rightarrow \quad F_2 &= \frac{R_B}{R_C} F_1 \\
&= \frac{R_B}{R_C} F_i .
\end{aligned}
$$



Filename:sfig4-mech-gear-b

Figure 6.44:

Finally, from the free-body diagram of the last gear D and the moment equilibrium about its center R, we get

$$
\begin{aligned}
-F_2 R_D \hat{k} + F_o R_D \hat{k} &= \vec{0} \\
\Rightarrow \quad F_o &= F_2 \\
&= \frac{R_B}{R_C} F_i \\
&= \frac{4\,\text{cm}}{8\,\text{cm}} \cdot 50\,\text{N} = 25\,\text{N}.
\end{aligned}
$$

$F_o = 25\,\text{N}$

SAMPLE 6.9 Find the force F to hold the 100 kg box shown in the figure in equilibrium. Assume $g \approx 10\,\mathrm{m/s^2}$.

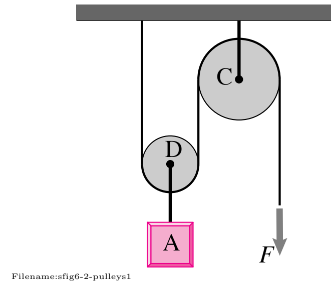


Figure 6.45:

Solution The free-body diagrams of the two pulleys are shown in fig. 6.44 where the tension in the rope running over the two pulleys has been assumed as T. For the lower pulley D, the force balance in the y-direction, $\sum F_y = 0$, requires

$$2T - mg = 0 \quad \Rightarrow \quad T = \frac{mg}{2}.$$

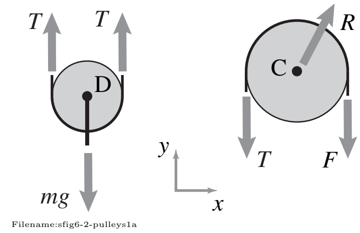


Figure 6.46: Free-body diagram of the two pulleys. We use C and D to designate both the pulleys and their respective centers.

The free-body diagram of the upper pulley C contains an unknown reaction force R at the attachment point C. However, if we write moment balance about point C, $\sum M_C = 0$, this unknown force contributes nothing. Let the radius of pulley C be r. Thus, the moment balance equation about C gives

$$\begin{aligned} Tr - Fr &= 0 \\ \Rightarrow \quad F &= T = \frac{mg}{2} \approx 500\,\mathrm{N}. \end{aligned}$$

$$\boxed{F \approx 500\,\mathrm{N}}$$

SAMPLE 6.10 A container box weighing 1 kN is dragged slowly and steadily along the floor with force F as shown in the figure. The coefficient of friction between the box and the floor is 0.6. Find the force required to pull the box and the force amplification obtained by the pulley arrangement.

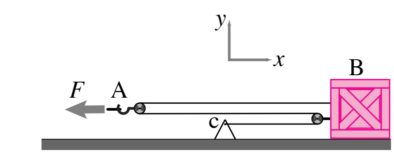


Figure 6.47:

Solution It is clear from the figure that the same rope passes over the two pulleys used in the arrangement to pull the box. Let the tension in the rope be T. A partial free-body diagram (that includes forces acting only in the x-direction) of the box along with the pulley attached to it is shown in fig. 6.46. The same figure also shows the free-body diagram of pulley A at the force end. From the force balance equation for the box in the x-direction, we get

$$f - 3T = 0 \quad \Rightarrow \quad T = \frac{f}{3} = \frac{\mu mg}{3}.$$

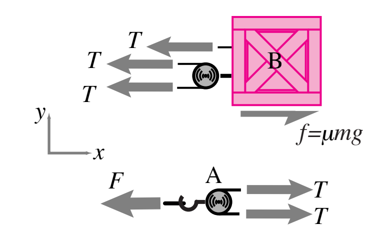


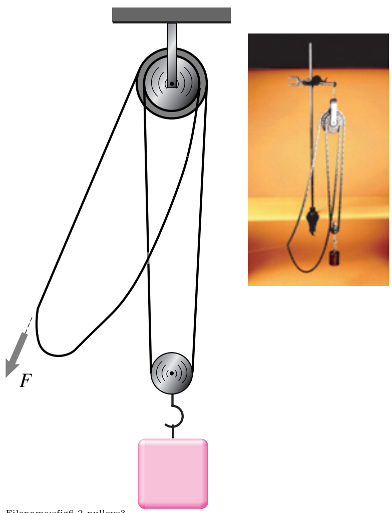
Figure 6.48:

Now, from the force balance of pulley A in the x-direction, we get

$$\begin{aligned} 2T - F &= 0 \\ \Rightarrow \quad F &= 2T = \frac{2\mu mg}{3} \\ &= \frac{2 \cdot (0.6) \cdot (1\,\mathrm{kN})}{3} = 400\,\mathrm{N}. \end{aligned}$$

Since the force of friction on the box while sliding is $f = \mu mg = 0.6(1\,\mathrm{kN}) = 600\,\mathrm{N}$ and the force applied at A to overcome this friction is 400 N, the force amplification is 1.5. That is, the pulley arrangement amplifies the input force (400 N) 1.5 times at the output end.

$$\boxed{F = 400\,\mathrm{N},\ \text{Force amplification} = 1.5}$$



Filename:sfig6-2-pulleys3

Figure 6.49:

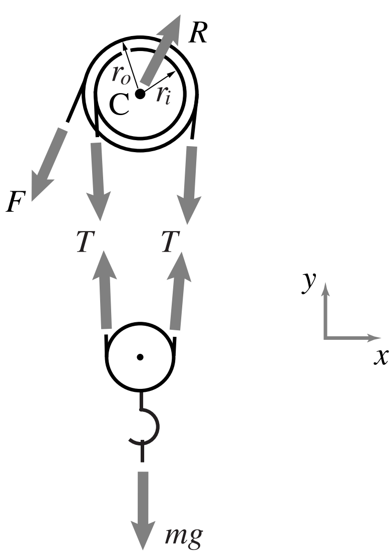


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Figure 6.50:

SAMPLE 6.11 A differential hoist is used to lift a crate of mass 500 kg. The hoist pulley uses two discs of radius 30 cm and 25 cm. Find the force F required to lift the crate steadily. Take $g \approx 10\,\mathrm{m/s^2}$.

Solution The free-body diagrams of the upper pulley and the lower pulley are shown in fig. 6.48. Since the lower pulley is slightly smaller than the upper pulley, the chain passing over the two pulleys is not exactly vertical but makes a small angle with the vertical. Thus the tension forces shown in the free-body diagrams are slightly off from the vertical direction. However, since the angle is very small, we can treat T to be essentially vertical.

For the lower pulley, the force balance in the y direction gives

$$\begin{aligned} 2T - mg &= 0 \\ \Rightarrow \quad T &= \frac{mg}{2}. \end{aligned}$$

Now the moment balance about point C, $\sum M_C = 0$, for the upper pulley gives

$$\begin{aligned} Fr_o + Tr_i - Tr_o &= 0 \\ \Rightarrow \quad F &= \left(\frac{r_o - r_i}{r_o}\right) T \\ &= \left(1 - \frac{r_i}{r_o}\right)\frac{mg}{2} \\ &= \left(1 - \frac{25\,\mathrm{cm}}{30\,\mathrm{cm}}\right)\frac{5000\,\mathrm{N}}{2} \\ &= 417\,\mathrm{N} \end{aligned}$$

Thus the force amplification in this case is about 12 (5000 N/417 N). From the analysis above, it is also clear that the ratio of the radii of the two disks used in the upper pulley decide this force amplification. One can get a big force amplification, at least theoretically, by making $r_i \approx r_o$. In this problem, for example, if $r_i = 29\,\mathrm{cm}$ rather than the given 25 cm, we get $r_i/r_o \approx 0.97$ giving $F \approx 83\,\mathrm{N}$ which corresponds to a force amplification of 60.

$$\boxed{F = 417\,\mathrm{N}}$$

SAMPLE 6.12 A wedge with friction. Consider the wedge described in the text (page 343), but now with friction μ between the blocks.

Consideration of friction qualitatively changes the behavior of the machine. For simplicity still take the wall and floor interactions to be frictionless.

1. What is the relation between F_A and F_B when block A is sliding down?
2. What is the relation between F_A and F_B when block A is sliding up?
3. Under what conditions is it impossible for F_B to slide block A up, even when F_A is vanishingly small? Such a case is called 'nonbackdriveable' or 'self-locking'.

How do your answers simplify for small wedge angle θ and small friction angle ϕ (with $\tan\phi = \mu$).

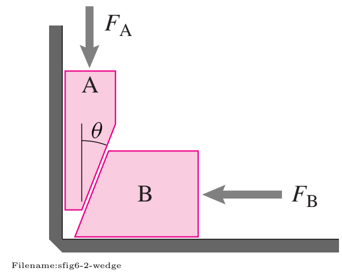


Figure 6.51: A wedge with friction μ between the blocks. Treat the walls as frictionless.

Solution Figure 6.50 shows free body diagrams of wedge blocks. We draw separate free body diagrams for the case when (a) block A is sliding down and block B to the right, and (b) block A is sliding up and block B to the left. In both cases the friction resists relative slip and obeys the sliding friction relation

$$F_f = \underbrace{\tan\phi}_{\mu}\; N$$

where fig. 6.50 shows the resultant contact force (normal component plus frictional component) and its angle ϕ to the surface normal.

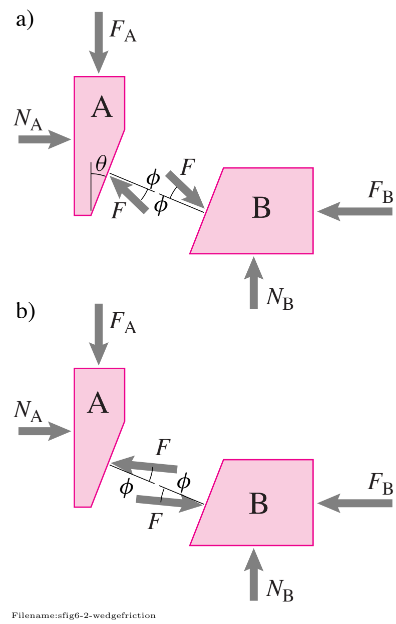


Figure 6.52: Free body diagrams of a wedge a) assuming A slides down, b) assuming A slides up

1. **Block A sliding down:** Assuming block A is sliding down we get from free body diagram 6.50a that

$$\begin{array}{llll} \text{For block A,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} & \Rightarrow & -F_A + F\sin(\theta+\phi) & = 0 \\ \text{For block B,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} & \Rightarrow & -F_B + F\cos(\theta+\phi) & = 0. \end{array}$$

Eliminating F we get,

$$F_B = \frac{1}{\tan(\theta+\phi)} F_A. \qquad (6.13)$$

$$\boxed{\text{When A slides down} \quad \Rightarrow \quad F_B = \frac{1}{\tan(\theta+\phi)} F_A.}$$

If we take a taper of 6° and a friction coefficient of $\mu = .3$ ($\Rightarrow \phi \approx 17^\circ$) we get that $F_B/F_A \approx 2.5$ instead of 10 as we got when neglecting friction. The wedge still serves as a way to multiply force, but substantially less so than the frictionless idealization led us to believe.

2. **Block A sliding up:** Now lets consider the case when force F_B is pushing block B to the left, pinching block A, and forcing it up. The only change in the calculation is the change in the direction of the friction interaction force. From free body diagram 6.50b

$$\begin{array}{llll} \text{For block A,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} & \Rightarrow & -F_A + F\sin(\theta-\phi) & = 0 \\ \text{For block B,} \left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} & \Rightarrow & -F_B + F\cos(\theta-\phi) & = 0 \end{array}$$

Eliminating F we get,

$$F_A = \tan(\theta-\phi) F_B. \qquad (6.14)$$

$$\boxed{\text{When A slides up} \quad \Rightarrow \quad F_B = \tan(\theta+\phi) F_A.}$$

Again using $\theta = 6^\circ$ and $\phi = 17^\circ$ we see that if $F_B = 100\,\text{lbf}$ then $F_A = \tan(-11^\circ) \cdot 100\,\text{lbf} \approx -20\,\text{lbf}$. That is, the 100 pounds doesn't push block A up at all, but even with no gravity you need to pull up with a 20 pound force to get it to move.

If we insist that the downwards force F_A is positive or zero, that the pushing force F_B is positive, and that block A is sliding up then there is no solution to the equilibrium equations whenever $\phi > \theta$. (Actually we didn't need to do this second calculation at all. Eqn 1 shows the same paradox when $\theta + \phi > 90^\circ$. Trying to squeeze block B to the right for large θ is exactly like trying to squeeze block A up for small θ.)

3. **Self-locking:** This *self-locking* situation is intuitive. In fact it's hard to picture the contrary, that pushing a block like B would lift block A. If you view this wedge mechanism as a transmission, it is said to be *non-backdrivable* whenever $\phi > \theta$. Even though pushing down on A can 'drive' block B to the right, but pushing to the left on block B cannot 'back-drive' block A up. Non-backdrivability is a feature or a defect depending on context②.

   The borderline case of backdrivability is when $\theta = \phi$ and $F_B = F_A/\tan 2\theta$. Assuming θ is a fairly small angle we get

$$F_B = \frac{F_A}{\tan 2\theta} \approx \frac{F_A}{2\theta} \approx \frac{1}{2}\frac{F_A}{\tan\theta}$$
$$\approx \frac{1}{2} \cdot \text{(the value of } F_B \text{ had there been no friction).}$$

②Standard car transmissions are back-driven when they are push-started and when a driver downshifts to slow the car instead of using the brakes. On the other hand most electric hand-mixers cannot be backdriven; you can't turn the motor by forcing the beater blade (*Unplug before trying.*)

Thus the design guideline:

> Non-back-drivable transmissions are generally 50% or less efficient, they transmit 50% or less of the force they would transmit if they were frictionless.

To use a wedge in this backwards way requires very low friction. A rare case where a narrow wedge is back drivable is with a fresh wet watermelon seed squeezed between two pinched fingers.

6.3 Mechanisms

We would now like to analyze things built of pieces that are connected in a way that amplifies, attenuates or redirects a force or moment.

For completeness, we present the statics recipe for machines, although it is an exact repeat of the recipe used for frames.

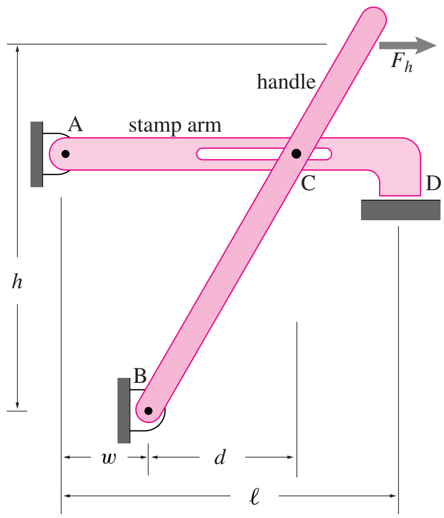
- Draw free body diagrams of
  - the whole machine; and
  - the separate parts of the machine; and
  - collections of parts of the machine if such seems likely to be fruitful;
  - Use the principal of action and reaction in the free body diagrams so that there is only one unknown force at a point where two bodies contact;
- for each free body diagram write equilibrium conditions. These should yield three independent scalar equations for each non-point part (in 2D)
- solve some or all of the equilibrium equations for desired unknowns

Some useful tricks and shortcuts include:

- for any two force bodies assign an equal valued tension to each end (thus eliminating any need or use for equilibrium equations for that object)
- To minimize calculation, look for a subset of the equilibrium equations that
  - contains your unknowns of interest, and
  - has as many uknowns as scalar equations, and
  - contains as few equations as possible.

Example: Stamp machine

Pulling on the handle (below) causes the stamp arm to press down with a force N at D. We can find N in terms of F_h by drawing free body diagrams of the handle and stamp arm, writing three equilibrium equations for each piece and then solving these 6 equations for the 6 unknowns (A_x, A_y, F_C, N, B_x, and B_y).



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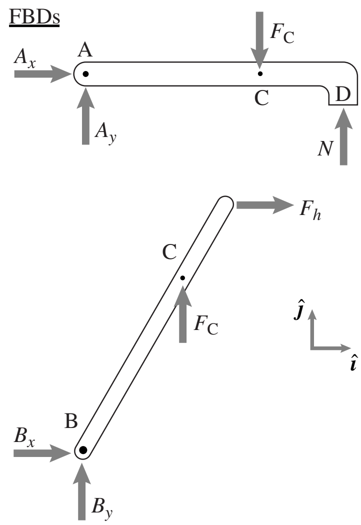

For this problem, the answer can be found more quickly with a judicious choice of equilibrium equations.

$$
\begin{aligned}
&\text{For the handle, } \left\{\sum \vec{M}_{/\mathrm{B}} = \vec{0}\right\} \cdot \hat{k} && \Rightarrow && -hF_h + dF_c &&= 0\\
&\text{For the stamp arm, } \left\{\sum \vec{M}_{/\mathrm{A}} = \vec{0}\right\} \cdot \hat{k} && \Rightarrow && -(d+w)F_c + \ell N &&= 0
\end{aligned}
$$

$$\text{eliminating } F_c \quad \Rightarrow \quad N = \frac{h(d+w)}{d\ell} F_h.$$

Note that the stamp force N can be made very large by making d small and thus the handle nearly vertical. Often in structural or machine design one or another force gets extremely large or small as the design is changed to put pieces in near alignment.

Example: Improved stamp machine

Figure 6.51 shows a stamp machine with all the same components. The method of analysis is identical. However the design represents an improvement 2 ways:

- The lever in the stamp arm amplifies rather then attenuates the stamp force.
- In the previous design it gets harder and harder to generate a given stamp force as the stamped object compresses. In this design the toggle mechanism associated with the lever arm and sliding pin is in compression. Thus as the stamping progresses and the handle becomes more vertical the stamping force increases for a fixed hand-force.

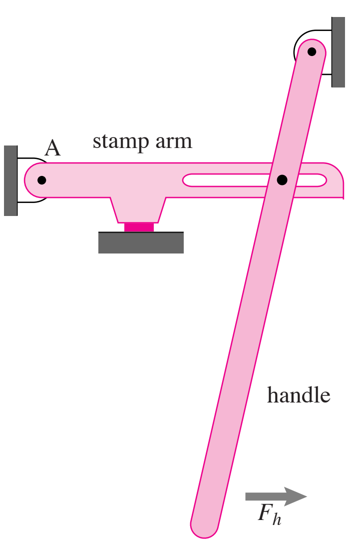

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Figure 6.53: An improved stamp machine.

Non-rigid structures are mechanisms

A non-rigid structure cannot carry all loads and, if not also redundant, has more equilibrium equations than unknown reaction or interaction force components. Such a structure is also called a mechanism. The stamp machine above is a mechanism if there is assumed to be no contact at D. In particular the equilibrium equations cannot be satisfied unless $F_h = 0$. Mechanisms have variable configurations. That is, the constraints still allow relative motion.

An attempt to design a rigid structure that turns out to be a mechanism is a design failure. But for machine design, the mechanism aspect of a structure is essential. Even though mechanisms are called 'statically indeterminate' because they cannot carry all possible loads, the desired forces can often be determined using statics. For the stamp machine above the equilibrium equations are made solvable by treating one of the applied forces, say N, as an unknown, and the other, F in this case, as a known. This is a common situation in machine design where you want to determine the loads at one part of a mechanism in terms of loads at another part. For the purposes of analysis, a trick is to make a mechanism determinate by putting a pin on rollers connection to ground at the location of any forces with unknown magnitudes but known directions.

Example: Stamp machine with roller
Putting a roller at D, the location of the unknown stamp force, turns the stamp machine into a determinate structure.

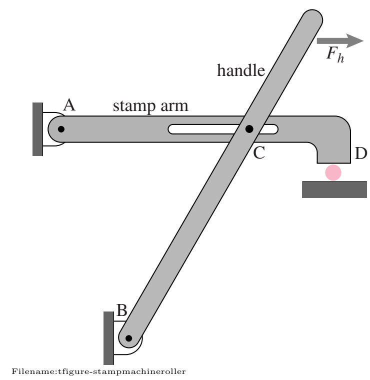


Figure 6.54: Stamp machine with roller.

Pulley and chain drives

Chain and pulley drives are kind of like spread out gears (fig. 6.53). The rotation of two shafts is coupled not by the contact of gear teeth but by a belt around a pulley or a chain around a sprocket. For simple analysis one draws free body diagrams for each sprocket or pulley with a little bit of chain as in fig. 6.53b. Note that $T_1 \neq T_2$, unlike the case of an ideal undriven pulley. Applying moment balance we find,

$$\begin{aligned}
&\text{For gear A,} \left\{\sum \vec{M}_{\mathrm{i/A}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad -R_{\mathrm{A}}(T_2 - T_1) + M_{\mathrm{A}} &= 0\\
&\text{For gear B,} \left\{\sum \vec{M}_{\mathrm{i/B}} = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad R_{\mathrm{B}}(T_1 - T_2) - M_{\mathrm{B}} &= 0
\end{aligned}$$

$$\text{eliminating } (T_2 - T_1) \quad \Rightarrow \quad M_B = \frac{R_B}{R_A} M_A \quad \text{or} \quad M_A = \frac{R_A}{R_B} M_B$$

exactly as for a pair of gears. Note that we cannot find T_2 or T_1 but only their difference. Typically in design if, say, M_A is positive, one would try to keep T_1 as small as possible without the belt slipping or the chain jumping teeth. If T_1 grows then so must T_2, to preserve their difference. This increase in tension increases the loads on the bearings as well as the chain or belt itself.

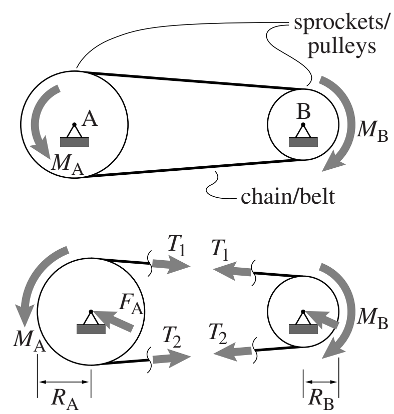
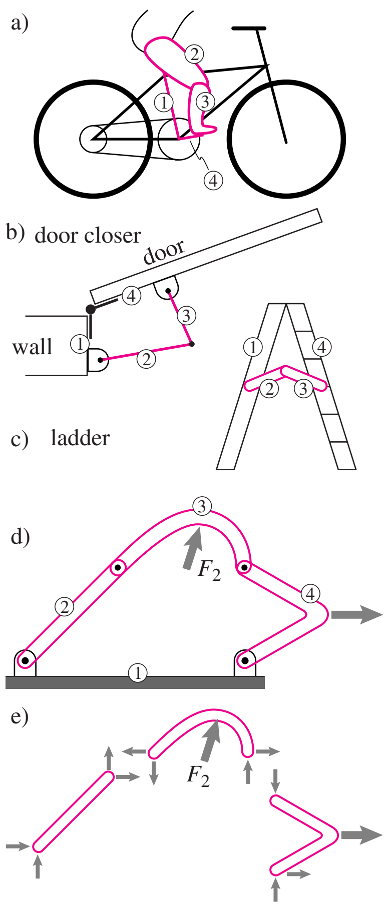


Figure 6.55: a) A chain or pulley drive involving two sprockets or pulleys and one chain or belt, b) free body diagrams of each of the sprockets/pulleys.

4-bar linkages

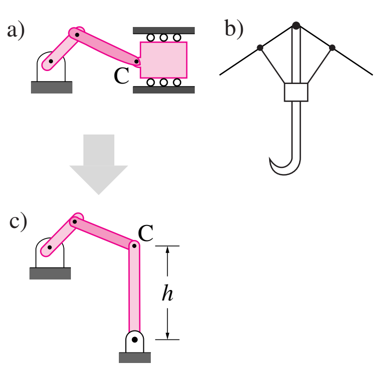
Four bar linkages often, confusingly, have 3 bars, the fourth piece is the something bigger. A planar mechanism with four pieces connected in a loop by hinges is a four bar linkage. Four bar linkages are remarkably common. After a single body connected at a hinge (like a gear or lever) a four bar linkage is one of the simplest mechanisms that can move in just one way (have just one degree of freedom).

A reasonable model of seated bicycle pedaling uses a 4-bar linkage (fig. 6.54a). The whole bicycle frame is one bar, the human thigh is the second, the calf is the third, and the bicycle crank is the fourth. The four



Filename:tfigure-fourbar

Figure 6.56: Four bar linkages. a) A bicycle, thigh, calf, and crank, b) a door closer, c) a folding ladder, d) a generic mechanism, e) free body diagrams of the parts of a generic mechanism.



Filename:tfigure-slidercrank

Figure 6.57: a) a slider crank, b) an umbrella has a slider-crank mechanism, c) the equivalent four-bar linkage, at least when $h \to \infty$.

hinges are the hip joint, the knee joint, the pedal axle, and the bearing at the bicycle crank axle. A more sophisticated model of the system would include the ankle joint and the foot would make up a fifth bar.

A standard door closing mechanism is part of a 4-bar linkage (fig. 6.54b). The door jamb and door are two bars and the mechanism pieces make up the other two.

A standard folding ladder design is, until locked open, a 4-bar linkage (fig. 6.54c).

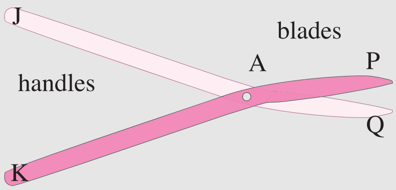
An abstracted 4-bar linkage with two loads is shown in fig. 6.54d with free body diagrams in fig. 6.54e. If one of the applied loads is given, then the other applied load along with interaction and reaction forces make up nine unknown components (after using the principle of action and reaction). With three equilibrium equations for each of the three bars, all these unknowns can be found.

Slider crank

A mechanism closely related to a four bar linkage is a slider crank (fig. 6.55a). An umbrella is one example (rotated 90° in fig. 6.55b). If the sliding part is replaced by a bar, as in fig. 6.55c, the point C moves in a circle instead of a straight line. If the height h is very large then the arc traversed by C is nearly a straight line so the motion of the four-bar linkage is almost the same as the slider crank. For this reason, slider cranks are sometimes regarded as a special case of a four-bar linkage in the limit as one of the bars gets infinitely long.

6.6 Shears with gears

Many cutters, pliers and shears are essentially two levers pivoting against each other. For example these shears

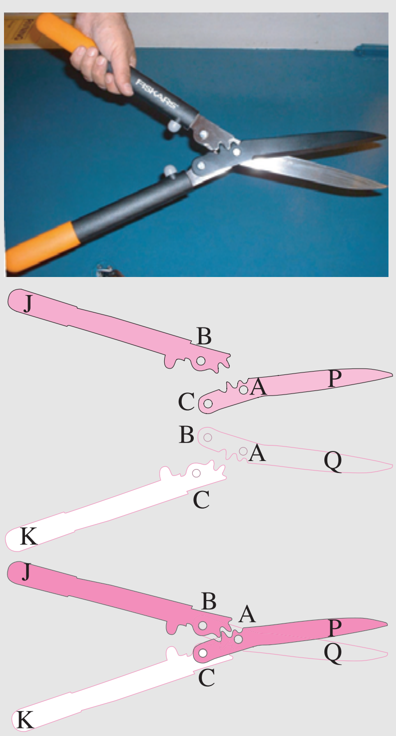


Filename:tfigure-FiskarsBefore

consist of two levers, JAQ and KAP, pivoted at A. The hands squeeze the handles at J and K causing a cutting force on an object between the blades at P and Q. The force at P, say, is $|KA|/|AP|$ times the force at K (from moment balance about A using a free body diagram of KAP). Two possible deficiencies of this bi-lever design are that

- One may want more mechanical advantage but not longer handles, and
- For a given hand strength (available force at J and K) the force at the cutting edge gets less and less as the location of the cut force at P and Q moves farther out on the blade, away from A.

The Fiskars company, known mostly for scissors using the basic design above, has some designs that address these defficiencies. The loppers in problem 6.3.9 use a 3 piece mechanism to address these issues. Here, even more elaborately, are Fiskars shears using 4 moving parts.



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The two identical blades AP and AQ are hinged at A. The two identical handles JB and and KC are hinged to the blades at B and C. Each handle also has gear teeth at the end that engage gear teeth on the opposite blade. Lets take P and Q to be the point of contact of the object being cut.

Lets try to understand the mechanism without detailed analysis (see homework problem 6.3.10). To start, forget handle KC and assume that blade BAQ is held firmly by something outside. Blade CAP is attached at A about which it is free to spin. Handle JB is attached at B about which it is free to spin. But JB and CAP roll against each other with engaged gear teeth. So if handle JB rotates counter-clockwise about B then CAP rotates clockwise about A.

Although the gear teeth are complex looking, there is always an *effective* contact point G between handle JB and blade CAP on the line segment AB. G is effectively a hinge between JB and CAP. You can think of the handle as a lever with force points at J, B and G. Thus blade CAP is closed by the force on the gear teeth at G. The shorter BG the bigger the forces at B and G.

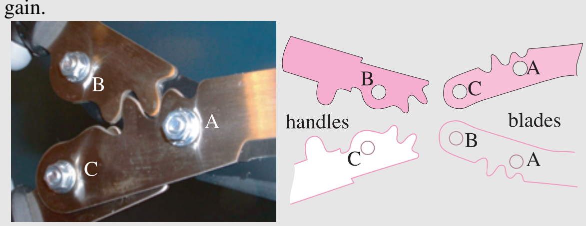
Simultaneously you could think of blade CAP as fixed with blade BAQ and handle KC hinged to it and geared to each other at G' (not shown). Thus blade CAP is also closed by an upwards force at C from handle KC. Similarly blade BAQ is closed by a downwards force at B from handle JB and a downwards force at G' from handle KC.

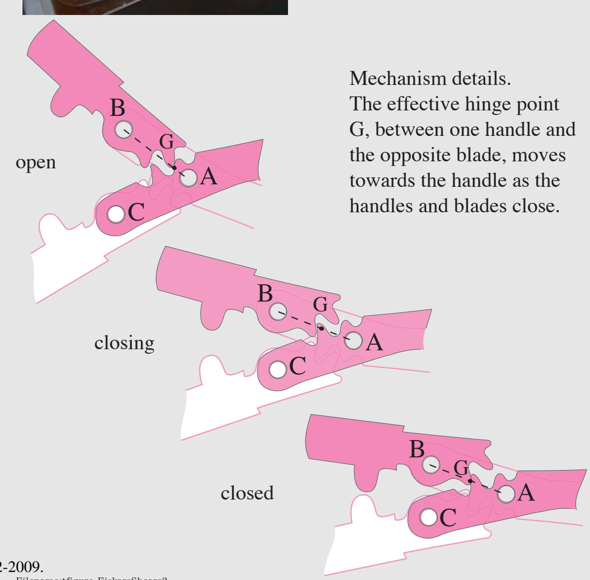
The effective hinges G and G' have locations which change as the blades close. When blades are wide open G and G' are near A. When the blades are closed G and G' have moved to about the midpoint between B and A and C and A, respectively.

If G was *at* A then this 4-piece design would be equivalent to a standard 2-piece cutter. Because BG is shorter than BA this design gives a bigger downwards force at B.

The shape of the geared curves makes the distance BG decrease, and the distance AG increase, as the blades close. Thus for given forces acting at J and Q, as the blades close the force at B increases, the force at G increases, and the lever-arm AG increases. These three effects partially compensate for the standard scissors problem, the decreasing mechanical advantage from the distance AP increasing as the blades close.

Another way to see the mechanical advantage of this design compared to the 2-piece design is to see that during a cut the handle angle decrease is greater than the blade angle decrease. Following the general rule for mechanisms, a motion attenuation is a force gain.





Mechanism details.
The effective hinge point G, between one handle and the opposite blade, moves towards the handle as the handles and blades close.

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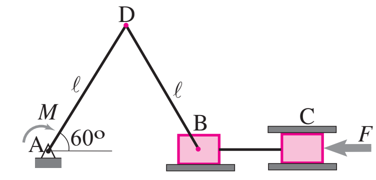

Figure 6.58:

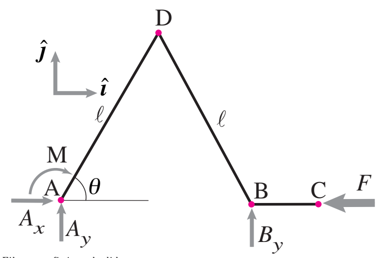

Figure 6.59:

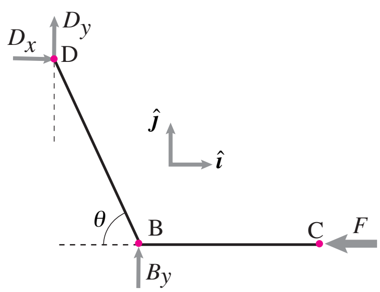

Figure 6.60:

SAMPLE 6.13 A slider crank: A torque $M = 20\,\text{N·m}$ is applied at the bearing end A of the crank AD of length $\ell = 0.2\,\text{m}$. If the mechanism is in static equilibrium in the configuration shown, find the load F on the piston.

Solution The free-body diagram of the whole mechanism is shown in Fig. 6.57. From the moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, we get

$$\begin{aligned}
\vec{M} + \vec{r}_{B/A} \times (\vec{B} + \vec{F}) &= \vec{0} \\
-M\hat{k} + 2\ell\cos\theta\hat{\imath} \times (B_y\hat{\jmath} - F\hat{\imath}) &= \vec{0} \\
(-M + 2B_y\ell\cos\theta)\hat{k} &= \vec{0} \\
\Rightarrow \quad B_y &= \frac{M}{2\ell\cos\theta}.
\end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned}
(A_x - F)\hat{\imath} + (A_y + B_y)\hat{\jmath} &= 0 \\
A_x &= F \\
A_y &= -B_y
\end{aligned}$$

Note that we still need to find F or A_x. So far, we have had only three equations in four unknowns (A_x, A_y, B_y, F). To solve for the unknowns, we need one more equation. We now consider the free-body diagram of the mechanism without the crank, that is, the connecting rod DB and the piston BC together. See Fig. 6.58. Unfortunately, we introduce two more unknowns (the reactions) at D. However, we do not care about them. Therefore, we can write the moment equilibrium equation about point D, $\sum \vec{M}_D = \vec{0}$ and get the required equation without involving D_x and D_y.

$$\begin{aligned}
\vec{r}_{B/D} \times (-F\hat{\imath} + B_y\hat{\jmath}) &= \vec{0} \\
\ell(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \times (-F\hat{\imath} + B_y\hat{\jmath}) &= \vec{0} \\
B_y\ell\cos\theta\hat{k} - F\ell\sin\theta\hat{k} &= \vec{0}.
\end{aligned}$$

Dotting the last equation with $\hat{k}$ we get

$$\begin{aligned}
F &= B_y\frac{\cos\theta}{\sin\theta} \\
&= \frac{M}{2\ell\cos\theta}\cdot\frac{\cos\theta}{\sin\theta} \\
&= \frac{M}{2\ell\sin\theta} \\
&= \frac{20\,\text{N·m}}{2\cdot 0.2\,\text{m}\cdot\sqrt{3}/2} \\
&= 57.74\,\text{N}.
\end{aligned}$$

$F = 57.74\,\text{N}$

Note that the force equilibrium carried out above is not really useful since we are not interested in finding the reactions at A. We did it above to show that just one free-body diagram of the whole mechanism was not sufficient to find F. On the other hand, writing moment equations about A for the whole mechanism and about D for the connecting rod plus the piston is enough to determine F.

SAMPLE 6.14 A flyball governor: A flyball governor is shown in the figure with all relevant masses and dimensions. The relaxed length of the spring is 0.15 m and its stiffness is 500 N/m.

1. Find the static equilibrium position of the center collar.
2. Find the force in the strut AB or CD.
3. How does the spring force required to hold the collar depend on θ?

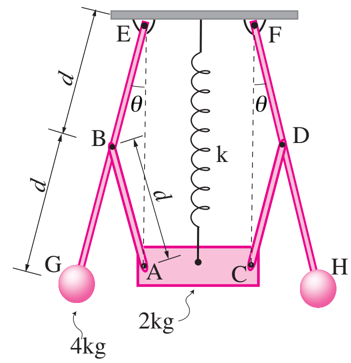

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Figure 6.61:

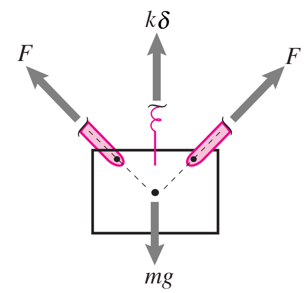

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Figure 6.62:

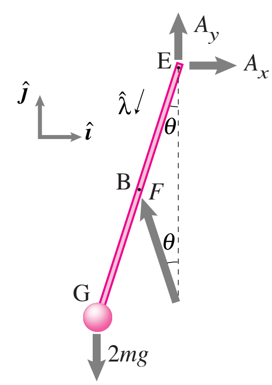

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Figure 6.63:

Solution Let $\ell_0 (= 0.15\,\mathrm{m})$ denote the relaxed length of the spring and let ℓ be the stretched length in the static equilibrium configuration of the flyball, *i.e.,* the collar is at a distance ℓ from the fixed support EF. Then the net stretch in the spring is $\delta \equiv \Delta\ell = \ell - \ell_0$. We need to determine ℓ, the spring force $k\delta$, and its dependence on the angle θ of the ball-arm.

The free-body diagram of the collar is shown in fig. 6.60. Note that the struts AB and CD are two-force bodies (forces act only at the two end points on each strut). Therefore, the force at each end must act along the strut. From geometry (AB = BE = d), then, the strut force F on the collar must act at angle θ from the vertical. Now, the force balance in the vertical direction, *i.e.,* $[\sum \vec{F} = \vec{0}] \cdot \hat{\jmath}$, gives

$$-2F\cos\theta + k\delta = mg. \tag{6.15}$$

Thus to find δ we need to find F and θ. Now we draw the free-body diagram of arm EBG as shown in fig. 6.61. From the moment balance about point E, we get

$$\begin{aligned}
\vec{r}_{G/E} \times (-2mg\hat{\jmath}) + \vec{r}_{B/E} \times \vec{F} &= \vec{0} \\
2d\hat{\boldsymbol{\lambda}} \times (-2mg\hat{\jmath}) + d\hat{\boldsymbol{\lambda}} \times F(-sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) &= \vec{0} \\
-4mgd(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{\jmath}}_{-\sin\theta\hat{\boldsymbol{k}}}) + Fd[-\sin\theta(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{\imath}}_{\cos\theta\hat{\boldsymbol{k}}}) + \cos\theta(\underbrace{\hat{\boldsymbol{\lambda}} \times \hat{\jmath}}_{-\sin\theta\hat{\boldsymbol{k}}})] &= \vec{0} \\
4mgd\sin\theta\hat{\boldsymbol{k}} + Fd(-\sin\theta\cos\theta\hat{\boldsymbol{k}} - \cos\theta\sin\theta\hat{\boldsymbol{k}}) &= \vec{0} \\
(4mgd\sin\theta - 2Fd\sin\theta\cos\theta)\hat{\boldsymbol{k}} &= \vec{0}.
\end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{k}}$ and assuming that $\theta \neq 0$, we get

$$2F\cos\theta = 4mg. \tag{6.16}$$

Substituting eqn. (6.16) in eqn. (6.15) we get

$$\begin{aligned}
k\delta &= mg + 2F\cos\theta = mg + 4mg = 5mg \\
\Rightarrow \quad \delta &= \frac{5mg}{k} = \frac{5 \cdot 2\,\mathrm{kg} \cdot 9.81\,\mathrm{m/s^2}}{500\,\mathrm{N/m}} = 0.196\,\mathrm{m}.
\end{aligned}$$

1. The equilibrium configuration is specified by the stretched length ℓ of the spring (which specifies θ). Thus,

$$\ell = \ell_0 + \delta = 0.15\,\mathrm{m} + 0.196\,\mathrm{m} = 0.346\,\mathrm{m}.$$

   Now, from $\ell = 2d\cos\theta$, we find that $\theta = 30.12°$.

2. The force in strut AB (or CD) is

$$F = 2mg/\cos\theta = 45.36\,\mathrm{N}.$$

3. The force in the spring $k\delta = 5mg$ as shown above and thus, it does not depend on θ! In fact, the angle θ is determined by the relaxed length of the spring.

(a) $\ell = 0.346\,\mathrm{m}$, (b) $F = 45.36\,\mathrm{N}$, (c) $k\delta \neq f(\theta)$

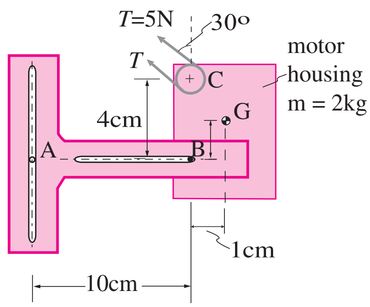

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Figure 6.64:

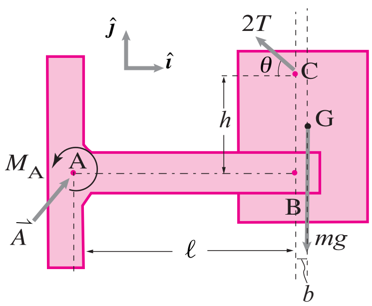

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Figure 6.65:

SAMPLE 6.15 : A motor housing support: A slotted arm mechanism is used to support a motor housing that has a belt drive as shown in the figure. The motor housing is bolted to the arm at B and the arm is bolted to a solid support at A. The two bolts are tightened enough to be modeled as welded joints (*i.e.,* they can also take some torque). Find the support reactions at A.

Solution Although the mechanism looks complicated, the problem is straightforward. We cut the bolt at A and draw the free-body diagram of the motor housing plus the slotted arm. Since the bolt, modeled as a welded joint, can take some torque, the unknowns at A are $\vec{A}(= A_x\hat{\imath} + A_y\hat{\jmath})$ and $\vec{M}_A$. The free-body diagram is shown in Fig. 6.63. Note that we have replaced the tension at the two belt ends by a single equivalent tension $2T$ acting at the center of the axle. Now taking moments about point A, we get

$$\vec{M}_A + \vec{r}_{C/A} \times 2\vec{T} + \vec{r}_{G/A} \times m\vec{g} = \vec{0}$$

where

$$\begin{aligned}
\vec{r}_{C/A} \times 2\vec{T} &= (\ell\hat{\imath} + h\hat{\jmath}) \times 2T(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
&= 2T(\ell\sin\theta + h\cos\theta)\hat{k} \\
\vec{r}_{G/A} \times m\vec{g} &= [(\ell + d)\hat{\imath} + (\text{anything})\hat{\jmath}] \times (-mg\hat{\jmath}) \\
&= -mg(\ell + d)\hat{k}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{M}_A &= -\vec{r}_{C/A} \times 2\vec{T} - \vec{r}_{G/A} \times m\vec{g} \\
&= -2T(\ell\sin\theta + h\cos\theta)\hat{k} + mg(\ell + d)\hat{k} \\
&= -2(5\,\mathrm{N})(0.1\,\mathrm{m}\cdot\sin 60^\circ + 0.04\,\mathrm{m}\cdot\cos 60^\circ)\hat{k} \\
&\quad + 2\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot(0.1 + 0.01)\,\mathrm{m}\hat{k} \\
&= 1.092\,\mathrm{N{\cdot}m}\hat{k}.
\end{aligned}$$

The reaction force $\vec{A}$ can be determined from the force balance, $\sum\vec{F} = \vec{0}$ as follows.

$$\begin{aligned}
\vec{A} + 2\vec{T} + m\vec{g} &= \vec{0} \\
\Rightarrow \quad \vec{A} &= -2\vec{T} - m\vec{g} \\
&= -10\,\mathrm{N}(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}) - (-19.62\,\mathrm{N}\hat{\jmath}) \\
&= 5\,\mathrm{N}\hat{\imath} + 10.96\,\mathrm{N}\hat{\jmath}.
\end{aligned}$$

$$\vec{M}_A = 1.092\,\mathrm{N{\cdot}m}\hat{k} \qquad \text{and} \qquad \vec{A} = 5\,\mathrm{N}\hat{\imath} + 10.96\,\mathrm{N}\hat{\jmath}$$

SAMPLE 6.16 Push-up mechanics: During push-ups, the body including the legs, usually moves as a single rigid unit; the ankle is almost locked, and the push-up is powered by the shoulder and the elbow muscles. A simple model of the body during push-ups is a four-bar linkage ABCDE shown in the figure. In this model, each link is a rigid rod, joint B is rigid (thus ABC can be taken as a single rigid rod), joints C, D, and E are hinges, but there is a motor at D that can supply torque. The weight of the person, $W = 150\,\text{lbf}$, acts through G. Find the torque at D for $\theta_1 = 30°$ and $\theta_2 = 45°$.

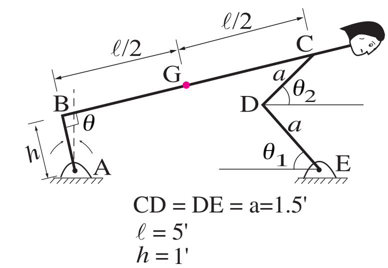

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Figure 6.66:

Solution The free-body diagram of part ABC of the mechanism is shown in Fig. 6.65. Writing moment balance equation about point A, $\sum \vec{M}_A = \vec{0}$, we get

$$\vec{r}_C \times \vec{C} + \vec{r}_G \times \vec{W} = \vec{0}.$$

Let $\vec{r}_C = r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}$ and $\vec{r}_G = r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}$ for now (we can figure it out later). Then, the moment equation becomes

$$(r_{C_x}\hat{\imath} + r_{C_y}\hat{\jmath}) \times (C_x\hat{\imath} + C_y\hat{\jmath}) + (r_{G_x}\hat{\imath} + r_{G_y}\hat{\jmath}) \times (-W\hat{\jmath}) = \vec{0}$$

$$[(C_y r_{C_x} - C_x r_{C_y})\hat{k} - W r_{G_x}\hat{k} = \vec{0}]$$

$$[\,]\cdot\hat{k} \quad \Rightarrow \quad C_y r_{C_x} - C_x r_{C_y} = W r_{G_x}. \tag{6.17}$$

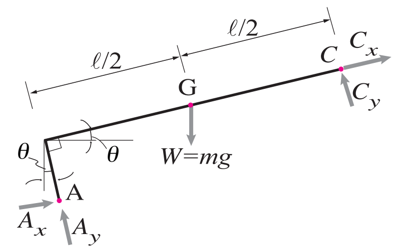

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Figure 6.67:

We now draw free-body diagrams of the links CD and DE separately (Fig. 6.66) and write the moment and force balance equations for them.

For link CD, the force equilibrium $\sum \vec{F} = \vec{0}$ gives

$$(-C_x + D_x)\hat{\imath} + (D_y - C_y)\hat{\jmath} = \vec{0}.$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$ gives

$$\begin{aligned} D_x &= C_x \\ D_y &= C_y \end{aligned} \tag{6.18}$$

and the moment equilibrium about point D, gives

$$\begin{aligned} M\hat{k} - a(\cos\theta_2\hat{\imath} + \sin\theta_2\hat{\jmath}) \times (-C_x\hat{\imath} - C_y\hat{\jmath}) &= \vec{0} \\ M\hat{k} + (C_y a\cos\theta_2 - C_x a\sin\theta_2)\hat{k} &= \vec{0}. \end{aligned} \tag{6.19}$$

Similarly, the force equilibrium for link DE requires that

$$\begin{aligned} E_x &= D_x \\ E_y &= D_y \end{aligned} \tag{6.20}$$

and the moment equilibrium of link DE about point E gives

$$-M + D_x a\sin\theta_1 + D_y a\cos\theta_1 = 0. \tag{6.21}$$

Now, from eqns. (6.18) and (6.21)

$$-M + C_x a\sin\theta_1 + C_y a\cos\theta_1 = 0. \tag{6.22}$$

Adding eqns. (6.19) and (6.22) and solving for C_x we get

$$C_x = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1} C_y.$$

For simplicity, let

$$f(\theta_1, \theta_2) = \frac{\cos\theta_1 + \cos\theta_1}{\sin\theta_2 - \sin\theta_1}$$

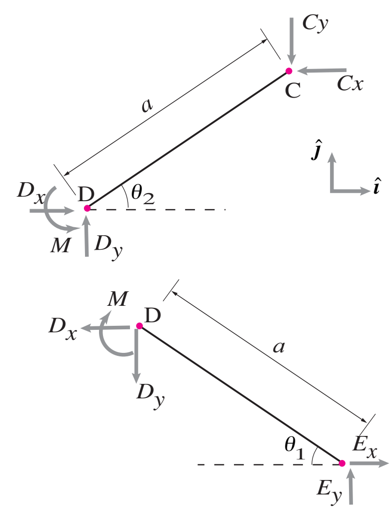

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Figure 6.68:

so that

$$C_x = f(\theta_1, \theta_2) C_y. \tag{6.23}$$

Now substituting eqn. (6.23) in (6.17) we get

$$C_y = \frac{r_{G_x}}{r_{C_x} - r_{C_y} f} W.$$

Now substituting C_y and C_x into eqn. (6.22) we get

$$M = \frac{r_{G_x} a(\cos\theta_1 + f \sin\theta_1)}{r_{C_x} - r_{C_y} f} W$$

where

$$\begin{aligned} r_{G_x} &= (\ell/2)\cos\theta - h\sin\theta \\ r_{C_x} &= \ell\cos\theta - h\sin\theta \\ r_{C_y} &= \ell\sin\theta + h\cos\theta. \end{aligned}$$

Now plugging all the given values: $W = 160\,\text{lbf}, \theta_1 = 30°, \theta_2 = 45°, \ell = 5\,\text{ft}, h = 1\,\text{ft}, a = 1.5\,\text{ft}$, and, from simple geometry, $\theta = 9.49°$,

$$\begin{aligned} f &= 7.60 \\ r_{C_x} &= 4.77\,\text{ft}, \quad r_{C_y} = 1.81\,\text{ft}, \quad r_{G_x} = 2.30\,\text{ft} \\ \Rightarrow \quad M &= -269.12\,\text{lb}\cdot\text{ft}. \end{aligned}$$

$$M = -269.12\,\text{lb}\cdot\text{ft}$$

SAMPLE 6.17 A spring and rod buckling model: A simple model of sideways buckling of a flexible (elastic) rod can be constructed with a spring and a *rigid* rod as shown in the figure. Assume the rod to be in static equilibrium at some angle θ from the vertical. Find the angle θ for a given vertical load P, spring stiffness k, and bar length ℓ. Assume that the spring is relaxed when the rod is vertical.

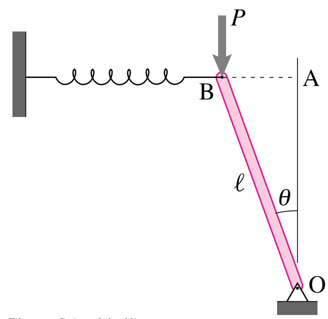


Figure 6.69:

Solution When the rod is displaced from its vertical position, the spring gets compressed or stretched depending on which side the rod tilts. The spring then exerts a force on the rod in the opposite direction of the tilt. The free-body diagram of the rod with a counterclockwise tilt θ is shown in Fig. 6.68. From the moment balance $\sum \vec{M}_O = \vec{0}$ (about the bottom support point O of the rod), we have

$$\vec{r}_B \times \vec{P} + \vec{r}_B \times \vec{F}_s = \vec{0}.$$

Noting that

$$\begin{aligned} \vec{r}_B &= \ell\hat{\boldsymbol{\lambda}}, \\ \vec{P} &= -P\hat{\boldsymbol{\jmath}}, \\ \text{and} \quad \vec{F}_s &= k(\vec{r}_A - \vec{r}_B) \\ &= k(\ell\hat{\boldsymbol{\jmath}} - \ell\hat{\boldsymbol{\lambda}}), \end{aligned}$$

we get

$$\begin{aligned} \ell\hat{\boldsymbol{\lambda}} \times (P\hat{\boldsymbol{\jmath}}) + \ell\hat{\boldsymbol{\lambda}} \times k\ell(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\lambda}}) &= \vec{0} \\ -P\ell(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) + k\ell^2(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}}) &= \vec{0}. \end{aligned}$$

Dotting this equation with $(\hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\jmath}})$ we get

$$\begin{aligned} -P\ell + k\ell^2 &= 0 \\ \Rightarrow \quad P &= k\ell. \end{aligned}$$

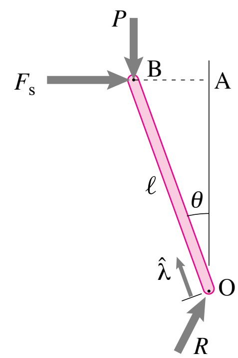


Figure 6.70:

Thus the equilibrium only requires that P be equal to $k\ell$ and it is independent of θ! That is, the system will be in static equilibrium at any θ as long as $P = k\ell$.

If $P = k\ell$, any θ is an equilibrium position.

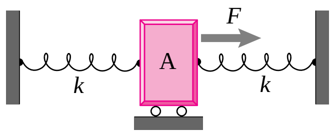
Problems for Chapter 6

Machines

6.1 Springs

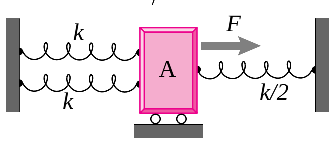
Preparatory Problems

6.1.1 Find the force F required to push the massless block by 1 cm to the right if $k = 500\,\text{N/m}$. *



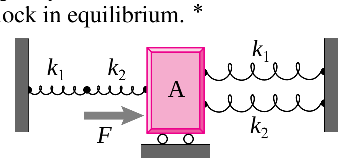
Filename:pfig-springs-2p
Problem 6.1.1

6.1.2 A force $F = 20\,\text{N}$ is applied on the massless block shown in the figure. Find the displacement of the block for equilibrium if $k = 100\,\text{N/cm}$. *



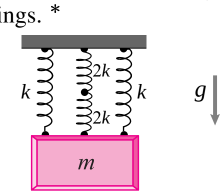
Filename:pfig-springs-3p
Problem 6.1.2

6.1.3 A network of relaxed springs holds a massless block as shown in the figure where $k_1 = 100\,\text{N/cm}$ and $k_2 = 400\,\text{N/cm}$. If the block is pushed to the right by 2 cm, find the force F to hold the block in equilibrium. *



Filename:pfig-springs-3p1s
Problem 6.1.3

6.1.4 A block of mass $m = 300\,\text{kg}$ hangs from the ceiling with the help of a network of springs in series and parallel as shown. Taking $k = 20\,\text{kN/m}$ and $g = 10\,\text{m/s}^2$, find the stretch in the two side (the left and right) springs. *



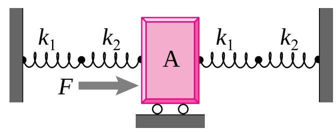
Filename:pfig-springs-3p1sv
Problem 6.1.4

6.1.5 For the arrangement of springs shown in the figure, $k_1 = 50\,\text{N/cm}$ and $k_2 = 100\,\text{N/cm}$. Find

a) the equivalent spring stiffness of the arrangement,

b) the displacement of the block if a force $F = 50\,\text{N}$ acts on the block.

*

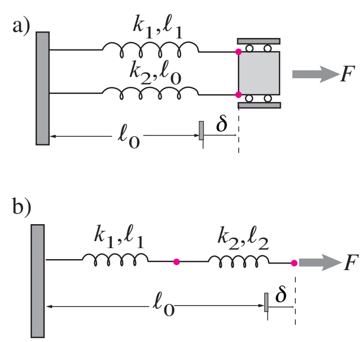


Filename:pfig-springs-2p2s
Problem 6.1.5

6.1.6 Find F in terms of some or all of k_1,ℓ_1, k_2, ℓ_2, ℓ_0 and δ. Note that F is generally not zero even if δ is zero.

a) Springs in parallel.

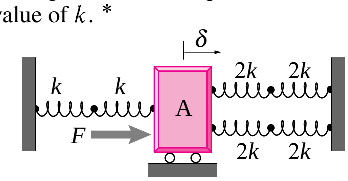
b) Springs in series.



Filename:pfigure-spring2
Problem 6.1.6

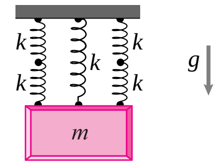
More-Involved Problems

6.1.7 A massless block is held in position by a network of springs shown in the figure. If the blcok is displaced to the right by 1 cm from the relaxed position of the springs, a force of $F = 50\,\text{N}$ is required to keep the block in equilibrium. Find the value of k. *



Filename:pfig-springs-3p3s
Problem 6.1.7

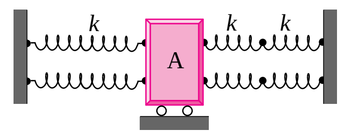
6.1.8 A box weighing 1000 N is hung from the ceiling using a network of springs, each with stiffness $k = 500\,\text{N/cm}$. Find the stretch in each spring. *



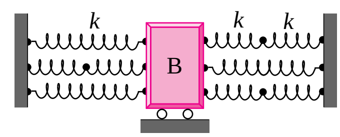
Filename:pfig-springs-3p2sv
Problem 6.1.8

6.1.9 For the netwrok of springs shown below, find the stiffness and strength of each netwrok if the stiffness and strength of individual springs are $k = 10\,\text{kN/m}$ and $T_0 = 2\,\text{kN}$, respectively.

(a)

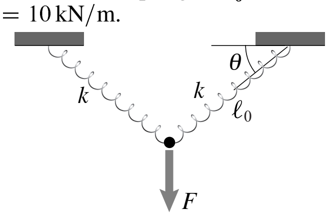


(b)



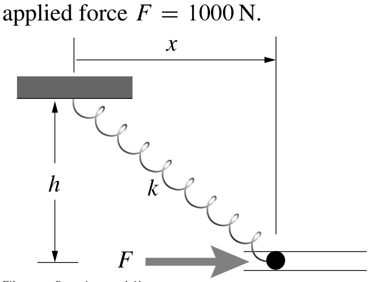
Filename:pfig-springs-strstf
Problem 6.1.9

6.1.10 Find the stretch in each spring to hold the pin in equilibrium for $F = 10\,\text{kN}$ if the relaxed length (in the horizontal position) of each spring is $\ell_0 = 15\,\text{cm}$ and $k = 10\,\text{kN/m}$.



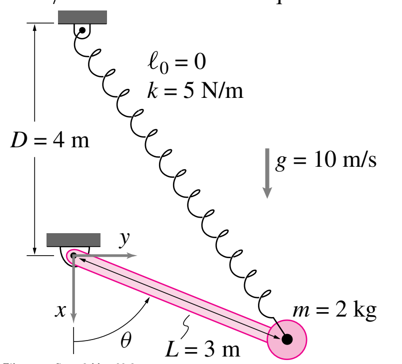
Filename:pfig-springs-angle2p
Problem 6.1.10

6.1.11 A pin is held in a horizontal track with a zero-length spring ($\ell_0 = 0$) of stiffness 50 kN/m. Find the horizontal position x of the pin if it is in equilibrium with an applied force $F = 1000\,\text{N}$.



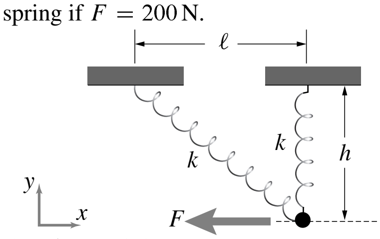
Filename:pfig-springs-angle1h
Problem 6.1.11

6.1.12 A zero length spring (relaxed length $\ell_0 = 0$) with stiffness $k = 5\,\mathrm{N/m}$ supports the pendulum shown. Assume $g = 10\ \mathrm{N/m}$. Find θ for static equilibrium. *



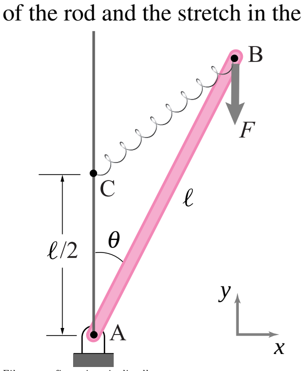
Filename:pfigure2-blue-80-2-a
Problem 6.1.12

6.1.13 In the figure shown, the two springs with $k_1 = 50\,\mathrm{N/cm}$ and $k_2 = 100\,\mathrm{N/cm}$ are in relaxed position when $h = 30\,\mathrm{cm}$ and $\ell = 40\,\mathrm{cm}$ (and, of course, $F = 0$). Find the position of the pin on the horizontal track and change in length of each spring if $F = 200\,\mathrm{N}$.



Filename:pfig-springs-angle2ph
Problem 6.1.13

6.1.14 In the mechanism shown, the relaxed length of the spring is $\ell/2$ and the length of the bar AB is $\ell = 2\,\mathrm{m}$. For $F = 500\,\mathrm{N}$, find the equilibrium angle θ of the rod and the stretch in the spring.

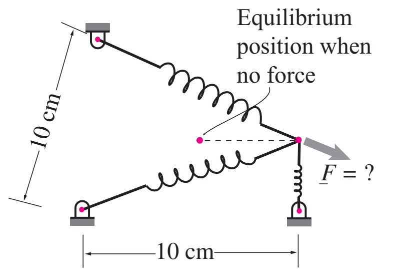


Filename:pfig-springs-inclinedbar
Problem 6.1.14

6.1.15 The ends of three identical springs are rooted at the corners of a 10 cm equilateral triangle with base that is in the $\hat{\imath}$ direction. Find the force $\vec{F}$ needed to hold the ends of the springs 5 cm to the right of the triangle center if

a) $\ell_0 = 0$, $k = 10\,\mathrm{N/\,cm}$?

b) $\ell_0 = 10/\sqrt{3}\,\mathrm{cm}$, $k = 10\,\mathrm{N/\,cm}$?

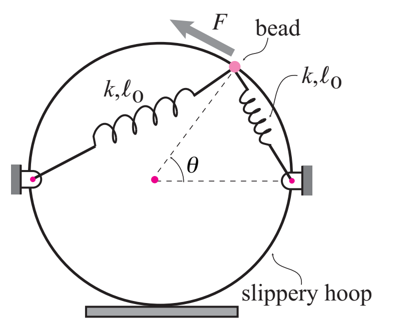


Filename:pfigure-threesprings
Problem 6.1.15

6.1.16

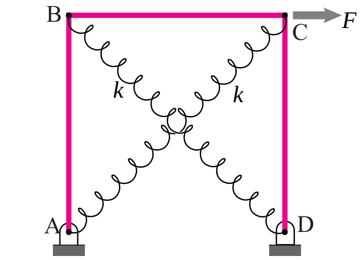
a) in terms of some or all of k, ℓ_0,r, and θ find F. The hoop is rigid, round and frictionless and the force is tangent to the hoop.

b) How does the answer above simplify in the special case that $\ell_0 = 0$? [You can do this by simplifying the expression above, or by doing the problem from scratch assuming $\ell_0 = 0$. In the latter case, an answer can be generated quickly if vector methods are used.]



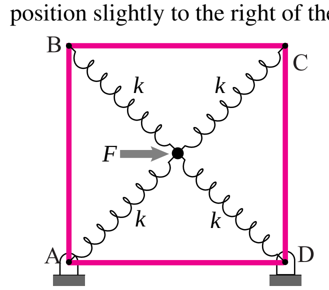
Filename:pfigure-spring4
Problem 6.1.16

6.1.17 The square box mechanism shown consists of three identical bars and two identical diagonal springs in their relaxed configuration. Each bar is 0.4 m long. A horizontal force $F = 100\,\mathrm{N}$ acts at C. Find the change in length of each spring if $k = 10\,\mathrm{kN/m}$.



Filename:pfig-springs-kinematics1
Problem 6.1.17

6.1.18 In the mechanism shown, the pin is held in the center of the square frame of side 1 m with relaxed springs of stiffness $k = 5\,\mathrm{kN/m}$ in the absence of any force. Find the change in length of each spring when an applied horizontal force $F = 50\,\mathrm{N}$ keeps the pin in equilibrium at a position slightly to the right of the center.



Filename:pfig-springs-kinematics2
Problem 6.1.18

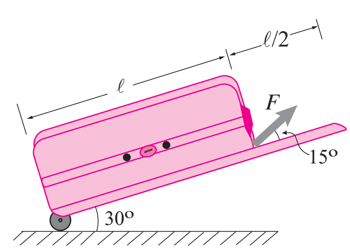
6.2 Levers, Wedges, Toggles, Gears and Pulleys

6.2.1 A suitcase of length $\ell = 0.5\,\mathrm{m}$ is pulled along steadily with a force $F = 100\,\mathrm{N}$ as shown in the figure.

a) Find the weight W of the suitcase.

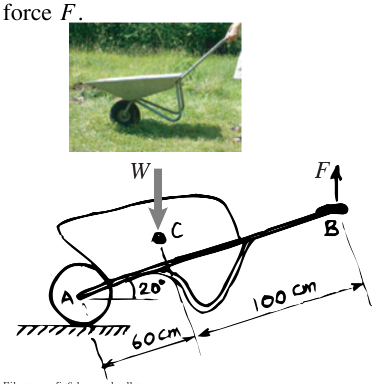
b) Find the ground force on the wheel (both magnitude and direction).

c) What is force amplification if you consider F as the input and W as the output.



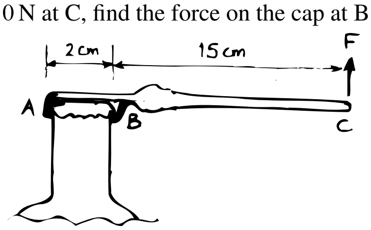
Filename:pfig6-lever-suitcase
Problem 6.2.1

6.2.2 A wheelbarrow containing 100 kg of this-n-that is wheeled steadily with a force F as shown in the figure. For the given geometry and $g \approx 10\,\mathrm{m/s^2}$, find the required force F.



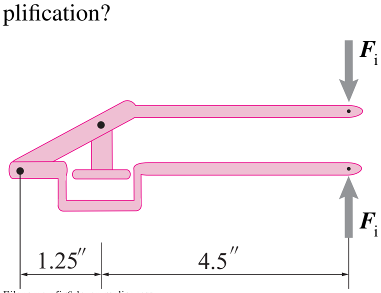
Filename:pfig6-lever-wheelbarrow
Problem 6.2.2

6.2.3 A bottle-opener ABC contains a cut-out AB of approximate diameter 2 cm that clamps on the bottle cap. The arm BC is approximately 15 cm long. If the cap is opened by applying a vertical force $F = 10\,\mathrm{N}$ at C, find the force on the cap at B.



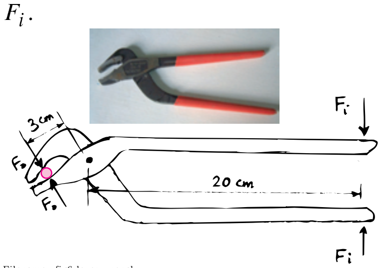
Filename:pfig6-lever-bottleopener
Problem 6.2.3

6.2.4 A cut-out view of a garlic press is shown in the figure. For an input force $F_i = 10\,\mathrm{lbf}$, find the output force F_o at the site of the press. What is the force amplification?



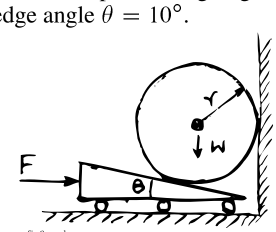
Filename:pfig6-lever-garlicpress
Problem 6.2.4

6.2.5 A simple wrench is shown in the figure along with the relevant dimensions. If the torque required on the approximately circular bolt of diameter 1 cm is 2 N·m and the coefficient of friction between the bolt and wrench is $\mu = 0.2$, find the input force F_i.



Filename:pfig6-lever-wrench
Problem 6.2.5

6.2.6 Assuming all frictionles contacts, find the force F on the wedge required to lift the the sphere weighing 500 N if the wedge angle $\theta = 10°$.

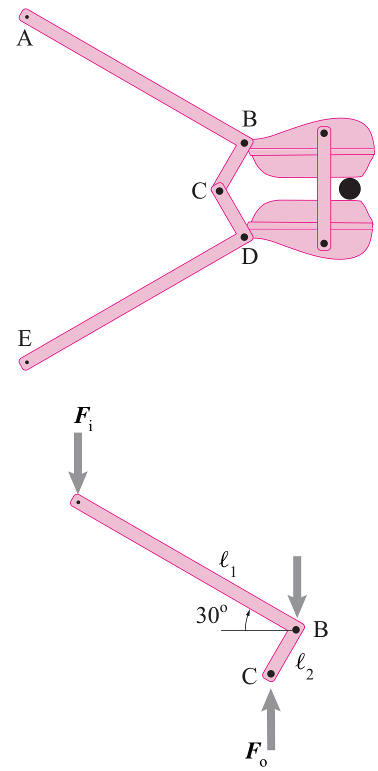


Filename:pfig6-wedge
Problem 6.2.6

6.2.7 A cutter, shown in the figure, uses a toggle mechanism BCD to get a big force amplification at the cutting edge. A partial free-body diagram of one of the arms of the cutter is shown in the figure. Assuming an input force of $F_i = 20\,\mathrm{N}$ at A, find the intermediate output force F_o at C when

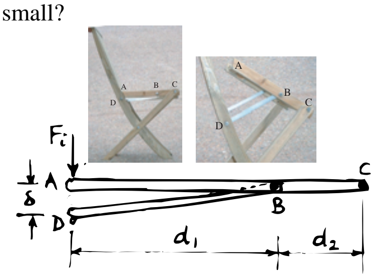
a) $\theta = 30°$,

b) $\theta = 10°$.



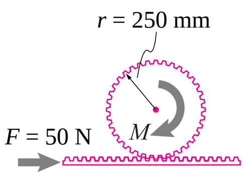
Filename:pfig6-toggle-cutter
Problem 6.2.7

6.2.8 A toggle-like mechanism is used in a folding chair shown in the pictures here. The metallic link DB gets almost parallel to the seating plank AC when the chair is open. Given the dimensions $d_1 = 30\,\mathrm{cm}$, $d_2 = 10\,\mathrm{cm}$, $\delta = 2\,\mathrm{cm}$ and the force at A, $F_i = 500\,\mathrm{N}$, find the tension in the link DB. Why is this force so big or small?



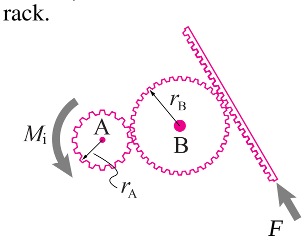
Filename:pfig6-toggle-chair
Problem 6.2.8

6.2.9 A gear of radius 250 mm is meshed in with a rack that carries a horizontal load $F = 50\,\mathrm{N}$. Find the torque M on the gear that is require for equilibrium.



Filename:pfig6-gears1
Problem 6.2.9

6.2.10 The input gear A of radius r_A = 10 cm drives gear B that is one and a half times bigger than gear A. Gear B, in turn, drives a rack. If the input torque on gear A is M_i = 30 N·m, find the load F on the rack.

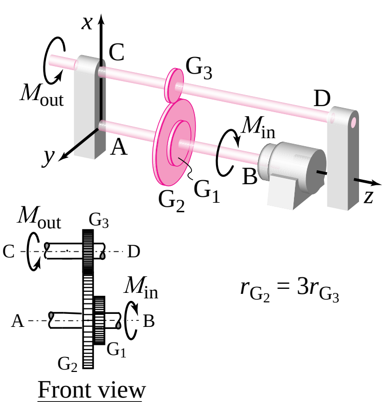


Filename:pfig6-gears2
Problem 6.2.10

6.2.11 In the gear arrangement shown, gears G_1 and G_2 are welded together. The output gear G_3 is one third the size of gear G_2.

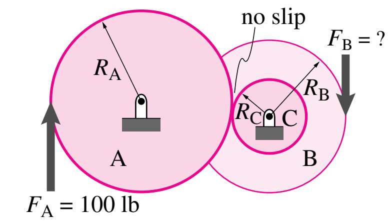
a) Is this gear train for torque amplification or for torque reduction?

b) If the input torque M_{in} on gear G_1 is 300 N·m, find the output torque M_{out}.



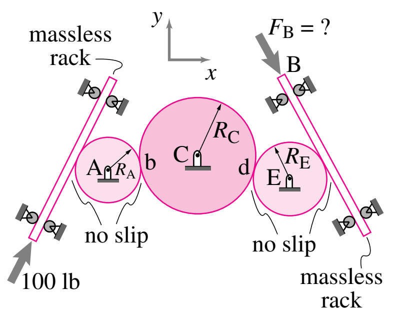
Filename:pfig6-gears3
Problem 6.2.11

6.2.12 At the input to a gear box, a 100 lbf force is applied to gear A. At the output, the machinery (not shown) applies a force of F_B to the output gear. Assume the system of gears is at rest. What is F_B?



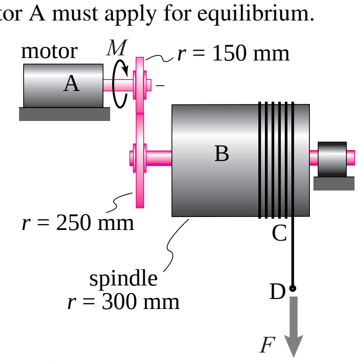
Filename:pg131-3-a
Problem 6.2.12

6.2.13 A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of F_B to the other rack. Assume the gear-train is at rest. What is F_B?



Filename:ch4-5-a
Problem 6.2.13

6.2.14 The gear train and spindle shown in the figure are used for hoisting heavy loads. For the dimensions given, if the load F = 2 kN, find the torque M that the motor A must apply for equilibrium.



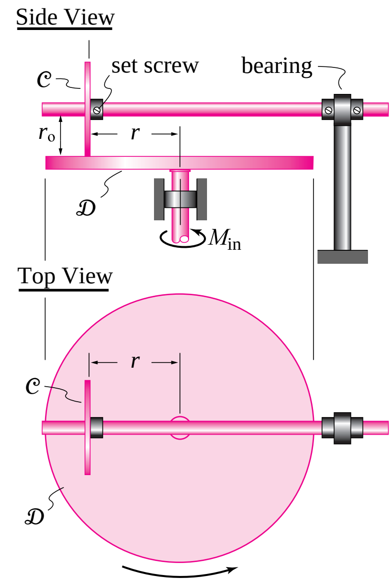
Filename:pfig6-gears4
Problem 6.2.14

6.2.15 The figure shows a brush gear (also called a crown wheel) where wheel $\mathcal{C}$ of radius r_o, rolls on the surface of wheel $\mathcal{D}$ without slipping. In addition, the position r of wheel $\mathcal{C}$ from the center of wheel D can be varied. Let the input torque on wheel $\mathcal{D}$ be M_i.

a) Find the output torque M_o on wheel $\mathcal{C}$ as a function of r.

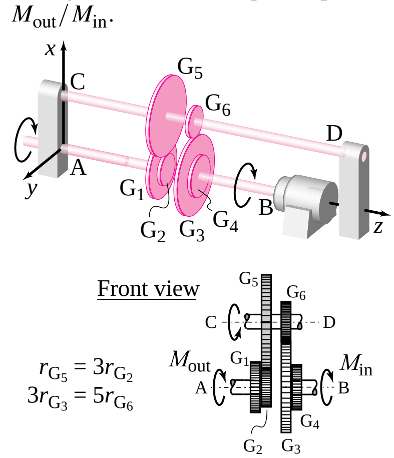
b) Find the output torque M_o when $r = r_o$ and when $r = r_o/4$.

c) If the output torque were not to exceed 100 times the input torque, where will you put safety latches on the axle of wheel $\mathcal{C}$?



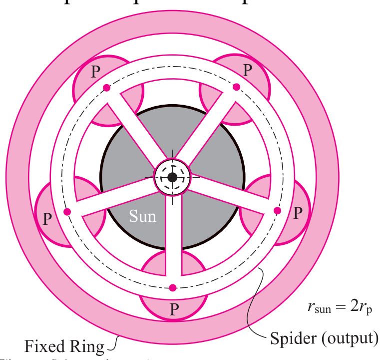
Filename:pfig6-gears5
Problem 6.2.15

6.2.16 For the gear train shown in the figure, find the torque amplification $M_{\text{out}}/M_{\text{in}}$.



Filename:pfig6-gears6
Problem 6.2.16

6.2.17 A torque amplifying planetary gear is shown in the figure where the sun-gear is free to rotate but the ring-gear is fixed. The sun-gear drives five planet-gears that drive the spider-gear through their axles housed in bearings in the spider. The radius of the planet-gears r_p = 50 mm and the radius of the sun-gear is twice as big. If the input torque on the sun-gear is 2000 N·m, find the output torque on the spider.

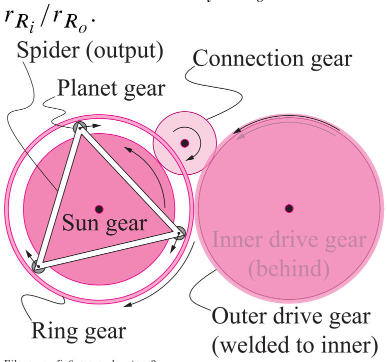


Filename:pfig6-gears-planetary1
Problem 6.2.17

6.2.18 Consider the high gear ratio planetary gear discussed on page 345 of the text. Let r_{D_i} and r_{D_o} be the inner and outer radii, respectively, of the drive gear, and r_{R_i} and r_{R_o} be the inner and outer radii, respectively, of the ring-gear. Let r_s and r_p denote the radii of the sun and the planet-gear respectively. SHow that the ratio of the output torque M_{out} on the spider-gear to the input torque M_{in} on the drive gear is approximately given by

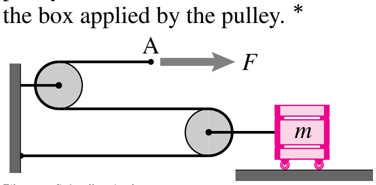
$$\frac{M_{\text{out}}}{M_{\text{in}}} \approx \frac{2}{\frac{R_D}{R_R} - 1}$$

where $R_D = r_{D_i}/r_{D_o}$ and $R_R = r_{R_i}/r_{R_o}$.



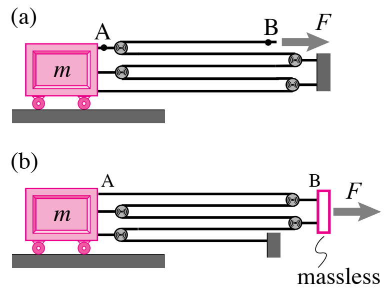
Filename:pfig6-gears-planetary2
Problem 6.2.18

6.2.19 A force $F = 100$ N acts at A. The pulleys are frictionless. Find the force on the box applied by the pulley. *



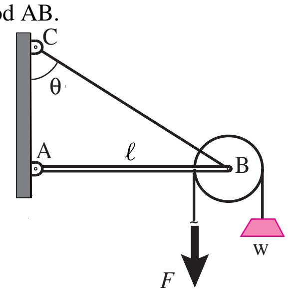
Filename:pfig6-pulley-simple
Problem 6.2.19

6.2.20 A force F is applied as shown in the pulley arrangements shown in (a) and (b). Which arrangement gives a bigger force amplification on the box?



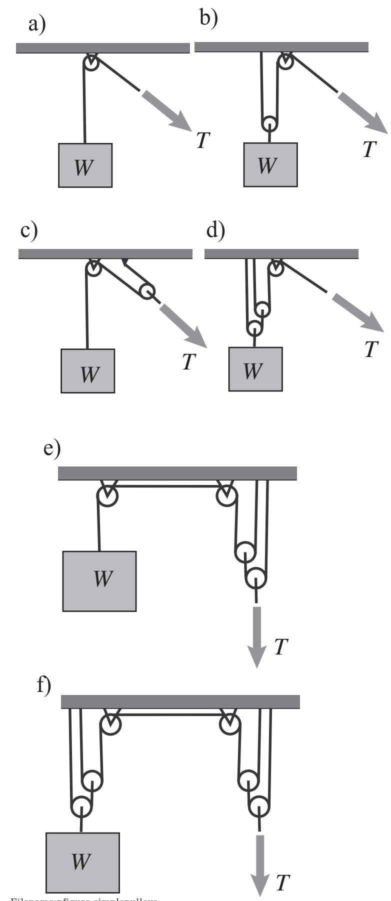
Filename:pfig6-pulley-friction
Problem 6.2.20

6.2.21 A weight W is held in place with a force $F = 100$ N applied through a massless pulley as shown in the figure. The pulley is attached to a rod AB which, in turn, is held horizontal with the help of a string CB. Find the tension (or compression) in rod AB.



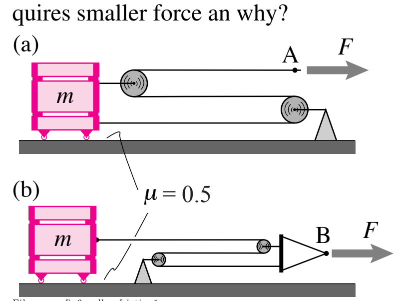
Filename:pfigure4-1-rp9
Problem 6.2.21

6.2.22 Given W and the frictionless pulleys shown find the tension T needed to lift the weight in the situations shown.



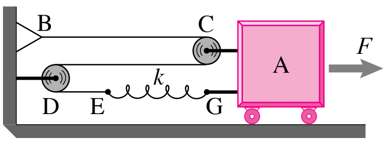
Filename:pfigure-simplepulleys
Problem 6.2.22

6.2.23 In the two cases shown in (a) and (b), find the maximum force F that can be applied before the box starts skidding on the ground. Take $m = 50$ kg and $g \approx 10\,\text{m/s}^2$. Which arrangement requires smaller force an why?



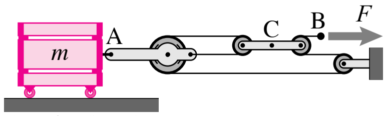
Filename:pfig6-pulley-friction1
Problem 6.2.23

6.2.24 The pulley arrangement shown in the figure uses a spring EG of stiffness $k = 200$ N/cm. If the spring is stretched by 1.5 cm under the application of force F for equilibrium, find F.



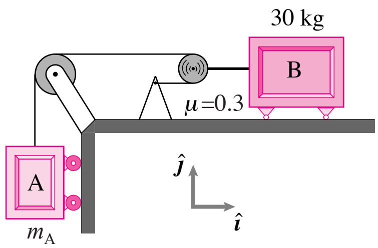
Filename:pfig6-pulley-spring
Problem 6.2.24

6.2.25 Find the force on the mass at A in terms of F and thus find the force amplification provided by the pulley arrangement used.



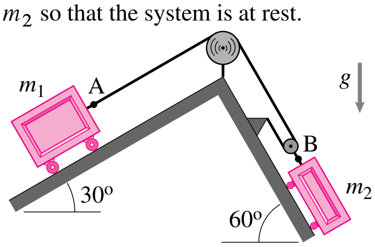
Filename:pfig6-pulley-complex
Problem 6.2.25

6.2.26 In the figure shown, there is no friction between block A and the vertical wall but there is friction ($\mu = 0.3$) between block B and the floor. If $m_B = 30$ kg, find the mass of block A for equilibrium.



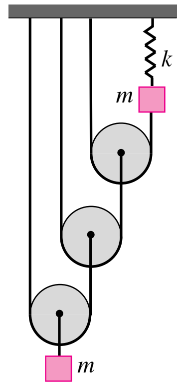
Filename:pfig6-pulley-gravity1
Problem 6.2.26

6.2.27 Find the ratio of the masses m_1 and m_2 so that the system is at rest.



Filename:pulley4-c
Problem 6.2.27

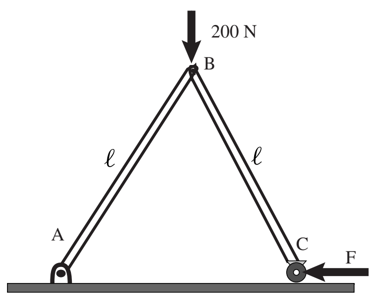
6.2.28 If the mass and pulley system shown in the figure is in equilibrium when the spring is stretched by 3 cm, find m, given $k = 500\,\text{N/m}$ and $g \approx 10\,\text{m/s}^2$.



Filename:pfig6-pulley-gravity3
Problem 6.2.28

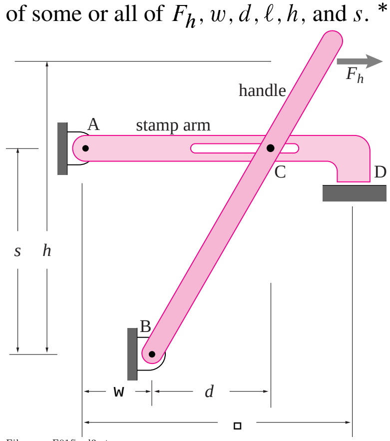
6.3 Machines

6.3.1 A simply supported two bar mechanism supports a load of 200 N at joint B with the help of a horizontal force F applied at joint C. Find F.



Filename:pfigure4-1-rp12
Problem 6.3.1

6.3.2 Pulling on the handle causes the stamp arm to press down at D. Neglect gravity and assume that the hinges at A and B, as well as the roller at C, are frictionless. Find the force N that the stamp machine causes on the support at D in terms of some or all of F_h, w, d, ℓ, h, and s. *



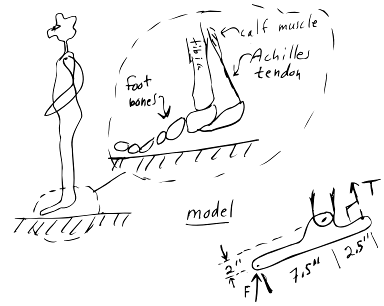
Filename:F01final3-stamp
Problem 6.3.2

More-Involved Problems

6.3.3 See Problem 4.2.17 on page 245. A person who weighs W stands on tiptoes on one foot. Assume the weight of the foot is negligible.

a) Draw a free-body diagram of the whole person and find the force of the ground on the foot front.

b) Draw a free body diagram of the foot and find the force of the calf on the foot at the ankle and the tension in the Achilles Tendon.

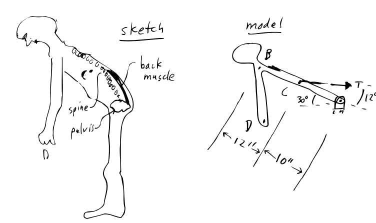


Filename:pfigureSoodak4-25
Problem 6.3.3

6.3.4 See Problem 4.2.17 on page 245. A person with weight $W = 140$ lbf has an upper body with weight $0.7W$ with center of mass at C. The back muscles are idealized as a single muscle with one end (the muscle *origin* also at C. Use the idealization and geometry shown.

a) Find the back-muscle tension and the force of the lower body on the upper body at the hips.

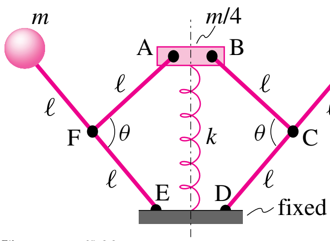
b) Repeat the problem but assume that the person is lifting a 30 lbf load at D.



Filename:pfigureSoodak4-26
Problem 6.3.4

6.3.5 In the flyball governor shown, the mass of each ball is $m = 5$ kg, and the length of each link is $\ell = 0.25$ m. There are frictionless hinges at points A, B, C, D, E, F where the links are connected. The central collar has mass $m/4$. Assuming that the spring of constant $k = 500$ N/m is uncompressed when $\theta =$

π radians, what is the compression of the spring?

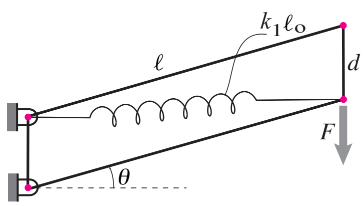


Filename:summer95p2-2-a
Problem 6.3.5

6.3.6

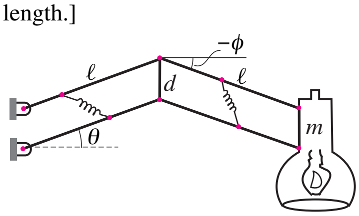
a) Find F for equilibrium for the parallelogram structure shown assuming the rest length of the spring is zero.

b) Comment on how your answer above depends on θ.



Filename:pfigure-spring5
Problem 6.3.6

6.3.7 A common lamp design is shown. In principle the lamp should be in equilibrium in all positions. According to the original patent from the 1930s it can be, even with no friction in the joints. Unfortunately, the recent manufacturers of this lamp seem to have lost the wisdom of the original patent. Show how to place what springs so this lamp is in equilibrium for all $\theta < \pi/2$ and $\phi < \pi/2$. [Hint: use springs with zero rest length.]

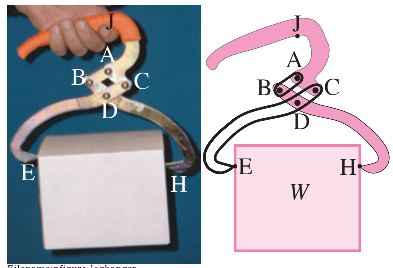


Filename:pfigure-spring6
Problem 6.3.7: Lamp.

6.3.8 Log carrier. This self-locking scissors-mechanism gadget is used to pick up logs and blocks of ice. The 5 cm wide and 3 $cm high$ diamond arrangement of hinges ABCD makes up a 4-bar linkage. The grips E and H are 16 $cm apart$ and 9 cm below D. The block weighs 250 N. Neglect the weight of the mechanism.

a) What is the horizontal component of the force on the block at E?

b) What is the minimum coefficient of friction μ for which this device self locks?



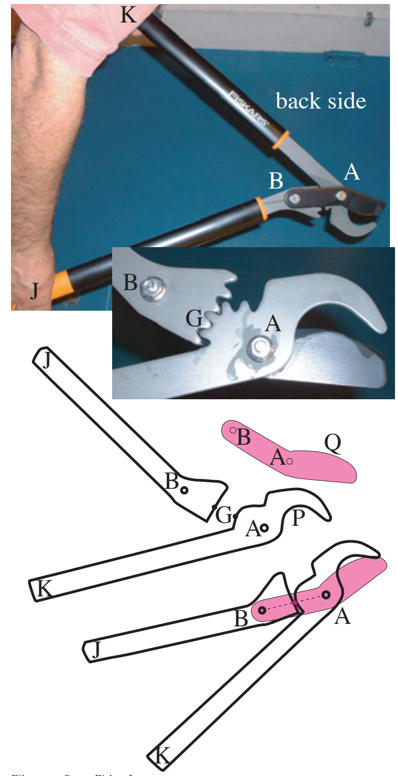
Filename:pfigure-loghanger
Problem 6.3.8

6.3.9 Gear teeth on handle JB mesh with teeth on handle-and-blade KAP at point G midway between hinges A and B. Assume that in the configuration of interest J, B and A are co-linear, that K, A and Q are co-linear and that the cutting contact points Q and P are effectively coincident, that angle JAK = 20°, $JB = 40$ cm, $BA = 6$ cm, $KA = 46$ cm, $AP = AQ = 3$ cm, and that the co-linear squeezing forces at J and K are 100 N.

a) Find the cutting force at Q and P.

b) Replace this design with one that has no tooth engagement at G. But instead handle JB and blade BAQ are welded together as one piece. Assuming the same geometry as before, what then is the cutting force at Q and P?

c) Without detailed calculations, explain the ratio of the two answers above. *



Filename:pfigure-FiskarsLoppers
Problem 6.3.9

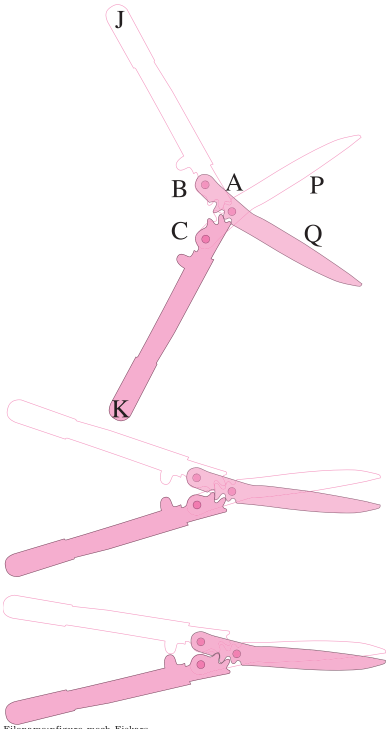
6.3.10 These 4 piece shears use the mechanism in problem 6.3.9 twice over. Co-linear hand forces F_{JK} are applied to handles JB and KC at J and K. Handle JB is hinged to blade BAQ at B. Handle KC is hinged to blade CAP at C. The blades are hinged to each other at A. Handle JB is effectively hinged to CAP, by means of gear teeth, at G, a point on the line segment AB. Similarly KC is effectively hinged to BAQ at point G' on the segment AC. The cut object presses with colinear forces F_{PQ} on the blades at P and Q. See box 6.6 on page 359 for more pictures of these shears.

Assume $F_{JK} = 100$ N, $JB = KC = 30$ cm, $AB = AC = 6$ cm, $AG = AG' = 3$ cm, and $AP = AQ = 20$ cm. Assume AB, BJ, AC,and CK all make angles of $\pm 20^\circ$ with a horizontal line. Assume P and Q are coincident and on a horizontal line extending from A.

a) Find F_{PQ}.

b) Replace this design with one where JB is welded to BAQ at B, KC is wleded to CAP at C, and there are no contacting gears. In this same geometry what is F_{PQ}?

c) Give a quantitative estimate, but not a detailed calculation that tells you the ratio of the forces in the above two problems? *



Filename:pfigure-mech-Fiskars
Problem 6.3.10

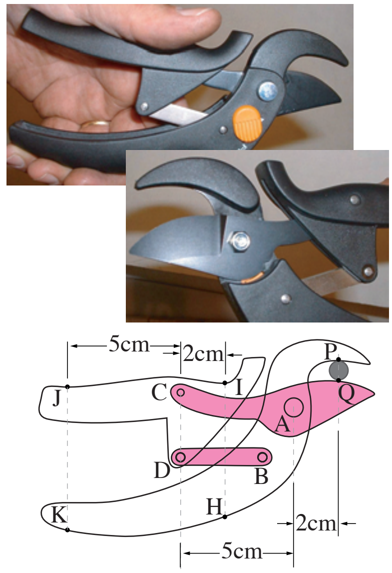
6.3.11 The garden cutters shown are a 4-bar linkage. Estimate the locations of points, as needed, using the given dimensions as a scale (the drawn clippers are shrunk slightly from reality to simplify the numbers).

a) If the handle is squeezed with a pair of 50 N forces at J and K what is the cutting force at P and Q? *

b) If the handle is squeezed with a pair of 50 N forces at I and H what is the cutting force at P and Q? *

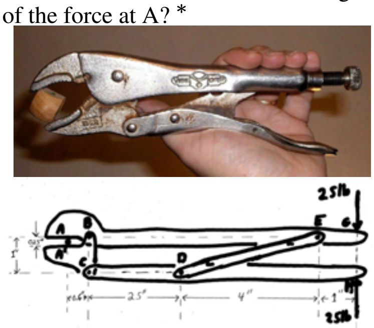
c) If this design was changed by eliminating link DB and welding handle JCDI to the blade CAQ, what would be the answers to the two questions above. *

d) Describe in words, the reasons for the similarities and differences between the answers above. *



Filename:pfigure-FiskarsCutter
Problem 6.3.11

6.3.12 For simplicity the vice grips shown in the photo are approximated as in the drawing. Round piece AA' is gripped between the upper handle/jaw ABEG and the lower jaw A'BC. The upper handle ABEG is pinned to the lower jaw A'BC at B. Handle CDH is pinned to the lower jaw at C and to the bar DE at D. Bar DE is pinned to the upper handle ABEG at E. The 25 lbf forces act at G and H as shown. Dimensions are as shown. What is the magnitude of the force at A? *



Filename:S03q4p8vicedrawing
Problem 6.3.12

6.3.13 Pipe wrench. A wrench is used to turn a pipe as shown in the figure. Neglecting the weight of the pipe, find

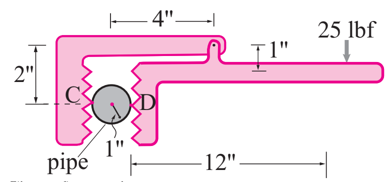
a) the torque of the pipe wrench forces about the center of the pipe

b) the forces on the pipe at C and D

c) the needed friction coefficient between the wrench and pipe for the wrench not to slip.

d) what design change would reduce this needed coefficient of friction (what change of dimensions)? *

e) given that the design change above is possible, why isn't it used? [hint: implement the design change and calculate the forces on the pipe.] *

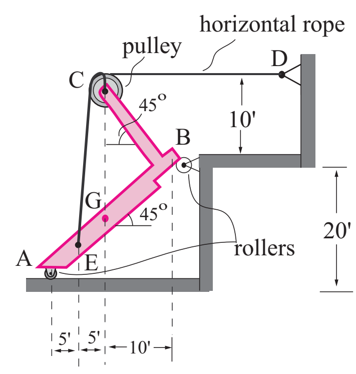


Filename:pfigure-wrench
Problem 6.3.13

6.3.14 The center of mass of 200 pound structure AEGBC is at G. It is held by rollers at A and B as well as with the rope which starts at E, wraps around the pulley at C, and ends at D.

a) Find the force of the ground on the structure at A. *

b) Find the tension in the rope. *



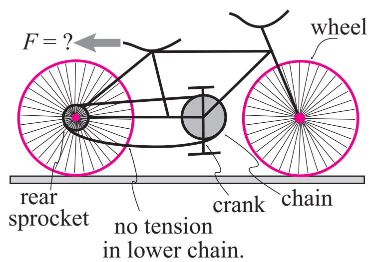
Filename:S03q2b-bentbar
Problem 6.3.14

6.3.15 Consider a bike on level ground that is held from falling sideways with forces that don't push it forward or back. Assume that all the bearings are ideal and that the wheels don't slip.
R_r = radius of rear wheel,
R_s = radius of rear sprocket,
R_p = crank length from crank-axle to pedal, and
R_c = radius of chain wheel (front sprocket).
What backwards force F on the seat is required to keep the bike from going forward (*i.e.,* to maintain static equilibrium) if

a) A person sits on the bike and pushes back on the bottom pedal with a force F_p? (is $F > 0$?)

b) A person standing next to the bike pushes back on the pedal with force F_p? (is $F > 0$?)

Your answer should be in terms of some or all of R_r, R_s, R_p, R_c, and F_p. Of great interest is whether F is bigger or less than zero. So pay close attention to signs.
To solve this problem you have to draw several free body diagrams: 1) of the whole bike and rider (if the rider is on the bike), 2) of the crank-pedal-chain-wheel system, with a little bit of chain, 3) The rear wheel and rear sprocket, with a little bit of chain.

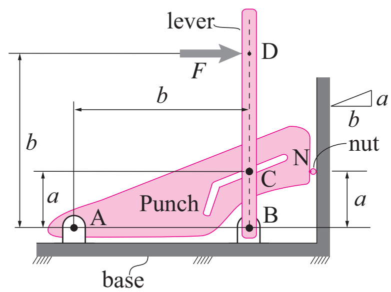


Filename:pfigure-bikepedalback
Problem 6.3.15

6.3.16 The proposed nutcracker design consists of two moving parts: a lever hinged to the fixed base at B and a punch hinged to the fixed base at A. All joints and slots are assumed to have negligible friction.
Mechanism and geometry clarifications: The vertical lever has a pin at C and a horizontal force F applied at D. The punch has a slot in which the lever pin slides at C. The slot is parallel to the line AC. The spherical nut is cracked by being squeezed between the vertical surface of the punch at N and the vertical surface attached to the base. Point N at the left edge of of the nut is level with the sliding pin at C. The horizontal distance from C to N does not enter the solution, but assume it is c if you need it for an intermediate calculation.
Quantities: $F = 10\,\text{lbf}$, $a = 2\,\text{in}$, $b = 10\,\text{in}$.

a) Find the force acting on the nut at N. A number is desired (i.e., so many lbf force). [Hint: Only substitute in numbers when you have a formula for your answer in terms of a, b and F.] *

b) The answer to (a) is conspicuous in its being either much smaller than F, very similar to F, or much bigger than F. Which is it? Explain, in words, why. The best possible answer will generate an approximate formula for the force at N using next-to-no equations. *



Filename:F02p1-3Nut
Problem 6.3.16

CHAPTER 7

Tension, shear and bending moment

The 'internal forces' tension, shear and bending moment can vary from point to point in long narrow objects. Here we introduce the notion of graphing this variation and noting the features of these graphs.

Contents

In Section 4.4 starting on page 225 we defined the notion of *'internal forces'*, especially *tension* T, *shear* V, and *bending moment* M. A common issue in structural mechanics is keeping track of how these internal forces, and other more advance internal force concepts (ie stress) vary from point to point in a structure. Commonly this understanding comes from 'finite-element-method' programs①. However, there are a variety of important engineering problems for which accurate and useful estimation of internal forces can be found using methods at the level of this book. These are problems where the structure of interest is long and narrow. For reasons like those discussed in the introductory paragraphs about trusses (e.g. the discussion of 'swiss cheese' page 256), long narrow objects are surprisingly common in engineered objects as well as in biologically evolved designs. Despite the availability of computers for analysis of these, the elementary methods we will introduce here are useful because,

① Part of the finite element method is the dividing of an object into a grid; dividing the object of interest into 'finite elements'. Film and brochure makers are magnetically attracted to these grids, like a bee to a flower. So even if you have never used one of these programs, you have seen signs of them, grids superposed on objects, in advertising, science, and science-fiction videos.

- For simple problems, they are easier than using a computer;
- The methods here help build understanding and intuition;
- The methods here can give formulas from which a design can be controlled more easily than by numerical parameter studies;
- For very narrow objects, the methods here are often more accurate than the computer solutions;
- To understand the vocabulary used in the output of the computer programs you need to understand the concepts associated with the methods here.

As for elementary truss analysis, the methods here are easily learned and pleasingly useful. For example, the formulas for bending moment in a simply-supported overhanging beam not only tell you the 'internal moment' for a giving loading, but how to space joists in a floor or the wing supports on a human-powered hydrofoil. And the capstain formula isn't just a way to calculate cable tensions. It tells you how to make a simple modification to many bicycles to improve the performance of their brakes and derailleurs.

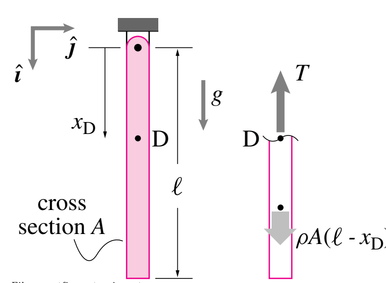


Figure 7.1: a) Rod hanging with gravity. b) free body diagram with cut at x_D.

7.1 Free body cuts at arbitrary locations

Tension, shear force, and bending moment diagrams

Engineers often want to know how the internal forces vary from point to point in a structure. If you want to know the internal forces at a variety of points you can draw a variety of free body diagrams with cuts at those points of interest. Another approach, which we present now, is to leave the position of the free body diagram cut a variable, and then calculate the internal forces in terms of that variable.

Example: Tension in a two-force body
Recall that in the first example of this section we found T without ever using information about the location of the free body diagram cut. So the location does not effect the tension. For a two force body the tension is a constant along the length.

Example: Tension in a rod from its own weight.
The uniform $1\,\mathrm{cm}^2$ steel square rod with density $\rho = 7.7\,\mathrm{gm/cm}^3$ and length $\ell = 100\,\mathrm{m}$ has total weight $W = mg = \rho\ell Ag$ (see fig. 7.1). What is the tension a distance x_D from the top? Using the free body diagram with cut at x_D we get:

$$
\begin{aligned}
&\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \\
\Rightarrow \quad T \quad &= \rho Ag(\ell - x_D) \\
&= (7.7\,\mathrm{gm/cm}^3)(1\,\mathrm{cm}^2)(9.8\,\mathrm{N/kg})(100\,\mathrm{m} - x_D) \\
&= 7.7 \cdot 9.8 \frac{\mathrm{gm\,N\,m}}{\mathrm{cm\,kg}}\left(100 - \frac{x_D}{\mathrm{m}}\right) \underbrace{\left(\frac{1\,\mathrm{kg}}{1000\,\mathrm{gm}}\right)}_{1} \underbrace{\left(\frac{100\,\mathrm{cm}}{1\,\mathrm{m}}\right)}_{1} \\
&= 7.5\left(100 - \frac{x_D}{\mathrm{m}}\right)\,\mathrm{N}.
\end{aligned}
$$

So, at the bottom end at $x_D = 100\,\mathrm{m}$ we get $T = 0$ and at the top end where $x_D = 0\,\mathrm{m}$ we get $T = 750\,\mathrm{N}$ and in the middle at $x_D = 50\,\mathrm{m}$ we get $T = 375\,\mathrm{N}$.

Because the free body diagram cut location is variable, we can plot the internal forces as a function of position. This is most useful in civil engineering where an engineer wants to know the internal forces in a horizontal beam carrying vertical loads. Common examples include bridge platforms and floor joists.

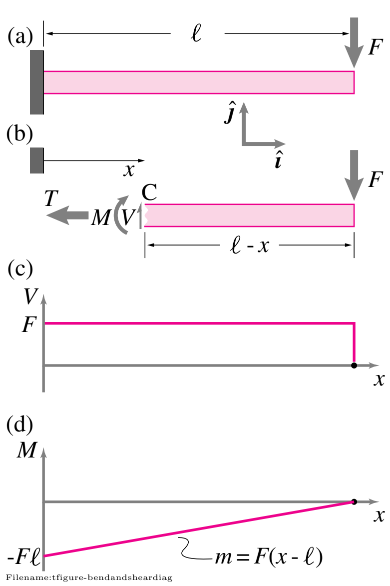


Figure 7.2: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

Example: Cantilever M and V diagram
A *cantilever* beam is mounted firmly at one end and has various loads orthogonal to its length, in this case a downwards load F at the end (fig. 7.2a). By drawing a free body diagram with a cut at the arbitrary point C (fig. 7.2b) we can find the internal forces as a function of the position of C.

$$
\begin{aligned}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} \quad &\Rightarrow \quad V = F \\
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\imath} \quad &\Rightarrow \quad T = 0 \\
\left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} \quad &\Rightarrow \quad M(x) = F(x - \ell).
\end{aligned}
$$

That the tension is zero in these problems is so well known that the tension is often not drawn on the free body diagram and not calculated. We can now plot $V(x)$ and $M(x)$ as in figs. 7.2c and 7.2d. In this case the shear force is a constant and the bending moment varies from its maximum magnitude at the wall ($M = -F\ell$) to 0 at the end. It is the big value of $|M|$ at the fixed support that makes cantilever beams typically break there.

Often one is interested in distributed loads from gravity on the structure itself or from a distribution (say of people on a floor). The method is the same.

Example: Distributed load

A cantilever beam has a downwards uniformly distributed load of w per unit length (fig. 7.4a). Using the free body diagram shown (fig. 7.4b) we can find:

$$
\begin{aligned}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} \quad \Rightarrow \quad & \{V(x)\hat{\jmath} + \textstyle\int d\vec{F}\} \cdot \hat{\jmath} = 0 \\
\Rightarrow \quad V(x) &= \textstyle\int_x^\ell w\, dx' \\
&= w \cdot (\ell - x)
\end{aligned}
$$

$$
\begin{aligned}
\left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} \quad \Rightarrow \quad & \left\{M(x)(-\hat{k}) + \textstyle\int \vec{r}_{/C} \times d\vec{F}\right\} \cdot \hat{k} = 0 \\
\Rightarrow \quad M(x) &= \textstyle\int_x^\ell (x' - x) w\, dx' \\
&= w \cdot (x'^2/2 - x'x)\Big|_x^\ell \\
&= (\ell^2/2 - \ell x) - (x^2/2 - x^2) \\
&= -w \cdot (\ell - x)^2/2.
\end{aligned}
$$

The integrals were used because of their general applicability for distributed loads. For this problem we could have avoided the integrals by using an equivalent downwards force $w \cdot (\ell - x)$ applied a distance $(\ell - x)/2$ to the right of the cut. Shear and bending moment diagrams are shown in figs. 7.4a and 7.4b.

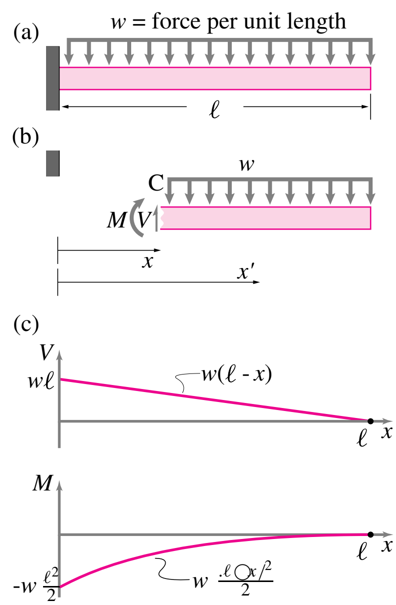


Figure 7.4: a) Cantilever beam, b) free body diagram, c) Shear force diagram, d) Bending moment diagram

As for all problems based on the equilibrium equations and a given geometry, the principle of superposition applies.

Example: Superposition

Consider a cantilever beam that simultaneously has both of the loads from the previous two examples. By the principle of superposition:

$$
\begin{aligned}
V &= F + w(\ell - x) \\
M(x) &= F(x - \ell) + -w(\ell - x)^2/2.
\end{aligned}
$$

The shear force at every point is the sum of the shear forces from the previous examples. The bending moment at every point is the sum of the bending moments.

If there are concentrated loads in the middle of the region of interest the calculation gets more elaborate; the concentrated force may or may not show up on the free body diagram of the cut bar, depending on the location of the cut.

Example: Simply supported beam with point load in the middle

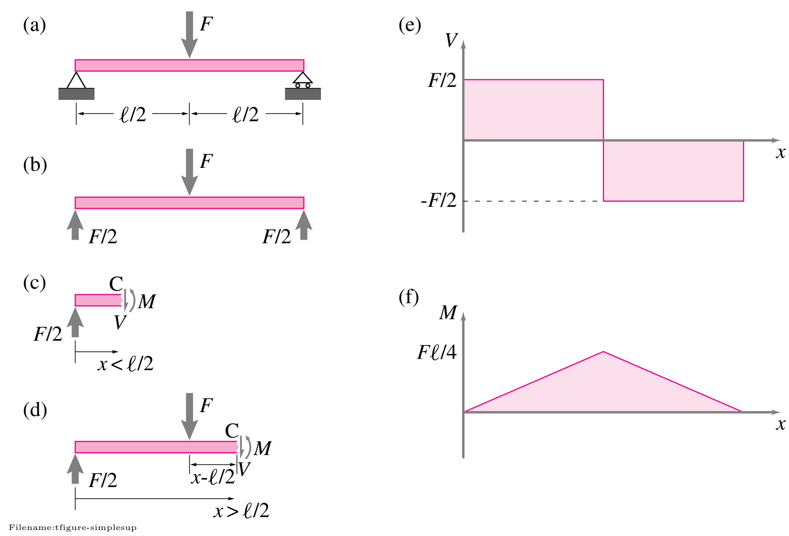


Figure 7.3: a) Simply supported beam, b) free body diagram of whole beam, c) free body diagram with cut to the left of the applied force, d) free body diagram with cut to the right of the applied force e) Shear force diagram, f) Bending moment diagram

A *simply supported* beam is mounted with pivots at both ends (fig. 7.3a). First we draw a free body diagram of the whole beam (fig. 7.3a) and then two more, one with a cut to the left of the applied force and one with a cut to the right of the applied force (figs. 7.3c and 7.3d). With the free body diagram 7.3c we can find $V(x)$ and $M(x)$ for $x < \ell/2$ and with the free body diagram 7.3d we can find $V(x)$ and $M(x)$ for $x > \ell/2$.

$$
\begin{array}{llrcll}
\left\{\sum \vec{F}_i = \vec{0}\right\} \cdot \hat{\jmath} & \Rightarrow & V & = & F/2 & \text{for } x < \ell/2 \\
 & & & = & -F/2 & \text{for } x > \ell/2 \\
\left\{\sum \vec{M}_C = \vec{0}\right\} \cdot \hat{k} & \Rightarrow & M(x) & = & Fx/2 & \text{for } x < \ell/2 \\
 & & & = & F(\ell - x)/2 & \text{for } x > \ell/2
\end{array}
$$

These relations can be plotted as in figs. 7.3e and 7.3f. Some observations: For this beam the biggest bending moment is in the middle, the place where simply supported beams often break. Instead of the free body diagram shown in (c) and (d) we could have drawn a free body diagrams of the bar to the right of the cut and would have got the same $V(x)$ and $M(x)$. We avoided drawing a free body diagram cut *at* the applied load where $V(x)$ has a discontinuity.

How to find T, V, and M

Here are some guidelines for finding internal forces and drawing shear and bending moment diagrams.

- Draw a free body diagram of the whole bar.
- Using the free body diagram above find the reaction forces .
- Draw a free body diagram(s) of the cut bar of interest.
  - For each region between concentrated loads draw one free body diagram.

  - Show the piece from the cut to one or the other end (So that all but the internal forces are known).
  - Don't make cuts *at* intermediate points of connection or load application.
- Use the equilibrium equations to find T, V, or M (Moment balance about a point at the cut is a good way to find M.)
- Use the results above to plot $V(x)$ and $M(x)$ ($T(x)$ is rarely plotted).
  - Use the same x scale for this plot as for the free body diagram of the whole bar.
  - Put the plots directly under the free body diagram of the bar (so you can most easily relate features of the loads to features of the V and M diagrams).

Stress is force per unit area

For a given load, if you replace one bar in tension with two bars side by side you would imagine the tension in each bar would go down by a factor of 2. Thus the pair of bars should be twice as strong as a single bar. If you glued these side by side bars together you would again have one bar but it would be twice as strong as the original bar. Why? Because it has twice the cross sectional area.

What makes a solid break is the force per unit area carried by the material. For an applied tension load T, the force per unit area on an interior free body diagram cut is T/A. Force per unit area normal to an internal free body diagram cut is called *tension stress* and denoted σ (lower case 'sigma', the Greek letter s).

$$\sigma = T/A$$

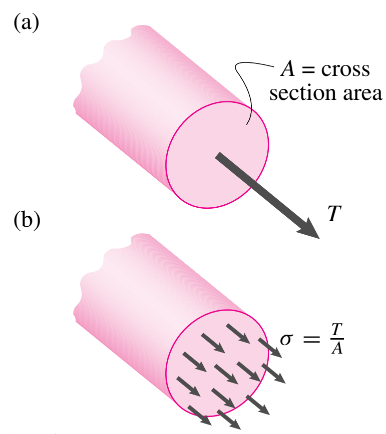


Figure 7.5: a) Tension on a free body diagram cut is equivalent to b) uniform tension stress.

Example: Stress in a hanging bar
Look at the hanging bar in the example on page 378. We can find the tension stress in this bar as a function of position along the bar as:

$$\sigma = \frac{T}{A} = \frac{\rho g A(\ell - x)}{A} = \rho g(\ell - x).$$

Note that the stress for this bar doesn't depend on the cross sectional area. The bigger the area the bigger the volume and hence the load. But also, the bigger the area on which to carry it.

For reasons that are beyond this book, the tension stress tends to be uniform in homogeneous (all one material) bars, no matter what their cross sectional shape, so that the average tension stress $\frac{T}{A}$ is actually the tension stress all across the cross section.

We can similarly define the average *shear stress* τ_{ave} ('tau') on a free body diagram cut as the average force per unit area tangent to the cut,

$$\tau_{\text{ave}} = \frac{V}{A}.$$

For reasons you may learn in a strength of materials class, shear stress is not so uniformly distributed across the cross section. But the average shear stress τ_{ave} does give an indication of the actual shear stress in the bar (*e.g.,* for a rectangular elastic bar the peak shear stress is 50% larger than τ_{ave}).

The biggest stresses typically come from bending moment. Motivating formulas for these stresses here is too big a digression. The formulas for the stresses due to bending moment are a key part of elementary strength of materials. But just knowing that these stresses tend to be big, gives you the important notion that bending moment is a common cause of structural failure.

Internal force summary

'Internal forces' are the scalars which describe the force and moment on potential internal free body diagram cuts. They are found by applying the equilibrium equations to free body diagrams that have cuts at the points of interest. The internal forces are intimately associated with the internal stresses (force per unit area) and thus are important for determining the strength of structures.

SAMPLE 7.1 Support reactions on a simply supported beam: A uniform beam of length 3 m is simply supported at A and B as shown in the figure. A uniformly distributed vertical load $q = 100\,\mathrm{N/m}$ acts over the entire length of the beam. In addition, a concentrated load $P = 150\,\mathrm{N}$ acts at a distance $d = 1$ m from the left end. Find the support reactions.

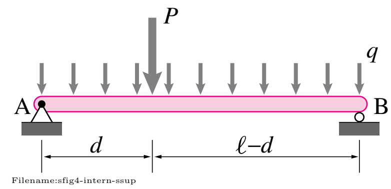


Figure 7.6:

Solution Since the beam is supported at A on a pin joint and at B on a roller, the unknown reactions are

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}, \qquad \vec{B} = B_y\hat{\jmath}.$$

The uniformly distributed load q can be replaced by an equivalent concentrated load $W = q\ell$ acting at the center of the beam span. The free-body diagram of the beam, with the concentrated load replaced by the equivalent concentrated load is shown in Fig. 7.7. The moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, gives

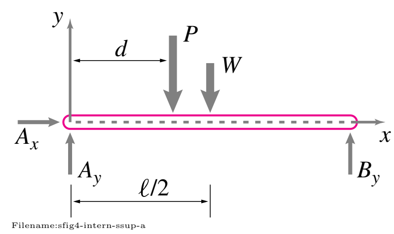


Figure 7.7:

$$(-Pd - W\frac{\ell}{2} + B_y\ell)\hat{k} = \vec{0}$$

$$\begin{aligned} \Rightarrow \quad B_y &= P\frac{d}{\ell} + \frac{1}{2}W \\ &= 150\,\mathrm{N}\cdot\frac{1}{3} + \frac{1}{2}\cdot 300\,\mathrm{N} = 200\,\mathrm{N}. \end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\vec{A} + B_y\hat{\jmath} - P\hat{\jmath} - W\hat{\jmath} = \vec{0}$$

$$\begin{aligned} \Rightarrow \quad \vec{A} &= (-B_y + P + W)\hat{\jmath} \\ &= (-200\,\mathrm{N} + 150\,\mathrm{N} + 300\,\mathrm{N})\hat{\jmath} = 250\,\mathrm{N}\hat{\jmath}. \end{aligned}$$

$$\boxed{\vec{A} = 250\,\mathrm{N}\hat{\jmath}, \quad \vec{B} = 200\,\mathrm{N}\hat{\jmath}}$$

SAMPLE 7.2 Support reactions on a cantilever beam: A 2 kN horizontal force acts at the tip of an 'L' shaped cantilever beam as shown in the figure. Find the support reactions at A.

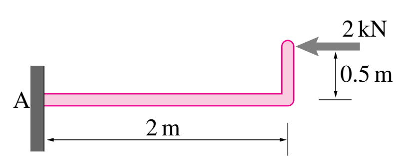


Figure 7.8:

Solution The free-body diagram of the beam is shown in Fig. 7.9. The reaction force at A is $\vec{A}$ and the reaction moment is $\vec{M} = M\hat{k}$. Writing moment balance equation about point A, $\sum \vec{M}_A = \vec{0}$, we get

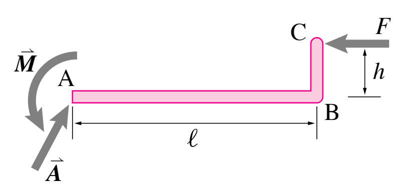


Figure 7.9:

$$\begin{aligned} \vec{M} + \vec{r}_{C/A} \times \vec{F} &= \vec{0} \\ \vec{M} + (\ell\hat{\imath} + h\hat{\jmath}) \times (-F\hat{\imath}) &= \vec{0} \\ \Rightarrow \quad \vec{M} &= -Fh\,\hat{k} \\ &= -2\,\mathrm{kN}\cdot 0.5\,\mathrm{m}\,\hat{k} \\ &= -1\,\mathrm{kN}\cdot\mathrm{m}\,\hat{k}. \end{aligned}$$

The force equilibrium, $\sum \vec{F} = \vec{0}$, gives

$$\begin{aligned} \vec{A} + \vec{F} &= \vec{0} \\ \Rightarrow \quad \vec{A} &= -\vec{F} = -(-2\,\mathrm{kN}\,\hat{\imath}) = 2\,\mathrm{kN}\,\hat{\imath}. \end{aligned}$$

$$\boxed{\vec{A} = 2\,\mathrm{kN}\,\hat{\imath}, \quad \vec{M} = -1\,\mathrm{kN}\cdot\mathrm{m}\,\hat{k}}$$

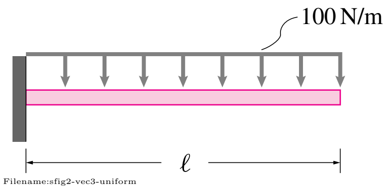


Figure 7.10:

SAMPLE 7.3 Net force of a uniformly distributed system: A uniformly distributed vertical load of intensity 100 N/macts on a beam of length $\ell = 2\,\mathrm{m}$ as shown in the figure.

1. Find the net force acting on the beam.
2. Find an equivalent force-couple system at the mid-point of the beam.
3. Find an equivalent force-couple system at the right end of the beam.

Solution

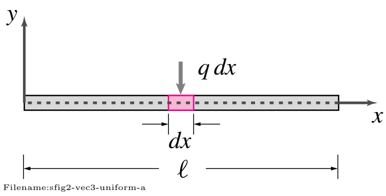


Figure 7.11:

1. **The net force:** Since the load is uniformly distributed along the length, we can find the total or the net load by calculating the load on an infinitesimal segment of length dx of the beam and then integrating over the entire length of the beam. Let the load intensity (load per unit length) be q ($q = 100\,\mathrm{N/m}$, as given). Then the vertical load on segment dx is (see Fig. 7.11),

$$d\vec{\boldsymbol{F}} \;=\; q\,dx(-\hat{\boldsymbol{\jmath}}).$$

Therefore, the net force is,

$$\vec{\boldsymbol{F}}_{\text{net}} \;=\; \int_0^{\ell} q\,dx(-\hat{\boldsymbol{\jmath}}) = q\,\ell\,\hat{\boldsymbol{\jmath}} = -100\,\mathrm{N/m}\cdot 2\,\mathrm{m}\,\hat{\boldsymbol{\jmath}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}.$$

$$\boxed{\vec{\boldsymbol{F}}_{\text{net}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}}$$

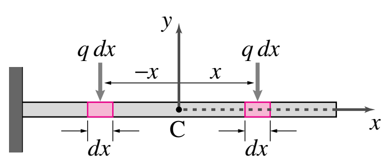


Figure 7.12:

2. **The equivalent system at the mid-point:** We have already calculated the net force that can replace the uniformly distributed load. Now we need to calculate the couple at the mid-point of the beam to get the equivalent force-couple system. Again, consider a small segment of the beam of length dx located at distance x from the mid-point C (see Fig. 7.12). The moment about point C due to the load on dx is $(q\,dx)x(-\hat{\boldsymbol{k}})$. But, we can find a similar segment on the other side of C with exactly the same length dx, at exactly the same distance x, that produces a moment of $(q\,dx)x(+\hat{\boldsymbol{k}})$. The two contributions cancel each other and we have a net zero moment about C. Now, you can imagine the whole beam made up of these pairs that contribute equal and opposite moment about C and thus the net moment about the mid-point is zero. You can also find the same result by straight integration:

$$\vec{\boldsymbol{M}}_{\mathrm{C}} = \int_{-\ell/2}^{+\ell/2} qx\,dx(-\hat{\boldsymbol{k}}) = \left.\frac{qx^2}{2}\right|_{-\ell/2}^{+\ell/2}(-\hat{\boldsymbol{k}}) = \vec{\boldsymbol{0}}.$$

$$\boxed{\vec{\boldsymbol{F}}_{\text{net}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}},\ \text{and}\ \vec{\boldsymbol{M}}_{\mathrm{C}} = \vec{\boldsymbol{0}}}$$

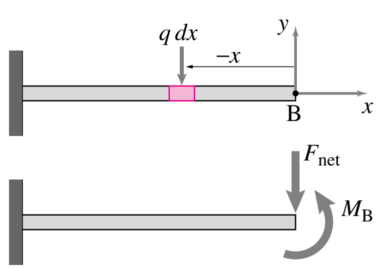


Figure 7.13:

3. **The equivalent system at the end:** The net force remains the same as above. We compute the net moment about the end point B, referring to Fig. 7.13, as follows.

$$\begin{aligned}
\vec{\boldsymbol{M}}_{\mathrm{B}} &= \int_0^{\ell}(-x\hat{\boldsymbol{\imath}})\times(-q\,dx\,\hat{\boldsymbol{\jmath}}) = -q\int_0^{\ell} x\,dx\,\hat{\boldsymbol{k}} \\
&= -\frac{q\ell^2}{2}\hat{\boldsymbol{k}} = -\frac{100\,\mathrm{N/m}\cdot 4\,\mathrm{m}^2}{2}\hat{\boldsymbol{k}} = -200\,\mathrm{N{\cdot}m}\hat{\boldsymbol{k}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{F}}_{\text{net}} = -200\,\mathrm{N}\,\hat{\boldsymbol{\jmath}}\ \text{and}\ \vec{\boldsymbol{M}}_{\mathrm{B}} = -200\,\mathrm{N{\cdot}m}\hat{\boldsymbol{k}}}$$

SAMPLE 7.4 For the uniformly loaded, simply supported beam shown in the figure, find the shear force and the bending moment at the mid-section c-c of the beam.

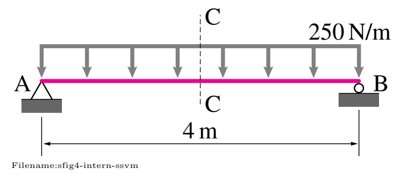


Figure 7.14:

Solution To determine the shear force V and the bending moment M at the mid-section c-c, we cut the beam at c-c and draw its free-body diagram as shown in fig. 7.15. For writing force and moment balance equations we use the second figure where we have replaced the distributed load with an equivalent single load $F = (q\ell)/2$ acting vertically downward at distance $\ell/4$ from end A.

The force balance, $\sum \vec{F} = \vec{0}$, implies that

$$A_x\hat{\imath} + A_y\hat{\jmath} - V\hat{\jmath} - F\hat{\jmath} = \vec{0}.$$

Dotting with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$\begin{aligned} A_x &= 0 \\ V &= A_y - F \qquad (7.1) \\ &= A_y - \frac{q\ell}{2}. \qquad (7.2) \end{aligned}$$

From the moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, we get

$$M\hat{k} - \left(\frac{q\ell}{2}\cdot\frac{\ell}{4}\right)\hat{k} - V\frac{\ell}{2}\hat{k} = 0$$

$$\Rightarrow \quad M = \frac{q\ell^2 + 4V\ell}{8}. \qquad (7.3)$$

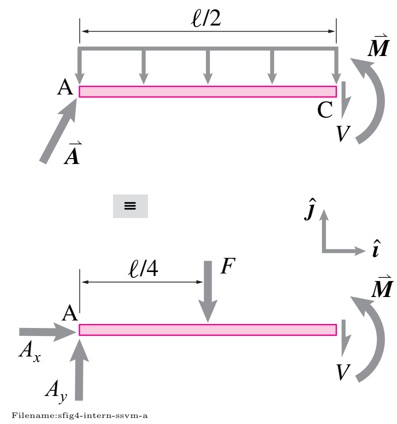


Figure 7.15:

Thus, to find V and M we need to know the support reaction $\vec{A}$. From the free-body diagram of the beam in fig. 7.16 and the moment equilibrium equation about point B, $\sum \vec{M}_B = \vec{0}$, we get

$$\begin{aligned} \vec{r}_{A/B} \times \vec{A} + \vec{r}_{C/B} \times \vec{Q} &= \vec{0} \\ (-A_y\ell + q\ell\frac{\ell}{2})\hat{k} &= \vec{0} \\ \Rightarrow \quad A_y = \frac{q\ell}{2} &= 500\,\text{N}. \end{aligned}$$

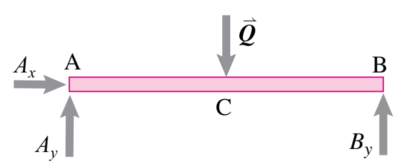


Figure 7.16:

Thus $\vec{A} = 500\,\text{N}\hat{\jmath}$. Substituting $\vec{A}$ in eqns. (7.2) and (7.3), we get

$$\begin{aligned} V &= 500\,\text{N} - 500\,\text{N} = 0 \\ M &= \frac{(250\,\text{N}\cdot 4\,\text{m})^2 + 0}{8} \\ &= 500\,\text{N·m}. \end{aligned}$$

$$\boxed{V = 0, \quad M = 500\,\text{N·m}}$$

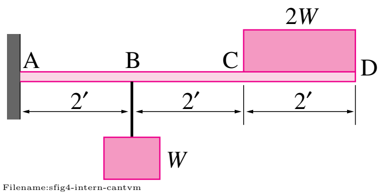


Figure 7.17:

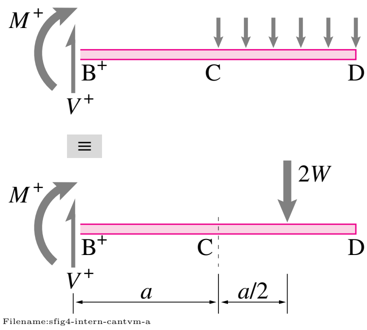


Figure 7.18:

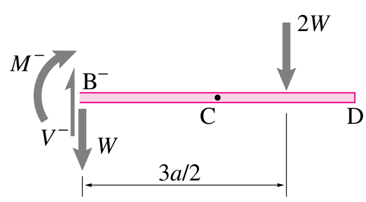


Figure 7.19:

SAMPLE 7.5 The cantilever beam AD is loaded as shown in the figure where $W = 200$ lbf. Find the shear force and bending moment on a section just left of point B and another section just right of point B.

Solution To find the desired internal forces, we need to make a cut at a section just to the left of B and one just to the right of B. We first take the one that is to the right of point B. The free-body diagram of the right part of the cut beam is shown in fig. 7.18. Note that if we selected the left part of the beam, we would need to determine support reactions at A. The uniformly distributed load $2W$ of the block sitting on the beam can be replaced by an equivalent concentrated load $2W$ acting at point E, at distance $a/2$ from the end D of the beam.

Let us denote the shear force by V^+ and the bending moment by M^+ at the section of our interest. Now, from the force equilibrium of the part-beam BD we get

$$\begin{aligned} V^+\hat{\boldsymbol{j}} - 2W\hat{\boldsymbol{j}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad V^+ &= 2W \\ &= 400\,\text{lbf}. \end{aligned}$$

The moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_B = \vec{\boldsymbol{0}}$, gives

$$\begin{aligned} -M^+\hat{\boldsymbol{k}} - 2W\cdot\frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad M^+ &= -3Wa \\ &= -1200\,\text{lb·ft}. \end{aligned}$$

Now, we determine the internal forces at a section just to the left of point B. Let the shear and bending moment at this section be V^- and M^-, respectively, as shown in the free-body diagram (fig. 7.19). Note that load W acting at B is now included in the free-body diagram since the beam is now cut just a teeny bit left of this load.

From the force equilibrium of the part-beam, we have

$$\begin{aligned} V^-\hat{\boldsymbol{j}} - W\hat{\boldsymbol{j}} - 2W\hat{\boldsymbol{j}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad V^- &= 3W \\ &= 600\,\text{lbf} \end{aligned}$$

and, from moment equilibrium about point B, $\sum \vec{\boldsymbol{M}}_B = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned} -M^-\hat{\boldsymbol{k}} - 2W\cdot\frac{3a}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\ \Rightarrow \quad M^- &= -3Wa \\ &= -1200\,\text{lb·ft}. \end{aligned}$$

$$\boxed{M^+ = M^- = -1200\,\text{lb·ft}, \quad V^+ = 400\,\text{lbf}, \quad V^+ = 600\,\text{lbf}}$$

Note that the bending moment remains the same on either side of point B but the shear force jumps by $V^+ - V^- = 200\,\text{lbf} = W$ as we go from right to the left. This jump is expected because a concentrated load W acts at B, in between the two sections we consider. Concentrated external forces cause a jump in shear, and concentrated external moments cause a jump in the bending moment.

SAMPLE 7.6 **Tension in a bar:** A T-shaped bar is fixed in a wall at one end and is acted by three forces as shown in the figure. Find the tension in the rod at

1. section a-a, and
2. section b-b.

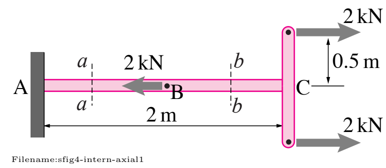


Figure 7.20:

Solution

1. Let us cut the bar at section a-a and consider the part of the bar to the right of the cut-section. The free-body diagram of this part of the bar is shown in fig. 7.21. The scalar force balance in the horizontal direction gives

$$\begin{aligned} -T - F + 2F &= 0 \\ \Rightarrow \quad T &= F \\ &= 2\,\text{kN}. \end{aligned}$$

At section a-a: $T = 2\,\text{kN}$

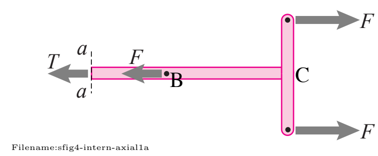


Figure 7.21:

2. Now, we cut the bar at section b-b and again consider the section of the bar to the right of the cut-section. The free-body diagram of this part of the bar is shown in fig. 7.22. Again, the force balance in the horizontal direction gives

$$\begin{aligned} -T + 2F &= 0 \\ \Rightarrow \quad T &= 2F \\ &= 4\,\text{kN}. \end{aligned}$$

At section b-b: $T = 4\,\text{kN}$

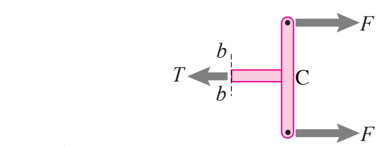


Figure 7.22:

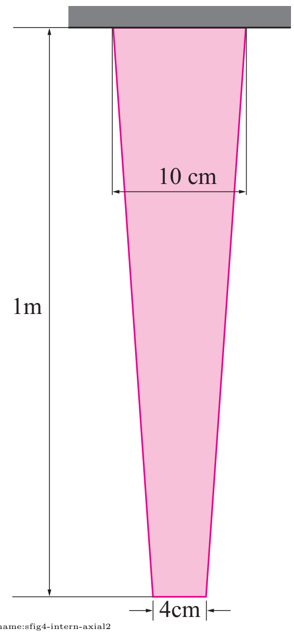


Figure 7.23:

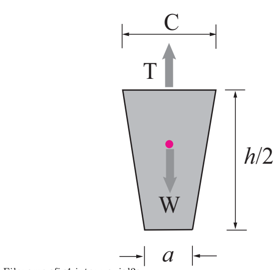


Figure 7.24:

SAMPLE 7.7 **Tension in a tapered bar due to self weight:** A tapered bar of height 1 m, base width 10 cm, top width 4 cm and uniform thickness 4 cm hangs upside down from a ceiling. If the density of the material is $7500\,\mathrm{kg/m^3}$, find the tension in the rod halfway from the top. You may take $g \approx 10\,\mathrm{m/s^2}$.

Solution Let us cut the bar at a section halfway from the top. The free-body diagram of the bar below the cut is shown in fig. 7.24. From the scalar force balance in the vertical direction, we have

$$T = W$$

where W is the weight of the lower part of bar below the cut section. Now, $W = \rho A t g$ where A is the frontal area, t is the thickness, and $\rho = 7500\,\mathrm{kg/m^3}$ is the density of the rod material. We need to compute W.

The width of the bar at the cut section is $c = (a+b)/2$ where $a = 4\,\mathrm{cm}$ and $b = 10\,\mathrm{cm}$. The frontal area of the bar-part is $A = (a+c)/2 \cdot (h/2)$ where $h = 1\,\mathrm{m}$. Thus,

$$\begin{aligned} W &= \rho \left(\frac{a+c}{2} \cdot \frac{h}{2} \right) t g \\ &= 7500\,\mathrm{kg/m^3} \left(\frac{0.04\,\mathrm{m} + 0.07\,\mathrm{m}}{2} \cdot \frac{1\,\mathrm{m}}{2} \cdot (0.04\,\mathrm{m}) \right) 10\,\mathrm{m/s^2} \\ &= 82.5\,\mathrm{N}. \end{aligned}$$

Thus, $T = 82.5\,\mathrm{N}$.

$$\boxed{T = 82.5\,\mathrm{N}}$$

SAMPLE 7.8 A simple frame: A 2 m high and 1.5 m wide rectangular frame ABCD is loaded with a 1.5 kN horizontal force at B and a 2 kN vertical force at C. Find the internal forces and moments at the mid-section e-e of the vertical leg AB.

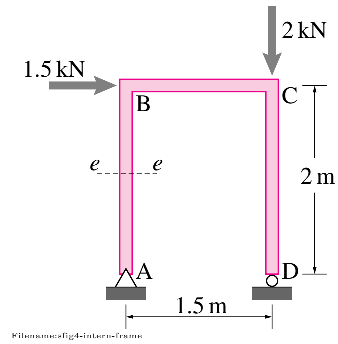


Figure 7.25:

Solution To find the internal forces and moments, we need to cut the frame at the specified section e-e and consider the free-body diagram of either AE or EBCD. No matter which of the two we select, we will need the support reactions at A or D to determine the internal forces. Therefore, let us first find the support reactions at A and D by considering the free-body diagram of the whole frame (fig. 7.26). The moment balance about point A, $\sum \vec{M}_A = \vec{0}$, gives

$$\begin{aligned}
\vec{r}_B \times \vec{F}_1 + \vec{r}_C \times \vec{F}_2 + \vec{r}_D \times \vec{D} &= \vec{0} \\
h\hat{\jmath} \times F_1\hat{\imath} + (h\hat{\jmath} + \ell\hat{\imath}) \times (-F_2\hat{\jmath}) + \ell\hat{\imath} \times D\hat{\jmath} &= \vec{0} \\
-F_1 h\hat{k} - F_2\ell\hat{k} + D\ell\hat{k} &= \vec{0} \\
\Rightarrow \quad D &= F_1\frac{h}{\ell} + F_2 \\
&= 1.5\,\text{kN} \cdot \frac{2}{1.5} + 2\,\text{kN} \\
&= 4\,\text{kN}.
\end{aligned}$$

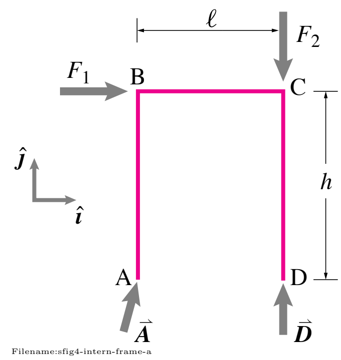


Figure 7.26:

From force equilibrium, $\sum \vec{F} = \vec{0}$, we have

$$\begin{aligned}
\vec{A} &= -\vec{F}_1 - \vec{F}_2 - \vec{D} \\
&= -F_1\hat{\imath} + F_2\hat{\jmath} - D\hat{\jmath} \\
&= -1.5\,\text{kN}\hat{\imath} - 2\,\text{kN}\hat{\jmath}.
\end{aligned}$$

Now we draw the free-body diagram of AE to find the shear force V, axial (tensile) force T, and the bending moment M at section e-e.

From the force equilibrium of part AE, we get

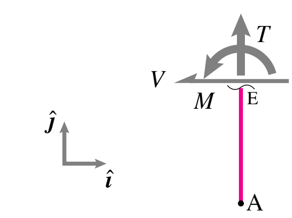


Figure 7.27:

$$\begin{aligned}
\vec{A} - V\hat{\imath} + T\hat{\jmath} &= \vec{0} \\
(A_x - V)\hat{\imath} + (A_y + T)\hat{\jmath} &= \vec{0} \\
\Rightarrow \quad V = A_x &= -1.5\,\text{kN} \\
T = -A_y &= 2\,\text{kN}.
\end{aligned}$$

From the moment equilibrium about point A, $\sum \vec{M}_A = \vec{0}$, we have

$$\begin{aligned}
M\hat{k} + \frac{h}{2}\hat{\jmath} \times (-V\hat{\imath}) &= \vec{0} \\
M\hat{k} + V\frac{h}{2}\hat{k} &= \vec{0} \\
\Rightarrow \quad M &= -V\frac{h}{2} \\
&= -(-1.5\,\text{kN}) \cdot \frac{2\,\text{m}}{2} \\
&= 1.5\,\text{k N·m}.
\end{aligned}$$

$$V = 1.5\,\text{kN}, \quad T = 2\,\text{kN}, \quad M = 1.5\,\text{k N·m}$$

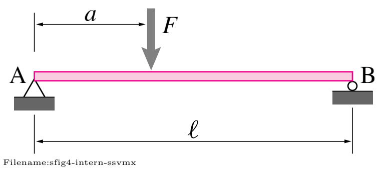


Figure 7.28:

SAMPLE 7.9 Shear force and bending moment diagrams: A simply supported beam of length $\ell = 2\,\mathrm{m}$ carries a concentrated vertical load $F = 100\,\mathrm{N}$ at a distance a from its left end. Find and plot the shear force and the bending moment along the length of the beam for $a = \ell/4$.

Solution We first find the support reactions by considering the free-body diagram of the whole beam shown in fig. 7.29. By now, we have developed enough intuition to know that the reaction at A will have no horizontal component since there is no external force in the horizontal direction. Therefore, we take the reactions at A and B to be only vertical. Now, from the moment equilibrium about point B, $\sum \vec{M}_B = \vec{0}$, we get

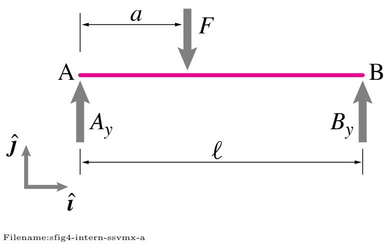


Figure 7.29:

$$
\begin{aligned}
F(\ell - a)\hat{\boldsymbol{k}} - A_y \ell \hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad A_y &= \frac{F(\ell - a)}{\ell} \\
&= F\left(1 - \frac{a}{\ell}\right)
\end{aligned}
$$

and from the force equilibrium in the vertical direction, $(\sum \vec{F} = \vec{0}) \cdot \hat{\jmath}$, we get

$$B_y = F - A_y = F\frac{a}{\ell}.$$

Now we make a cut at an arbitrary (variable) distance x from A where $x < a$ (see fig. 7.30). Carrying out the force balance and the moment balance about point A, we get, for $0 \le x < a$,

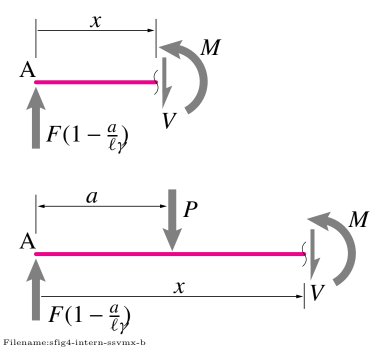


Figure 7.30:

$$
\begin{aligned}
V &= A_y = F\left(1 - \frac{a}{\ell}\right) &\quad (7.4)\\
M &= Vx = F\left(1 - \frac{a}{\ell}\right)x &\quad (7.5)
\end{aligned}
$$

Thus V is constant for all $x < a$ but M varies linearly with x.

Now we make a cut at an arbitrary x to the right of load F, *i.e.,* $a < x \le \ell$. Again, from the force balance in the vertical direction, we get

$$V = -F + F\left(1 - \frac{a}{\ell}\right) = -F\frac{a}{\ell} \qquad (7.6)$$

and from the moment balance about point A,

$$
\begin{aligned}
M &= Fa + Vx \\
&= Fa - F\frac{a}{\ell}x \\
&= Fa\left(1 - \frac{x}{\ell}\right). &\quad (7.7)
\end{aligned}
$$

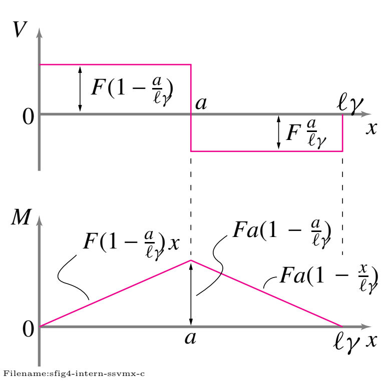


Figure 7.31:

Although eqn. (7.5) is strictly valid for $x < a$ and eqn. (7.7) is strictly valid for $x > a$, sustituting $x = a$ in these two equations gives the same value for $M(= Fa(1 - a/\ell))$ as it must because there is no reason to have a jump in the bending moment at any point along the length of the beam. The shear force V, however, does jump because of the concentrated load F at $x = a$.

Now, we plug in $a = \ell/4 = 0.5\,\mathrm{m}$, and $F = 100\,\mathrm{N}$, in eqns. (7.4)–(7.7) and plot V and M along the length of the beam by varying x. The plots of $V(x)$ and $M(x)$ are shown in fig. 7.31.

SAMPLE 7.10 Shear force and bending moment diagrams by superposition: For the cantilever beam and the loading shown in the figure, draw the shear force and the bending moment diagrams by

1. considering all the loads together, and
2. considering each load (of one type) at a time and using superposition.

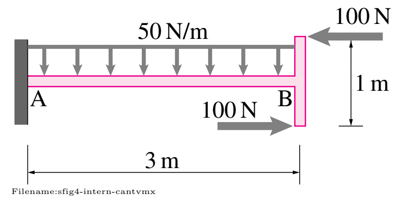


Figure 7.32:

Solution

1. *$V(x)$ and $M(x)$ with all forces considered together:* The horizontal forces acting at the end of the cantilever are equal and opposite and, therefore, produce a couple. So, we first replace these forces by an equivalent couple $M_{\text{applied}} = 100\,\text{N} \cdot 1\,\text{m} = 100\,\text{N·m}$. Since we have a cantilever beam, we can consider the right hand side of the beam after making a cut anywhere for finding V and M without first finding the support reactions.

   Let us cut the beam at an arbitrary distance x from the right hand side. The free-body diagram of the right segment of the beam is shown in fig. 7.33. From the force balance, $\sum \vec{F} = \vec{0}$, we find that

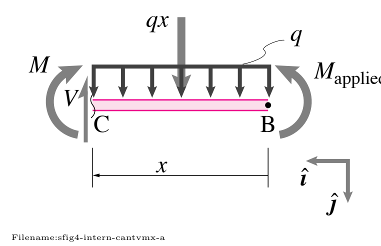


Figure 7.33:

$$\begin{aligned} -V\hat{\jmath} + qx\hat{\jmath} &= \vec{0} \\ \Rightarrow \quad V &= qx \qquad (7.8) \\ &= (50\,\text{N/m})x. \end{aligned}$$

   Thus the shear force varies linearly along the length of the beam with

$$\begin{aligned} V(x=0) &= 0, \\ \text{and} \qquad V(x=3\,\text{m}) &= 150\,\text{N}. \end{aligned}$$

   The moment balance about point C, $\sum \vec{M}_C = \vec{0}$, gives

$$-M\hat{k} - qx \cdot \frac{x}{2}\hat{k} + M_{\text{applied}}\hat{k} = \vec{0}$$

   where the moment due to the distributed load is most easily computed by considering an equivalent concentrated load qx acting at $x/2$ from the end B. Thus,

$$\Rightarrow \quad M = M_{\text{applied}} - q\frac{x^2}{2} \qquad (7.9)$$

$$= 100\,\text{N·m} - 50\,\text{N/m} \cdot \frac{x^2}{2}. \qquad (7.10)$$

   Thus, the bending moment varies quadratically with x along the length of the beam. In particular, the values at the ends are

$$\begin{aligned} M(x=0) &= 100\,\text{N·m} \\ \text{and} \qquad M(x=3\,\text{m}) &= -125\,\text{N·m}. \end{aligned}$$

   The shear force and the bending moment diagrams obtained from eqns. (7.8) and (7.9) are shown in fig. 7.34. Note that $M = 0$ at $x = 2\,\text{m}$ as given by eqn. (7.9).

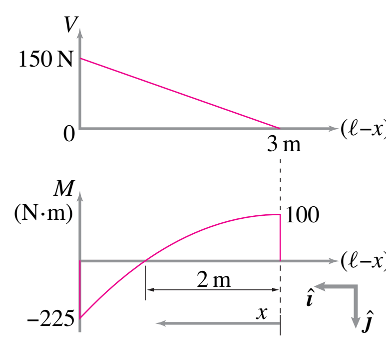


Figure 7.34:

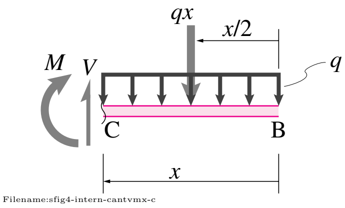


Figure 7.35:

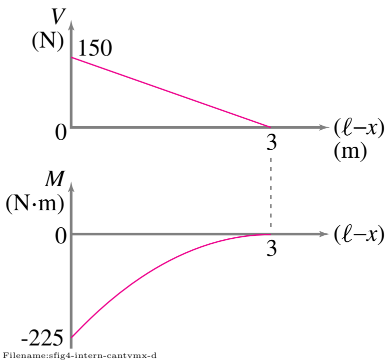


Figure 7.36:

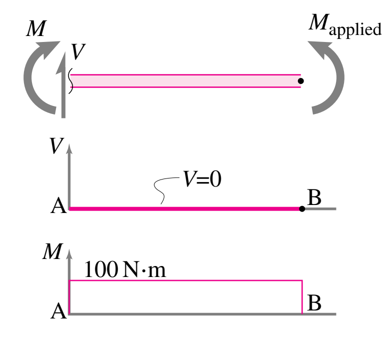


Figure 7.37:

2. $V(x)$ *and* $M(x)$ *by superposition:* Now we consider the cantilever beam with only one type of load at a time. That is, we first consider the beam only with the uniformly distributed load and then only with the end couple. We draw the shear force and the bending moment diagrams for each case separately and then just *add them up*. That is superposition.

So, first let us consider the beam with the uniformly distributed load. The free-body diagram of a segment CB, obtained by cutting the beam at a distance x from the end B, is shown in fig. 7.35. Once again, from force balance, we get

$$V = qx \qquad \text{for } 0 \leq x \leq \ell \tag{7.11}$$

and from the moment balance about point C, $\sum \vec{M}_C = \vec{0}$, we get

$$M = -qx \cdot \frac{x}{2} = -q\frac{x^2}{2} \qquad \text{for } 0 \leq x \leq \ell. \tag{7.12}$$

Figure 7.36 shows the plots of V and M obtained from eqns. (7.11) and (7.12), respectively, with the values computed from $x = 0$ to $x = 3\,\text{m}$ with $q = 50\,\text{N/m}$ as given.

Now we take the beam with only the end couple and repeat our analysis. A cut section of the beam is shown in fig. 7.37. In this case, it should be obvious that from force balance and moment balance about any point, we get

$$\begin{aligned} V &= 0 \\ \text{and} \qquad M &= M_{\text{applied}}. \end{aligned}$$

Thus, both the shear force and the bending moment are constant along the length of the beam as shown in fig. 7.37.

Now superimposing (adding) the shear force diagrams from Figs. 7.36 and 7.37, and similarly, the bending moment diagrams from Figs. 7.36 and 7.37, we get the same diagrams as in fig. 7.38.

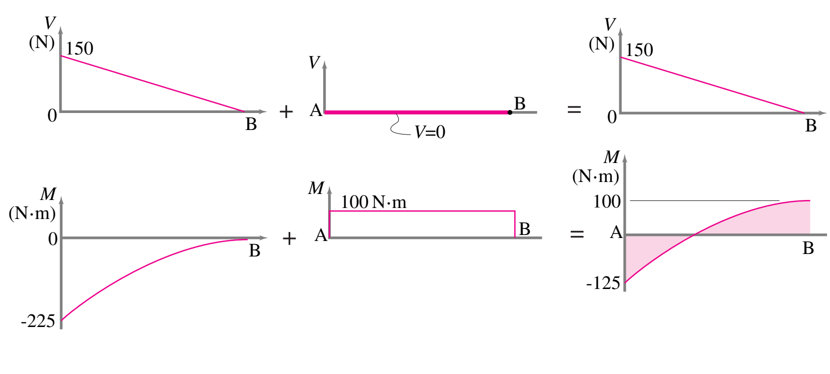


Figure 7.38:

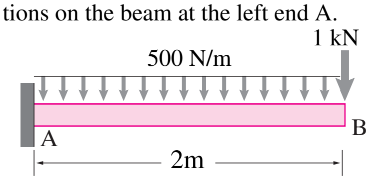
Problems for Chapter 7

Tension, shear, and bending diagrams

7.1 Shear force, bending moment and tension diagrams

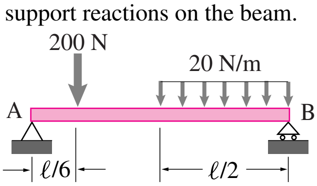
Preparatory Problems

7.1.1 A cantilever beam AB is loaded as shown in the figure. Find the support reactions on the beam at the left end A.



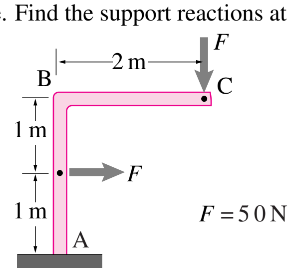
Filename:pfigure4-4-beamcl1
Problem 7.1.1

7.1.2 A simply supported beam AB of length $\ell = 6\,\mathrm{m}$ is partly loaded with a uniformly distributed load as shown in the figure. In addition, there is a concentrated load acting at $\ell/6$ from the left end A. Find the support reactions on the beam.



Filename:pfigure4-4-beamss2
Problem 7.1.2

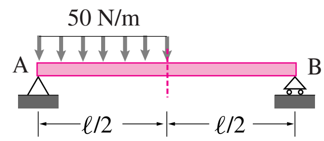
7.1.3 An (inverted) L-shaped frame is loaded with two equal concentrated forces of magnitude 50 N each as shown in the figure. Find the support reactions at A.



Filename:pfigure4-4-frame3
Problem 7.1.3

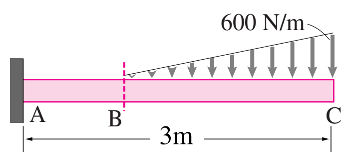
More-Involved Problems

7.1.4 Find the shear force and the bending moment at the mid section of the simply supported beam shown in the figure.



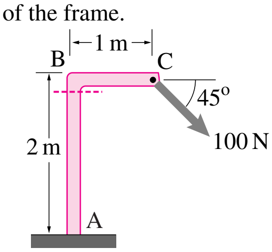
Filename:pfigure4-4-beamss4
Problem 7.1.4

7.1.5 A cantilever beam ABC is loaded with a linearly variable distributed load along two thirds of its span. The intensity of the load at the right end is $600\,\mathrm{N}/m$. Find the shear force and the bending moment at section B of the beam.



Filename:pfigure4-4-beamcl5
Problem 7.1.5

7.1.6 Analyze the frame shown in the figure and find the shear force and the bending moment at the end of the vertical section of the frame.

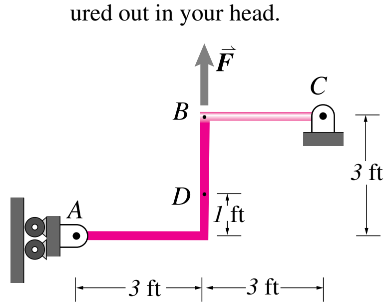


Filename:pfigure4-4-frame6
Problem 7.1.6

7.1.7 A force $F = 100\,\mathrm{lbf}$ is applied to the bent rod shown. Before doing any calculations, try to figure out the tension at D in your head.

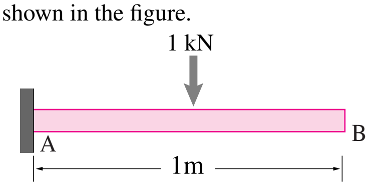
a) Find the reactions at A and C.

b) Find the tension, shear and bending moment at the section D. Check your answer against what you figured out in your head.



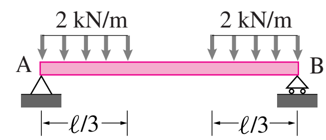
Filename:pfigure-bentrod
Problem 7.1.7

7.1.8 Draw the shear force and the bending moment diagram for the cantilever beam shown in the figure.



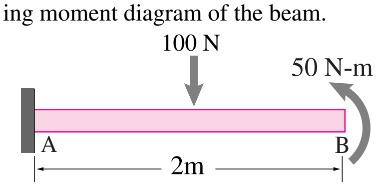
Filename:pfigure4-4-beamcl7
Problem 7.1.8

7.1.9 A simply supported beam AB is loaded along one thirds of its span from both ends by a uniformly distributed load of intensity $2\,\mathrm{kN/m}$. Draw the shear force and the bending moment diagram of the beam.



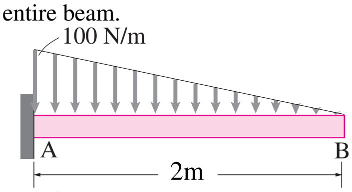
Filename:pfigure4-4-beamss8
Problem 7.1.9

7.1.10 The cantilever beam shown in the figure is loaded with a concentrated load and a concentrated moment as shown in the figure. Draw the shear force and the bending moment diagram of the beam.



Filename:pfigure4-4-beamcl9
Problem 7.1.10

7.1.11 A cantilever beam AB is loaded with a triangular shaped distributed load as shown in the figure. Draw the shear force and the bending moment diagrams for the entire beam.

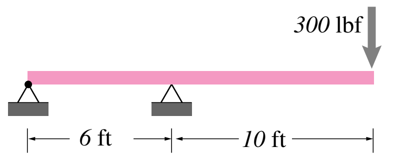


Filename:pfigure4-4-beamcl10
Problem 7.1.11

7.1.12 A regulation 16 ft diving board is supported as shown.

a) Where is the bending moment the greatest and how big is it there?

b) Draw a bending moment diagram for this board.



Filename:pfigure-divingboard
Problem 7.1.12

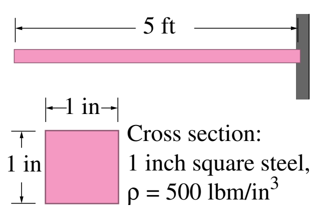
7.1.13 The cantilever steel beam is loaded by its own weight.

a) Find the bending moment and shear force at the free and at the clamped end.

b) Draw a shear force diagram

c) Draw a bending moment diagram

d) The tension stress σ in the beam at the top edge where it is biggest is given by $\sigma = 12M/h^3$ where $h = 1$ in for this beam. The strength (the maximum tension stress the material can bear) of soft steel is about $\sigma_{max} = 30,000\,\mathrm{lbf/in^2}$. What is the longest a beam with this cross section be made and still not fail?

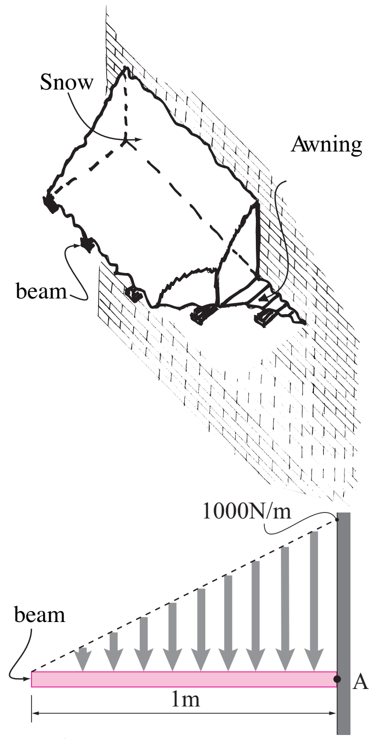


Filename:pfigure-bendingownweight
Problem 7.1.13

7.1.14 A snow loaded bus-stop awning (shown partially cut away) on the side of a building is supported by horizontal, cantilevered, beams. The loading that is carried by one beam is as shown below.

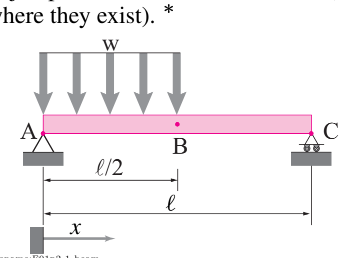
a) Find the reaction force and couple at the wall at A (the force and moment acting on one beam from the wall). *

b) Draw shear force and bending moment diagrams for the beam.



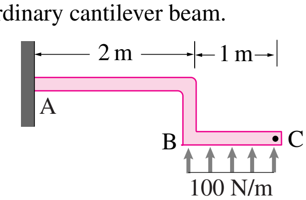
Filename:F02final4-awning
Problem 7.1.14

7.1.15 Draw shear and bending moment diagrams of the beam shown. Clearly label the values of the heights of the curves at jumps, kinks and local maxima (if and where they exist). *



Filename:F01p2-1-beam
Problem 7.1.15

7.1.16 A frame ABC is much like a cantilever beam with a short bent section of length 0.5 m. The frame is loaded as shown in the figure. Draw the shear force and the bending moment diagrams of the entire frame indicating how it differs from an ordinary cantilever beam.

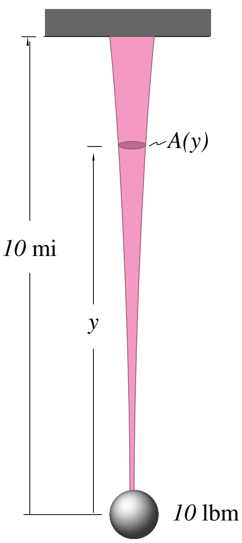


Filename:pfigure4-4-frame11
Problem 7.1.16

7.1.17 A 10 pound ball is suspended by a long steel wire. The wire has a density of about $500\,\mathrm{lbm/ft^3}$. The strength of the wire (the maximum force per unit area it can carry) is about $\sigma_{max} = 60,000\,\mathrm{lbf/in^2}$.

a) First, neglecting the weight of the wire in the calculation of stress, what is the weight of wire needed to hold the weight?

b) Taking account the weight of the wire in the load calculation, what is the weight of wire needed to hold the weight? *



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Problem 7.1.17

CHAPTER 8

Hydrostatics

Hydrostatics concerns the equivalent force and moment due to distributed pressure on a surface from a still fluid. Pressure increases with depth. With constant pressure the equivalent force has magnitude = pressure times area, acting at the centroid. For linearly-varying pressure on a rectangular plate the equivalent force is the average pressure times the area acting 2/3 of the way down. The net force acting on a totally submerged object in a constant density fluid is the displace weight acting at the centroid.

Contents

Hydrostatics is primarily concerned with finding the net force and moment of still fluid on a surface. The surfaces are typically the sides of a pool, dam, container, or pipe, or the outer surfaces of a floating object such as a boat or of a submerged object like a toilet bowl float. Finally, one is sometimes concerned with the force on an imagined surface that separates some water of interest from the other water. Although the hydrostatics of air helps explain the floating of hot air balloons, dirigibles, and chimney smoke; and the hydrostatics of oil is important for hydraulics (hydraulic brakes for example), often the fluid of concern for engineers is water. So, as in the title of the chapter ('hydro'), we often use the word 'water' as an informal synonym for 'fluid.'

Besides the utility of the subject in applications, hydrostatics is also a good introduction to distributed forces and continuum mechanics.

8.1 Fluid pressure

Besides the basic laws of mechanics that you already know, elementary hydrostatics is based on the following two constitutive assumptions (see page 25):

1) The force of water on a surface is perpendicular to the surface; and
2) The density of water, ρ (pronounced 'row') is a constant (doesn't vary with depth or pressure),

Sometimes we use the weight density $\gamma = g\rho$ (pronounced 'gammuh equals gee row'), the weight per unit volume . The first assumption, that all static water forces are perpendicular to surfaces on which they act, can be restated:

Still water cannot carry any shear stress.

For near-still water this constitutive assumption is abnormally accurate (compared to most constitutive assumptions for materials), approximately as good as the laws of mechanics.

The assumption of constant density is called *incompressibility* because it corresponds to the idea that water does not change its volume (compress) much under pressure. This assumption is reasonable for most purposes. At the bottom of the deepest oceans, for example, the extreme pressure (about 800 atmospheres) causes water to increase its density only about 4% from that of water at the surface①

We also assume that the direction and magnitude of the local gravitational constant is, well, constant. This assumption becomes inaccurate when considering, say, the hydrostatics of whole oceans (the direction of the gravity force changes as you go around the word, this helps keep the Australians in place), or of the upper atmosphere (the magnitude of the gravity decays with distance from the center of the earth).

① That fluid density does depend on salinity, temperature and pressure is sometimes important in hydrostatics. In particular for determining which water floats on which other water. This is important in the ecology of lakes, the effects of the oceans on climate, and in air for the stability of the atmosphere, and the mechanics of fireplace chimneys.

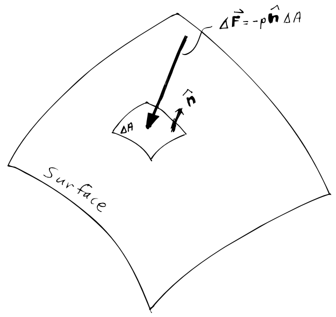

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Figure 8.1: A bit of area ΔA on a surface on which pressure p acts. The outward (into the water) normal of the surface is $\hat{\boldsymbol{n}}$ so the force is $\Delta\vec{\boldsymbol{F}} = -p\hat{\boldsymbol{n}}\Delta A$.

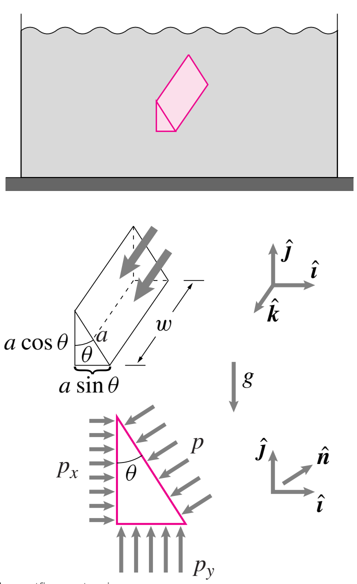

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Figure 8.2: A small prism of water is isolated from some water in equilibrium. The free body diagram does not show the forces in the z direction. Force balance applied to this free body diagram shows that $p_x = p_y = p$, pressure is the same in all directions.

Surface area A, outward normal $\hat{\boldsymbol{n}}$, pressure p, and force $\vec{\boldsymbol{F}}$

We are going to be generalizing the high-school physics fact

$$\text{force} = \text{pressure} \times \text{area}$$

to take account that force is a vector, that pressure varies with position, and that not all surfaces are flat. So we need a clear notation and sign convention. The area of a surface is A which we can think of as being the sum of the bits of area ΔA that compose it:

$$A = \int dA.$$

Every bit of surface area has an *outer* normal $\hat{\boldsymbol{n}}$ that points from the surface out into the fluid. The (scalar) force per unit area on the surface is called the pressure p, so that the force on a small bit of surface is

$$\Delta\vec{\boldsymbol{F}} = p\,(-\hat{\boldsymbol{n}})\,(\Delta A)$$

pointing into the surface, assuming positive pressure, and with magnitude proportional to both pressure and area. Thus the total force and moment due to pressure forces on a surface :

$$\begin{aligned} \vec{\boldsymbol{F}} &= \int d\vec{\boldsymbol{F}} &&= -\int_A p\,\hat{\boldsymbol{n}}\;dA \\ \vec{\boldsymbol{M}}_{\mathrm{C}} &= \int_A d\vec{\boldsymbol{M}}_{/\mathrm{C}} &&= -\int_A \vec{\boldsymbol{r}}_{/\mathrm{C}} \times (p\,\hat{\boldsymbol{n}})\;dA \end{aligned} \tag{8.1}$$

Hydrostatics is the evaluation of the (intimidating-at-first-glance) integrals 8.1 and their role in equilibrium equations. In the rest of this section we consider a variety of important special cases.

Water in equilibrium with itself

Before we worry about how water pushes on other things, let's first understand what it means for water to be in static equilibrium. These first important facts about hydrostatics follow from drawing free body diagrams of various chunks of water and assuming static equilibrium (see box 8.2 on page 401).

1. Pressure is the same in every direction, $p_x = p_y = p$.
2. Pressure doesn't vary with side to side position, $p(x, y, z) = p(y)$.
3. Pressure varies linearly with depth, $p = \rho g h = \gamma h$.

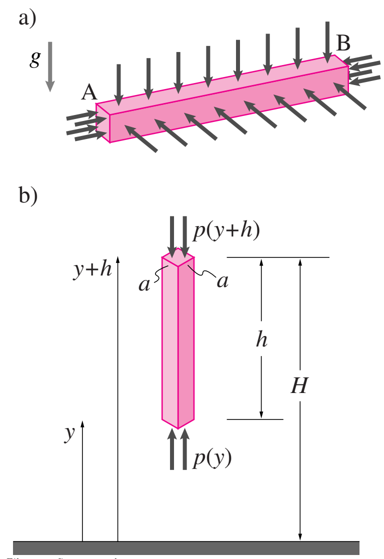


Figure 8.3: Free body diagrams of aligned boxes of water cut out of a bigger body of water. a) a horizontally aligned box, b) a vertically aligned box. Force balance on applied to these free body diagrams shows that $p = p(y)$.

The buoyant force of water on water.

In a place under water in a still swimming pool where there is nothing but water, imagine a chunk of water the shape of a sea monster. Now draw a free body diagram of that water. Because your sea monster is in equilibrium, force balance and moment balance must apply. The only forces are the complicated distribution of pressure forces and the weight of water. The pressure forces must exactly cancel the weight of the water and, to satisfy moment balance, must pass through the center-of-mass of the water monster. So, in static equilibrium:

The pressure forces acting on a surface enclosing a volume of water is equivalent to the negative weight passing through the center-of-mass of the water.

The force of water on submerged and floating objects

The net pressure force and moment on a still object surrounded by still water can be found by a clever argument credited to Archimedes. The pressure at any one point on the outside of the object does not depend on what's inside. The pressure is determined by how far the point of interest is below the surface by eqn. 8.2②. So if you can find the resultant force on any object that is the shape of the submerged object, but replacing the submerged object, it tells you what you want to know.

The clever idea is to replace your object with water. In this new system the water is in equilibrium, so the pressure forces exactly balance the weight. We thus obtain Archimedes' Principle:

The resultant of all pressure forces on a totally submerged object is an upwards force with the same magnitude as the weight of the displaced water. The resultant acts at the centroid of the displaced volume:

$$\vec{F}_{\text{buoyancy}} = \gamma V \hat{\jmath} \qquad \text{acting at} \qquad \vec{r} = \frac{\int \vec{r}_{/0}\, dV}{V}.$$

②If there is no column of water from the point up to the surface it is still true that the pressure is γh, as you can figure out by tracking the pressure changes along on a staircase-like path from the surface to that point.

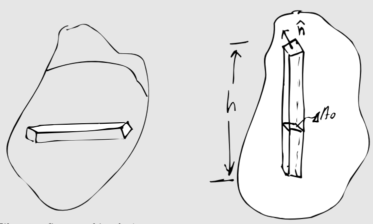
8.1 Adding forces to derive Archimedes' principle

(One can do most hydrostatics calculations, say typical homework problems, without being able to reproduce the derivations here.)

Archimedes' principle follows from adding up all the pressure forces on the outer surfaces of an arbitrarily shaped submerged solid, say something potato shaped.

First we find the answer by cutting the potato into french fries. This approach is effectively a derivation of a theorem in vector calculus. After that, for those who have the appropriate math background, we quote the vector calculus directly.

First cut the potato into horizontal french-fries (horizontal prisms) and look at the forces on the end caps (there are no water forces on the sides since those are inside the potato).



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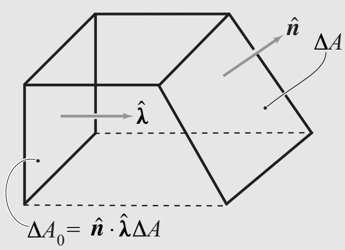
The pressure on two ends is the same (because they have the same water depth). The areas on the two ends are probably different because your potato is probably not box shaped. But the area is bigger at one end if the normal to the surface is more oblique compared to the axis of the prism. If the cross sectional area of the prism is ΔA_0 then the area of one of the prism caps is



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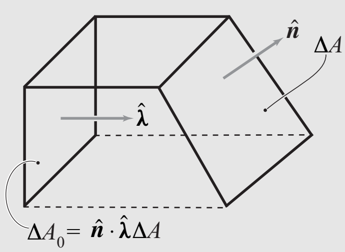
$\Delta A = \Delta A_0 / (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$ where $\hat{\boldsymbol{\lambda}}$ is along the axis of the prism and $\hat{\boldsymbol{n}}$ is the outer unit normal to the end cap (Note $\Delta A \geq \Delta A_0$ because $\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}} \leq 1$).



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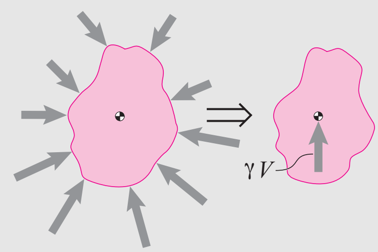
So the net force on the cap is $-p\Delta A_0 \hat{\boldsymbol{n}} / (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})$. The component of the force along the prism is $\left[-p\Delta A_0 \hat{\boldsymbol{n}} / (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\lambda}})\right] \cdot \hat{\boldsymbol{\lambda}}$ which is $-p\Delta A_0$. An identical calculation at the other end of the french fry gives minus the same answer. So the net force of the water pressure along the prism is zero for this and every prism and thus the whole potato. Likewise for prisms with any horizontal orientation. Thus the net sideways force of water on any submerged object is zero.

To find the net vertical force on the potato we cut it into vertical french fries. The net forces on the end caps are calculated just as in the above paragraph but taking account that the pressure on the bottom of the french fry is bigger than at the top. The sum of the forces of the top and bottom caps is an upwards force that is

$$
\begin{aligned}
\text{net upwards force on vertical french fry} &= \Delta p \Delta A_0 \\
&= (\gamma h) \Delta A_0 \\
&= \gamma (h \Delta A_0) \\
&= \gamma \, \Delta V_0
\end{aligned}
$$

where ΔV_0 is the volume of the french fry. Adding up over all the french fries that make up the potato one gets that the net upwards force is γV The net result, summarized by the figure below, is that the resultant of the pressure forces on a submerged solid is an upwards force whose magnitude is the weight of the displaced water. The location of the force is the centroid of the displaced volume. (Note that the centroid of the displaced volume is not necessarily at the center of mass of the submerged object.)



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A vector calculus derivation

Here is a derivation of Archimedes' principle, at least the net force part, using multi-variable integral calculus. Only read on if you have taken a math class that covers the divergence theorem. The net pressure force on a submerged object is

$$
\begin{aligned}
\vec{\boldsymbol{F}}_{\text{buoyancy}} &= -\textstyle\int_A \quad p \quad \hat{\boldsymbol{n}} \quad dA \\
&= -\textstyle\int_S \quad p \quad \hat{\boldsymbol{n}} \quad dS \\
&= -\textstyle\int_S \quad (H-z)\gamma \quad \hat{\boldsymbol{n}} \quad dS \\
&= -\textstyle\int_V \quad \vec{\nabla}((H-z)\gamma) \quad dV \\
&= -\textstyle\int_V \quad (-\hat{\boldsymbol{k}})\,\gamma \quad dV \\
&= \textstyle\int_V \quad \gamma \quad dV \quad \hat{\boldsymbol{k}} \\
&= \text{(weight of displaced water)} \quad \hat{\boldsymbol{k}}.
\end{aligned}
$$

In this derivation we first changed from calling bits of surface area $dA$to dS because that is a common notation in calculus books. The depth from the surface, of a point with vertical component z from the bottom, is $H - z$. The $\vec{\nabla}$ symbol indicates the gradient and its place in this equation is from the divergence theorem:

$$
\int_S (\text{any scalar}) \hat{\boldsymbol{n}} \, dS = \int_V \vec{\nabla}(\text{the same scalar}) \, dV.
$$

The gradient of $(H - z)\gamma$ is $-\hat{\boldsymbol{k}}\gamma$ because H and γ are constants. Note, where we write $\int_S$ some books would write $\iint_S$, and where we write $\int_V$ some books would write $\iiint_V$.

8.2 Pressure doesn't depend on direction or horizontal position and increases linearly with depth

We assume that the pressure p does not vary too wildly from point to point, thus if we look at a small enough region we can think of the pressure as constant in that region. If we draw a free body diagram of a little triangular prism of water the net forces on the prism must add to zero (see fig. 8.2 on page 8.2). For each surface the magnitude of the force is the pressure times the area of the surface and the direction is minus the outward normal of the surface. We assume, for the time being, that the pressure is different on the differently oriented surfaces. So, for example, because the area of the left surface is $a\cos\theta w$ and the pressure on the surface is p_x, the net force is $a\cos\theta w p_x\hat{\imath}$. Calculating similarly for the other surfaces:

$$\begin{aligned}
\vec{\mathbf{0}} &= \sum \vec{F}_i \\
&= \underbrace{(a\cos\theta)w\; p_x\,\hat{\imath} + (a\sin\theta)w\; p_y\,\hat{\jmath} - a\,w\,p\,\hat{n}}_{\text{pressure terms}} \\
&\qquad - \underbrace{\frac{a^2\cos\theta\sin\theta w}{2}\rho g}_{\text{weight}}\,\hat{\jmath} \\
&= aw\left(\cos\theta\; p_x\,\hat{\imath} + \sin\theta\; p_y\,\hat{\jmath} - p\,\underbrace{(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath})}_{\hat{n}} - \frac{a\cos\theta\sin\theta}{2}\rho g\,\hat{\jmath}\right)
\end{aligned}$$

If a is arbitrarily small, the weight term drops out compared to the pressure terms. Dividing through by aw we get

$$\vec{\mathbf{0}} = \cos\theta p_x\hat{\imath} + \sin\theta\; p_y\,\hat{\jmath} - p\,(\cos\theta\hat{\imath} + \sin\theta)\,\hat{\jmath}.$$

Taking the dot product of both sides of this equation with $\hat{\imath}$ and $\hat{\jmath}$ gives that

$$p = p_x = p_y.$$

Since θ could be anything, force balance for the free body diagram of a small prism tells us that for a fluid in static equilibrium

> pressure is the same in every direction.

[Other free body diagrams can be used. That pressure has to be the same in any pair of directions could also be found by drawing a prism with a cross section which is an isosceles triangle. The prism is oriented so that two surfaces of the prism have equal area and have the desired orientations. Force balance along the base of the triangle gives that the pressures on the equal area surfaces are equal. The argument that pressure must not depend on direction in 3D is generally based on equilibrium of a small tetrahedron.]

Pressure doesn't vary with side to side position Consider the equilibrium of a horizontally aligned box of water cut out of a bigger body of water (fig. 8.3a on page 399). The forces on the end caps at A and B are the only forces along the box. Therefor they must cancel. Since the areas at the two ends are the same, the pressure must be also. This box could be anywhere and at any length and any horizontal orientation. Thus for a fluid in static equilibrium

> pressure doesn't depend on horizontal position.

If we take the $\hat{\jmath}$ or y direction to be up (fig. 8.3 on page 399), then we have

$$p(x, y, z) = p(y).$$

Pressure increases linearly with depth Consider the vertically aligned box of fig. 8.3b.

$$\begin{aligned}
\left\{\sum \vec{F}_i = \vec{\mathbf{0}}\right\}\cdot\hat{\jmath} \quad &\Rightarrow \quad \underbrace{p(y)a^2 - p(y+h)a^2}_{\text{pressure terms}} - \underbrace{\rho g a^2 h}_{\text{weight}} = 0 \\
&\Rightarrow \quad p_{\text{bottom}} - p_{\text{top}} = \rho g h.
\end{aligned}$$

So the pressure increases linearly with depth. If the top of a lake, say, is at atmospheric pressure p_a then we have that

$$p = p_a + \rho g h = p_a + \gamma h = p_a + (H - y)\gamma$$

where h is the distance down from the surface, H is the depth to some reference point underwater and y is the distance up from that reference point (so that $h = H - y$). Neglecting atmospheric pressure at the top surface we have the useful and easy to remember formula:

$$p = \gamma h. \tag{8.2}$$

Because the pressure at equal depths must be equal and because the pressure at the top surface must be equal to atmospheric pressure, the top surface must be flat and level. Thus waves and the like are a definite sign of static disequilibrium as are any bumps on the water surface even if they don't seem to move (as for a bump in the water where a stream goes steadily over a rock).

The result can also be found by adding the effects of all the pressure forces on the outside surface (see box 8.1 on page 400).

For floating objects, the same argument can be carried out, but since the replaced fluid has to be in equilibrium we cannot replace the whole object with fluid, but only the part which is below the level of the water surface.

Displaced fluid

Sometimes people discuss Archimedes' principle in terms of the displaced fluid. A floating object in equilibrium displaces an amount of fluid with the same weight as the object; this is also the amount of volume of the floating object that is below the water level. On the other hand an object that is totally under water, for whatever reason (it is resting on the bottom, or it is being held underwater by a string, etc), displaces as much fluid as the space it occupies. Putting these two ideas together one can remember that

> A floating object displaces its weight, a submerged object displaces its volume.

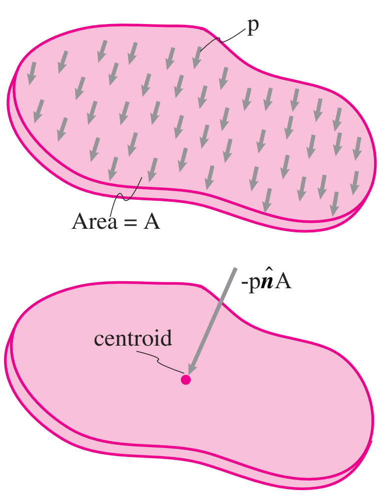

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Figure 8.4: The resultant force from a constant pressure p on a flat plate is $\vec{F} = -pA\hat{n}$ acting at the centroid of the plate.

The force of constant pressure on a totally immersed object

When there is no gravity, or gravity is neglected, the pressure in a static fluid is the same everywhere. Exactly the same argument we have just used shows that the resultant of the pressure forces is zero. We could derive this result just by setting $\gamma = 0$ in the formulas above.

The force of constant pressure on a flat surface

The net force of constant pressure on one flat surface (not all the way around a submerged volume) is the pressure times the area acting normal to the surface at the centroid of the surface:

$$\begin{aligned} \vec{F}_{\text{net}} &= \int_A -p\,\hat{n}\,dA \\ &= -pA\hat{n}. \end{aligned}$$

That this force acts at the centroid can be checked by calculating the moment of the pressure forces relative to the centroid C,

$$\begin{aligned} \vec{M}_{/\text{C,net}} &= \int_A \vec{r}_{/\text{C}} \times (-p\,\hat{n}\,dA) \\ &= \underbrace{\left(\int_A \vec{r}_{/\text{C}}\,dA\right)}_{0} \times (-p\,\hat{n}) \\ &= 0\ . \end{aligned}$$

where the zero follows from the position of the center-of-mass relative to the center-of-mass being zero.

The force of water on a rectangular plate

Consider a rectangular plate with width into the page w and length ℓ. Assume the water-side normal to the plate is $\hat{\boldsymbol{n}}$ and that the top edge of the plate is horizontal. Take $\hat{\boldsymbol{\jmath}}$ to be the up direction with y being distance up from the bottom and the total depth of the water is H. Thus the area of the plate is $A = \ell w$. If the bottom and top of the plate are at y_1 and y_2 the net force on the plate can be found as:

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -\int_A p\hat{\boldsymbol{n}}\,dA \\
&= -\int_A \gamma(H-y)\hat{\boldsymbol{n}}\,dA \\
&= -w\int_0^\ell \gamma(H-y(s))\hat{\boldsymbol{n}}\,ds \\
&= -w\int_0^\ell \gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,s)\right)\hat{\boldsymbol{n}}\,ds \\
&= -w\gamma\left(H\ell-y_1\ell-\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell^2/2\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\jmath}}\,\ell/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\gamma\left(H-(y_1+(y_2-y_1)/2)\right)\hat{\boldsymbol{n}} \\
&= -w\ell\left(\gamma(H-y_1)/2+\gamma(H-y_2)/2\right)\hat{\boldsymbol{n}}
\end{aligned}$$

So

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{net}} &= -w\ell\frac{p_1+p_2}{2}\,\hat{\boldsymbol{n}}. \\
&= -(\text{area})(\text{average pressure})(\text{outwards normal direction}).
\end{aligned}$$

The net water force is the same as that of the average pressure acting on the whole surface. To find where it acts it is easiest to think of the pressure distribution as the sum of two different pressure distributions. One is a constant over the plate at the pressure of the top of the plate. The other varies linearly from zero at the top to $\gamma(y_2-y_1)$ at the bottom.

$$p = \gamma(H-y) = \underbrace{\gamma(H-y_2)}_{\text{Constant pressure, the pressure at the top edge.}} + \underbrace{\gamma(y_2-y)}_{\text{Varies linearly from 0 at the top to } \gamma(y_2-y_1) \text{ at the bottom.}}$$

The first corresponds to a force of $w\ell\gamma(H-y_2)$ acting at the middle of the plate. The second corresponds to a force of $w\ell\gamma\frac{y_2-y_1}{2}$ acting a third of the way up from the bottom of the plate.

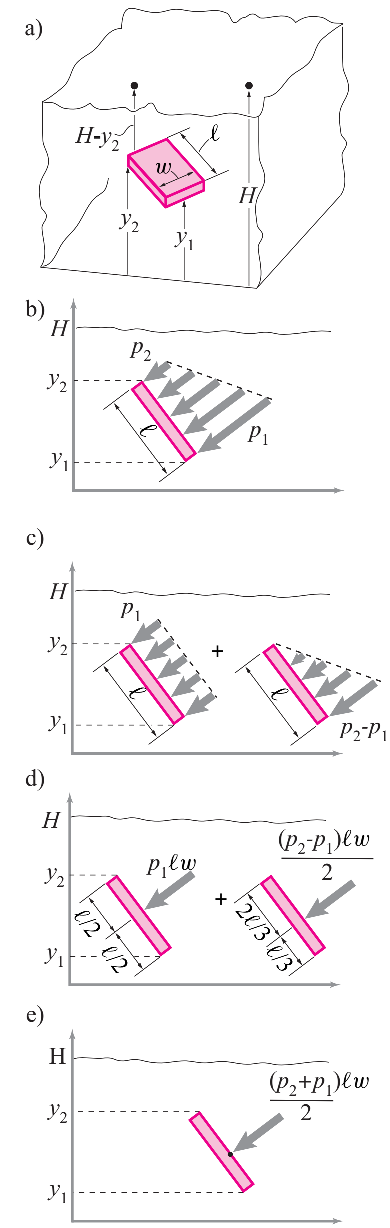

Figure 8.5: The resultant force from a constant depth-increasing pressure on a rectangular plate.

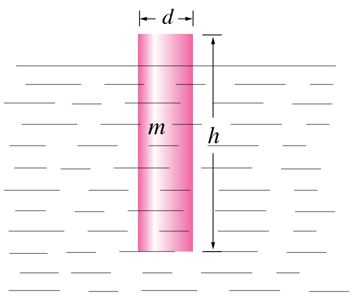

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Figure 8.6:

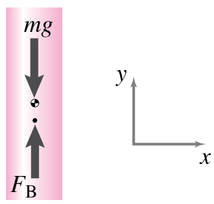

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Figure 8.7:

SAMPLE 8.1 A uniform solid cylinder of mass $m = 12\,\text{kg}$, diameter $d = 0.1\,\text{m}$ and height $h = 2\,\text{m}$ floats in water (density $\rho = 1000\,\text{kg/m}^3$).

1. Assuming the cylinder floats vertically, find the submerged height of the cylinder.
2. If the cylinder floats longitudinally (its longitudinal axis parallel to the water surface), what will be the submerged section of the cylinder?

Solution

1. **Cylinder floating vertically:** Let h_s be the submerged height of the cylinder and $r = d/2$ be its radius. Then the force of buoyancy F_B is equal to the weight of water replaced by the submerged volume of the cylinder. Thus,

$$\vec{F}_B = \overbrace{\pi r^2 h_s}^{\text{volume}} \gamma\,\hat{\jmath}.$$

From the force balance on the cylinder (see the free-body diagram in fig. 8.7),

$$\begin{aligned} \vec{F}_B - mg\hat{\jmath} &= \vec{0} \\ \Rightarrow \quad (\pi r^2 h_s \gamma - mg)\hat{\jmath} &= \vec{0} \\ \Rightarrow \quad h_s &= \frac{mg}{\pi r^2 \gamma} = \frac{m}{\pi r^2 \rho} \\ &= \frac{12\,\text{kg}}{\pi \cdot (0.05\,\text{m})^2 \cdot 1000\,\text{kg/m}^3} = 1.53\,\text{m}. \end{aligned}$$

$$\boxed{h_s = 1.53\,\text{m}}$$

2. **Cylinder floating horizontally:** No matter how the cylinder floats, the force of buoyancy has to equal the weight of the cylinder. This force is equal to the weight of the displaced water. Thus, the volume of displaced water has to be the same no matter what the orientation of the cylinder is with respect to the water surface. Therefore, the submerged volume of the cylinder while floating longitudinally must equal the volume submerged while floating vertically. That is (see fig. 8.8),

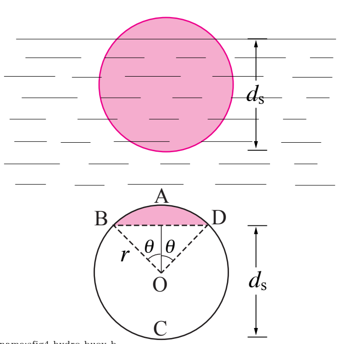
$$\text{area of BCD} \cdot h = \pi r^2 h_s \quad \Rightarrow \quad \text{area of BCD} = \pi r^2 (h_s/h) = 0.006\,\text{m}^2.$$

Now we can figure out what d_s should be so that the submerged area is 76% of the total cross sectional area. This is an exercise in geometry. Since, area of BCD $= \pi r^2 -$ area of ABD, area of ABD $= \pi r^2 -$ area of BCD $= 0.018\,\text{m}^2$. But the area of ABD is the area of the circular sector OBAD ($r^2\theta$) minus the area of triangle OBD ($\frac{1}{2} \cdot r\cos\theta \cdot 2r\sin\theta$). Thus,

$$\begin{aligned} \text{area of ABD} \equiv r^2\theta - \frac{1}{2}r^2 \sin 2\theta &= 0.018\,\text{m}^2 \\ \Rightarrow \quad \theta - \frac{1}{2}\sin 2\theta - 0.738 &= 0. \end{aligned}$$

We need to solve this nonlinear equation. Using trial and error or root finding on a computer or a graphical method, we find $\theta = 1.126\text{rad} = 64.5°$. Using this value, we get, $d_s = r + r\cos\theta = 0.07\,\text{m}$.

$$\boxed{d_s = 0.07\,\text{m}}$$



Filename:sfig4-hydro-buoy-b

Figure 8.8:

SAMPLE 8.2 **The force due to varying hydrostatic pressure:** The hydrostatic pressure distribution on the face of a wall submerged in water up to a height $h = 10\,\text{m}$ is shown in the figure. Find the net force on the wall from water. Take the length of the wall (into the page) to be 1 m.

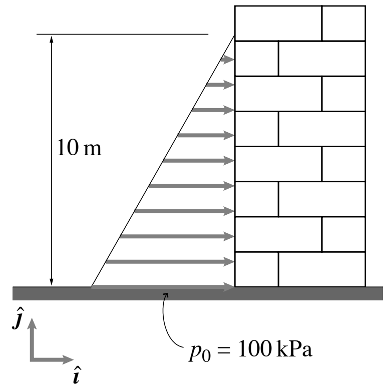


Figure 8.9:

Solution Since the pressure varies across the height of the submerged part of the wall, let us take an infinitesimal strip of height dy along the full length ℓ of the wall as shown in fig. 8.10. Since the height of the strip is infinitesimal, we can treat the water pressure on this strip to be essentially constant and equal to $p_0 \frac{y}{h}$. Then the force on the strip (of area ℓdy) due to the constant water pressure $p(y) = p_0 y/h$ is

$$d\vec{F} = (p(y) \cdot \overbrace{\ell dy}^{\text{area}})\,\hat{\imath} = p_0 \frac{y}{h}\,\ell\,dy\,\hat{\imath}.$$

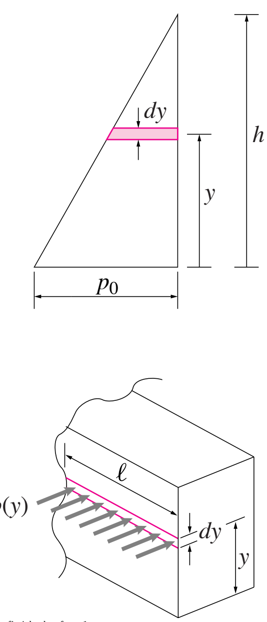


Figure 8.10:

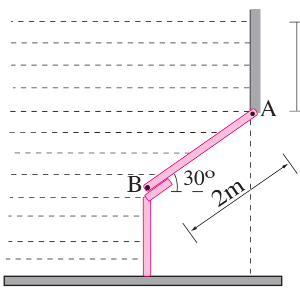
The net force due to the pressure distribution on the whole wall can now be found by integrating $d\vec{F}$ along the wall.

$$\begin{aligned}
\vec{F} &= \int d\vec{F} = \int_0^h p_0 \frac{y}{h}\,\ell\,dy\,\hat{\imath} \\
&= \left(p_0 \frac{\ell}{h} \int_0^h y\,dy\right)\hat{\imath} = p_0 \frac{\ell}{h}\frac{h^2}{2}\hat{\imath} \\
&= \frac{1}{2} p_0 h \ell\,\hat{\imath} \\
&= \frac{1}{2} \cdot (100 \frac{\text{kN}}{\text{m}^2}) \cdot (10\,\text{m}) \cdot (1\,\text{m})\hat{\imath} \\
&= (500\,\text{kN})\,\hat{\imath}.
\end{aligned}$$

$$\vec{F} = 500\,\text{kN}\,\hat{\imath}$$

Alternatively, the net force can be computed by calculating the area of the pressure triangle and multiplying by the unit length ($\ell = 1\,\text{m}$), *i.e.,*

$$\begin{aligned}
\vec{F} &= \overbrace{\frac{1}{2} \cdot h \cdot p_0}^{\text{triangle area}}\ \ell\,\hat{\imath} \\
&= \left(\frac{1}{2} \cdot 10\,\text{m} \cdot 100 \frac{\text{kN}}{\text{m}^2} \cdot 1\,\text{m}\right)\hat{\imath} \\
&= 500\,\text{kN}\,\hat{\imath}.
\end{aligned}$$



Filename:sfig4-hydro-gate

Figure 8.11:

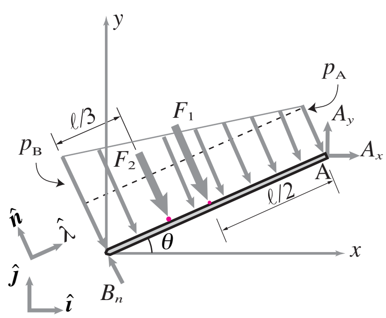


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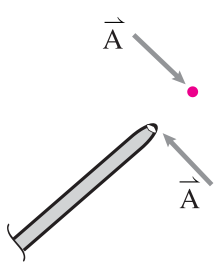
Figure 8.12: The free body diagram of the gate (plate AB) with distributed water pressure.



Filename:sfig4-hydro-gate-b

Figure 8.13: The free body diagram of the pin at A.

SAMPLE 8.3 Forces on a submerged sluice gate: A rectangular plate is used as a gate in a tank to prevent water from draining out. The plate is hinged at A and rests on a frictionless surface at B. Assume the width of the plate to be 1 m. The height of the water surface above point A is h. Ignoring the weight of the plate, find the forces on the hinge at A as a function of h. In particular, find the vertical pull on the hinge for $h = 0$ and $h = 2\,\text{m}$.

Solution Let $\gamma = \rho g$ be the weight density (weight per unit volume) of water. Then the pressure due to water at point A is $p_A = \gamma h$ and at point B is $p_B = \gamma(h + \ell \sin\theta)$. The pressure acts perpendicular to the plate and varies linearly from p_A at A to p_B at B. The free-body diagram of the plate is shown in fig. 8.12. Let $\hat{\boldsymbol{\lambda}}$ be a unit vector along BA and $\hat{\boldsymbol{n}}$ be a unit vector normal to BA. For computing the reaction forces on the plate at points A and B, we first replace the distributed pressure on the plate by two equivalent concentrated forces F_1 and F_2 by dividing the pressure distribution into a rectangular and a triangular region and finding their resultants.

$$F_1 = p_A \ell = \gamma h \ell, \quad F_2 = (p_B - p_A)\frac{\ell}{2} = \frac{1}{2}\gamma \ell^2 \sin\theta.$$

Now, we carry out moment balance about point A, $\sum \vec{\boldsymbol{M}}_{\text{A}} = \vec{\boldsymbol{0}}$, which gives

$$\begin{aligned}
\vec{r}_{\text{B/A}} \times \vec{\boldsymbol{B}} + \vec{r}_{\text{D/A}} \times \vec{\boldsymbol{F}}_2 + \vec{r}_{\text{C/A}} \times \vec{\boldsymbol{F}}_1 &= \vec{\boldsymbol{0}} \\
-\ell\hat{\boldsymbol{\lambda}} \times B_n\hat{\boldsymbol{n}} - \frac{2\ell}{3}\hat{\boldsymbol{\lambda}} \times (-F_1\hat{\boldsymbol{n}}) - \frac{\ell}{2}\hat{\boldsymbol{\lambda}} \times (-F_2\hat{\boldsymbol{n}}) &= \vec{\boldsymbol{0}} \\
-B_n\ell\hat{\boldsymbol{k}} + F_1\frac{2\ell}{3}\hat{\boldsymbol{k}} + F_2\frac{\ell}{2}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{0}} \\
\Rightarrow \quad B_n = \frac{2F_1}{3} + \frac{F_2}{2} = \gamma\ell(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta)
\end{aligned}$$

and, from force balance, $\sum \vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$, we get

$$\begin{aligned}
\vec{\boldsymbol{A}} &= -B_n\hat{\boldsymbol{n}} + F_1\hat{\boldsymbol{n}} + F_2\hat{\boldsymbol{n}} \\
&= \left(-\gamma\ell(\frac{2}{3}h + \frac{1}{4}\ell\sin\theta) + \gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} \\
&= \left(\frac{1}{3}\gamma h\ell + \frac{1}{2}\gamma\ell^2\sin\theta\right)\hat{\boldsymbol{n}} = \gamma\ell(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta)\hat{\boldsymbol{n}}.
\end{aligned}$$

The force $\vec{\boldsymbol{A}}$ computed above is the force exerted by the hinge at A on the plate. Therefore, the force on the hinge, exerted by the plate, is $-\vec{\boldsymbol{A}}$ as shown in fig. 8.13. From the expression for this force, we see that it varies linearly with h.

Let the vertical pull on the hinge be A_{hinge_y}. Then

$$A_{\text{hinge}_y} = -\vec{\boldsymbol{A}} \cdot \hat{\boldsymbol{\jmath}} = -\gamma\ell(\frac{1}{3}h + \frac{1}{2}\ell\sin\theta)\overbrace{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{\jmath}}}^{\cos\theta} = \frac{1}{4}\gamma\ell\sin 2\theta + (\frac{1}{3}\gamma\ell\cos\theta)h.$$

Now, substituting $\gamma = 9.81\,\text{kN/m}^3$, $\ell = 2\,\text{m}$, $\theta = 30^\circ$, the two specified values of h, and multiplying the result (which is force per unit length) with the width of the plate (1 m) we get,

$$A_{\text{hinge}_y}(h = 0) = 4.25\,\text{kN}, \quad A_{\text{hinge}_y}(h = 2\,\text{m}) = 15.58\,\text{kN}.$$

$$\boxed{A_{\text{hinge}_y}|_{h=0} = 4.25\,\text{kN}, \quad A_{\text{hinge}_y}|_{h=2\,\text{m}} = 15.58\,\text{kN}}$$

SAMPLE 8.4 Tipping of a dam: The cross section of a concrete dam is shown in the figure. Take the weight-density $\gamma(=\ \rho g)$ of water to be $10\,\mathrm{kN}/\,\mathrm{m}^3$ and that of concrete to be $25\,\mathrm{kN}/\,\mathrm{m}^3$. For the given design of the cross-section, find the ratio h/H that is safe enough for the dam to not tip over (about the downstream edge E).

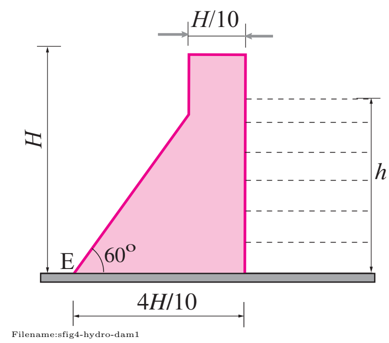


Figure 8.14:

Solution Let us imagine the critical situation when the dam is just about to tip over about edge E. In such a situation, the dam bottom would almost lose contact with the ground except along edge E. In that case, there is no force along the bottom of the dam from the ground except at E.③ With this assumption, the free-body diagram of the dam is shown in fig. 8.15.

③ This assumption is valid only if water does not leak through the edge B to the bottom of the dam. If it does, there would be some force on the bottom due to the water pressure. See the following sample where we include the water pressure at the bottom in the analysis.

To compute all the forces acting on the dam, we assume the width w (into the paper) to be unit (*i.e.,* $w = 1\,\mathrm{m}$). Let γ_w and γ_c denote the weight-densities of water and concrete, respectively. Then the resultant force from the water pressure is

$$F = \frac{1}{2}\gamma_w h \cdot h \cdot w = \frac{1}{2}\gamma_w h^2 w.$$

This is the horizontal force (in the -$\hat{\imath}$ direction) that acts through the centroid of triangle ABC.

To compute the weight of the dam, we divide the cross-section into two sections — the rectangular section CDGH and the triangular section DEF. We compute the weight of these sections separately by computing their respective volumes:

$$\begin{aligned} W_1 &= \underbrace{\alpha H^2 \cdot w}_{\text{volume}} \cdot \gamma_c = \gamma_c \alpha H^2 w \\ W_2 &= \underbrace{\frac{1}{2}\cdot 3\alpha H \cdot 3\alpha H \tan\theta \cdot w}_{\text{volume}} \cdot \gamma_c = \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta. \end{aligned}$$

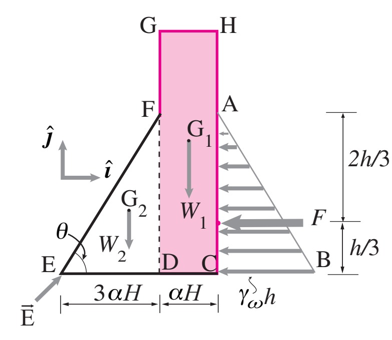


Figure 8.15:

Now we apply moment balance about point E, $\sum \vec{M}_{\mathrm{E}} = \vec{0}$, which gives

$$\begin{aligned} \vec{r}_{\mathrm{G}_1} \times \vec{W}_1 + \vec{r}_{\mathrm{G}_2} \times \vec{W}_2 + \vec{r}_{\mathrm{G}_3} \times \vec{F} &= \vec{0} \\ -(3\alpha H + \frac{1}{2}\alpha H) W_1 \hat{k} - \frac{2}{3}(3\alpha H) W_2 \hat{k} + \frac{h}{3} F \hat{k} &= \vec{0}. \end{aligned}$$

Dotting this equation with $\hat{k}$, we get

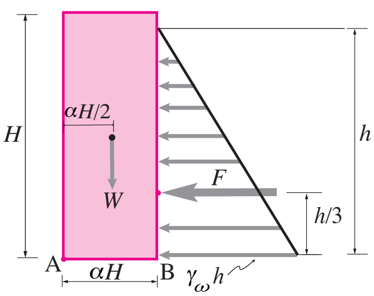
$$\begin{aligned} \frac{h}{3}F &= (3\alpha H + \frac{1}{2}\alpha H)\cdot \gamma_c \alpha H^2 w + \frac{2}{3}(3\alpha H)\cdot \frac{9}{2}\gamma_c \alpha^2 H^2 w \tan\theta \\ \frac{1}{2}\gamma_w \frac{h^3}{3} &= 9\gamma_c \alpha^3 H^3 \tan\theta + \frac{7}{2}\gamma_c \alpha^2 H^3 \\ \Rightarrow \quad \left(\frac{h}{H}\right)^3 &= \frac{\gamma_c}{\gamma_w}(54\alpha^3 \tan\theta + 21\alpha^2) \\ &= 2.5(54 \cdot 0.1^3 \cdot \sqrt{3} + 21 \cdot 0.1^2) = 0.7588 \\ \Rightarrow \quad \frac{h}{H} &= 0.91. \end{aligned}$$

Thus, for the dam to not tip over, $h \le 0.91H$ or 91% of H.

$$\frac{h}{H} \le 0.91$$

SAMPLE 8.5 Dam design: You are to design a dam of rectangular cross section ($b \times H$), ensuring that the dam does not tip over even when the water level h reaches the top of the dam ($h = H$). Take the specific weight of concrete to be 3. Consider the following two scenarios for your design.

1. The downstream bottom edge of the dam is plugged so that there is no leakage underneath.
2. The downsteram edge is not plugged and the water leaked under the dam bottom has full pressure across the bottom.



Filename:sfig4-hydro-dam2-a

Figure 8.16:

Solution Let γ_c and γ_w denote the weight densities of concrete and water, respectively. We are given that $\gamma_c/\gamma_w = 3$. Also, let $b/H = \alpha$ so that $b = \alpha H$. Now we consider the two scenarios and carry out analysis to find appropriate cross-section of the dam. In the calculations below, we consider unit length (into the paper) of the dam.

1. *No water pressure on the bottom:* When there is no water pressure on the bottom of the dam, then the water pressure acts only on the downstream side of the dam. The free-body diagram of the dam, considering critical tipping (just about to tip), is shown in fig. 8.16 in which F is the resultant force of the triangular water pressure distribution. The known forces acting on the dam are $W = \gamma_c \alpha H^2$, and $F = (1/2)\gamma_w h^2$. The moment balance about point A gives

$$
\begin{aligned}
F \cdot \frac{h}{3} &= W \cdot \frac{\alpha H}{2} \\
\frac{1}{2}\gamma_w \frac{h^3}{3} &= \gamma_c \frac{\alpha^2 H^3}{2} \\
\Rightarrow \quad \alpha^2 &= (1/3)(\gamma_w/\gamma_c)(h/H)^3.
\end{aligned}
$$

Considering the case of critical water level up to the height of the dam, *i.e.,* $h/H = 1$, and substituting $\gamma_c/\gamma_w = 3$, we get

$$\alpha^2 = 1/9 \quad \Rightarrow \quad \alpha = 1/3 = 0.333.$$

Thus the width of the cross-section needs to be at least one-third of the height. For example, if the height of the dam is 9 m then it needs to be at least 3 m wide.

$$\boxed{b/H = 0.33}$$

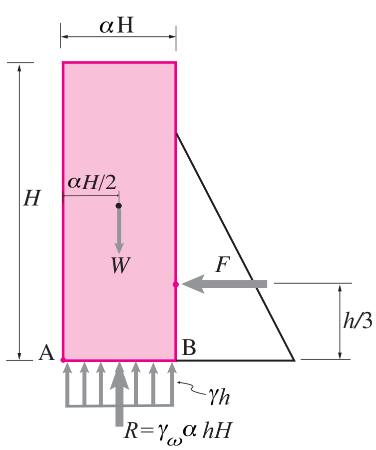


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Figure 8.17:

2. *Full water pressure on the bottom:* In this case, the water pressure on the bottom is uniformly distributed and its intensity is the same as the lateral pressure at B, *i.e.,* $p = \gamma_w h$. The free-body diagram diagram is shown in fig. 8.17 where the known forces are $W = \gamma_c \alpha H^2$, $F = (1/2)\gamma_w h^2$, and $R = \gamma_w \alpha h H$. Again, we carry out moment balance about point A to get

$$
\begin{aligned}
F \cdot \frac{h}{3} &= (W - R) \cdot \frac{\alpha h}{2} \\
\gamma_w h^3 &= 3(\gamma_c \alpha H^2 - \gamma_w \alpha h H)\alpha H \\
\alpha^2 &= \frac{(h/H)^3}{3(\gamma_c/\gamma_w - h/H)}.
\end{aligned}
$$

Once again, substituting the given values and $h/H = 1$, we get

$$\alpha^2 = 1/6 \quad \Rightarrow \quad \alpha = 0.408.$$

Thus the width in this case needs to be at least 0.41 times the height H, slightly wider than the previous case.

$$\boxed{b/H \geq 0.41}$$

Problems for Chapter 8

Statics

8.1 Net force and moments in hydrostatics

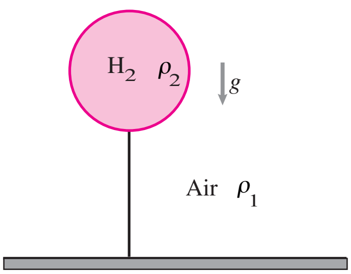
Preparatory Problems

More-Involved Problems

8.1.1 A balloon with volume V, whose membrane has negligible mass, holds a gas with density ρ_2. It is surrounded by a gas with density ρ_1.

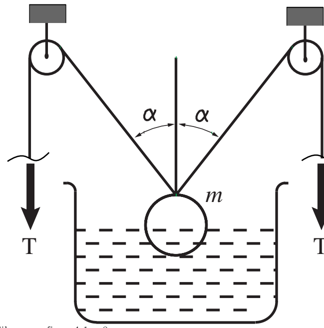
a) In terms of ρ_1, ρ_2, g, and V, find the tension in the string.

b) By some means look up the density of Helium and air at atmospheric temperature and pressure and calculate the volume, in cubic feet and in cubic meters, of a helium balloon that could lift 75 kg.



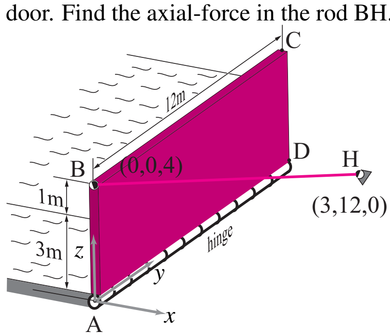
Filename:pfigure-hydro10
Problem 8.1.1: Balloon

8.1.2 A spherical body of mass $m = 10\,\mathrm{kg}$ and radius $R = 100\,\mathrm{mm}$ hangs from a continuous string as shown in the figure. The body is partially submerged in water and angle $\alpha = 45^\circ$ (fixed). If the force of buoyancy is $\rho V g$ where $\rho = 1000\,\mathrm{kg/m^3}$ = density of water, V is the submerged volume of the body, and g is the usual g; find the tension in the string as a function of the submerged volume V. Find the maximum and the minimum tension corresponding to fully and zero submerged volume of the body respectively.



Filename:pfigure4-1-rp8
Problem 8.1.2

8.1.3 A 4-meter-high 'door' holds back a stream ($\gamma = 1000\,\mathrm{N/m^3}$) that is 3m deep and 12m wide. The door is hinged along its bottom and is propped up by a thin rod B that goes from a ball joint at H at () 3,12,0) to a boll joint at the upper left corner B of the door at (0,0,4). Neglect the mass of the door. Find the axial-force in the rod BH. *

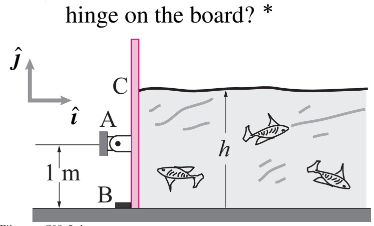


Filename:F02final2-dam
Problem 8.1.3

8.1.4 Water is held in a reservoir by a board with negligible weight that is 5 meters long. It is hinged 1 meter off the bottom at A and kept from leaking by a seal at B. Assume $\rho = 1000\,\mathrm{kg/m^3}$, $g = 10\,\mathrm{N/kg}$.

a) What is h when the board starts to pull away from the stop at B? *

b) At that h what is the force of the hinge on the board? *

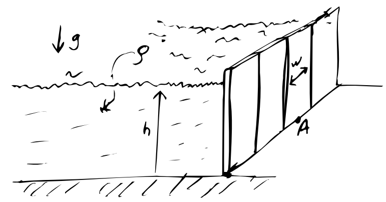


Filename:S03q5-dam
Problem 8.1.4

8.1.5 The side of a pool is made of vertical boards which are stuck in the ground. Assuming that the boards, on average, get no support from their neighbors, and neglect the weight of the board itself,

a) calculate the force and moment from the ground on one board (answer in terms of some or all of w, h, ρ, and g.

b) For a one foot board and 8 foot deep pool, find the size of a force, and its location, so the force is equivalent to the water pressure on the board (answer in lbf).

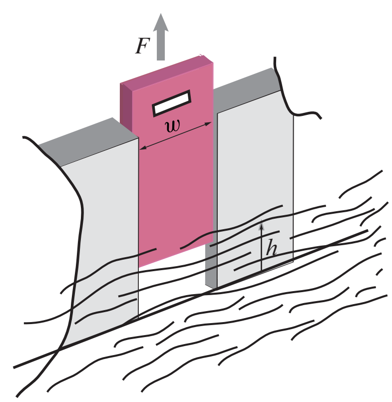


Filename:pfigure-hydro1
Problem 8.1.5

8.1.6 A sluice gate is a dam that can be opened. Sometimes it is just a board in a slot that is opened by pulling up the board. For water with density ρ and depth h pressing against a board with width w pressing against one face of the slot (the face away from the water) with coefficient of friction μ

a) find the force F needed to pull up the board in terms of g, ρ, h, and w.

b) Find the force in pounds force and Newtons assuming $g = 10\,\mathrm{m/s^2}$, $h = 1\,\mathrm{m}$, $w = 1\,\mathrm{m}$, and $\rho = 1000\,\mathrm{kg/m^3}$.



Filename:pfigure-hydro2
Problem 8.1.6

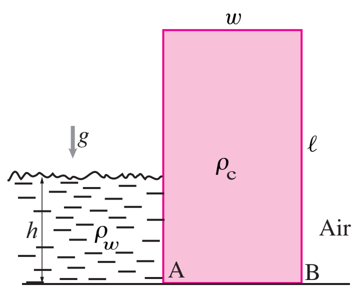
8.1.7 A concrete (density = ρ_c) wall with height ℓ, width w and length (into the paper) d rests on a flat rigid floor and serves as a damn for water with depth h and density ρ_w. Assume the wall only makes contact at edges A and B.

a) Assume there is a seal at A, so no water gets under the damn. What is the coefficient of friction needed to keep the block from sliding?

b) What is the maximum depth of water before the block tips?

c) Assume that there is a seal at B and that water gets under the block. What is the coefficient of friction needed to keep the block from sliding?

d) What is the maximum depth of water before the block tips?

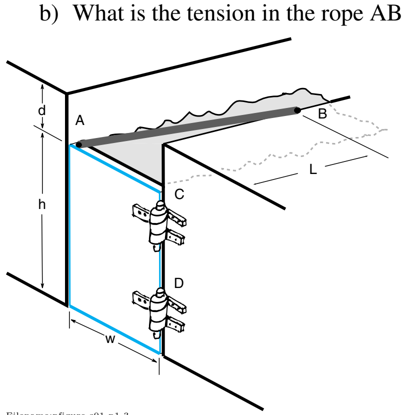


Filename:pfigure-hydro8
Problem 8.1.7

8.1.8 A door holds back the water at a lock on a canal. The water surface is at the top of the door. The rope AB keeps it from swinging open. The door has hinges at C and D. The height of the door is h, the width w. The point B is a distance d above the top of the door and is set back a distance L. The weight density of the water is γ.

a) What is the total force of the water on the door?

b) What is the tension in the rope AB?



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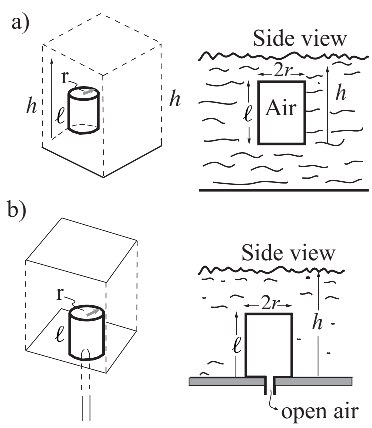
Problem 8.1.8

8.1.9 This problem somewhat explains the workings of some toilet valves. Open the tank of a toilet and look at the rubber piece at the bottom that sits on the bottom but then floats after initially lifted by the turning of the flush lever. The puzzle this problem solves is this: Why does the valve stick to the bottom, but then float when lifted.

a) A hollow cylinder with an open bottom (like an upside down but open can) is filled with air but is under water. What force is required to hold it under water (in terms of ρ, r, h, and g? *

b) The same can is on the bottom of a tank of water and its edges are sealed. The bottom is open to atmospheric air. How much force is needed to hold the can down now (so there is no force from the bottom of the tank onto the edges of the cylinder)? *



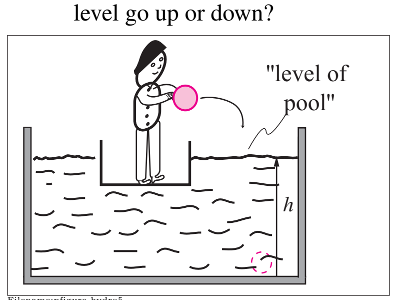
Filename:pfigure-hydro3
Problem 8.1.9

• • •

Some puzzles. The following puzzles are sometimes presented as brain teasers. You should be able to reason the answers carefully and irrefutably. **8.1.10** A person is in a boat in a pool with surface area A. She is holding a ball with volume V and mass m in a still pool. The ball is then thrown into the pool, no water is splashed out and the pool comes to rest again.

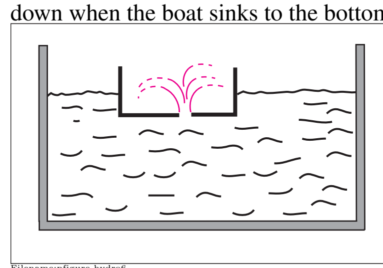
a) Assuming the ball floats, by how much does the pool level go up or down?

b) Assuming the ball sinks to the bottom by how much does the pool level go up or down?



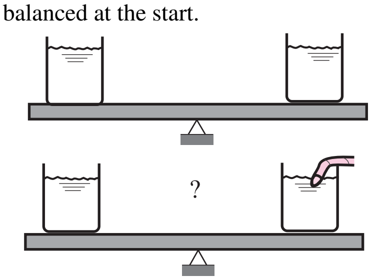
Filename:pfigure-hydro5
Problem 8.1.10

8.1.11 A steel boat with mass m and density ρ_s is floating in a pool of water with density ρ_w and cross sectional area A. By how much does the pool level go up or down when the boat sinks to the bottom?



Filename:pfigure-hydro6
Problem 8.1.11

8.1.12 Two cups of water are balanced. You then gently stick your finger into one of them. Does this upset the balance? This experiment can be set up with two cups and a hexagonal-cross-section pencil. The cups need not be identical, they just need to be balanced at the start.



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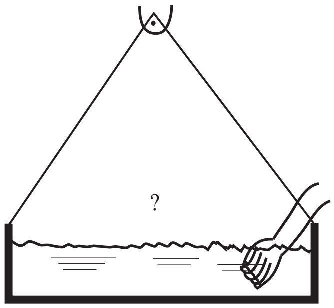
Problem 8.1.12

8.1.13 A tray of water is suspended and level.

a) A hand is gently placed in the tray but does not touch the edges or bottom. Is the level of the tray upset.

b) *Challenge:* Assuming the tray is massless with width w and water depth h, how high must be the hinge so the equilibrium is stable. That is, imagine the tray is rotated slightly about the hinge, the water pressure should cause a torque which tends to restore the vertical orientation shown.



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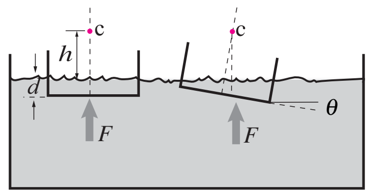
Problem 8.1.13



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Problem 8.1.14: Ship stability.

8.1.14 *Challenge:* This challenge problem is closely related to the challenge problem above, but is much more famous. It seems to have been first solved by Leonard Euler and Pierre Bouguer in about 1735. This solution seems to be the first mechanics problem in which the significance of the area moment of inertia was appreciated [this is a hint].

For simplicity assume that a boat is shaped like a box with width h and and length into the paper of b. Assume that the boat floats with its bottom a depth d under water. Now rotate the boat about an axis at the surface of the water and along its length (into the paper). Imagine that giant hands hold the boat in this position. In this rotated position the effect of the water pressure on the boat is a buoyant force and moment. This is equivalent to a force that is displaced slightly sideways.

Your goal is to find the height of the point that the line of action of this force intersects a mast of the boat. For small angles of boat tip the location is independent of the amount of tip.

This point is called the *metacenter* of the hull, and its distance up from the centroid of the boat's submerged volume is the hull's *metacentric height*. The condition of boat stability is that the metacenter be above the center of mass of the boat (thus the moment of the buoyant forces about the center of mass will tend to restore the boat to level).

Euler and Bouguer did the calculation you are asked to do here *after*, *e.g.,* , the launching of the 'great' Swedish ship Vasa, which capsized in the harbor on day 1. This was unfortunate for Sweden at the time, but fortunate now, because the brand-new 375 year old ship is a see-worthy tourist attraction in Stockholm.

CHAPTER 9

Dynamics in 1D

The scalar equation $F = ma$ introduces the concepts of motion and time derivatives to mechanics. In particular the equations of dynamics are seen to reduce to ordinary differential equations, the simplest of which have memorable analytic solutions. The harder differential equations need be solved on a computer. We explore various concepts and applications involving momentum, power, work, kinetic and potential energies, oscillations, collisions and multi-particle systems.

Contents

We now progress from statics to dynamics. As the names imply, statics generally concerns things that don't move, or at least don't accelerate much, whereas dynamics concerns things whose motion is of central interest. In statics we neglected inertial (terms involving acceleration of mass). So, in statics the linear and angular momentum balance equations were reduced to force and moment balance. In dynamics the inertial terms in the momentum balance equations are important. In statics all the forces and moments cancel each other. In dynamics the forces and moments add up to cause the acceleration of mass.

Once you have mastered free-body diagrams and statics, the hard part of dynamics is learning how to keep track of motion. Keeping track of motion, without yet worrying about the forces involved, is called *kinematics*. Kinematics is the study of geometry in motion. When we pay attention to the forces that cause the kinematics we are doing dynamics, in the mechanics sense of the word①. Dynamics is called *kinetics*. We will develop our understanding of dynamics (kinetics) by considering progressively more complex geometry of motion (kinematics).

This first dynamics chapter is limited to the unconstrained dynamics of one or more particles. What is a particle?

> A *particle* is a system idealized as being totally characterized by its position (as a function of time) and its (fixed) mass (read more on page 186).

Further, in this chapter we limit our attention to forces and motion in one spatial dimension (1D); each particle moves along a straight line and not on a planar or spatial curve. And all forces are also along the same line.

Unconstrained motion. Finally, in this chapter we only consider cases where the applied forces are either given as a function of time or can be determined from the positions and velocities of the particles. The time-varying thrust from an engine might be thought of as a force given as a function of time. Gravity and springs cause forces which are functions of position. And the drag on a particle as it moves through air or water can be modeled as a force depending on velocity. The forces we do not consider until the next chapter are forces caused by geometric constraints, for example the forces

①Outside of mechanics the word *dynamics* means the study of any system which changes in time. For example, "relationship dynamics" concerns how people's interpersonal relations change in time, and the "dynamics of the market place" concerns the change of the prices of things due to supply, demand and so on. In Electro-dynamics one studies how voltage varies in time and dynamics of the hormonal system concerns how hormone levels go up and down.

Systems with this more general dynamics are sometimes called "dynamical systems". The classic dynamical systems equation is

$$\frac{d}{dt}z = f(z)$$

where z is a list of numbers describing the state of the system and $f(z)$ is the set of rules that dictate how the system changes due to its present state. One sub-class of dynamical systems are the mechanical systems in this book where

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

is often eventually written in the form $\dot{z} = f(z)$, as you will see.

between particles connected by strings or rods. Such *constraint* forces need to be solved-for using dynamics, and cannot be found *apriori* from position, velocity or time.

Kinematics, acceleration and calculus. As mentioned, the main new concept here, which stays with us until the end of the book, is that things change with time. We keep track of that change using calculus. In particular, the equation $F = ma$ is a differential equation because

$$a \equiv \frac{d^2}{dt^2}x = \ddot{x} \ .$$

That is, any equation containing a is an equation containing a term with a second derivative in time. And any equation that has terms which are derivatives of functions is a differential equation.

The organization of this chapter

The first three sections are a review and deepening of material covered in freshman physics: $F = ma$, energy methods, and the harmonic oscillator. The last three sections concern multi-particle systems, collisions and more advanced vibration analysis.

Before going on please get the lay of the land by reading the summary of mechanics on the inside cover and the general introduction to mechanics in Chapter 1.

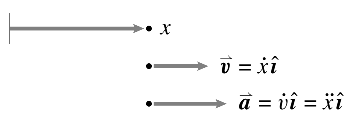

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Figure 9.1: One-dimensional position, velocity, and acceleration. Positive values for x, $\dot{x}$ and $\ddot{x}$ describe positions, velocities and accelerations in the in the positive x direction.

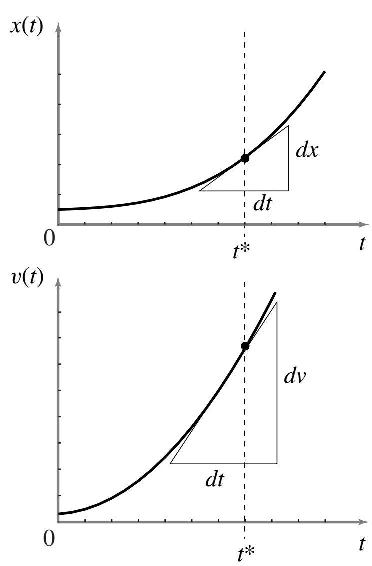

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Figure 9.2: Graphs of $x(t)$ and $v(t) = \frac{dx}{dt}$ versus time. The slope of the position curve dx/dt at t^* is $v(t^*)$. And the slope of the velocity curve dv/dt at t^* is $a(t^*)$.

9.1 Force and motion in 1D

Now we focus on a special case of particle motion: one particle moves on a given straight line. For this class of problems, with motion in only one direction, the kinematics is particularly simple. It's essentially a rehash of freshman calculus. Even in 1D, vectors can be useful because of their help with signs. But vectors are not really needed and we will not be zealous in their use (in this one chapter). As mentioned, we only look at forces along that line.

Position, velocity, and acceleration in one dimension

If, say, we call the direction of motion the $\hat{\imath}$ direction, then we can call x the position of the particle (see fig. 9.1). Even though we are neglecting the spatial extent of the particle, to be precise we can define x to be the x coordinate of the particle's center-of-mass. We can write the position $\vec{r}$, velocity $\vec{v}$ and acceleration $\vec{a}$ as

$$\vec{r} = x\hat{\imath}, \quad \vec{v} = v\hat{\imath} = \frac{dx}{dt}\hat{\imath} = \dot{x}\hat{\imath} \qquad \text{and} \qquad \vec{a} = a\hat{\imath} = \frac{dv}{dt}\hat{\imath} = \frac{d^2x}{dt^2}\hat{\imath} = \ddot{x}\hat{\imath}.$$

Figure 9.2 shows example graphs of $x(t)$ and $v(t)$ versus time.

Signs. Without vectors we need to be careful with signs; when in doubt, we will take v and a to be positive if they have the same direction as increasing x (or y or whatever coordinate describes position). Even though we pedantically declare that 'velocity is a vector' and 'acceleration is a vector', we will loosely use the words 'velocity' and 'acceleration' to stand for the coefficients of $\hat{\imath}$ in the vector expressions above.

Example: Position, velocity, and acceleration in one dimension
If position is given as

$$x(t) = 3e^{4t/\mathrm{s}}\,\mathrm{m}$$

then $v(t) = dx/dt = 12e^{4t/\mathrm{s}}\,\mathrm{m/s}$
and $a(t) = dv/dt = 48e^{4t/\mathrm{s}}\,\mathrm{m/s^2}$.
So at, say, time $t = 2\,\mathrm{s}$ the acceleration is

$$a|_{t=2\,\mathrm{s}} = 48e^{4\cdot 2\,\mathrm{s}/\mathrm{s}}(\mathrm{m/s^2}) = 48\cdot e^8\,\mathrm{m/s^2} \approx 1.43\cdot 10^5\,\mathrm{m/s^2}.$$

Units. Note that, in the example above, the unit inverse-seconds 1/s is part of the argument of the exponential function. Thus when the exponential $e^{4t/\mathrm{s}}$ is differentiated with respect to time t the 1/s is carried along with the 4; the coefficient of t in the exponential is 4/s, so that same factor comes out front in the differentiation. If you treat units as quantities manipulated like all others, as we have done in the example above, the units come out right. Note how the units cancel in the last line when the dimensional quantity (2 s) is substituted in for the variable t. For more on units see appendix A.

1D kinematics ⇔ calculus. One-dimensional kinematics problems can include almost all of the skills in elementary calculus. In kinematics you are often given position, velocity or acceleration as function of time and you have to differentiate or integrate to find one of the other quantities. For example, if you are given the velocity $v(t)$ as a function of time and are asked to find the acceleration $a(t)$, you have to differentiate. If instead you were asked to find the position $x(t)$, you would calculate an integral (see fig. 9.3). Using the fundamental theorem of calculus, we get the integral versions of the relations between position, velocity, and acceleration (see fig. 9.3).

$$\begin{aligned} x(t) &= x_0 + \int_{t_o}^{t} v(\tau)\,d\tau \qquad \text{with} \qquad x_0 = x(t_0), \text{ and} \\ v(t) &= v_0 + \int_{t_o}^{t} a(\tau)\,d\tau \qquad \text{with} \qquad v_0 = v(t_0). \end{aligned}$$

With more indefinite notation, these equations can also be written as:

$$\begin{aligned} x &= \int v\,dt \\ v &= \int a\,dt. \end{aligned}$$

If acceleration is given as a function of time, then position is found by integrating twice.

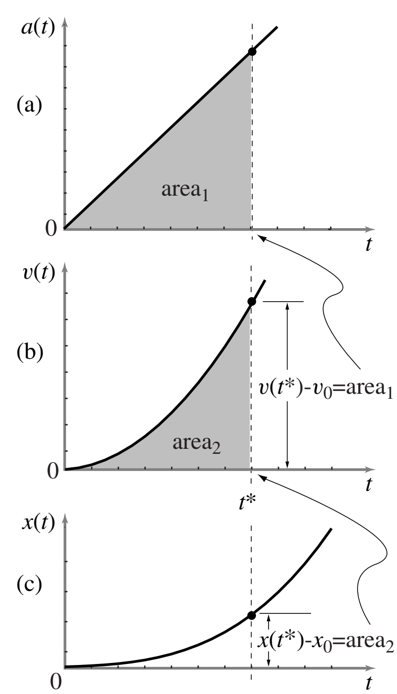

Figure 9.3: One-dimensional kinematics of a particle: (a) is a graph of the acceleration of a particle $a(t)$; (b) is a graph of the particle velocity $v(t)$ and the integral of $a(t)$ from $t_0 = 0$ to t^*, the shaded area under the acceleration curve; (c) is the position of the particle $x(t)$ and the integral of $v(t)$ from $t_0 = 0$ to t^*, the shaded area under the velocity curve.

1D kinematics, bicycles and calculus. To put it another way, almost every calculus question could be phrased as a question about a bicycle speedometer①. With a bicycle speedometer (which includes a distance-measuring odometer) you can read your speed and distance travelled as functions of time. And given one of those two functions you could find the other using calculus. Acceleration is also of interest, but few bicycle speedometers also measure acceleration.

① To cover the range of calculus problems you need to be a very good rider, however, and be able to ride frontwards, backwards, at zero speed and infinitely fast.

Differential equations

A *differential equation* is an equation that involves derivatives. Thus the equation relating position to velocity is

$$\frac{dx}{dt} = v \qquad \text{or, more explicitly} \qquad \frac{dx(t)}{dt} = v(t),$$

is a differential equation. An *ordinary differential equation* (ODE) is an equation that contains some terms that are ordinary derivatives (as opposed to partial derivatives and partial differential equations which we don't use in this book).

Example: Calculating a derivative solves an ODE
Given that the height of an elevator as a function of time on its 5 seconds long 3 meter trip from the first to second floor is

$$y(t) = (3\,\mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}$$

we can solve the differential equation $v = \frac{dy}{dt}$ by differentiating to get

$$v = \frac{dy}{dt} = \frac{d}{dt}\left[(3\,\mathrm{m})\frac{\left(1-\cos\left(\frac{\pi t}{5\,\mathrm{s}}\right)\right)}{2}\right] = \frac{3\pi}{10}\sin\left(\frac{\pi t}{5\,\mathrm{s}}\right)\ \mathrm{m/s}$$

Note: this would be a harsh elevator because of the jump in the acceleration (not calculated above) at the start and stop.

A little less trivial is the case when you want to find a function when you are given the derivative.

Example: Integration solves a simple ODE
Assume that you start at home ($x = 0$) and, over about 30 seconds, you accelerate towards a steady-state speed of 4 m/s according to (see fig. 9.4)

$$v(t) = 4(1 - e^{-t/(30\,\mathrm{s})})\,\mathrm{m/s}.$$

Your ride lasts 1000 seconds. We can find how far you go by solving

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0.$$

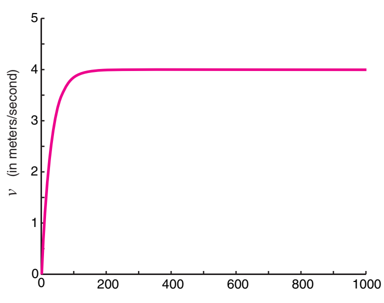

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Figure 9.4: Velocity (speed) vs time for a 1000 s bike trip. The rider increases speed to exponentially approach 4 m/s.

This is simply solved by integration. Say, after 1000 seconds

$$\begin{aligned}
x(t=1000\,\mathrm{s}) &= \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} 4(1-e^{-t/(30\,\mathrm{s})})(\mathrm{m/s})\,dt \\
&= \left(4t + (120\,\mathrm{s})e^{-t/(30\,\mathrm{s})}\right)\Big|_0^{1000\,\mathrm{s}}\ \mathrm{m/s} \\
&= \left((4\cdot 1000\,\mathrm{s} + (120\,\mathrm{s})e^{-100/3}) - (0 + (120\,\mathrm{s})e^{0})\right)\ \mathrm{m/s} \\
&= \left(4000 - 120 + 120e^{-100/3}\right)\ \mathrm{m} \\
&\approx 3880\,\mathrm{m} \qquad \text{(to within an angstrom or so)}
\end{aligned}$$

This is only 120 m less than if the whole trip was travelled at a steady 4 m/s (then $x = 4\,\mathrm{m/s}\times 1000\,\mathrm{s} = 4000\,\mathrm{m}$).

Unlike the integral above, many integrals cannot be evaluated by hand (analytically).

Example: An ODE that leads to an intractable integral
Assume now that

$$v(t) = \frac{4t}{t + e^{-t/(30\,\mathrm{s})}\,\mathrm{s}}\ \mathrm{m}.$$

Again we have a bike trip where you start at zero speed and approach a steady speed of 4 m/s. So your position as a function of time should be similar. Following the last example, we have

$$\dot{x} = v(t) \quad \text{with the initial condition} \quad x(0) = 0$$

with the given $v(t)$. The integral for position is then

$$\begin{aligned}
x(t=1000\,\mathrm{s}) &= \int_0^{1000\,\mathrm{s}} v(t)\,dt = \int_0^{1000\,\mathrm{s}} \tfrac{4t}{t+e^{-t/(30\,\mathrm{s})}\,\mathrm{s}}\ \mathrm{m}\,dt \\
&= \ldots
\end{aligned} \tag{9.1}$$

which is the kind of thing you have nightmares about seeing on an exam. You couldn't solve this integral if your life depended on it. No one could. There is no formula for $x(t)$ that solves the differential equation, unless you regard eqn. (9.1) as a formula. In days of old they would say 'the problem has been reduced to quadrature' meaning that the remaining work was evaluating an integral(2), even if they didn't know how to evaluate it exactly.

(2) Literally *quadrature* means finding a square (a 'quad') with area equal to the area under a given curve. The phrase 'reduced to quadrature' is used more generally to mean integration, even if the integration is of several variables.

Just because a differential equation can't be solved analytically with pencil and paper doesn't mean it can't be solved numerically. Most often the setup for numerical solution is not that difficult. Note that for numerical solution you either need dimensionless calculations, or at least need all variables to be in consistent units.

Example: Numerical solution of ODE
One of many ways to evaluate the integral of the above example numerically is by the following pseudo code.

```
ODE = { xdot = 4 t / (t+e^(-t/30)) }
IC  = { x(0) = 0  }
solve ODE with IC and evaluate at t=1000
```

The result is $x \approx 3988\,\mathrm{m}$ which is also, as expected because of the similarity with the previous example, only slightly shy of the steady-speed approximation of 4000 m. ③

More differential equations.

As mentioned, because dynamics equations contain derivatives they are all differential equations. A catalogue of the simplest differential equations and their solutions is given in box 9.3 on page 424.

The equations of dynamics

We want to understand kinetics (mechanics, dynamics), not just kinematics. The subject of mechanics is held up by the three pillars of material properties, geometry, and 'Newton's laws' (see page 26). Here we begin to flesh out the 'Newton's laws' pillar beyond statics (the first 8 chapters of this book), using kinematics (we just started with that above) to the the third pillar, dynamics.

Linear momentum balance

For a particle moving in the x direction the velocity and acceleration are $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. Thus the linear momentum and its rate of change are

$$\begin{aligned} \vec{\boldsymbol{L}} &\equiv \sum m_i \vec{\boldsymbol{v}}_i &= m\vec{\boldsymbol{v}} = mv\hat{\boldsymbol{\imath}}, \text{ and} \\ \dot{\vec{\boldsymbol{L}}} &\equiv \sum m_i \vec{\boldsymbol{a}}_i &= m\vec{\boldsymbol{a}} = \ ma\hat{\boldsymbol{\imath}}. \end{aligned}$$

Using any of the free body diagrams in fig. 9.5, where $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$, the equation of linear momentum balance, ④ eqn. I from the front inside cover, or equation 10.1 reduces to:

$$F\hat{\boldsymbol{\imath}} = ma\hat{\boldsymbol{\imath}} \tag{9.2}$$

which in scalar form is the central subject of this section.

$$F = ma$$

In scalar form, F is the net force to the right and a is the acceleration to the right. For the equation $F = ma$ to have content each of the terms must have some meaning in other contexts. And, at least intuitively, each does (see box 9.1 on page 419).

Force. The force F could come from a spring, or a fluid or from your hand pushing the thing to the right or left, or any combination of these things. The most general case we want to consider here is that the force is determined by the position and velocity of the particle as well as the present time. Thus

$$F = f(x, v, t). \tag{9.3}$$

What do we mean 'determined by'? We mean that we have an independent

③**Numerical error vs real difference.** When you notice such small differences (12m out of 4000m) based on computer calculation you need to question whether the difference is something real in the problem or, rather, is due to numerical errors. Two ways to check are with so-called convergence tests and by using your canned package's error estimate. We checked that the numerical integration is accurate to about 10^{-3}m, less than the 1m resolution that we printed (no need to type lots of digits with little information). Thus the $\approx$ 12m difference between the constant v solution and the solution where v approaches a constant, is a real difference. The startup is genuinely quicker in the second example (12 m behind constant v versus 120 m behind constant v), and not a numerical artifact.

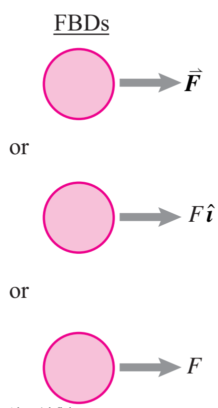


Figure 9.5: Three possible free body diagrams for 1D particle motion. It is assumed that $\vec{\boldsymbol{F}}$ represents the sum of all forces acting on the particle. Note, even though the inertial term ma is of special interest, it does not show on the FBD. See the free body diagram Chapter 3 for the prescription about this. See box 9.4 on page 427 for a further warning about D'Alambert 'forces'.

④**Angular momentum?** We skip thinking about angular momentum balance in this section. Why? If we pick an origin on the line of travel, all terms on both sides of all angular momentum balance equations are zero, and the equation $0 = 0$ tells us nothing new.

way of knowing the force from its position, velocity and time, even without thinking yet about the linear momentum balance equation $F = ma$. Special cases would be, say,

$$\begin{array}{llll} F = & f(t) & = F_0 \sin(\beta t) & \text{an oscillating load,} \\ F = & mg & & \text{the force of earth's gravity, } x \text{ pointing down} \\ F = & f(v) & = -cv & \text{a linear viscous drag,} \\ F = & f(x) & = -kx & \text{a linear spring, and} \\ F = & f(x,v,t) & = -kx - cv + F_0 \sin(\beta t) & \text{a combination of forces.} \end{array}$$

All elementary 1D particle mechanics problems can be reduced to the solution of this pair of coupled first order differential equations,

$$\begin{array}{lll} \frac{dv}{dt} = & \underbrace{f(x,v,t)/m}_{a(t)} & (a) \\ \frac{dx}{dt} = & v(t) & (b) \end{array} \qquad (9.4)$$

where the function $f(x,v,t)$ is given and $x(t)$ and $v(t)$ are to be found.

9.1 What do the terms in $F = ma$ mean?

Some say '$F = ma$ is just a definition of force'. This is a legitimate point of view. But it is not useful if we care about force for other reasons, which we do. For the equation $F = ma$ to have useful content, we need to be able to think about all of the terms indepently.

$m =$ **Mass.** Now that we know about atoms (these centuries) and what they are made of (these decades) we can approximately (about one percent accuracy) define the mass of a system by (in principle) counting up the total number of protons and neutrons and multiplying by $1.67 \cdot 10^{-27}$ kg. That is, mass is a measure of the extent of matter. Given that we think of mass as the amount of matter, we could more accurately and more easily use a reference volume of a pure chemical substance as a reference. This way, with a good balance and some trouble, we could get an accuracy of parts per million. Officially, mass is measured in comparison to a fancy piece of metal locked in a box in some basement in some government building. That calibrated kilogram is accurate to about a part in 20,000,000 (See Appendix A starting on page 974). We can find the mass of a more complicated thing using that reference mass and a (very good) balance.

$a =$ **Acceleration.** Because this is a course in mechanics and not in philosophy of science, we will just accept the concepts of space (x) and time (t) as given and measurable (using rulers and clocks). So acceleration is operationally well defined (d^2x/dt^2) with no use of $F = ma$. At least to about one part per billion for most engineering purposes (see page 31).

$F =$ **Force.** Here is where people argue. It's easiest define force using the deformation of solids. When one thing pushes on another, think of your little finger as caught in between. How much your finger is squeezed, as measured by how loud you yell, is a measure of force. More technically, we could look at the small amounts of deformation occurring where the bodies contact, and use the deformation as a measure of force. Or, more practically, we could interpose a calibrated material and measure its deformation. Such a chunk of material with deformation-measuring electronics is called a *load cell*. Load cells are sold by the millions (say, in bathroom scales). A load cell uses nothing about $F = ma$ to operate accurately.

One reason it is nice to think of force as having a life away from $F = ma$ is that the whole coherent and useful subject of statics, useful for designing bridges and other things, has no use for $F = ma$. Alternatively, still without thinking about $F = ma$, one could define force in terms of the net effect of earth's gravitational pull on a calibrated mass at some officially-ordained location like Potsdam.

However you like to define force, the great result is that with any one of several definitions things mostly work out. Miraculously, the same concept F works in all three of these contexts,

$$1) F = mg \quad \text{and} \quad 2) F = kx \quad \text{and} \quad 3) F = ma,$$

at least for engineering purposes. Pick your favorite as fundamental and use the others with confidence anyway.

Know a solution when you see one. How can you tell if candidate functions solve a differential equation? First you can tell that the initial conditions are satisfied by evaluating the expressions at $t = 0$. To check that the differential equations are satisfied, you plug the candidate solutions into the equation and see that an identity results. Differential equations are satisfied when the unknown functions therein are replaced with specific functions that make the equations correct.

Viscous drag. If the only applied force is a viscous drag, $F = -cv$ (see fig. 9.6), then linear momentum balance ($F = ma$) would be $-cv = ma$ and Eqns. 9.4 are

$$\begin{aligned} \frac{dv}{dt} &= -cv/m \\ \frac{dx}{dt} &= v \end{aligned}$$

where c and m are constants and $x(t)$ and $v(t)$ are yet to be determined functions of time. Because the force only slows the particle there is be no motion unless the particle has some initial velocity. In general, you need to specify the initial position and velocity to find a solution. So we complete the problem statement with the initial conditions

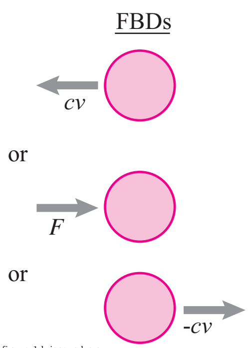


Figure 9.6: Three acceptable free body diagrams showing the viscous drag on a particle moving to the right at speed v. The second FBD depends on being told on the side, say here, that $F = -cv$.

Example: Slowing with viscous drag
Find $x(t)$ given that

$$x(0) = x_0 \quad \text{and} \quad v(0) = v_0$$

where x_0 and v_0 are given constants. Before worrying about how to solve such equations, you should know how to recognize a solution. The following two functions, assume for now that they fell from the sky, solve the differential equations.

$$\begin{aligned} v(t) &= v_0 e^{-ct/m}, \quad \text{and} \\ x(t) &= x_0 + mv_0(1 - e^{-ct/m})/c \end{aligned}$$

Plugging this presumed solution for $v(t)$ into $\dot{v} = -cv/m$ gives, and this is what we want, $0 = 0$. And similarly, when the presumed solution for $x(t)$ is plugged in to $\dot{x} = v$ you also get the 'satisfying' result that $0 = 0$.

Replacing the unknown functions $v(t)$ and $x(t)$ with the given formulas gives an identity. Thus the given formulas satisfy (or solve) the differential equations. Just like the case of integration (or equivalently the solution for x of the ODE $\dot{x} = v(t)$), one often cannot find formulas for the solutions of differential equations.

Example: A dynamics problem with no pencil and paper solution
Consider the following case which models a particle in a sinusoidal force field with a second applied force that oscillates in time. Using the dimensional constants c, d, F_0, β, and m,

$$\begin{aligned} \frac{dv}{dt} &= \left(c\sin(x/d) + F_0\sin(\beta t)\right)/m \\ \frac{dx}{dt} &= v \end{aligned}$$

with initial conditions $x(0) = 0 \quad \text{and} \quad v(0) = 0.$

There is no known formula for $x(t)$ that solves this ODE.

Just writing the ordinary differential equations and initial conditions is analogous to setting up an integral in freshman calculus. The solution is reduced to quadrature. Because numerical solution of sets of ordinary differential equations is a standard part of all modern computation packages you are in some sense done when you get this far. A computer can finish up for you.

Some special cases in 1D mechanics

There are various special cases of eqn. (9.4) which have simple solutions.

Example: The simplest dynamics problem.

9.2 The units of force

The simplest way to measure force is with the "metric"/SI convention and to use Newtons (N) where

$$1\,N \equiv \frac{1\,\text{kg}\,\text{m}}{\text{s}^2}.$$

Unfortunately, to everyone's confusion, there are other units of force.

Kilogram force. One confusing force unit is the kilogram of force, 1 kgf = g× 1 kg ≈ 9.81N. One kgf is the force of gravity at the earth's surface. Another name for kilogram force is kilopond, kp. How much do you weigh? In Europe most often people give their weight (weight is a force) in "kilograms" which means kgf. Basically we advise against using kgf for any purpose. But you should know that some people use it. To keep things accurate and simple, in Europe people should go on diets to lose mass.

Pound force. Analogous to the kilogram force is the pound force lbf. One lbf is the force of gravity (at the earth's surface) on one pound mass. Thus $1\,\text{lbf} = g \times 1\,\text{lbm} \approx 32.2\,\text{lbm}\,\text{ft/ s}^2$.

Example: What is the force required to accelerate 10 lbm an amount of 5 ft/s^2?

$$\begin{aligned} F &= ma \\ &= (10\,\text{lbm})(5\,\text{ft/ s}^2) \\ &= (10\,\text{lbm})(5\,\text{ft/ s}^2)\cdot \underbrace{\left(\frac{1\,\text{lbf}}{1\,\text{lbm}\cdot g}\right)}_{1}\underbrace{\left(\frac{g}{32\,\text{ft/ s}^2}\right)}_{1}. \end{aligned}$$

All of the units in the above expression cancel (appear an equal number of times on the top as the bottom of fractions), except for lbf. So

$$F \approx (10\cdot 5/32)\,\text{lbf} \approx 1.6\,\text{lbf}.$$

As shown in this example, the surest way to know whether to multiply or divide by g is by systematic multiplication by 1. If you pick the wrong version of the number 1 you still get the right answer, but in a strange mixture of units.

The poundal. If the English system imitated the metric system it would have a unit for the force needed to accelerate one lbm one ft/s^2. And it does. Its called the poundal, abreviated as pdl. 1 pdl = 1 lbm ft/s^2. Poundals are as sensible to the English system as Newtons are in SI, but they are rarely used. Because the poundal is unfamiliar, unfamiliar things are strange and strange things are confusing, the poundal is generally catalogued as confusing. However, the poundal is as simple as the Newton.

Attempts at using the number 1. In the metric system the standard unit of force (N) is 1 (one) times the standards for mass, distance and inverse-time squared: 1 N = 1 kg m/s^2. One is a nice number. An attempt to get mass and force related by the number one, an attempt that has failed in the market place of engineering practice, uses the poundal. A second failed attempt defines a new unit of mass, the slug: 1 lbf = 1 slug ft/s^2. Slugs are also rarely used. But in principle a slug is just as simply related to a lbf as is a kg is to a N.

Europe is dynamic the USA static. The standard units used in Europe (meters, kilograms, Newtons & seconds) are easy if you are mostly studying dynamics: in Europe, a unit of mass is accelerated a unit amount with a unit force. The standard units in the USA are easiest for gravitational loads: the unit of mass has one unit of gravitational force on it.

Most people studying mechanics try to avoid all this confusion by sticking with SI, that's the real SI that has no such thing as a kilogram force (kgf). Still now in the 21st century, in the USA and some other places, we have to learn to live with pound force and pound mass. At least we can be thankful that most of us can avoid dealing with the kilogram force (or kilopond), the poundal and the slug. Read more about such issues in the appendix on units (974).

1D, particle, no force. Formally working out the details,

$$\begin{array}{llrcl} F = ma \text{ with } F = 0 & \Rightarrow & 0 & = & ma \\ \text{definition of } a & \Rightarrow & \dot{v} & = & 0 \\ \text{integrating} & \Rightarrow & v & = & v_0 \quad (= \text{any constant}) \\ \text{integrating again} & \Rightarrow & x & = & x_0 + v_0 t \end{array}$$

In a sense we have thus derived Newton's first law, 'an object in motion tends to stay in motion unless acted upon by a force'.

Constant force. Another simple case is constant force F which leads to constant acceleration $a = F/m$. Using calculus you should know well by now, you get the following formulas:

$$\begin{aligned} a = const \Rightarrow \quad x &= x_0 + v_0 t + at^2/2 \\ a = const \Rightarrow \quad v &= v_0 + at \\ a = const \Rightarrow \quad v &= \pm\sqrt{v_0^2 + 2ax}. \end{aligned}$$

These are much seen in high school physics because, by permuting what is given and what is unknown, one can make up 100 homework problems that can be solved with these formulas and without calculus.

Force given as a function of time. Say F is given as $F = F(t)$. This general case shows up when some kind of motor force is controlled by a human or computer to vary in time is some predetermined manner.

$$\begin{array}{llrcl} F = ma \text{ with } F = F(t) & \Rightarrow & F(t) & = & ma \\ \text{definition of } a & \Rightarrow & \dot{v} & = & F(t)/m \\ \text{integrating} & \Rightarrow & v & = & v_0 + \frac{1}{m}\int_0^t F(\tau)\,d\tau \end{array}$$

And we have to integrate once again to get position.

Example: Ramping up the acceleration at the start

If you get a car going by gradually depressing the 'accelerator' so that its acceleration increases linearly with time, we have

$$\begin{array}{llllll} & a & = & ct & & \text{(take } t = 0 \text{ at the start)} \\ \Rightarrow & v(t) & = & \int_0^t a\,d\tau + v_0 & = & \int_0^t c\tau\,d\tau \quad = \quad ct^2/2 \\ & & & & & \text{(since } v_0 = 0) \\ \Rightarrow & x(t) & = & \int_0^t v\,d\tau + x_0 & = & \int_0^t (c\tau^2/2)\,d\tau \quad = \quad ct^3/6 \\ & & & & & \text{(since } x_0 = 0). \end{array}$$

The distance the car travels is proportional to the cube of the time that has passed from dead stop.

The overall subject of 'vibrations' is in some sense about what happens when something is shaken. We can think of 'shaking' as applying a force which varies sinusoidally in time.

Example: Force varies sinusoidally in time.
Assume a 1 kg mass starts from rest and has a force of $F = 2\cos(2\pi t/\mathrm{s})\,\mathrm{N}$ applied. That's a force that oscillates once per second with an amplitude of 2 N. What is the position at $t = 10\,\mathrm{s}$?

$$\begin{array}{lclcl}
F = ma \text{ with } F = F(t) & \Rightarrow & 2\cos(2\pi t/\mathrm{s})\,\mathrm{N} & = & m\dot{v} \\
\text{integrating} & \Rightarrow & v & = & v_0 + \frac{1}{m}\int_0^t 2\cos(2\pi t/\mathrm{s})\,\mathrm{N}\,d\tau \\
\text{using freshman calculus} & \Rightarrow & v & = & v_0 + \frac{1}{\pi m}\sin(2\pi t/\mathrm{s})\,\mathrm{N\,s} \\
\text{IC: } v(0) = 0 \Rightarrow v_0 = 0 & \Rightarrow & v & = & \frac{1}{\pi m}\sin(2\pi t/\mathrm{s})\,\mathrm{N\,s} \\
\text{integrate again, using } \dot{x} = v & \Rightarrow & x & = & x_0 - \frac{1}{2\pi^2 m}\cos(2\pi t/\mathrm{s})\,\mathrm{N\,s^2} \\
\text{IC: } x(0) = 0 \Rightarrow x_0 = \frac{1}{2\pi^2 m} & \Rightarrow & x & = & \frac{1}{2\pi^2 m}\left(1 - \cos(2\pi t/\mathrm{s})\right)\,\mathrm{N\,s^2}
\end{array}$$

Now we can substitute in $m = 1\,\mathrm{kg}$ and $t = 10\,\mathrm{s}$ to get $x = 0.0\,\mathrm{m}$. The algebraic cancellation of units came about naturally from substituting in the definition of a Newton $1\,\mathrm{N} = 1\,\mathrm{kg\,m/s^2}$. We carried the units through even though the final answer was 0.

Force depends on velocity. This case is encountered when, say, an object moves through a fluid and other forces, say gravity, are negligible. Here we have

$$F = ma \quad \Rightarrow \quad F(v) = m\dot{v}.$$

This is solved by multiplying both sides by dt and dividing both sides by $F(v)$ and integrating to get

$$\int dt = m\int \frac{dv}{F(v)} \quad \Rightarrow \quad t = m\int_{v_0}^{v} \frac{dv'}{F(v')}$$

If we want to know position vs time we have to integrate once again.

Example: The slowing of a bullet.
The main force on a bullet after it leaves the gun and before it hits its mark is from air drag. This drag is roughly proportional to the speed squared, thus

$$F = ma \quad \Rightarrow \quad -cv^2 = m\dot{v} \quad \Rightarrow \quad -c\int dt = m\int_{v_0}^{v} \frac{dv'}{v'^2}.$$

Carrying out the integrals ($\int dt = t$ and $\int v^{-2}\,dv = -v^{-1}$) we get

$$ct = m\left(\frac{1}{v} - \frac{1}{v_0}\right) \quad \Rightarrow \quad v = \frac{v_0}{cv_0 t/m + 1}$$

To get position we would integrate again to get:

$$x = \int_0^t v(t')\,dt' = \int_0^t \frac{v_0}{cv_0 t'/m + 1}\,dt' = (m/c)\ln(1 + cv_0 t/m)$$

Interestingly, according to this equation (which becomes less and less accurate as the bullet slows and gravity and eventually viscous forces become important) the bullet goes an infinite distance before stopping.

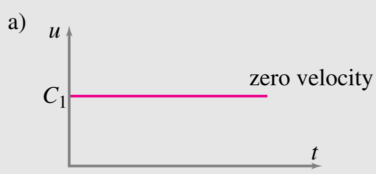
9.3 The simplest ODEs, their solutions, and heuristic explanations

This box is *not* an aside. Know it well.

Here are some of the simplest useful ordinary differential equations (ODEs) and their general solutions. Think of u as the distance an object has moved to the right of its 'home' at $u = 0$ in time t. The velocity and acceleration to the right are $du/dt = \dot{u}$ and $d^2u/dt^2 = \ddot{u}$. If $\dot{u} < 0$ the particle is moving to the left. If $\ddot{u} < 0$ the particle is accelerating to the left. In all cases A and B are constants and λ is a positive constant. C_1, C_2, C_3, and C_4 are arbitrary (undetermined) constants in the solutions that get pinned down (determined) by fixing the initial conditions.

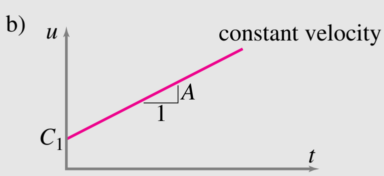
a) $\dot{u} = 0 \ \Rightarrow\ u = C_1.$

$\dot{u} = 0$ means that the velocity is zero. This equation would arise in dynamics if a particle has no initial velocity and no force is applied to it. The particle doesn't move. Its position must be constant. But it could be anywhere, say at position C_1. Hence the general solution $u = C_1$, as can be found by direct integration.



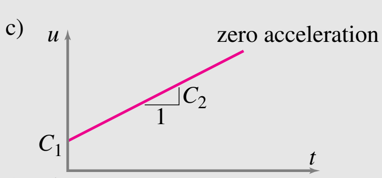
b) $\dot{u} = A \ \Rightarrow\ u = At + C_1.$

$\dot{u} = A$ means the object has constant speed. This equation describes the motion of a particle that starts with speed $v_0 = A$ and because it has no force acting on it continues to move at constant speed. How far does it go in time t? It goes $v_0 t$. Where was it at time $t = 0$? It could have been anywhere then, say C_1. So where is it at time t? It's at its original position plus how far it has moved, $u = v_0 t + C_1$, as can also be found by direct integration.



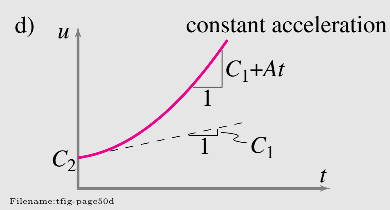
c) $\ddot{u} = 0 \ \Rightarrow\ u = C_1 + C_2 t.$

$\ddot{u} = 0$ means the acceleration is zero. That is, the rate of change of velocity is zero. This constant-velocity motion is the general equation for a particle with no force acting on it. The velocity, if not changing, must be constant. What constant? It could be anything, say C_2. Now we have the same situation as in case (b). So the position as a function of time is anything consistent with an object moving at constant velocity: $u = C_1 + C_2 t$, where the constants C_1 and C_2 depend on the initial position and initial velocity. If you know that the position at $t = 0$ is u_0 and the velocity at $t = 0$ is v_0, then the position is $u = u_0 + v_0 t$.



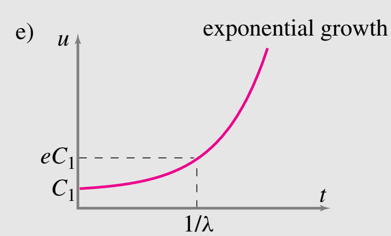
d) $\ddot{u} = A$
$\quad\Rightarrow\ u = At^2/2 + C_1 t + C_2.$

This constant acceleration A, constant rate of change of velocity, is the classic (all-too-often studied) case. This situation arises for vertical motion of an object in a constant gravitational field as well as in problems of constant acceleration or deceleration of vehicles. The velocity increases in proportion to the time that passes. The change in velocity in a given time is thus At and the velocity is $v = \dot{u} = v_0 + At$ (given that the velocity was v_0 at $t = 0$). Because the velocity is increasing constantly over time, the average velocity in a trip of length t occurs at $t/2$ and is $v_0 + At/2$. The distance traveled is the average velocity times the time of travel so the distance of travel is $t \cdot (v_0 + At/2) = v_0 t + At^2/2$. The position is the position at $t = 0$, u_0, plus the distance traveled since time zero. So $u = u_0 + v_0 t + At^2/2 = C_2 + C_1 t + At^2/2$. This solution can also be found by direct integration.



e) $\dot{u} = \lambda u \ \Rightarrow\ u = C_1 e^{\lambda t}.$

The displacement u grows in proportion to its present size. This equation describes the initial falling of an inverted pendulum in a thick viscous fluid. The bigger the u, the faster it moves. Such situations are called exponential growth (as in population growth or monetary inflation) for a good mathematical reason. The solution u is an exponential function of time: $u(t) = C_1 e^{\lambda t}$, as can be found by separating variables or guessing.

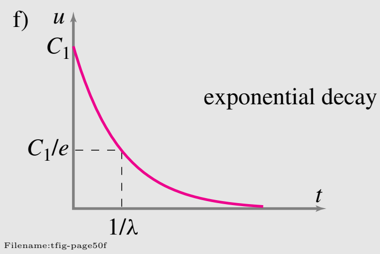


f) $\dot{u} = -\lambda u \ \Rightarrow\ u = C_1 e^{-\lambda t}.$

The smaller u is, the more slowly it gets smaller. u gradually tapers towards nothing: u decays exponentially. The solution to the equation is: $u(t) = C_1 e^{-\lambda t}$. This expression is essentially the same equation as in (e) above.

(continued...)

9.3 The simplest ODEs, their solutions, and heuristic explanations (continued)

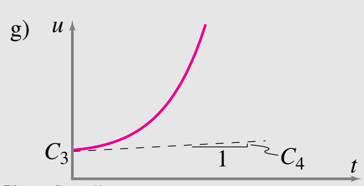


g) $\ddot{u} = \lambda^2 u$

$$\Rightarrow u = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$$

or $$\Rightarrow u = C_3 \cosh(\lambda t) + C_4 \sinh(\lambda t).$$

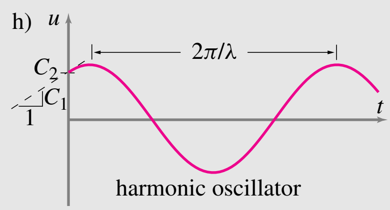
Note, sinh and cosh are just combinations of exponentials. For $\ddot{u} = \lambda^2 u$, the point accelerates more and more away from the origin in proportion to the distance from the origin. This equation describes the falling of a nearly vertical inverted pendulum when there is no friction. Most often, the solution of this equation gives roughly exponential growth. The pendulum accelerates away from being upright. The reason there is also an exponentially decaying solution to this equation is a little more subtle to understand intuitively: if a not quite upright pendulum is given just the right initial velocity it will slowly approach becoming just upright with an exponentially decaying displacement. This decaying solution is not easy to see experimentally because without the perfect initial condition the exponentially growing part of the solution eventually dominates and the pendulum accelerates away from being just upright.



h) $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$

$$\Rightarrow \quad u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t).$$

This equation describes a mass that is restrained by a spring which is relaxed when the mass is at $u = 0$. When u is positive, $\ddot{u}$ is negative. That is, if the particle is on the right side of the origin it accelerates to the left. Similarly, if the particle is on the left it accelerates to the right. In the middle, where $u = 0$, it has no acceleration, so it neither speeds up nor slows down in its motion whether it is moving to the left or the right. So the particle goes back and forth: its position oscillates. A function that correctly describes this oscillation is $u = \sin(\lambda t)$, that is, sinusoidal oscillations. The oscillations are faster if λ is bigger. Another solution is $u = \cos(\lambda t)$. The general solution is $u = C_1 \sin(\lambda t) + C_2 \cos(\lambda t)$. A plot of this function reveals a sine wave shape for any value of C_1 or C_2, although the phase depends on the relative values of C_1 and C_2. The equation $\ddot{u} = -\lambda^2 u$ or $\ddot{u} + \lambda^2 u = 0$ is called the 'harmonic oscillator' equation and is important in almost all branches of science. The solution may be found by guessing or other means (which are usually guessing in disguise). In the context of this equation, λ is called the (angular) frequency of oscillation.



More ODEs Besides the ODEs listed here there are a few others that are often solved by hand rather than with numerical simulation. Most famously there is the forced, damped oscillator equation $A\ddot{u} + B\dot{u} + Cu = F \sin(Dt)$ which gets its own section.

With the exception of the damped or forced oscillator, most engineers now-a-days will use numerical integration if they want to solve an ODE not in this box.

Conversely,

> Anyone competent at dynamics knows all the equations and solutions on these two pages outside, inside, and inside-out.

Whether you have or have not learned these in a calculus course you should learn them every different way that you can.

Force varies with position. This case, where $F = F(x)$ will be treated in some detail in the next section on energy.

The simplest ODEs

The simplest and most common ODEs in dynamics and the rest of science and engineering are

Linear: e.g., no functions squared.

First or second order: Have only first or second derivatives, respectively, and

Constant coefficient: All multiples of the derivatives are constants, not functions of time.

As mentioned previously, some special cases of these ODEs are listed and discussed in box 9.3 page 424.

9.4 D'Alembert's mechanics: beginners beware

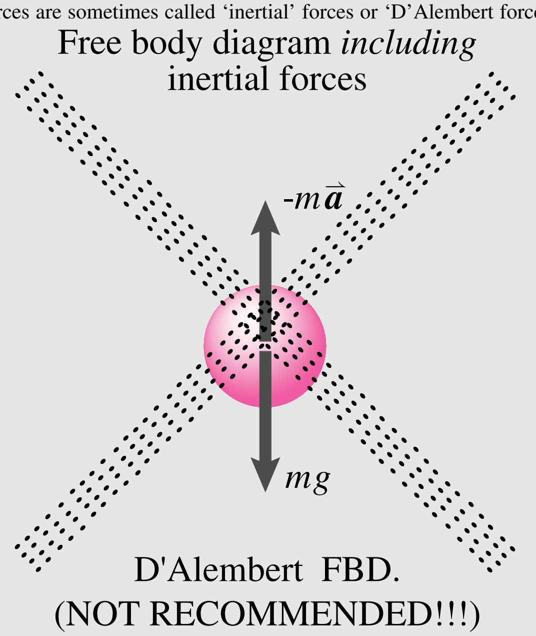
This box *is* an aside. It will not help you do dynamics problems. As warned on page 151, its worse than that. The material in this box usually harms more than it helps. The D'Alembert's approach to mechanics, described here, cannot be well-absorbed by beginners. Students attempting to use D'Alembert methods make frequent mistakes. We advise against the use of D'Alembert mechanics for beginners. We don't allow its use in homework and exams.

But you might be curious about this forbidden fruit. To demystify the taboo, we briefly describe the approach. You might as well learn it here instead of somewhere else.

The D'Alembert approach has an intuitive appeal to experts. And the D'Alembert equations are the first step in deriving the more advanced (*e.g.,* Lagrangian, Hamiltonian, 'method of virtual speed', and 'Kane') approaches to dynamics.

How does it go?

First, label the free body diagram: *'free body diagram including inertial forces.'* Then, in addition to the applied forces draw pseudo-forces equal to $-m\vec{a}$ for every mass particle m. These pseudo-forces are shown in the 'FBD' of a falling ball. The pseudo-forces are sometimes called 'inertial' forces or 'D'Alembert forces.



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Instead of momentum balance equations you write 'pseudo-statics' equations of 'force' balance and 'moment' balance

$$\overbrace{\underbrace{\sum \vec{F}}_{\text{including inertial forces}} = \vec{0}}^{\text{pseudo-force balance}}$$

$$\overbrace{\underbrace{\sum \vec{M}_C}_{\text{including torques from inertial forces}} = \vec{0}}^{\text{pseudo-moment balance}}$$

These equations include the actual forces as well as the 'inertial' forces shown on the 'D'Alembert free body diagram'.

By these means, the dynamics equations have been reduced to statics equations. Linear momentum balance is replaced by pseudo-statics force balance. Angular momentum balance is replaced by pseudo-statics moment balance.

The moving of the inertial terms from the right side of the equation to the left leads to both conceptual simplicity *and* puts the equations of dynamics in a form that is closer to most people's intuitions. The simplification is not so great as it may seem at first sight. Accelerations still need to be calculated and the sums involved in calculation of rate of change of linear and angular momentum still need to be calculated, only now they are sums of pseudo *inertial* forces.

Consider the example of sitting in a car as the car rounds a corner to the left. In the momentum balance approach, we write

$$\vec{F} = \underbrace{m\vec{a}}_{\dot{\vec{L}}}$$

and say the force from the car on you to the left is equal to the rate of change of your linear momentum as you accelerate to the left. In the D'Alambert approach, we write

$$\vec{F} - \underbrace{m\vec{a}}_{\text{inertia force}} = \vec{0}$$

and think the inertia force to the right is balanced by the interaction force of the car on your body to the left.

It is a puzzle of human consciousness why such a trivial algebraic manipulation, namely,

$$\vec{F} = \mathrm{m}\vec{a} \;\Rightarrow\; \vec{F} - m\vec{a} = \vec{0}$$

should lead to such a great conceptual confusion. But, it is an empirical fact that most of us are susceptible to this confusion.

That is, if you follow your likely first intuition and think of $m\vec{a}$ as a force you will probably join the ranks of many other talented students who consequently make many sign errors.

Every teacher of mechanics has encountered the confusion in their students about whether $-m\vec{a}$ is or is not a force (and most likely in themselves as well.) To avoid such confusion, many teachers or texts take a firm stand and say

- '$m\vec{a}$ *is not* a force!'; but, as if believing in a different god, others will say with equal conviction

- '-$m\vec{a}$ *is* a force!'.

In this book, we take the former approach. We take the equation

$$\vec{F} = \mathrm{m}\vec{a}$$

to mean:

forces from interactions $= m\cdot$ (acceleration of mass).

If you insist on working with the D'Alambert approach instead, you must do so confidently and clearly. To repeat,

- instead of labeling your free body diagram '*FBD*', label it '*FBD including inertial forces*',
- instead of using '*Linear Momentum Balance*', use '*Pseudo-Force Balance*', and
- instead of using '*Angular Momentum Balance*' use '*Pseudo-Moment Balance*'

We do *not* recommend D'Alembert mechanics to beginners, but if you insist, good luck to you and don't blame us for your (almost inevitable) sign errors!

SAMPLE 9.1 Time derivatives: The position of a particle varies with time as $\vec{r}(t) = (C_1 t + C_2 t^2)\hat{\imath}$, where $C_1 = 4\,\mathrm{m/s}$ and $C_2 = 2\,\mathrm{m/s^2}$.

1. Find the velocity and acceleration of the particle as functions of time.
2. Sketch the position, velocity, and acceleration of the particle against time from $t = 0$ to $t = 5\,\mathrm{s}$.
3. Find the position, velocity, and acceleration of the particle at $t = 2\,\mathrm{s}$.

Solution

1. We are given the position of the particle as a function of time. We need to find the velocity (time derivative of position) and the acceleration (time derivative of velocity).

$$\vec{r} = (C_1 t + C_2 t^2)\hat{\imath} = (4\,\mathrm{m/s}\,t + 2\,\mathrm{m/s^2}\,t^2)\hat{\imath} \tag{9.5}$$

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(C_1 t + C_2 t^2)\hat{\imath}$$

$$= (C_1 + 2C_2 t)\hat{\imath} = (4\,\mathrm{m/s} + 4\,\mathrm{m/s^2}\,t)\hat{\imath} \tag{9.6}$$

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d}{dt}(C_1 + 2C_2 t)\hat{\imath}$$

$$= 2C_2\hat{\imath} = (4\,\mathrm{m/s^2})\hat{\imath} \tag{9.7}$$

$\vec{v} = (4\,\mathrm{m/s} + 4\,\mathrm{m/s^2}\,t)\hat{\imath},\ \vec{a} = (4\,\mathrm{m/s^2})\hat{\imath}.$

Thus, we find that the velocity is a linear function of time and the acceleration is time-independent (a constant).

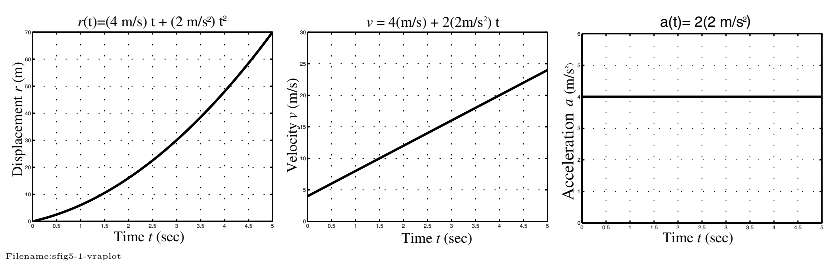


Figure 9.7:

2. We plot eqns. (9.5, 9.6, and 9.7) against time by taking 100 points between $t = 0$ and $t = 5\,\mathrm{s}$, and evaluating $\vec{r}$, $\vec{v}$ and $\vec{a}$ at those points. The plots are shown in fig. 9.7.

3. We can find the position, velocity, and acceleration at $t = 2\,\mathrm{s}$ by evaluating their expressions at the given time instant:

$$\vec{r}|_{t=2\,\mathrm{s}} = [(4\,\mathrm{m/s})\cdot(2\,\mathrm{s}) + (2\,\mathrm{m/s^2})\cdot(2\,\mathrm{s})^2]\hat{\imath}$$

$$= (16\,\mathrm{m})\hat{\imath}$$

$$\vec{v}|_{t=2\,\mathrm{s}} = [(4\,\mathrm{m/s}) + (2\,\mathrm{m/s^2})\cdot(2\,\mathrm{s})]\hat{\imath}$$

$$= (8\,\mathrm{m/s})\hat{\imath}$$

$$\vec{a}|_{t=2\,\mathrm{s}} = (2\,\mathrm{m/s^2})\hat{\imath} = \vec{a}(\text{for all t})$$

At $t = 2\,\mathrm{s}$, $\vec{r} = (16\,\mathrm{m})\hat{\imath}$, $\vec{v} = (8\,\mathrm{m/s})\hat{\imath}$, $\vec{a} = (2\,\mathrm{m/s^2})\hat{\imath}$.

SAMPLE 9.2 Math review: Solving simple differential equations. For the following differential equations, find the solution for the given initial conditions.

1. $\frac{dv}{dt} = a,\ v(t=0) = v_0$, where a is a constant.
2. $\frac{d^2x}{dt^2} = a,\ x(t=0) = x_0,\ \dot{x}(t=0) = \dot{x}_0$, where a is a constant.

Solution

1.

$$\begin{aligned} \frac{dv}{dt} &= a \quad \Rightarrow \quad dv = a\,dt \\ \text{or} \qquad \int dv &= \int a\,dt = a\int dt \\ \text{or} \qquad v &= at + C, \quad \text{where C is a constant of integration} \end{aligned}$$

Now, substituting the initial condition into the solution,

$$v(t=0) \equiv v_0 = a \cdot 0 + C \quad \Rightarrow \quad C = v_0.$$

Therefore,

$$v = at + v_0.$$

$$\boxed{v = v_0 + at}$$

Alternatively, we can use definite integrals:

$$\int_{v_0}^{v} dv = \int_0^t a\,dt \quad \Rightarrow \quad v - v_0 = at \quad \Rightarrow \quad v = v_0 + at.$$

2. This is a second order differential equation in x. We can solve this equation by first writing it as a first order differential equation in $v \equiv dx/dt$, solving for v by integration, and then solving again for x in the same manner.

$$\begin{aligned} \frac{d^2x}{dt^2} &= a \quad \text{or} \quad \frac{dv}{dt} = a \\ \text{or} \qquad \int dv &= \int a\,dt \\ \Rightarrow \quad v \equiv \dot{x} &= at + C_1 && (9.8) \\ \text{but, } v \equiv \frac{dx}{dt}, \quad \Rightarrow \quad \int dx &= \int at\,dt + \int C_1\,dt \\ \text{or} \qquad x &= \frac{1}{2}at^2 + C_1 t + C_2, && (9.9) \end{aligned}$$

where C_1 and C_2 are constants of integration. Substituting the initial condition for $\dot{x}$ in Eqn. (9.8), we get

$$\dot{x}(t=0) = \dot{x}_0 = a \cdot 0 + C_1 \quad \Rightarrow \quad C_1 = \dot{x}_0.$$

Similarly, substituting the initial condition for x in Eqn. (9.9), we get

$$x(t=0) = x_0 = \frac{1}{2}a \cdot 0 + \dot{x}_0 \cdot 0 + C_2 \quad \Rightarrow \quad C_2 = x_0.$$

Therefore,

$$x(t) = x_0 + \dot{x}_0 t + \frac{1}{2}at^2.$$

$$\boxed{x(t) = x_0 + \dot{x}_0 t + \tfrac{1}{2}at^2}$$

SAMPLE 9.3 Constant speed motion: A ship cruises at a constant speed of 15 knots (15 nautical miles per hour) due Northeast. It passes a lighthouse at 8:30 am. The next lighthouse is approximately 35 nautical miles straight ahead. At what time does the ship pass the next lighthouse?

Solution We are given the distance s and the speed of travel v. We need to find how long it takes to travel the given distance.

$$\begin{aligned} s &= vt \\ \Rightarrow \quad t &= \frac{s}{v} = \frac{35\text{nautical miles}}{15\text{(nuatical miles)/hour}} = 2.33 \text{ hrs.} \end{aligned}$$

Now, the time at $t = 0$ is 8:30 am. Therefore, the time after 2.33 hrs (2 hours 20 minutes) will be 10:50 am.

$\boxed{10:50 \text{ am}}$

SAMPLE 9.4 Constant velocity motion: A particle travels with constant velocity $\vec{\boldsymbol{v}} = 5\,\text{m/s}\hat{\boldsymbol{\imath}}$. The initial position of the particle is $\vec{\boldsymbol{r}}_0 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$. Find the position of the particle at $t = 3$ s.

Solution Here, we are given the velocity, *i.e.,* the time derivative of position:

$$\vec{\boldsymbol{v}} \equiv \frac{d\vec{\boldsymbol{r}}}{dt} = v_0\hat{\boldsymbol{\imath}}, \qquad \text{where } v_0 = 5\,\text{m/s}.$$

We need to find $\vec{\boldsymbol{r}}$ at $t = 3$ s, given that $\vec{\boldsymbol{r}}$ at $t = 0$ is $\vec{\boldsymbol{r}}_0$.

$$\begin{aligned} d\vec{\boldsymbol{r}} &= v_0\hat{\boldsymbol{\imath}}dt \\ \Rightarrow \quad \int_{\vec{\boldsymbol{r}}_0}^{\vec{\boldsymbol{r}}(t)} d\vec{\boldsymbol{r}} &= \int_0^t v_0\hat{\boldsymbol{\imath}}dt = v_0\hat{\boldsymbol{\imath}}\int_0^t dt \\ \vec{\boldsymbol{r}}(t) - \vec{\boldsymbol{r}}_0 &= v_0\hat{\boldsymbol{\imath}}t \\ \vec{\boldsymbol{r}}(t) &= \vec{\boldsymbol{r}}_0 + v_0 t\hat{\boldsymbol{\imath}} \\ \vec{\boldsymbol{r}}(3\,\text{s}) &= (2\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}) + (5\,\text{m/s})\cdot(3\,\text{s})\hat{\boldsymbol{\imath}} \\ &= 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}. \end{aligned}$$

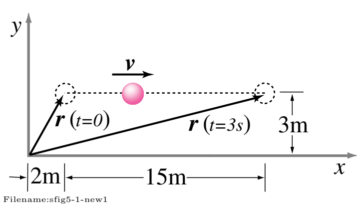


Figure 9.8:

$\boxed{\vec{\boldsymbol{r}} = 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}}$

Comments: We could solve this problem more compactly by working with scalars or components. It is given that the velocity is constant and is only in the x-direction. Therefore, the y-component of particle position will remain the same, *i.e.,* $r_y = r_{0y} = 3\,\text{m}$, and $r_x = r_{0x} + v_x t = 2\,\text{m} + (5\,\text{m/s})\cdot(3\,\text{s}) = 17\,\text{m}$. Thus, $\vec{\boldsymbol{r}}(3\,\text{s}) = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} = 17\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$.

SAMPLE 9.5 Constant acceleration: A 0.5 kg mass starts from rest and attains a speed of $20\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$ in 4 s. Assuming that the mass accelerates at a constant rate, find the force acting on the mass.

Solution Here, we are given the initial velocity $\vec{v}(0) = \vec{0}$ and the final velocity $\vec{v}$ after $t = 4\,\mathrm{s}$. We have to find the force acting on the mass. The net force on a particle is given by $\vec{F} = m\vec{a}$. Thus, we need to find the acceleration $\vec{a}$ of the mass to calculate the force acting on it. Now, the velocity of a particle under constant acceleration is given by

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t.$$

Therefore, we can find the acceleration $\vec{a}$ as

$$\begin{aligned} \vec{a} &= \frac{\vec{v}(t) - \vec{v}(0)}{t} \\ &= \frac{20\,\mathrm{m/s}\hat{\boldsymbol{\imath}} - \vec{0}}{4\,\mathrm{s}} \\ &= 5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}. \end{aligned}$$

The force on the particle is

$$\vec{F} = m\vec{a} = (0.5\,\mathrm{kg}) \cdot (5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}) = 2.5\,\mathrm{N}\hat{\boldsymbol{\imath}}.$$

$$\boxed{\vec{F} = 2.5\,\mathrm{N}\hat{\boldsymbol{\imath}}}$$

SAMPLE 9.6 Time of travel for a given distance: A ball of mass 200 gm falls freely under gravity from a height of 50 m. Find the time taken to fall through a distance of 30 m, given that the acceleration due to gravity $g = 10\,\mathrm{m/s^2}$.

Solution The entire motion is in one dimension — the vertical direction. We can, therefore, use scalar equations for distance, velocity, and acceleration. Let y denote the distance travelled by the ball. Let us measure y vertically downwards, starting from the height at which the ball starts falling (see fig. 9.9). Under constant acceleration g, we can write the distance travelled as

$$y(t) = y_0 + v_0 t + \frac{1}{2}gt^2.$$

Note that at $t = 0$, $y_0 = 0$ and $v_0 = 0$. We are given that at some instant t (that we need to find) $y = 30\,\mathrm{m}$. Thus,

$$\begin{aligned} y &= \frac{1}{2}gt^2 \\ t &= \sqrt{\frac{2y}{g}} = \sqrt{\frac{2 \times 30\,\mathrm{m}}{10\,\mathrm{m/s^2}}} = 2.45\,\mathrm{s}. \end{aligned}$$

$$\boxed{t = 2.45\,\mathrm{s}}$$

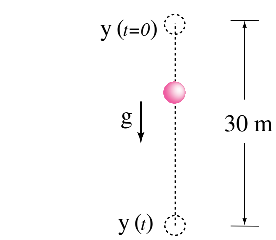


Figure 9.9:

SAMPLE 9.7 Time varying acceleration: A force $F(t) = F_0 \sin \lambda t$ acts on an initially still cart of mass m in a particular direction. Find the speed and the distance travelled by the cart as functions of time. Plot the acceleration, the speed and the displacement of the cart against time for $0 \leq t \leq \pi\,\mathrm{s}$, assuming $\lambda = 1/\mathrm{s}$. What are the speed and the displacement of the cart at $t = \pi\,\mathrm{s}$ if $F_0 = 1\,\mathrm{N}$ and $m = 1\,\mathrm{kg}$?

Solution We are given the applied force and the mass of the cart. Therefore, we know the acceleration ($a = F/m$). Thus,

$$\begin{aligned} a \equiv \frac{dv}{dt} &= \frac{F_0}{m}\sin\lambda t \\ \Rightarrow \quad dv &= a_0 \sin\lambda t\, dt \end{aligned}$$

where $a_0 = F_0/m$. Hence,

$$\begin{aligned} \int_0^{v(t)} dv &= \int_0^t a_0 \sin\lambda\tau\, d\tau \\ \Rightarrow \quad v(t) &= -\frac{a_0}{\lambda}(\cos\lambda t - 1) \\ &= \frac{a_0}{\lambda}(1 - \cos\lambda t). \end{aligned}$$

Since the speed $v = \frac{dx}{dt}$, we have,

$$\begin{aligned} dx &= \frac{a_0}{\lambda}(1-\cos\lambda t)dt \\ \int_0^{x(t)} dx &= \int_0^t \frac{a_0}{\lambda}(1-\cos\lambda\tau)d\tau \\ \Rightarrow \quad x(t) &= \frac{a_0}{\lambda}\left(t - \frac{1}{\lambda}\sin\lambda t\right). \end{aligned}$$

$$\boxed{v(t) = \frac{F_0/m}{\lambda}(1-\cos\lambda t), \quad x(t) = \frac{F_0/m}{\lambda}\left(t - \tfrac{1}{\lambda}\sin\lambda t\right)}$$

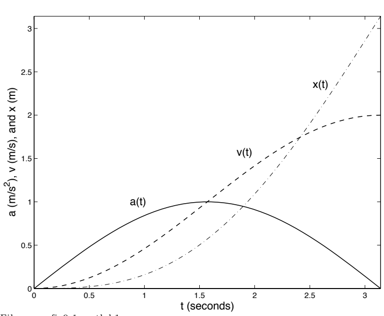


Figure 9.10: Plots of acceleration $a(t)$, speed $v(t)$ and displacement $x(t)$ as functions of time: $a(t) = a_0 \sin\lambda t$, $v(t) = \frac{a_0}{\lambda}(1-\cos\lambda t)$, and $x(t) = \frac{a_0}{\lambda}(t - \frac{1}{\lambda}\sin\lambda t)$ where $a_0 = 1\,\mathrm{m/s^2}$ and $\lambda = 1/\mathrm{s}$.

Substituting $a_0 = F_0/m = 1\,\mathrm{N}/1\,\mathrm{kg} = 1\,\mathrm{m/s^2}$, $\lambda = 1/\mathrm{s}$ and $t = \pi\,\mathrm{s}$ in the expressions for v and x above, we find the speed and the displacement (distance travelled by the cart) at $t = \pi$ seconds as follows.

$$\begin{aligned} v(t = \pi\,\mathrm{s}) &= \frac{1\,\mathrm{m/s^2}}{1/\mathrm{s}}(1-\cos\pi) \\ &= 2\,\mathrm{m/s}, \\ x(t=\pi\,\mathrm{s}) &= 1\,\mathrm{m/s}\left(\pi\,\mathrm{s} - \frac{1}{1/\mathrm{s}}\sin(\frac{1}{\mathrm{s}}\cdot\pi\,\mathrm{s})\right) \\ &= \pi\,\mathrm{m}. \end{aligned}$$

$$\boxed{\text{At } t = \pi\,\mathrm{s}, \quad v = 2\,\mathrm{m/s}, \quad x = \pi\,\mathrm{m}}$$

The graph of $a(t)$, $v(t)$, and $x(t)$ are shown in fig. 9.10 for $0 \leq t \leq \pi\,\mathrm{s}$ assuming the given values of m, λ, and F_0. Note the behavior of $v(t)$ and $x(t)$ close to $t = 0$. Since the cart starts from rest, the speed builds up slowly, and the displacement builds up even more slowly because the speed is very low in the beginning.

SAMPLE 9.8 Numerical integration of ODE's:

1. Write the second order linear nonhomogeneous differential equation, $\ddot{x} + c\dot{x} + kx = a_0 \sin\omega t$, as a set of first order equations that can be used for numerical integration.
2. Write the second order nonlinear homogeneous differential equation, $\ddot{x} + c\dot{x}^2 + kx^3 = 0$, as a set of first order equations that can be used for numerical integration.
3. Solve the nonlinear equation given in (b) by numerical integration taking $c = 0.05$, $k = 1$, $x(0) = 0$, and $\dot{x}(0) = 0.1$. Compare this solution with that of the linear equation in (a) by setting $a_0 = 0$ and taking other values to be the same as for (b).

Solution

1.

$$\begin{array}{lrcl} \text{If we let} & \dot{x} & = & y, \\ \text{then} & \dot{y} & = & \ddot{x} = -c\dot{x} - kx + a_0 \sin\omega t \\ & & = & -cy - kx + a_0 \sin\omega t \end{array}$$

$$\text{or} \quad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -c \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} + \begin{Bmatrix} 0 \\ a_0 \sin\omega t \end{Bmatrix}. \tag{9.10}$$

Equation (9.10) is written in matrix form to show that it is a set of *linear* first-order ODE's. In this case linearity means that the dependent variables only appear linearly, not as powers *etc.*

2.

$$\begin{array}{lrcl} \text{If} & \dot{x} & = & y \\ \text{then} & \dot{y} & = & \ddot{x} = c\dot{x}^2 - kx^3 = -cy^2 - kx^3 \end{array}$$

$$\text{or} \quad \begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{Bmatrix} y \\ -cy^2 - kx^3 \end{Bmatrix}. \tag{9.11}$$

Equation (9.11) is a set of *nonlinear* first order ODE's. It cannot be arranged as Eqn. 9.10 because of the nonlinearity in x and y. It is, however, in an appropriate form for numerical integration.

3. Now we solve the set of first order equations obtained in (b) using a numerical ODE solver with the following pseudocode.

```
ODEs = {xdot = y,  ydot = -c y^2 - k x^3}
IC   = {x(0) = 0, y(0) = 0.1}
Set  k=1, c=0.05
Solve ODEs with IC for t=0 to t=200
Plot x(t) and y(t)
```

The plot obtained from numerical integration using a Runge-Kutta based integrator is shown in fig. 9.11. A similar program used for the equation in (a) with $a_0 = 0$ gives the plot shown in fig. 9.12. The two plots show how a simple nonlinearity changes the response drastically.

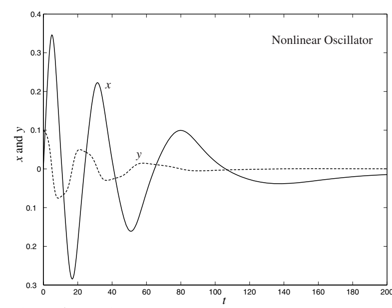


Figure 9.11: Numerical solution of the nonlinear ODE $\ddot{x} + c\dot{x}^2 + kx^3 = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

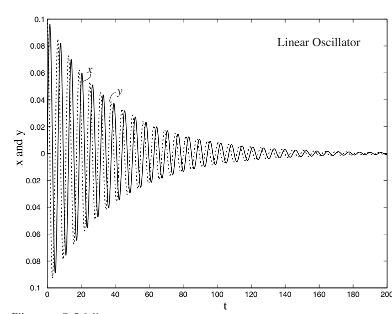


Figure 9.12: Numerical solution of the linear ODE $\ddot{x} + c\dot{x}^2 + kx = 0$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0.1$.

9.2 Energy methods in 1D

Energy is an important concept in science and engineering. Energy is also a kind of currency in human trade. Energy is also a concept that is somewhat bigger than can be defined inside classical mechanics, when we look at, say, the chemical energy cost of various mechanical tasks.

For a student learning mechanics, energy is first a method, or trick, for solving some simple problems of the type assigned in elementary courses like this one. As problems become more difficult (have more degrees of freedom or include, say, more-than-just-constant friction) energy becomes less useful as a problem solving technique. However, in more advanced mechanics, energy gets a central role again: energy is the central concept in some advanced ways to write equations of motion and for some methods of understanding stability.

Power, work, kinetic energy and potential energy

Before we get to the facts and theorems, we start with some definitions. Here are four words. We will use these definitions, or generalizations of them, throughout dynamics.

Power. The power of a force F is its product with the velocity v of the point on which it is acting,

$$\text{Power} = P = Fv.$$

This is the 1D version of the more general $P = \vec{F} \cdot \vec{v}$ which we will use once we go on to 2D and 3D dynamics. In full generality, power is a scalar (not a vector). The common units for power are watts ($1\,\text{W} = \text{N m/s} = \text{J/s}$), kilowatts ($1\text{KW} = 10^3\ \text{W}$), lbf ft/ s (no special name) and horsepower ($1\,\text{hp} \equiv 550\,\text{lbf ft/s} \approx 745.7 \approx 746$ ① watts).

①
In fourteen hundred and ninety-two,
Columbus sailed the ocean blue,
And that's twice the watts
in a horsepower too.

$$\begin{array}{r} 746 \\ \underline{\times 2} \\ 1492 \end{array}$$

Example:
A 5 N force acting on a particle moving 3 m/s has a power of
$P = Fv = (5\,\text{N})(3\,\text{m/s}) = 15\,\text{N m/s} = 15\,\text{W} \approx 0.02\,\text{hp}$.

Work. The work W of a force is most easily defined incrementally (ΔW) for small motions Δx of a particle; motions so small that variations in force can be neglected and the force viewed as constant,

$$\text{increment of work} = \Delta W = F\Delta x.$$

This is a special 1D reduction of the more general $\Delta W = \vec{F} \cdot \Delta\vec{r}$. Even in 2D and 3D work is a scalar. Often we want to know the work for larger (non-infinitesimal) displacements. We do this by adding up the increments. Using sloppy calculus (implicitly taking the limit of a Reimann sum):

$W = \sum \Delta W = \int dW = \int F\,dx$, which we can write more definitely as

$$\text{Work} = W = \int_{x_0}^{x} F(x')\,dx'.$$

which is the 1D version of the more general $W = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}$. Common units of work are Joules ($1\,\text{J} = 1\,\text{N m} = 1\,\text{kg m}^2/\,\text{s}^2$), foot-pounds ($= 1\,\text{ft lbf}$) and kilo-watt hours ($= 3.6 \cdot 10^6\,\text{J}$).

Example:

The force $F = F_0 \sin(cx)$ pushes a mass from $x_0 = 0\,\text{m}$ to $x_1 = \pi\,\text{m}$ where $F_0 = 7\,\text{N}$ and $c = 1/\,\text{m}$. Then

$$W = \int_{x_0}^{x_1} F\,dx = \int_0^{\pi\,\text{m}} F_0 \sin(cx)\,dx = -(F_0/c)\cos(cx)|_0^{\pi\,\text{m}} = 14\,\text{N}$$

Kinetic energy. The kinetic energy quantifies the motion a little differently than momentum does. In kinetic energy high speed gets extra credit (v^2 instead of just v). Further, for kinetic energy we don't worry about which way a particle moves. The kinetic energy E_K of a particle in 1D is

$$\text{kinetic energy} = E_\text{K} = \frac{1}{2}mv^2$$

In two and three dimensions the formula above applies for one particle (taking $v = |\vec{\boldsymbol{v}}|$). For a collection of particles E_K is defined as a sum of E_K for each particle separately. In full generality work is a scalar. The units of work (force$\times$distance) and of kinetic energy (mass$\times$speed^2) are the same (mass$\times$distance^2/ time^2) and so are the common measures, namely Joules, foot-pounds and kilo-watt hours.

Example:

A 3 kg mass moving at a speed of 4 m/s has a kinetic energy of $E_\text{K} = mv^2/2 = (3\,\text{kg})(4\,\text{m/s})^2/2 = 24\,\text{kg m}^2/\,\text{s}^2 = 24\,\text{J}$.

Potential energy. This is the most abstract of the definitions. The potential energy E_P associated with a force F is defined as that function of x with these properties

$$E_\text{P}(x) = -\int_{x_0}^{x} F(x')\,dx' \qquad \text{and} \qquad F(x) = -\frac{d}{dx}E_\text{P}(x)$$

which people write more indefinitely as $E_\text{P} = -\int F\,dx$ and $F = -E_\text{P}{}'$. In two and three dimensions the concept of potential energy is more subtle still, being defined by a path integral which may or may not be sensible. But it is still a scalar.

Example:

The force $F = c/x^2$ is associated with the potential energy $E_\text{P} = -\int F\,dx = c/x + C_0$.

The datum for potential energy. Potential energy always has an undetermined, and generally irrelevant, integration constant. The integration constant is irrelevant because usually we care about *changes* in energy. So in the example above we could set $C_0 = 0$ and write $E_P = -\int F\,dx = c/x$. In general we define the *datum* for potential energy as that position where we set the potential energy to zero.

- For near-earth gravity the datum is usually set at the height of the ground (so that $E_P = mgh$), a launch point, or of a conspicuous physical point (say the hinge of a pendulum).
- For inverse-square gravity the datum is usually set at $r = \infty$ so that formulas are most simple.
- For springs that datum is usually set at the position where the spring is 'relaxed' (unstretched and at its rest-length), again simplifying the terms in energy equations.

Potential energy is a shortcut for calculating work. From the definition of potential energy we can calculate work of a force in moving a particle from one place to another as:

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{P2} - E_{P1}\right).$$

Of course you need to know, or find, $E_P(x)$ first in order to use this shortcut.

Example:
The work of $F = c/x^2$ in moving a mass from x_1 to x_2 is

$$\text{work} = \int_{x_1}^{x_2} F(x')\,dx' = -\left(E_{P2} - E_{P1}\right) = c/x_1 - c/x_2$$

Where we used that $E_P = c/x$ has the needed property that $F = -\dfrac{d}{dx}E_P$.

Why all this new language? All of the words above are defined in terms of position, velocity and force. So anything we say about power, work and kinetic and potential energies we could say already using x, v and F. More particularly, we already have two ways of quantifying the motion of a particle, v and $L = mv$. Why do we need a third, $E_K = mv^2/2$? The answer is this, to simplify the solution of some problems. Various facts and theorems are simpler if commonly appearing groups of terms are given names. And all of the definitions above are common groups. Then, luckily, some of them turn out to be more general than just 1D particle mechanics.

The new vocabulary makes thinking easier. Various so-called 'one degree of freedom' problems can be solved by noting that energy is conserved. And features of solutions of more-complex problems can be extracted or checked by making sure that energy balance comes out right.

Power and work

The simplest relation between the quantities we have defined above is that between Power and work:

$$W = \int F\,dx = \int Fv\frac{dx}{v} = \int Fv\,dt$$

or more definitely

$$W = \int_{x_0}^{x_1} F\,dx = \int_{t_0}^{t_1} Fv\,dt = \int_{t_0}^{t_1} P\,dt$$

Example: Integrate power to get work.
If the power of a force acting on a particle is $P = P_0(ct^2)$ where $P_0 = 10\,\mathrm{W}$ and $c = 3/\,\mathrm{s}^2$ then over 3 seconds the work done by the force is:

$$W = \int_{t_0}^{t_1} P\,dt = \int_{t_0}^{t_1} P_0(ct^2)\,dt = P_0ct^3/3\Big|_{t_0}^{t_1} = (10\,\mathrm{W})(3/\,\mathrm{s}^2)t^3/3\Big|_0^{3\,\mathrm{s}}$$
$$= 270\,\mathrm{W\,s} = 270\,\mathrm{J}$$

Power and rate-of-change of kinetic energy

On the inside cover the third basic law of mechanics is energy balance. Energy balance takes a number of different forms, depending on context. The power balance equation from the front cover and simplified for a particle is

$$P = \dot{E}_{\mathrm{K}},$$

where, recall, $P = Fv$ is the power of the applied force F. The derivation of this result from $F = ma$ for a particle is simple enough, and is good to know. First note the following result from using the chain of differentiation:

$$\frac{d}{dt}\left(v^2\right) = 2v\frac{dv}{dt} = 2v\dot{v} = 2va.$$

When we need to call on this simple kinematics (calculus) result it usually comes to us the other way around. So what you should remember is this formula, one of the basic tricks of the trade:

$$va = \frac{d}{dt}\left(\frac{v^2}{2}\right).$$

Multiplying both sides by m and substituting in $F = ma$ we get our 1D power balance equation:

$$F v = \frac{d}{dt}\left(\frac{mv^2}{2}\right).$$

The power of a given force depends on the speed of the object to which it is applied. When a finite non-zero force is applied to a stationary object the power of the force is zero and so is the rate of change of kinetic energy. If the object accelerates, its speed is increasing, but when the speed is zero, $\dot{E}_K = 0$.

Example:
A constant force F is applied to an initially stationary mass m starting at $t = 0$. Then $v = Ft/m$, $E_K = mv^2/2 = F^2t^2/(2m)$ and $P = Fv = F^2t/m$. Note that $\dot{E}_K = P$ and both are zero at $t = 0$.

Work is change of kinetic energy

Integrating the power balance equation in time we get

$$\int P\,dt = \int \dot{E}_K\,dt = \Delta E_K \tag{9.12}$$

More definitely, and also using the work integral, we have that the work of the net force on a particle is the change of its kinetic energy:

$$\int_{t_1}^{t_2} \underbrace{F v}_{P}\,dt = \int_{x_1}^{x_2} F\,dx = E_{K2} - E_{K1}$$

Once we remember that

work is change in kinetic energy,

we can use it without deriving it every time from $F = ma$ or from more general energy balance equations.

Example:
A force applied to a particle m varies sinusoidally with position according to $F = F_0 \cos(cx)$. At $x = 0$ the particle has speed $v = v_0$. Then

$$\begin{aligned} W = \Delta E_K \quad &\Rightarrow \quad \int_0^x F(x')\,dx' = \Delta\left(mv^2/2\right) \\ &\Rightarrow \quad F_0 \sin(cx)/c = mv^2/2 - mv_0^2/2 \\ \text{so} \quad & \quad v = m\sqrt{v_0^2 + 2F_0 \sin(cx)/(mc)} \end{aligned}$$

The above example illustrates three points you should remember:

- The work-energy equations always leave the *sign* of the velocity unknown. You can see this because the derivation involves v^2. You can also see it in formulas you get for velocity. They involve a square root, and thus, implicitly a $\pm$. Whether one, the other or both roots are relevant depends on reasoning that lies outside the energy equation itself.
- The work-energy equations can generate formulas that, in certain situations, are nonsense: If the initial speed v_0 is not high enough the particle will not get very far. In particular if $v_0^2 < 2F_0/(mc)$ the inside of the square root will be negative for some x and the "answer" will be imaginary. These are values of x that the particle will never reach.
- Here we have apparently solved for something about the motion of a particle. And we have, partially. But to find the $x(t)$ we would have to integrate again. And that next integral is hard. That is, energy balance lets us solve for some aspects of the motion, namely speed vs position, without ever needing to know in detail how position varies with time.

Conservation of energy

Many people leave high-school physics loving conservation of energy. It makes certain special homework problems easy. In the real world the principle is also useful for building intuition, and sometimes also for problem solving. In 1D particle mechanics energy conservation is a theorem②.

② Be forwarned, generally we cannot think of energy conservation as necessarily applicable nor, if applicable, as derivable from the equations of mechanics.

Recall that if a particle is acted on by a force that varies with position, $F = F(x)$, then we can define a potential energy $E_{\mathrm{P}} = -\int F\,dx$ and that the work done by the force when the particle moves from x_1 to x_2 is

$$-(E_{\mathrm{P}2} - E_{\mathrm{P}1}) = -\Delta E_{\mathrm{P}}.$$

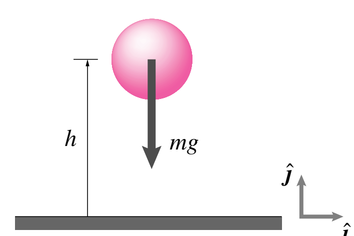
That is, the decrease in E_{P} is the amount of work that the force does. Or, in other words, E_{P} represents a potential to do work. Because work causes an increase in kinetic energy, E_{P} is called the *potential energy* of the force field. Now we can compare this result with the work-energy equation 9.12 to find that

$$-\Delta E_{\mathrm{P}} = \Delta E_{\mathrm{K}} \qquad \Rightarrow \qquad 0 = \Delta\,\underbrace{(E_{\mathrm{P}} + E_{\mathrm{K}})}_{E_{\mathrm{T}}}.$$

The *total energy* E_{T} doesn't change ($\Delta E_{\mathrm{T}} = 0$) and thus is a constant. In other words,

> as a particle moves in the presence of a force field with a potential energy, the total energy $E_{\mathrm{T}} = E_{\mathrm{K}} + E_{\mathrm{P}}$ is constant.

This fact goes by the name of *conservation of energy*.



Filename:tfigure1-falling-ball

Figure 9.13: Free body diagram of a falling ball, assuming gravity is the only significant external force acting on the ball.

Example: Falling ball
Consider the ball in the free body diagram 9.13. If we define gravitational potential energy as minus the work gravity does on a ball while it is lifted from the ground,

then

$$E_{\mathrm{P}} = -\int_0^y (-mg)\,dy' = mgy = mgh.$$

For vertical motion

$$E_{\mathrm{K}} = \frac{1}{2}m\dot{y}^2.$$

So conservation of energy says that in free fall:

$$\text{Constant} = E_{\mathrm{P}} + E_{\mathrm{K}} = mgy + m\dot{y}^2/2$$

which you could also derive directly from $m\ddot{y} = -mg$.

Using conservation of energy to find equations of motion. On the one hand conservation of energy sometimes gives us a (partial) solution to a mechanics problem. On the other, we can use conservation of energy to find the "equations of motion". The basic strategy is to take the derivative of the conservation of energy equation.

Example: Falling ball eqns. from energy.

$$\begin{aligned} E_{\mathrm{T}} = \text{constant} \quad \Rightarrow \quad 0 &= \tfrac{d}{dt}E_{\mathrm{T}} \\ &= \tfrac{d}{dt}(E_{\mathrm{P}} + E_{\mathrm{K}}) \\ &= \tfrac{d}{dt}(mgy + m\dot{y}^2/2) \\ &= (mg\dot{y} + m\dot{y}\ddot{y}) \\ \Rightarrow \quad m\ddot{y} &= -mg. \end{aligned}$$

We had to assume (and this is just a technical point) that $\dot{y} \neq 0$ in one of the cancellations. We have used energy balance to derive linear-momentum balance.

One can also find equations of motion starting with power balance.

$$P = \dot{E}_{\mathrm{K}}$$

as derived here in detail here for the case of gravity acting on a particle.

$$\begin{array}{rcll} P &=& \frac{d}{dt}\,(E_{\mathrm{K}}) & \text{(Power balance)} \\ \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} &=& \frac{d}{dt}\,(E_{\mathrm{K}}) & \text{(Power of external force)} \\ (-mg\,\hat{\boldsymbol{\jmath}}) \cdot (\dot{y}\,\hat{\boldsymbol{\jmath}}) &=& \frac{d}{dt}\left[\frac{1}{2}mv^2\right] & \text{(expanding terms)} \\ -mg\dot{y} &=& \frac{1}{2}m\frac{d}{dt}\left(\dot{y}^2\right) & \text{(evaluate dot product, substitute for } v\text{)} \\ -mg\dot{y} &=& \frac{1}{2}m\,(2\dot{y}\cdot\ddot{y}) & \text{(the chain rule)} \\ \ddot{y} &=& -g & \text{(cancel terms, switch sides),} \end{array} \tag{9.13}$$

The potential energy of a spring is $k(\Delta\ell)^2/2$. Besides near-earth gravity, which we already covered ($E_{\mathrm{P}} = mgh$), the main elementary use of potential energy is for the stretch of a linear spring. Integrating $dW = F dx$ for a linear spring with force on an object $F = -kx$, where x is the spring stretch, from the rest length, we get

$$E_{\mathrm{P}} = -\int_0^x F(x')\,dx' = -\int_0^x -kx'\,dx' = \frac{1}{2}kx^2. \tag{9.14}$$

In the above example we measured x from the rest position of one end of the spring. But often the natural x coordinate will not be so nicely set up. It is safer to remember the spring's potential energy in terms of its stretch:

$$E_P = \frac{k\,\Delta\ell^2}{2}, \tag{9.15}$$

where we measure $\Delta\ell = \ell - \ell_0$ where ℓ_0 is the spring's rest length ($\ell_0 =$ length when the tension is zero).

Thus for a spring and mass oscillator, the subject of the next section, conservation of energy tells us that $mv^2/2 + kx^2/2 = constant$.

Is energy balance a principle or a calculation trick?

For one dimensional particle motion, momentum balance, power balance, and energy balance can each be derived from either of the others. If we take $F = ma$ as primary, energy calculations are just a convenience of notation or, in the case of the work-energy relation, a useful calculation technique (trick).

Historically, conservation of energy was first noted in particle mechanics problems. But because the position-dependent forces of springs and gravity seemed so fundamental, that they had a description as the derivative of a potential gave the energy relations the smell of something more fundamental. And so it has turned out that energy is an important topic for chemistry, thermodynamics, electrodynamics and sub-atomic physics. Its not just an analogy, its the same energy. Thus energy is the primary currency of exchange between, say, the superficially disparate chemical and mechanical systems.

The exchange of energy between these forms, in the context of particle mechanical models, can give the sense that we are doing the same 1D momentum based mechanics calculations when actually we are using more general energy balance equations, equations that cannot be derived from $F = ma$.

Terrestrial locomotion: Trains, cars, bicycles and animals

A free body diagram of an accelerating car, treated as a 1D particle system, is shown in fig. 9.5 on page 418. The point represents the car, the force is the propulsion force from the wheel-ground interaction, and for now, we have neglected air friction. Without worrying about details we could say then that the power of the propulsion force is equal to the rate of change of kinetic energy.

Example: Accelerating car
An aggressive 1 ton car can accelerate with $a = 0.5g$ while going 60 mph. Neglecting friction and air resistance, the power of the propulsion force is

$$
\begin{aligned}
P &= Fv = mav = (1\,\text{ton})(0.5g)(60\,\text{mi/hr}) \\
&= (1\,\text{ton})(0.5g)(60\,\text{mi/hr}) \underbrace{\left(\frac{2000\,\text{lbm}}{\text{ton}}\right)}_{1} \underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)}_{1} \underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)}_{1} \underbrace{\left(\frac{1\,\text{lbf}}{g\cdot\text{lbm}}\right)}_{1} \underbrace{\left(\frac{1\,\text{hp}}{550\,\text{ft}\cdot\text{lbf/s}}\right)}_{1} \\
&= (0.5\cdot 60\cdot 2000\cdot 5280)/(3600\cdot 500)\,\text{hp} \approx 160\,\text{hp}
\end{aligned}
$$

Note the judicious multiple multiplication by 1 so that all units cancel but for horsepower; ton cancels ton, hr cancels hr, g cancels g and so on. The car engine needs to supply this 160 hp plus any internal transmission dissipation. And more still to cover the tire drag and air drag *etc.*

Such calculations are deceptively simple. Some apparent paradoxes:

- The propulsive force on the car comes from the interaction of the ground with the car. Are we saying that the (dead-as-a-doormat) ground supplies a power of, say, 160 hp to an accelerating car?
- The point of application of the force on the car is at the bottom of the tire. That point has no velocity. So the actual power of the ground force on the car (tire) is zero. How is that reconciled with, say, the 160 hp that we get from particle mechanics.

These are legitimate concerns which are discussed further in box 9.5 on page 445. The bottom line is that the calculation turns out, perhaps by the demands of dimensional consistency, to be useful and correct.

Drag power. The drag force of air on moving things has an effect on the energy balance. Air drag is important for cars, bicycles and animals that are moving quickly (say, running people). The air drag is proportional to

$$F_d = \frac{1}{2}\rho C_d A v^2$$

What are the proportionalities in the drag formula?

- The cross-sectional area (the area visible from directly in front) A. The bigger the area the more air has to be pushed out of the way. For a car $A \approx 2\,\text{m}^2$
- The density of air ρ. The more mass has to be pushed out of the way, the bigger the force. For rough calculations one can remember that the density of air is about one thousandth that of water $\rho_{air} \approx 1\,\text{kg/m}^3$. But the density varies in human environments from about $1.1\,\text{kg/m}^3$ in high-altitude (low pressure), high-temperature (gas expands when hot), humid (water vapor is lighter than air) environments up to about $1.4\,\text{kg/m}^3$ in low-lying cold dry places.
- The relative speed squared v^2. The faster you are moving the more air per unit time you must displace, and each bit of air gets displaced with a bigger speed. Typical highway speeds are about $v = 30\,\text{m/s}$

($\approx$ 67 mph) and a typical human walking speed is about $v = 1\,\text{m/s}$ (about 10% over $\approx$ 2 mph).

- A shape coefficient, sometimes called a drag coefficient C or C_d. Different shapes of the same size, can displace the air more or less as the vehicle passes through. Streamlined shapes have small C_d
- One half (1/2). Convention has a factor of 1/2. This simplifies the power interpretation.

The drag power is

$$P = F_d v = \frac{1}{2}\rho C A v^3$$

which is a key result: increasing the speed 1% increases the power demand by 3% and doubling the speed multiplies the power demand by a factor of 8. This huge dependence of power on speed motivates smug energy-conservers to drive annoyingly slowly on highways.

Another way of writing the drag equation is

$$P = C_d \times (\text{ The relative kinetic energy swept by the vehicle per unit time})$$

How's that? The volume "swept" by the vehicle per unit time is its area times speed vA. The air mass swept per unit time is thus $\rho v A$. The kinetic energy of the air, measured as moving relative to the vehicle, is $v^2/2$ per unit mass. Putting this together we get the swept kinetic energy of the air per unit time is $\rho A v^3/2$.

How big is the drag coefficient C_d? When in doubt take dimensionless constants as 1 and you are usually not too far off. At one extreme, a flat plate has $C_d \approx 1.25$ and good airfoils have $C_d \approx 0.05$. People, animals, and bicyclists all have C_d close to 1.

Drag on cars. For the worst cars C_d is actually almost 1. For typical cars on the street $C_d \approx 0.35$. For the best high-efficiency cars on the market in 2007, $C_d \approx 0.25$. Real marketed cars may one day get drag as low as $C_d \approx 0.2$. And concept cars that are shaped like trout can have drag as low as $C_d \approx 0.1$.

The drag power of a $2\,\text{m}^2$ car going $30\,\text{m/s}$ ($\approx$ 67 mph $=$ 108 km/ hr) is about

$$\begin{aligned} P_{drag} &= \frac{1}{2}C\rho A v^3 \approx \frac{1}{2}\cdot 0.35 \cdot (1\,\text{kg/ m}^3)\cdot(2\,\text{m}^2)\cdot(30\,\text{m/s})^3 \\ &= 9450\,\text{kg m}^2/\text{ s}^3 = 9.45\,\text{KW} \approx 13\,\text{hp} \end{aligned}$$

That is, comparing with the example above, a car that needs 160 hp extra to make a zippy pass needs only 13 hp to move steadily along at a typical highway speed. For the units conversion we used $1\,\text{N} = 1\,\text{kg m/ s}^2$, $1\,\text{J} = 1\,\text{N m}$, $1\,\text{W} = 1\,\text{J/ s}$, and $1\,\text{KW} \approx 1.34\,\text{hp}$.

Caveat on the drag "law". While there is some physics in the reasoning behind the drag law $F_d = \rho C A v^2/2$ the emphasis should be on the word “some”, the whole chaotic nature of turbulent flow is not captured. The quadratic drag law is an empirical fit. For a given shape the C_d actually depends on the surface texture. And for a given shape and texture the C_d depends on v, the v^2 doesn’t capture all of the velocity dependence. None-the-less, the drag law is a reasonable approximation for most engineering purposes where drag is important.

Summary

There are two basic types of energy problems

- Problems where force or acceleration is given as a function of position ($a = a(x)$ or $F = F(x)$) and energy methods are basically a trick for finding $v(x)$.
- Problems where work, energy or power is of interest for its own sake because of, say, interest in engine power, dissipated energy, etc.

Of course the two problem types can also overlap.

9.5 Energetics of locomotion: using particle equations for non-particle systems

On page 442 we showed a naive locomotion power example in which we used

$$P = Fv$$

where v was the car velocity, F the thrust on the car, and P was 'the power' of the locomotion force. We pointed out two issues.

- How does it make sense for the passive ground, the source of the propulsive force, to supply power?
- The point of application of the ground force on the car is at the bottom of a wheel, a point that is not moving ($v = 0$). So how can Fv be other than zero?

The basic issue is that a car is not a particle, it has many moving parts and also some chemistry, so particle equations need to be interpreted with some care.

Particle equations are exact for non-particle systems

The most general form for linear momentum balance, as applied to a complicated system moving and deforming in complicated ways, reduces to equation $\vec{F} = m\vec{a}$. That is, so long as we interpret $\vec{F}$ to be the total force on the sytem, $\vec{a}$ to be the acceleration of the center of mass, and m to be the total mass of the system.

The power and energy equations in this chapter have been based on $\vec{F} = m\vec{a}$ (or their 1D scalar version $F = ma$) so apply to *any* system. But the terms P and E_K have meanings that go beyond particle mechanics. So while it is correct that (we derived it from $F = ma$),

$$Fv = \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

> for non-particle systems it is *not correct* that Fv is the actual power of the force applied nor that $mv^2/2$ is the kinetic energy of the system.

To understand the situation depends on understanding multi-body systems where we will see that the power of a force is $\vec{F}\cdot\vec{v}_P$ where $\vec{v}_P$ is the velocity of the material point to which the force is applied; and the kinetic energy is larger than $\frac{1}{2}mv^2$ because of motion relative to the average motion. Remember to reconsider these issues when you know more.

More general energy balance equations

Without worrying about what we can derive from what, there is no doubt that for any closed system we can write the energy balance equation from the front inside cover of the book, the first law of thermodynamics, as:

$$\dot{Q} + P = \dot{E}_\mathrm{K} + \dot{E}_\mathrm{P} + \dot{E}_\mathrm{int}.$$

About the ever-shifting sign conventions, here we use $\dot{Q}$ as the heat flow *in* to the system, P is the power of external forces *on* the system, $\dot{E}_\mathrm{K}$ and $\dot{E}_\mathrm{P}$ are the rate of *increase* of the kinetic and potential energies of the system, and $\dot{E}_\mathrm{int}$ is the rate of *increase* of internal energy. We can consider an accelerating car using this energy equation. For simplicity assume that no external forces do work on the car (the ground certainly does no work, and let's neglect air friction for now). We can also look at a car on level ground so there are no changes in gravitational potential energy. Finally, even though a car has many moving parts, the bulk of the material goes at the speed of a typical point on the body of the car. Thus the particle formula for kinetic energy is reasonably accurate. Putting this altogether we have

$$\dot{Q} + \underbrace{P}_{0} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \underbrace{E_\mathrm{P}}_{0} + \dot{E}_\mathrm{int}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\dot{E}_\mathrm{int} + \dot{Q}$$

The rate of loss of chemical potential energy $-\dot{E}_\mathrm{int}$ less the heat flow out $-\dot{Q}$ is what we call the power of the engine. Say chemical energy is being lost (used up) at a rate of $-\dot{E}_\mathrm{int} \doteq 40\,\mathrm{KW}$. Say the heat flow out the exhaust is $-\dot{Q} = 30\,\mathrm{KW}$. Then, with that $40\,\mathrm{KW}$ of fuel use and that 25% efficient engine, we would have

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = -\dot{E}_\mathrm{int} + \dot{Q} = 40\,\mathrm{KW} - 30\,\mathrm{KW} = 10\,\mathrm{KW} \approx 13\,\mathrm{hp}.$$

But even this is not quite right because it does not take account the flow of gases in and out of the car. Things can be messy if you look carefully.

What's the bottom line? In the end, with some sloppiness of thought but not much inaccuracy, we are not far off thinking that the change of kinetic energy of the car has to come from some place. And that place is the work of the engine as supplied by the decrease in chemical potential energy of the fuel. When we write $P = \dot{E}_\mathrm{K}$ for a car, the P in that equation is the force applied to the car times the velocity of the car. But that P is not the power supplied by the outside agents on the car (e.g., the passive ground). Rather it is the power of forces inside the car. Never mind that we're modeling a car as a particle with no internal structure, at least for momentum-balance purposes.

This whole situation can only be properly clarified when we look at the power of internal and external forces in multi-body systems.

SAMPLE 9.9 Which is the best bicycle helmet? Assume a bicyclist moves with speed 25 mph when her head hits a brick wall. Assume her head is rigid and that it has constant deceleration as it travels through the 2 inches of the bicycle helmet. What is the deceleration? What force is required? (Neglect force from the neck on the head.)

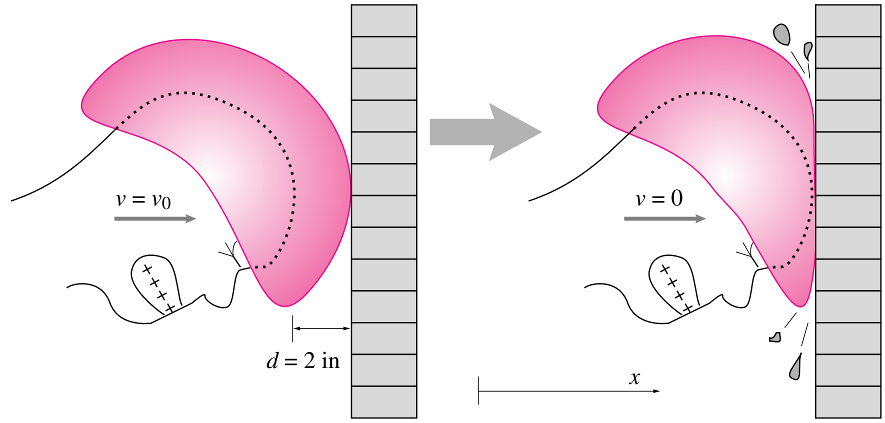


Figure 9.14:

Solution

Solution 1 – Kinematics method 1: We are given the initial speed of V_0, a final speed of 0, and a constant acceleration a (which is negative) over a given distance of travel d. If we call t_c the time when the helmet is fully crushed,

$$\begin{aligned}
v(t) &= v_0 + \int_0^{t_c} a(t')dt' = v_0 + at_c \\
0 = v(t_c) &= v_0 + at_c \quad \Rightarrow \quad t_c = -v_0/a \qquad (9.16)\\
x(t) &= x_0 + \int_0^{t_c} v(t')dt' = 0 + \int_0^{t_c} (v_0 + at)dt \\
d = x(t_c) &= 0 + v_0 t_c + a t_c^2/2 \\
d &= v_0\left(\frac{-v_0}{a}\right) + a\left(\frac{v_0}{a}\right)^2/2 \quad \Rightarrow \quad d = \frac{-v_0^2}{2a} \qquad \text{(using (9.16))}\\
\Rightarrow \quad a &= \frac{-v_0^2}{2d} \\
&= \frac{-(25\,\text{mph})^2}{2\cdot(2\,\text{in})} \\
&= \frac{-25^2}{4}\cdot\frac{\text{mi}^2}{\text{hr}^2\cdot\text{in}}\cdot\underbrace{\left(\frac{5280\,\text{ft}}{\text{mi}}\right)^2}_{1}\cdot\underbrace{\left(\frac{1\,\text{hr}}{3600\,\text{s}}\right)^2}_{1}\cdot\underbrace{\left(\frac{12\,\text{in}}{\text{ft}}\right)}_{1}\cdot\underbrace{\left(\frac{1g}{32.2\,\text{ft}/\text{s}^2}\right)}_{1} \\
&= \frac{-25}{4}\cdot\frac{5280^2}{3600^2}\cdot 12\cdot\frac{1}{32.2}g \\
a &= -125g
\end{aligned}$$

To stop from 25 mph in 2 inches requires an acceleration that is 125 times that of gravity.

Solution 2 – Kinematics method 2:

$$
\begin{array}{rcll}
\frac{dv}{dt} & = & a & \Rightarrow \quad dv = a\,dt \\
\Rightarrow \quad v\,dv & = & a\,v\,dt & \Rightarrow \quad v\,dv = a\frac{dx}{dt}dt \\
\Rightarrow \quad v\,dv & = & a\,dx & \\
\Rightarrow \quad \int v\,dv & = & \int a\,dx & \\
\Rightarrow \quad \Delta\frac{v^2}{2} & = & ax & \qquad \text{(since } a = \text{constant)} \\
\Rightarrow \quad 0 - \frac{v_0^2}{2} & = & ad & \Rightarrow \quad a = \frac{-v_0^2}{2d} \qquad \text{(as before)}
\end{array}
$$

Solution 3 – Quote formulas:

$$
\begin{array}{rcll}
\text{“}v & = & \sqrt{2ad}\text{”} & \\
\Rightarrow \quad a & = & \frac{v^2}{2d} & \quad \text{which is right if you know how to interpret it!}
\end{array}
$$

Solution 4 – Work-Energy:
Constant acceleration $\Rightarrow$ constant force

$$
\begin{array}{rcll}
\text{Work in} & = & \Delta E_{\text{K}} & \\
-Fd & = & 0 - \frac{mv_0^2}{2} & \Rightarrow \quad F = \frac{mv_0^2}{2d} \\
\text{But } \vec{F} = m\vec{a} & \Rightarrow & -F\hat{\imath} = ma\hat{\imath} \Rightarrow a = \frac{-F}{m} & \\
\text{So } a & = & \frac{-v_0^2}{2d} & \qquad \text{(again)}
\end{array}
$$

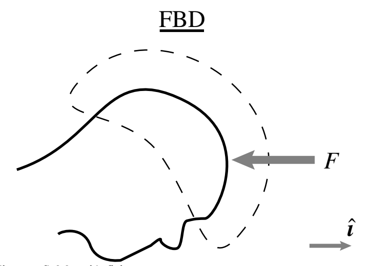


Figure 9.15: F is the force of the helmet on the moving head.

Assuming a head mass of 8 lbm, the force on the head during impact is

$$
|F| = \frac{mv_0^2}{2d} \quad = \quad ma \quad = \quad 8\,\text{lbm} \cdot 125g.
$$

$$
\boxed{|F| = 1000\,\text{lbf}}
$$

During a collision in which an 8 lbm head decelerates from 25 mph to 0 in 2 inches, the force applied to the head is 1000 lbf.

Note 1: The way to minimize the peak acceleration when stopping from a given speed over a given distance is to have constant acceleration. The ‘best’ possible helmet, the one we assumed, causes constant deceleration. There is no helmet of any possible material with 2 in thickness that could make the deceleration for this collision less than $125g$ or the peak force less than 1000 lbf.

Note 2: Collisions with head decelerations of $250g$ or greater are often fatal. Even $125g$ usually causes brain injury. So, the best possible helmet does not insure against injury for fast riders hitting solid objects.

Note 3: Epidemiological evidence suggests that, on average, chances of serious brain injury are decreased by about a factor of 5 by wearing a helmet.

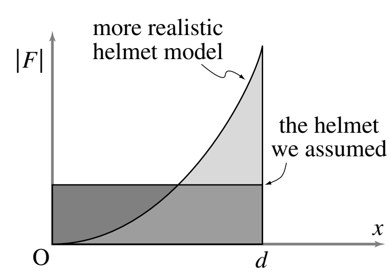


Figure 9.16:

SAMPLE 9.10 Dissipated energy in viscous drag: A ball of mass $m = 1\,\mathrm{kg}$ is dropped from rest from a height $h = 100\,\mathrm{m}$ under gravity. The air resistance on the ball is modeled as viscous drag $F_s = cv$ where v is the speed of the ball and $c = 0.25\,\mathrm{kg/\,s}$ is the drag coefficient. Find the energy dissipated in overcoming the air resistance during the entire flight of the ball.

Solution There are various ways in which we could calculate the energy dissipated in viscous drag. The most straightforward way is to compute the work done by the drag force on the body, $\int F_s dx$ during the entire flight. This calculation will be very easy if we knew the drag force as a function of position, that is, if we have $F_s(x)$. Unfortunately, we have $F_s \equiv F_s(v) = cv$ and we do not know v as a function of position. However, we can find the speed v as a function of time by solving the equation of motion $F = ma$ and determine the speed just before the ball hits the ground. Now, we can find the energy of the ball in two positions — just when it starts falling and just before it hits the ground. The difference between the two energies is what is lost or dissipated in the overcoming the air resistance. Let 'A' denote position-1 from where the ball is dropped, *i.e.,* $y_A = h$, and 'B' denote position-2, a hair above the ground, *i.e.,* $y_B = 0$. Taking the ground as the datum for potential energy, we have,

$$\begin{aligned} E_A &= (E_\mathrm{K} + E_\mathrm{P})_A = \frac{1}{2}m \overbrace{v_A^2}^{0} + mg \overbrace{y_A}^{h} \\ &= mgh \\ E_B &= (E_\mathrm{K} + E_\mathrm{P})_B = \frac{1}{2}m \overbrace{v_B^2}^{?} + mg \overbrace{y_B}^{0} \\ &= \frac{1}{2}mv_B^2. \end{aligned}$$

Therefore, the energy dissipated in air resistance is

$$E_\mathrm{drag} = \Delta E = E_A - E_B = mgh - \frac{1}{2}mv_B^2 \tag{9.17}$$

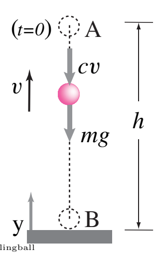


Figure 9.17: Free body diagram of the falling ball. Note that the drag force $F_s = cv$ is shown acting downwards. This is because we have assumed v to be positive upwards and the drag force always acts in the opposite direction of the velocity.

Now, we just need to find v_B. From the free-body diagram shown in fig. 9.17, we have,

$$\begin{aligned} m \overbrace{\ddot{y}}^{\dot{v}} &= -mg - cv \\ \text{or} \quad \frac{dv}{dt} &= -g - \frac{c}{m}v \\ \Rightarrow \quad \int_0^{v(t)} \frac{dv}{\tilde{c}v + g} &= -\int_0^t d\tau \end{aligned}$$

where $\tilde{c} = \frac{c}{m}$. Thus,

$$\begin{aligned} \frac{1}{\tilde{c}} \ln(\tilde{c}v + g)\Big|_0^{v(t)} &= -t \\ \Rightarrow \quad \ln \frac{\tilde{c}v(t) + g}{g} &= -\tilde{c}t \\ \Rightarrow \quad v(t) &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t} - 1). \end{aligned} \tag{9.18}$$

So, we have solved for $v(t)$. Unfortunately, we cannot find v_B from this expression because we do not know what t when the ball reaches the ground. Thus we need to first find t_B and then substitute it in eqn. (9.18) to find v_B. From eqn. (9.18), we have,

$$\begin{aligned}
\frac{dy}{dt} &= \frac{g}{\tilde{c}}(e^{-\tilde{c}t} - 1)\\
\Rightarrow \quad \int_h^0 dy &= \frac{g}{\tilde{c}}\int_0^{t_B}(e^{-\tilde{c}t} - 1)dt\\
\Rightarrow \quad -h &= \frac{g}{\tilde{c}}\left(\frac{e^{-\tilde{c}t}}{-\tilde{c}} - t\right)\Bigg|_0^{t_f}\\
&= \frac{g}{\tilde{c}}\left(-\frac{e^{-\tilde{c}t_f}}{\tilde{c}} - t_f + \frac{1}{\tilde{c}}\right)\\
\text{or} \quad \frac{\tilde{c}h}{g} &= \frac{1}{\tilde{c}}(e^{-\tilde{c}t_f} - 1) + t_f. \qquad (9.19)
\end{aligned}$$

This turns out to be a transcendental equation③ with no simple solution for t_f. We can, however, solve it numerically (either using a computer program, or by trial and error). For the given values of m, c, and h, we solve eqn. (9.19) by trial and error (to locate zero crossing), and find that $t_f = 5.5495\,\mathrm{s}$ (see fig. 9.18). Substituting $t = t_f$ in eqn. (9.18), we get

$$\begin{aligned}
v_B &= \frac{mg}{c}(e^{-\frac{c}{m}t_f} - 1)\\
&= \frac{1\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}}{0.25\,\mathrm{kg/\,s}}(e^{-\frac{0.25\,\mathrm{kg/\,s}}{1\,\mathrm{kg}}\cdot 5.5495\,\mathrm{s}} - 1)\\
&= -29.44\,\mathrm{m/s}.
\end{aligned}$$

Note that v comes out to be negative, which is expected because we assumed v to be positive upwards. The velocity is clearly directed downwards once the ball starts falling. Now, substituting the values of m, g, h, and v_B in eqn. (9.17), we get

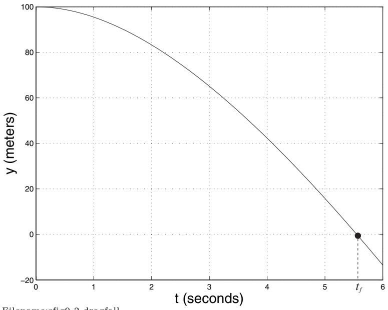
$$\begin{aligned}
E_{\text{drag}} &= mgh - \frac{1}{2}mv_B^2\\
&= 1\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot 100\,\mathrm{m} - \frac{1}{2}\cdot 1\,\mathrm{kg}\cdot(29.44\,\mathrm{m/s})^2\\
&= 547.64\,\mathrm{N{\cdot}m}
\end{aligned}$$

Thus more than half of the initial energy is dissipated in air friction. If there was no viscous drag on the ball, its speed just before hitting the ground would be

$$v_B = \sqrt{2gh} = 44.29\,\mathrm{m/s}.$$

$$\boxed{E_{\text{drag}} = 547.64\,\mathrm{N{\cdot}m}}$$

③A transcendental equation in t is one where t appears both as an argument of a trigonometric or exponential function and elsewhere. Such equations can almost never be solved by hand in closed form.



Filename:sfig9-2-dragfall

Figure 9.18: A graph of $y(t) = h - \frac{mg}{c}\left(\frac{m}{c}e^{-\frac{c}{m}t} + t - \frac{m}{c}\right)$ for determining t_f when $y = 0$. This is the same as solving eqn. (9.19) for t_f.

SAMPLE 9.11 How much time does it take for a car of mass 800 kg to go from 0 mph to 60 mph, if we assume that the engine delivers a constant power P of 40 horsepower during this period. (1 horsepower = 745.7 W)

Solution

$$\begin{aligned}
P &= \dot{W} \equiv \frac{dW}{dt} \\
dW &= P\,dt \\
W_{12} &= \int_{t_0}^{t_1} P\,dt = P(t_1 - t_0) = P\Delta t \\
\Delta t &= \frac{W_{12}}{P}.
\end{aligned}$$

Now, from IIIa in the inside front cover,

$$\begin{aligned}
W_{12} &= (E_K)_2 - (E_K)_1 \\
&= \frac{1}{2} m (v_2^2 - v_1^2) \\
&= \frac{800\,\mathrm{kg}[(60\,\mathrm{mph})^2 - 0]}{2} \\
&= \frac{1}{2} \cdot 800\,\mathrm{kg} \left(60 \frac{\mathrm{mi}}{\mathrm{hr}} \cdot \frac{1.61 \times 10^3\,\mathrm{m}}{1\,\mathrm{mi}} \cdot \frac{1\,\mathrm{hr}}{3600\,\mathrm{s}} \right)^2 \\
&= 288.01 \times 10^3\ \mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{m/s^2} \\
&= 288\ \mathrm{KJoule}.
\end{aligned}$$

Therefore,

$$\Delta t = \frac{288 \times 10^3\,\mathrm{J}}{40 \times 745.7\,\mathrm{W}} = 9.66\,\mathrm{s}.$$

Thus it takes about 10 s to accelerate from a standstill to 60 mph.

$$\boxed{\Delta t = 9.66\,\mathrm{s}}$$

Note 1: This model gives a roughly realistic answer *but* it is not a realistic model, at least at the start, at time t_0. In the model here, the acceleration is infinite at the start (the power jumps from zero to a finite value at the start, when the velocity is zero), something the finite-friction tires would not allow.

Note 2: We have been a little sloppy in quoting the energy equation. Since there are no external forces doing work on the car, somewhat more properly we should perhaps have written

$$0 = \dot{E}_K + \dot{E}_{int} + \dot{E}_P$$

and set $-(\dot{E}_{int} + \dot{E}_P) =$ 'the engine power' where the engine power is from the decrease in gasoline potential energy ($-\dot{E}_P$ is positive) less the increase in 'heat' ($\dot{E}_{int}$) from engine inefficiencies.

SAMPLE 9.12 **Energy of a mass-spring system.** A mass $m = 2\,\text{kg}$ is attached to a spring with spring constant $k = 2\,\text{kN/m}$. The relaxed (unstretched) length of the spring is $\ell = 40\,\text{cm}$. The mass is pulled up and released from rest at position A shown in Fig. 9.19. The mass falls by a distance $h = 10\,\text{cm}$ before reaching position B, which is the relaxed position of the spring. Find the speed at point B.

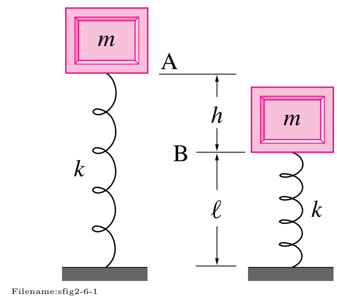


Figure 9.19:

Solution The total energy of the mass-spring system at any instant or position consists of the energy stored in the spring and the sum of potential and kinetic energies of the mass. For potential energy of the mass, we need to select a datum where the potential energy is zero. We can select any horizontal plane to be the datum. Let the ground support level of the spring be the datum. Then, at position A,

$$\begin{aligned}
\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = \frac{1}{2}kh^2 \quad \text{(see eqn. (9.15), page 441)}\\
\text{Energy of the mass} &= E_\text{K} + E_\text{P} = \frac{1}{2}m\underbrace{v_A^2}_{0} + mg(\ell+h) = mg(\ell+h).
\end{aligned}$$

Therefore, the total energy at position A

$$E_A = \frac{1}{2}kh^2 + mg(\ell+h).$$

Let the speed of the mass at position B be v_B. When the mass is at B, the spring is relaxed, i.e., there is no stretch in the spring. Therefore, at position B,

$$\begin{aligned}
\text{Energy in the spring} &= \frac{1}{2}k\,(\text{stretch})^2 = 0\\
\text{Energy of the mass} &= E_\text{K} + E_\text{P} = \frac{1}{2}mv_B^2 + mg\ell,
\end{aligned}$$

and the total energy

$$E_B = \frac{1}{2}mv_B^2 + mg\ell.$$

Because the net change in the total energy of the system from position A to position B is

$$\begin{aligned}
0 &= \Delta E\\
&= E_A - E_B = \frac{1}{2}kh^2 + mg(\ell+h) - \frac{1}{2}mv_B^2 - mg\ell\\
&= \frac{1}{2}(kh^2 - mv_B^2) + mgh\\
\Rightarrow \quad v_B^2 &= kh^2/m + 2gh\\
\Rightarrow \quad |v_B| &= \left(kh^2/m + 2gh\right)^{1/2}\\
&= \left((2000\,\text{N/m}\cdot(0.1\,\text{m})^2/2\,\text{kg}) + 2\cdot 9.81\,\text{m/s}^2\cdot 0.1\,\text{m}\right)^{1/2}\\
&= 3.46\,\text{m/s}.
\end{aligned}$$

$$\boxed{|v_B| = 3.46\,\text{m/s}}$$

9.3 Vibrations: mass, spring and dashpot

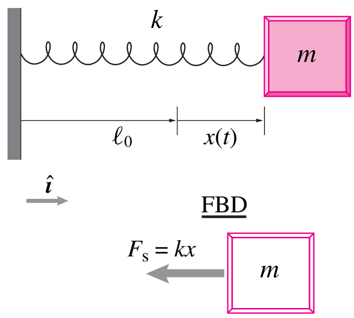


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Figure 9.20: A spring mass system. For simplicity in this 1D analysis, we don't show forces that have no component in the relevant direction; not shown are the force of gravity or the (assumed massless and frictionless) support wheels. It would be more complete to show those forces, but it would add clutter.

When the mass in the spring-mass system of fig. 9.20 is to the right ($x > 0$) of the rest position ($x = 0$) it accelerates to the left; when the mass is to the left ($x < 0$) it accelerates to the right. So it goes back and forth.

> Oscillations (vibrations) can occur whenever a spring-like 'restoring' force pulls a mass to a center position from both right and left.

Any system with mass and elasticity can oscillate. Because essentially all engineering materials are elastic (spring-like) under working conditions and all real things have mass, most any thing will oscillate if you hit it.

Some special examples are the strings of cellos and sitars; the air columns in clarinets, trumpets and organ pipes and the wood blocks in a marimba. So a system vibrating like this, whether literally a spring and mass or something more subtle, is called a 'harmonic oscillator'. More specifically, a harmonic oscillator has persistent (non-decaying) oscillations which are sinusoidal in time ($x = \sin t$).

With varying degrees of approximation, car suspensions, buildings responding to earthquakes, earthquake faults themselves, quartz timing chrystals and vibrating machines of all kinds are also modeled as mass-spring harmonic oscillators. The subject of 'vibration theory' is based on the harmonic oscillator.

> The unforced oscillation of a spring and mass is the basic model for all vibrating systems.

Filename:tfig-blowinggrass0949

Figure 9.21: **Blowing grass.** If you squeeze a blade of grass between your parallel thumbs and then blow in the slot between the left and right thumb joints, at the grass, you can make a squeal sound that is similar (kind of) to that of a clarinet. These are self-excited oscillations. The elastic restoring force is related to the tension in the grass which you can control by straightening and bending your thumbs. With practice you can adjust the pitch. Music, kind of. Photo by Katja Ojala.

Even structures which you think of as rigid for a statics analysis will vibrate if encouraged to do so by the shaking of an unbalanced motor, the rumbling of a truck, a party upstairs, or the ground motion of an earthquake. And the vibrations of one thing can excite oscillations of another. For example a vibrating bridge can excite oscillations in the air flowing by, which in turn can excite the bridge oscillate more; this mutual excitement of fluids and solids causes vibrations in a clarinet reed (fig. 9.21). Such mutually excited oscillations also caused the wild oscillations leading up to the infamous Tacoma Narrows bridge collapse.

Even if coming from an electrical device, all music is mechanical vibrations (at least of the air in your ear), and so are all annoying sounds. They are the main function of a vibrating massager, and the main defect of a squeaking hinge. Mechanical vibrations in pendula or quartz crystals are used to measure time, but vibrations can cause a machine to go out of control (e.g., bicycle shimmy), or a bridge to collapse. So the study of vibrations, good vibrations and bad vibrations, is a common use of dynamics.

Because the motions associated with vibrations have features which are common over all manner of structures and machines, a special vocabulary and special methods of approach have been developed. Some key words to learn here are *natural frequency*, *damping*, *resonance*, *normal modes*, and *frequency response*.

In this section we cover the math of the harmonic oscillator. At the end of the section we add friction.

The harmonic oscillator

The mother of all vibrating machines is the simple harmonic oscillator of fig. 9.20. The mass slides on a frictionless surface. The spring is relaxed at $x = 0$. The spring is thus stretched from ℓ_0 to $\ell_0 + \Delta\ell$, a stretch of $\Delta\ell = x$. A free body diagram of the mass, cut 'free' from the spring in its extended state, is shown in the lower part of fig. 9.20. Linear momentum balance in the x direction ($\{\sum \vec{\boldsymbol{F}} = \dot{\vec{\boldsymbol{L}}}\} \cdot \hat{\boldsymbol{\imath}}$) gives:

$$\begin{aligned} \sum F_x &= \dot{L}_x \\ -kx &= m\ddot{x}. \end{aligned}$$

Rearranging, we get one of the most famous and useful differential equations of all time①:

$$\ddot{x} + \frac{k}{m}x = 0. \qquad (9.20)$$

Eqn. (9.20) appears in many contexts both in and out of dynamics. In non-mechanical contexts the variable x and the parameter combination k/m are replaced by other physical quantities (> 0). In an electrical circuit, for example, x might represent a voltage and the term corresponding to k/m might be $1/LC$, where C is a capacitance and L an inductance. But even in dynamics the equation appears with other physical quantities besides k/m multiplying the x, and x itself could represent rotation, say, instead of displacement. In order to avoid being specific about the physical system being modeled, the harmonic oscillator equation is often written with no k or m as

$$\ddot{x} + \lambda^2 x = 0. \qquad (9.21)$$

The constant λ^2 in front of the x is used instead of just, say, λ ('lambda')②, for two reasons:

1. This convention shows that λ^2 is positive,
2. In the solution we need the square root of this coefficient, so it is convenient to have $\sqrt{\lambda^2} = \lambda$.

For the spring-block system, λ^2 is k/m and in other problems λ^2 is a combination of other physical quantities. A solution is a function $x(t)$ whose second derivative is the negative of the original function multiplied by the constant λ^2.

①**Caution:** If you make a sign error in your setup you can get $\ddot{x} - (k/m)x = 0$ or $\ddot{x} = (k/m)x$. This is a very different equation with a very different solution. See box 9.3 on page 424.

②**Notation.** Most books use p^2 or ω^2 in the place we have put λ^2. Using ω ('omega') can lead to confusion because we will later use ω for angular velocity. If one is studying vibrations of a rotating shaft then there would be two very different ω's in the problem. One, the coefficient of a differential equation and, the other, the angular velocity. To add to the confusion, simple harmonic oscillations and circular motion have a deep connection, so the coincidence of notation is not accidental. Deep connection or not, the ω in the harmonic oscillator equation is not the same thing as the ω describing angular motion of a physical object. We avoid this confusion by using λ instead of ω. (This λ is unrelated to the magnitude of the unit vector $\hat{\boldsymbol{\lambda}}$).

Solution of the harmonic oscillator differential equation

A typical ODE math class will spend some time deriving the solution of the harmonic oscillator differential equation 9.21③. It's best to just remember the solution, using the intuition from the first sentences of this section. Until you know it (which should be soon) you can look it up in box 9.3 on page 424, namely

$$\begin{array}{rrcl}
 & x(t) & = & A\,\cos(\lambda\,t) + B\,\sin(\lambda\,t), \\
\text{or} & x(t) & = & C_1\cos(\lambda\,t) + C_2\sin(\lambda\,t).
\end{array} \qquad (9.22)$$

This sum of two sine waves④ and is a solution of differential equation 9.21 for any values of the constants A (or C_1) and B (or C_2).

What does it means to say "$u = C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)$ satisfies the equation: $\ddot{u} = -\lambda^2 u$?" You can satisfy a differential equation by feeding it a function that fully eliminates. If you plug a candidate solution into a differential equation and get $0 = 0$ you have satisfied (solved) the equation.

Although you need not derive the solution, you should remember it and be able to check it.

Checking the solution in detail

To check if a function is a solution, plug it into the differential equation and see if an identity is obtained.

Is this true for the given $u(t)$?

$$\begin{array}{rcl}
\dfrac{d^2}{dt^2}u & = & -\lambda^2 u \\
\dfrac{d^2}{dt^2}\overbrace{[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]}^{u(t)} & \stackrel{?}{=} & -\lambda^2\overbrace{[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]}^{u(t)} \\
\dfrac{d}{dt}\left(\dfrac{d}{dt}[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]\right) & \stackrel{?}{=} & -\lambda^2[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)] \\
\dfrac{d}{dt}[C_1\lambda\cos(\lambda\,t) - C_2\lambda\sin(\lambda\,t)] & \stackrel{?}{=} & -\lambda^2[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)] \\
\overbrace{-C_1\lambda^2\sin(\lambda t) - C_2\lambda^2\cos(\lambda\,t)}^{\ddot{u}} & \stackrel{?}{=} & -\lambda^2\overbrace{[C_1\sin(\lambda\,t) + C_2\cos(\lambda\,t)]}^{u(t)}
\end{array}$$

The equation $\ddot{u} = -\lambda^2 u$ *does* hold with the given $u(t)$. Right and left sides match. This shows at a glance in the next line.

$$0 \stackrel{:)}{=} 0 \qquad \text{(Satisfied.)}$$

Whatever the constants C_1 and C_2, the proposed solution eqn. (9.22) satisfies

③**Solving the harmonic oscillator equation.** We don't especially advise it, but here's how. The energy equation gives (using C^2 to show that the energy is positive)

$$\begin{array}{rrcl}
 & \dot{x}^2/\lambda^2 + x^2 & = & C^2 \\
\text{so} & dx/\sqrt{C^2 - x^2} & = & \lambda dt.
\end{array}$$

Then integrate (how? substitute $x = C\sin\theta$ or guess), or else look it up on your symbolic calculator or a symbolic math program and get

$$\begin{array}{rrcl}
 & \cos^{-1}(x/C) & = & \lambda t - c_2 \\
\text{so} & x & = & C\cos(\lambda t - c_2)
\end{array}$$

where C and c_2 are arbitrary constants. We picked the signs of arbitrary constants to please us. That's one form of the general solution of the harmonic oscillator equation. You can plug it back into eqn. (9.21) and see that you get $0 = 0$ for all values of C, c_2 and t.

④A cosine function is also a sine wave.

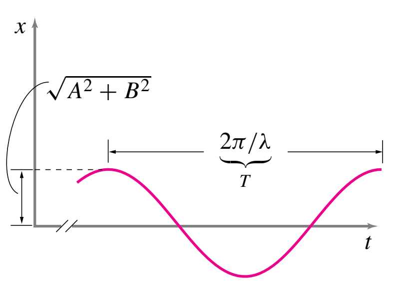

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Figure 9.22: **Position versus time for an undamped, unforced harmonic oscillator.** x is the position of the mass, t is time.

the differential equation eqn. (9.21).

⑤

Uniqueness. We have not proved 'uniqueness', that there are not other solutions to this differential equation than eqn. (9.22). There are not, as the mathematics-for-its-own-sake inclined student can learn to prove elsewhere.

⑤A plausibility argument for uniqueness goes like this. If you release a mass from a given position x_0 at a given speed v_0 it will move in a definite way and no other way. This is a special case of what is called "determinism". But all solutions have some position and speed at $t = 0$ and we can find a C_1 and C_2 in eqn. (9.22) to match each such. Thus we have found *the* motion for every possible situation, and there can be no others.

Interpreting the solution of the harmonic oscillator equation

We could use a coiled metal spring and a block on low friction rollers to make a machine schematically like the system shown in fig. 9.20. We could then watch it move. We would see (approximately) that

$$x(t) = A\cos(\lambda\, t) + B\sin(\lambda\, t),$$

as shown in the graph in fig. 9.22.

Angular frequency, period, and frequency

Three related measures of the rate of oscillation are angular frequency, period, and frequency. The simplest of these is *angular frequency* $\lambda = \sqrt{(k/m)}$, sometimes called *circular frequency*. The period T is the amount of time that it takes to complete one oscillation. One oscillation of both the *sine* function and the *cosine* function occurs when the argument of the function advances by 2π, that is when

$$\lambda T = 2\pi, \qquad \text{so} \qquad T = \frac{2\pi}{\lambda} = \frac{2\pi}{\sqrt{(k/m)}},$$

Some people memorize these formulas in high school. The *natural frequency* f is the reciprocal of the period

$$f = \frac{1}{T} = \frac{\lambda}{2\pi} = \frac{\sqrt{(k/m)}}{2\pi}.$$

Typically, natural frequency f is measured in cycles per second or *Hertz* and the angular frequency λ in radians per second. A computer or watch quartz timing crystal has mechanical vibrations at a frequency of millions of cycles per second, some molecules about a million times faster than that. On the other extreme, the free vibrations of the whole earth have frequencies of thousandths of a cycle per second (i.e. thousands of seconds per cycle). The slowest vibration mode of the earth has a period of about 54 minutes⑥.

⑥Why does the earth oscillate? First because it can. It has both mass and an 'elastic restoring force' The elastic restoring force comes from a combination two things: 1) the elasticity of rock and 2) the self-gravitation of the earth trying to bundle itself into a ball. What gets the earth started oscillating? Mostly big earthquakes.

Amplitude. The amplitude of the sine wave that results from the addition of the *sine* function and the *cosine* function is given by the square root of the sum of the squares of the two amplitudes. That is, the amplitude of the resulting sine wave is $\sqrt{A^2 + B^2}$. Another way of describing this sum is through the trigonometric identity:

$$A\cos(\lambda\, t) + B\sin(\lambda\, t) = R\cos(\lambda\, t - \phi), \tag{9.23}$$

where $R = \sqrt{A^2 + B^2}$ and $\tan\phi = B/A$ (see box 9.6 on page 456). So,

> the only possible motion of a spring and mass is a sinusoidal oscillation. You can think of this either as the sum of a *cosine* function and a *sine* function or as a single *cosine* function with *phase shift* ϕ.

Initial conditions determine the constants A and B

The constants A and B in equation 9.22 could have any value. Or, equivalently, the amplitude R and phase ϕ in equation 9.23 could be anything. These are determined by the way motion is started, the *initial conditions*. The following two special initial conditions are worth getting a feel for.

9.6 Derivation and visualization of the formula $A\cos(\lambda t) + B\sin(\lambda t) = R\cos(\lambda t - \phi)$

Here is a demonstration that the sum of a *cosine* function and a *sine* function is a new sine wave. By sine wave we mean a function whose shape is the same as the sine function, though it may be displaced along the time axis. For example $\cos t$ and $\cos(t - const)$ are both sinewaves.

The trig identity approach. The quickest approach is to start with the function $f(t) = R\cos(\lambda t - \phi)$ and use the trig addition identity for cosines

$$\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi.$$

Thus:

$$\begin{aligned} R\cos(\lambda t - \phi) &= R\cos\lambda t\cos\phi + R\sin\lambda t\theta\sin\phi \\ &= A\sin\lambda t + B\cos\lambda t. \end{aligned}$$

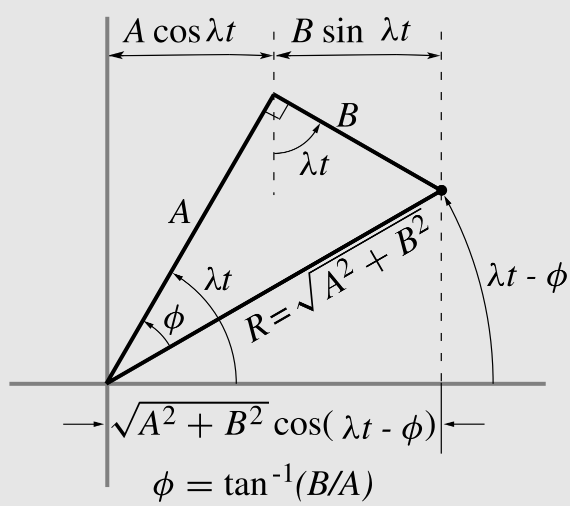
We can run the reasoning from right to left and set $A = R\cos\phi$ and $B = R\sin\phi$ and then solve for R and ϕ in terms of A and B. Thus demonstrating the formula titling this box. If you have trouble remembering the trig identity, one derivation uses the picture to the right. Trigonometry is full of such circular reasoning.

The geometric approach. Consider the line segment A spinning in circles about the origin at rate λ; that is, the angle the segment makes with the positive x axis is λt. The projection of that segment onto the x axis is $A\cos(\lambda t)$, a sine wave. Now consider the segment labeled B in the figure, glued at a right angle to A. The length of its projection on the x-axis is $B\sin(\lambda t)$. So, the sum of these two projections is $A\cos(\lambda t) + B\sin(\lambda t)$. The two segments A and B make up a right triangle with diagonal $R = \sqrt{A^2 + B^2}$.

The projection or 'shadow' of R on the x axis is the same as the sum of the shadows of A and B. The angle it makes with the x axis is $\lambda t - \phi$ where one can see from the triangle drawn that $\phi = \arctan(B/A)$. So, by adding the shadow lengths, we see

$$A\cos(\lambda t) + B\sin(\lambda t) = \sqrt{A^2 + B^2}\cos(\lambda t - \phi)\,.$$

The function $f(t) = R\cos(\lambda t - \phi)$ is a sine wave. In particular it is the cosine function with a maximum at $\lambda t - \phi$.



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Release from rest

The simplest motion is *release from rest*, meaning the initial velocity of the mass is zero. We find the motion from the general solution

$$x(t) = A\cos(\sqrt{(k/m)}\,t) + B\sin(\sqrt{(k/m)}\,t).$$

At $t = 0$, this general solution has to agree with the initial condition that $x(0) = x_0$ and the initial velocity is $v(0) = v_0 = 0$. In this case

$$x(0) = x_0 \quad \text{and} \quad v(0) = 0 \quad \Rightarrow \quad A = x_0 \quad \text{and} \quad B = 0.$$

The next example shows the details.

Example:
The mass in fig. 9.20 is 0.5 kg, the spring constant is $k = 50\,\mathrm{N/m}$, and the initial displacement is 1 cm, then

$$x(0) = A\underbrace{\cos(0)}_{1} + B\underbrace{\sin(0)}_{0} = A \qquad \Rightarrow \qquad A = 1\,\mathrm{cm}.$$

The initial velocity must also match, so first we find the velocity by differentiating the position to get

$$v(t) = \dot{x}(t) = -A\sqrt{(k/m)}\sin(\sqrt{(k/m)}\,t) + B\sqrt{(k/m)}\cos(\sqrt{(k/m)}\,t).$$

Now, we evaluate this expression at $t = 0$ and set it equal to the given initial velocity which in this case was zero:⑦

$$v(0) = -A\sqrt{(k/m)}\underbrace{\sin(0)}_{0} + B\sqrt{(k/m)}\underbrace{\cos(0)}_{1} = B\sqrt{(k/m)} \qquad \Rightarrow \qquad B = 0.$$

Substituting in the values for $k = 50\,\mathrm{N/m}$ and $m = 0.5\,\mathrm{kg}$, we get

$$x(t) = 1\cdot\cos\left(\underbrace{\sqrt{\left(\frac{0.5\,\mathrm{kg}}{50\,\mathrm{N/m}}\right)}}_{0.01\,\mathrm{s}^{-1}}\;t\right)\mathrm{cm} = 1\cos(0.1t/\mathrm{s})\,\mathrm{cm}$$

which is plotted in fig. 9.23.

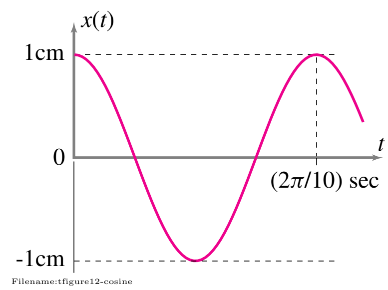


Figure 9.23: The position of a mass as a function of time if $k = 50\,\mathrm{N/m}$, $m = 0.5\,\mathrm{kg}$, $x(0) = 1$ cm and $v(0) = 0$.

⑦It is tempting, but wrong, to evaluate $x(t)$ at $t = 0$ and then differentiate to get $v(0)$. This procedure is wrong because $x(0)$ is just a number, differentiating it would always give zero, even when the initial velocity is not zero.

Initial velocity with no spring stretch

Assume the mass has some initial velocity but the spring has no initial stretch (say, just after a resting mass is hit by a hammer).

Example:
Using the same 0.5 kg mass and $k = 50\,\mathrm{N/m}$ spring, we now consider an initial displacement of zero but an initial velocity of $10\,\mathrm{cm/s}$. We can find the motion for this case from the general solution by the same procedure we just used. We get

$$x(t) = B\sin(\sqrt{(k/m)}\,t)$$

with $B\sqrt{(k/m)} = 10\,\mathrm{cm/s} \qquad \Rightarrow \qquad B = 1\,\mathrm{cm}.$

The resulting motion, $x(t) = (1\,\mathrm{cm})\cdot\sin\left(\frac{0.1t}{s}\right)$, is shown in fig. 9.24.

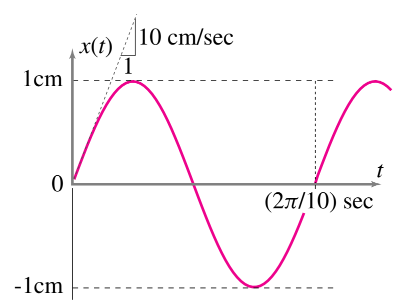


Figure 9.24: The position of a mass as a function of time if $k = 50\,\mathrm{N/m}$, $m = 0.5\,\mathrm{kg}$, $x(0) = 0$ and $v(0) = 10\ \mathrm{cm/s}$.

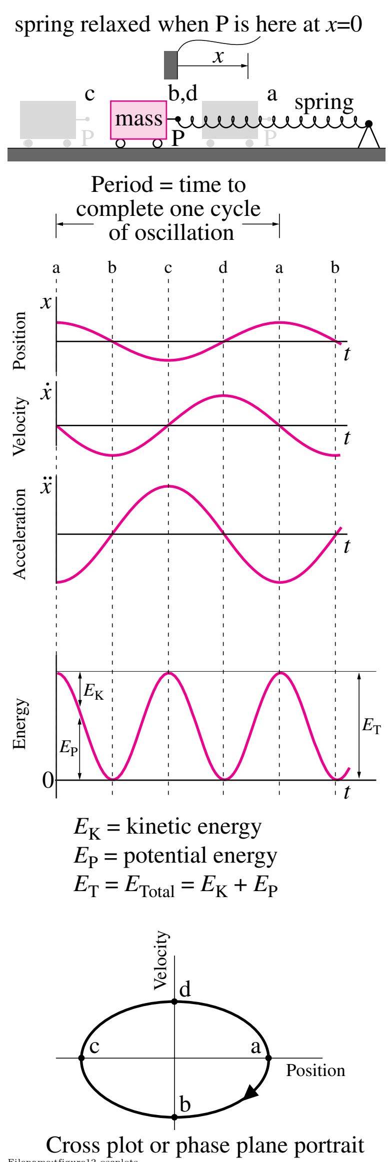


Figure 9.25: **Harmonic oscillator.** At $t = 0$ the mass is released from rest at a. The spring is relaxed at $x = 0$ (pts b and d). Some things to note: The acceleration curve is proportional to the negative of the displacement curve; The displacement is at a maximum or minimum when the velocity is zero; The velocity is at a maximum or minimum when the displacement is zero; The kinetic and potential energy fluctuate at twice the frequency as the position; The motion is an ellipse in the cross plot of velocity *vs.* position.

Conservation of energy

We have neglected all dissipation in the harmonic oscillator. So the total mechanical energy, the sum of the kinetic energy $E_K = \frac{1}{2}mv^2$ and the potential energy (from eqn. (9.15)) $E_P = \frac{1}{2}k(\Delta L)^2$, is constant in time.

$$E_T = E_K + E_P = \text{constant}.$$

As the mass moves, energy is exchanged back and forth between kinetic and potential energy. At the extremes in the displacement, the spring is most stretched, the potential energy is at a maximum and the kinetic energy is zero. When the mass passes through the center position the spring is relaxed, the potential energy is at a minimum (zero) and the mass is at its peak speed, and the kinetic energy is at its peak.

Although energy conservation is a basic principle, this is a case where it can be easily checked. Using the special case where the motion starts from rest (i.e., $x(t) = A\cos(\sqrt{k/m}\,t)$), we can check that the total energy really is constant.

$$\begin{aligned}
E_T &= E_P + E_K \\
&= \frac{1}{2}kx^2 + \frac{1}{2}mv^2 \\
&= \frac{1}{2}k(\underbrace{A\cos(\sqrt{k/m}t)}_{x})^2 + \frac{1}{2}m(\underbrace{A\sqrt{k/m}\sin(\sqrt{k/m}t)}_{v})^2 \\
&= \frac{1}{2}kA^2\underbrace{\{\cos^2(\sqrt{k/m}t) + \sin^2(\sqrt{k/m}t)\}}_{1} \\
&= \frac{1}{2}kA^2 = \quad \text{initial energy in spring}
\end{aligned}$$

which does not change with time.

Using energy to derive the oscillator equation

We could use start with energy balance and derive the equations of motion. Starting from E_T = constant, we get

$$\begin{aligned}
0 &= \frac{d}{dt}E_T = \frac{d}{dt}(E_P + E_K) \\
&= \frac{d}{dt}(\frac{1}{2}kx^2 + \frac{1}{2}mv^2) \\
&= kx\underbrace{\dot{x}}_{v} + mv\underbrace{\dot{v}}_{a} \\
&= kx\cancel{v} + m\cancel{v}\underbrace{a}_{\ddot{x}} \\
0 &= kx + m\ddot{x}
\end{aligned}$$

which is the harmonic oscillator equation. A technical defect of this derivation is that it does not apply at the instants when $v = 0$ (that is, $0 \cdot x = 0 \cdot y$

does not imply that $x = y$). Thus, technically, from this derivation we only know the differential equation holds for those times when $v \neq 0$. Nonetheless, it gives the right equation for all times.

Similarly, power balance also leads to the harmonic oscillator equation. Referring to the FBD in fig. 9.20, the equation of power balance for the block during its motion after release is:

$$\underbrace{P}_{\text{Power in}} = \underbrace{\dot{E}_\mathrm{K}}_{\text{Rate of change of kinetic energy}}$$

$$\begin{aligned}
\vec{F}_{spring} \cdot \vec{v}_A &= \frac{d}{dt}\left(\frac{1}{2}mv_A^2\right) \\
-kx_A\hat{\imath} \cdot \dot{x}_A\hat{\imath} &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}_A^2\right) \\
-kx_A\dot{x}_A &= m\dot{x}_A\ddot{x}_A \,.
\end{aligned}$$

Dividing both sides by $\dot{x}_A$ (assuming it is not zero), we again get our friend,

$$-kx_A = m\ddot{x}_A \qquad \text{or} \qquad m\ddot{x}_A + kx_A = 0.$$

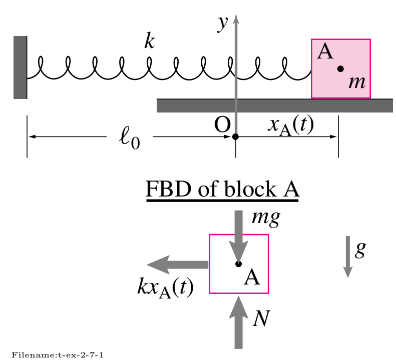


Figure 9.26:

Energy oscillations. Let's assume the block in fig. 9.26 is released from rest at $x = x_A > 0$. The mass begins to move to the left and the spring does positive work on the mass since the motion and the force are in the same direction. After the block passes through the rest point $x = O$, it does work on the spring until it comes to rest at its left extreme. The spring then commences to do work on the block again as the block gains kinetic energy in its rightward motion. The block then passes through the rest position and does work on the spring until its kinetic energy is all used up and it is back in its rest position. Note that the potential and kinetic energy each have a two local maxima and minima for each oscillation of the mass, thus their plots are sine-waves with twice the frequency of the basic oscillation.

A spring-mass system with gravity

When a mass is attached to a spring but gravity also acts one has to take some care to get things right (see fig. 9.27). Once a good free body diagram is drawn using well defined coordinates, all else follows easily.

Note that there are three natural choices for measuring the position of the mass in fig. 9.27. y measures position from the fixed end of the spring, x measures from the position of the mass when the spring is relaxed, and z measures from the position of the mass when it is in static equilibrium (with the gravity force balancing the spring compression). See more discussion of constant forcing in section 9.6 on page 502.

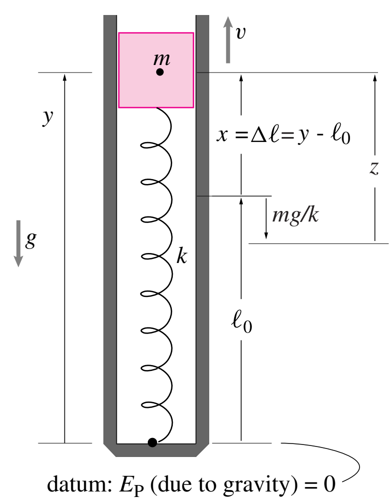


Figure 9.27: Spring and mass with gravity.

Damping

Dashpots are used to absorb energy. One is shown schematically in fig. 9.28. Often springs and dashpots are light in comparison to the machinery to which they are attached so their mass and weight are neglected. They are usually attached with pin joints, ball and socket joints, or other kinds of flexible connections so only forces are transmitted. Because they only have forces at their ends they are 'two-force' bodies so (see section 4.2) the forces at their ends are equal, opposite, and along the line of connection. The most familiar examples are the shock absorbers of a car or the damper for screen doors. The symbol for a dashpot shown in fig. 9.28.

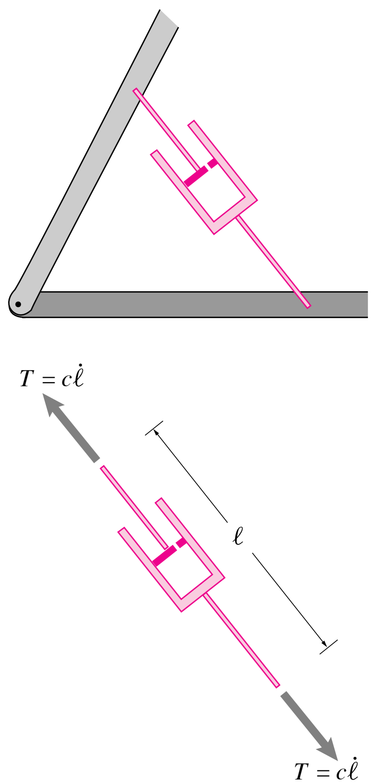


Figure 9.28: **A dashpot.** A dashpot (or damper) is shown here connecting two parts of a mechanism. The tension in the dashpot is proportional to the rate at which it lengthens. The symbol shown represents any device which resists the relative motion of its endpoints. The schematic is supposed to suggest a plunger in a cylinder. For the plunger to move, fluid must leak around the cylinder. This leakage happens for either direction of motion. Thus the damper resists relative motion in either direction; i. e., for $\dot{\ell} > 0$ and $\dot{\ell} < 0$.

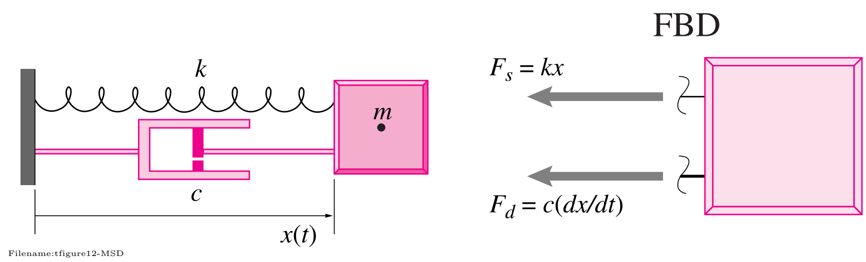


Figure 9.29: A mass spring dashpot system, or damped harmonic oscillator. Also shown is a free body diagram of the mass.

The dashpot provides resistance to motion by drawing air or oil in and out of the cylinder through a small opening. Due to the viscosity of the air or oil, a pressure drop is created across the opening that is related to the speed of the fluid flowing through. Ideally, this viscous resistance produces linear damping, meaning that the force is exactly proportional to the velocity. The relation is assumed to hold for negative lengthening as well. So the compression (negative tension) is also proportional to the rate at which the dashpot shortens (negative lengthens).

The tension in the dashpot is usually assumed to be proportional to the rate at which it lengthens, although this approximation is not especially accurate for most dampers one can buy. In a physical dashpot nonlinearities, from the fluid flow and from friction between the piston and the cylinder, are often significant. Also, dashpots that use air as a working fluid may have compressibility that introduces extra springiness to the system.

The defining equation for an ideal linear dashpot is:

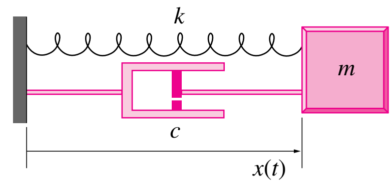
$$T = C\,\dot{\ell}$$

where C is the dashpot constant.

Damped oscillations

We now add a dashpot in parallel with the spring of a mass-spring system creates a *mass-spring-dashpot* system, or *damped harmonic oscillator*. The system is shown in fig. 9.29. Also in fig. 9.29 is a free body diagram of the mass. It has two forces acting on it, neglecting gravity:

$F_s \; = \; kx$		is the spring force, assuming a linear spring, and
$F_d \; = \; c\, dx/dt = c\dot{x}$		is the dashpot force assuming a linear dashpot.



The system is a one degree of freedom system because a single coordinate x is sufficient to describe the complete motion of the system. The equation of motion for this system is

$$m\ddot{x} = -F_d - F_s \quad \text{where} \quad \ddot{x} = d^2x/dt^2. \tag{9.24}$$

Assuming a linear spring and a linear dashpot this expression becomes

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{9.25}$$

We have taken care with the signs of the various terms. Make sure you can confidently derive equation 9.25 without introducing sign errors. The analytical solution of the damped-oscillator equation is in box 9.7. Some qualitative features of the damped solutions are shown in fig. 9.30

For given k and m we can think of the damping c as adjustable. A system which has small damping (small c) is *under-damped* and does not come to equilibrium quickly because oscillations last for a long time. A system which has a lot of damping (big c) is *over-damped* does not come to equilibrium quickly because the dashpot doesn't leak fast enough. A system which is in-between, *critically-damped* comes to equilibrium most quickly. The purpose of damping is often to purge motion after a disturbance. If the only design variable available for adjustment is the damping, then the quickest purge is accomplished with critical damping, $c = \sqrt{(4km)}$. In practice, any damping value close to critical is often used, more or less depending on whether a little oscillation is tolerable or not⑧.

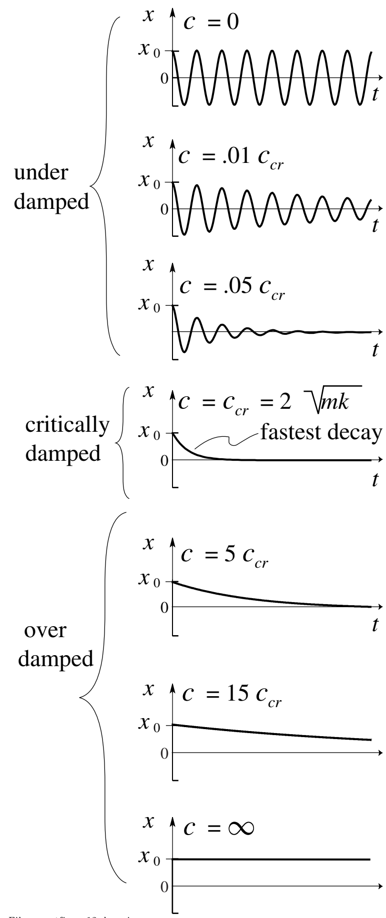


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Figure 9.30: The effect of varying the damping with a fixed mass and spring. In all the plots the mass is released from rest at $x = x_0$. In the case of under-damping, oscillations persist for a long time, forever if there is no damping. In the case of over-damping, the dashpot doesn't relax for a long time; it stays locked up forever in the limit of $c \to \infty$. The fastest relaxation occurs for critical damping.

⑧Stereotypically, the suspension of an overloaded old-fashioned luxury car is underdamped, imagine it bouncing along after a bump. And the suspension of a tight sports car is underdamped.

Summary of equations for the unforced harmonic oscillator

- $\ddot{x} + \frac{k}{m}x = 0$, mass-spring equation
- $\ddot{x} + \lambda^2 x = 0$, harmonic oscillator equation
- $x(t) = A\cos(\lambda t) + B\sin(\lambda t)$, general solution to harmonic oscillator equation
- $x(t) = R\cos(\lambda t - \phi)$, amplitude-phase version of solution to harmonic oscillator solution, $R = \sqrt{A^2 + B^2}, \phi = \tan^{-1}(\frac{B}{A})$ (See box on page 456).
- $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$, mass-spring-dashpot equation (see equations 9.26-9.29 for solutions)

9.7 Solution of the damped-oscillator equations

Here are some mathematical details you can use for reference. These details are of much lower status than those in box 9.3 on page 424. Only vibrations experts remember these formulas in detail.

Even if we make the common assumptions that m, c, and k are all positive, the whole nature of the solution of (9.25) depends on the values of those constants. The three types of solutions are categorized as follows:

- *Under-damped*: $c^2 < 4mk$. In this case the damping is small and oscillations persist forever, though their amplitude diminishes exponentially in time. The general solution for this case is:

$$x(t) = e^{(-\frac{c}{2m})t}[A\cos(\lambda_d t) + B\sin(\lambda_d t)], \quad (9.26)$$

where λ_d is the damped natural frequency and is given by

$$\lambda_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}. \quad (9.27)$$

- *Critically damped*: $c^2 = 4mk$. In this case the damping is at a critical level that separates the cases of under-damped oscillations from the simply decaying motion of the over-damped case. The general solution is:

$$x(t) = Ae^{(-\frac{c}{2m})t} + Bte^{(-\frac{c}{2m})t}. \quad (9.28)$$

- *Over-damped*: $c^2 > 4mk$. Here there are no oscillations, just a simple return to equilibrium with at most one crossing through the equilibrium position on the way to equilibrium. The general solution in the over-damped case is:

$$x(t) = Ae^{(-\frac{c}{2m}+\sqrt{(\frac{c}{2m})^2-\frac{k}{m}})t} + Be^{(-\frac{c}{2m}-\sqrt{(\frac{c}{2m})^2-\frac{k}{m}})t}. \quad (9.29)$$

The solution 9.29 actually includes equations 9.28 and 9.26 as special cases (To interpret (9.29) as the general solution, for the cases when the square root is imaginary you need to use the Euler equation $e^{i\lambda t} = \cos i\lambda t + \sin i\lambda t$).

Measurement of damping. In the under-damped case, the damping constant c can be found by measuring the rate of decay of unforced oscillations using the 'logarithmic decrement'. The logarithmic decrement is the natural logarithm of the ratio of the amplitude of any two successive peaks. The larger the damping, the greater the rate of decay and the bigger the decrement:

$$\text{logarithmic decrement} \equiv D = \ln\left(\frac{x_n}{x_{n+1}}\right) \quad (9.30)$$

where x_n and x_{n+1} are the heights of two successive peaks in the figure below (also seen on the 2nd and 3rd figures in fig. 9.30 on page 461). Because of the exponential envelope (bounding curve), $x_n = (\text{const.})e^{-(\frac{c}{2m})t_n}$ and $x_{n+1} = (\text{const.})e^{-(\frac{c}{2m})t_n+T}$.

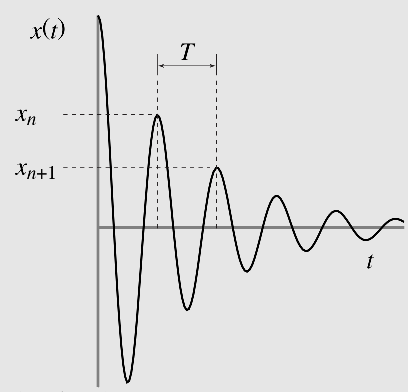
$$D = \ln[(e^{-(\frac{c}{2m})t_n})/(e^{-(\frac{c}{2m})t_n+T})]$$

Simplifying this expression, we get that

$$D = \frac{cT}{2m}$$

where T is the period of oscillation. Thus, measuring the logarithmic decrement D and the period of oscillation T determines c as

$$c = \frac{2mD}{T}.$$



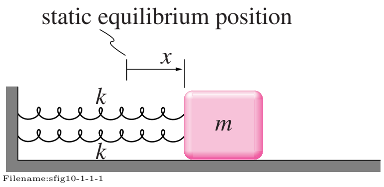


Figure 9.31:

SAMPLE 9.13 A block of mass $m = 20\,\text{kg}$ is attached to two identical springs each with spring constant $k = 1\,\text{kN/m}$. The block slides on a horizontal surface without any friction.

1. Find the equation of motion of the block.
2. What is the oscillation frequency of the block?
3. How much time does the block take to go back and forth 10 times?

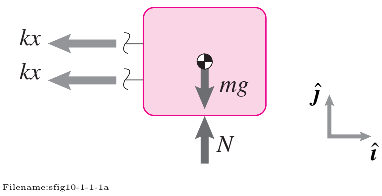


Figure 9.32:

Solution

1. The free body diagram of the block is shown in Figure 9.32. The linear momentum balance, $\sum \vec{F} = m\vec{a}$, for the block gives

$$-2kx\hat{\imath} + (N - mg)\hat{\jmath} = m\vec{a}$$

Dotting both sides with $\hat{\imath}$ we have,

$$\begin{aligned} -2kx = ma_x &= m\ddot{x} && (9.31)\\ \text{or}\quad m\ddot{x} + 2kx &= 0 && (9.32)\\ \text{or}\quad \ddot{x} + \frac{2k}{m}x &= 0. && (9.33) \end{aligned}$$

$$\boxed{\ddot{x} + \frac{2k}{m}x = 0}$$

2. Comparing Eqn. (9.33) with the standard harmonic oscillator equation, $\ddot{x} + \lambda^2 x = 0$, where λ is the oscillation frequency, we get

$$\begin{aligned} \lambda^2 &= \frac{2k}{m}\\ \Rightarrow \quad \lambda &= \sqrt{\frac{2k}{m}}\\ &= \sqrt{\frac{2\cdot(1\,\text{kN/m})}{20\,\text{kg}}}\\ &= 10\,\text{rad/s}. \end{aligned}$$

$$\boxed{\lambda = 10\,\text{rad/s}}$$

3. Time period of oscillation $T = \frac{2\pi}{\lambda} = \frac{2\pi}{10\,\text{rad/s}} = \frac{\pi}{5}$ s. Since the time period represents the time the mass takes to go back and forth just once, the time it takes to go back and forth 10 times (*i.e.,* to complete 10 cycles of motion) is

$$t = 10T = 10\cdot\frac{\pi}{5}\,\text{s} = 2\pi\,\text{s}.$$

$$\boxed{t = 2\pi\,\text{s}}$$

SAMPLE 9.14 Simple harmonic motion of a buoy. A cylinder of cross sectional area A and mass M is in static equilibrium inside a fluid of specific weight γ when L_o length of the cylinder is submerged in the fluid. From this position, the cylinder is pushed down vertically by a small amount x and let go. Assume that the only forces acting on the cylinder are gravity and the buoyant force and assume that the buoy's motion is purely vertical. Derive the equation of motion of the cylinder using Linear Momentum Balance. What is the period of oscillation of the cylinder?

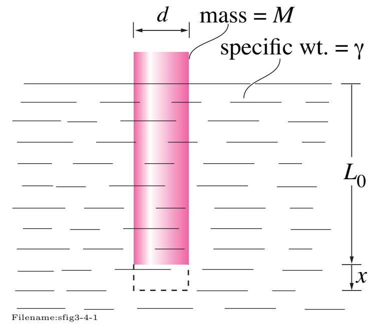


Figure 9.33:

Solution The free body diagram of the cylinder is shown in Fig. 9.34 where F_B represents the buoyant force (see the hydrostatics chapter starting on 396). Before the cylinder is pushed down by x, the linear momentum balance of the cylinder gives

$$F_B - Mg = M \underbrace{a}_{0} = 0 \qquad \Rightarrow \qquad F_B = Mg$$

Now F_B = (volume of the displaced fluid)$\cdot$ (its specific weight) $= AL_o\gamma$. Thus,

$$AL_o\gamma = Mg. \tag{9.34}$$

Now, when the cylinder is pushed down by an amount x,

$$F_B' = \text{ new buoyant force } = (L_o + x)A\gamma.$$

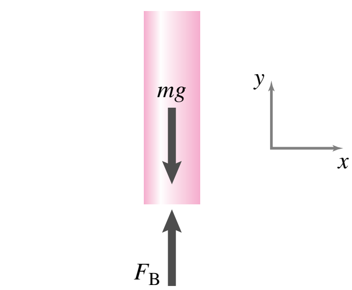


Figure 9.34:

Therefore, from LMB we get

$$\begin{array}{rrcl} & F_B' - Mg &=& -M\ddot{x} \\ \text{or} & (L_o + x)A\gamma - Mg &=& -M\ddot{x} \\ \text{or} & M\ddot{x} + A\gamma x &=& \overbrace{-AL_o\gamma + Mg}^{=0 \text{ from } (9.34).} \\ \text{or} & M\ddot{x} + A\gamma x &=& 0 \\ \text{or} & \ddot{x} + \frac{A\gamma}{M}x &=& 0. \end{array}$$

$$\boxed{\ddot{x} + \frac{A\gamma}{M}x = 0}$$

Comparing this equation with the standard simple harmonic equation (*e.g.*, eqn.(g), in the box on ODE's on page 424).

$$\text{The circular frequency } \lambda = \sqrt{\frac{A\gamma}{M}},$$

$$\text{Therefore,} \qquad \text{the period of oscillation } T = \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{M}{A\gamma}}$$

.

$$\boxed{T = 2\pi\sqrt{\frac{M}{A\gamma}}}$$

Comments: This calculation inaccurately uses fluid statics to calculate the dynamics of a buoy; the pressure used in this calculation assumes fluid statics when actually the fluid is moving. One common partial correction is to use 'added mass' to account for fluid that moves more-or-less with the cylinder. The added mass is usually something like one-half the mass of the displaced fluid, that is one half the mass of the bouy. Another missing effect is the fluid damping. This would be added as a drag force proportional to the velocity or the velocity squared.

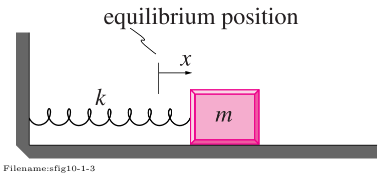


Figure 9.35:

SAMPLE 9.15 A spring-mass system executes simple harmonic motion: $x(t) = A\cos(\lambda t - \phi)$. The system starts with initial conditions $x(0) = 25\,\text{mm}$ and $\dot{x}(0) = 160\,\text{mm/s}$ and oscillates at the rate of 2 cycles/sec.

1. Find the time period of oscillation and the oscillation frequency λ.
2. Find the amplitude of oscillation A and the phase angle ϕ.
3. Find the displacement, velocity, and acceleration of the mass at $t = 1.5\,\text{s}$.
4. Find the maximum speed and acceleration of the system.
5. Draw an accurate plot of displacement vs. time of the system and label all relevant quantities. What does ϕ signify in this plot?

Solution

1. We are given $f = 2\,\text{Hz}$. Therefore, the time period of oscillation is

$$T = \frac{1}{f} = \frac{1}{2\,\text{Hz}} = 0.5\,\text{s},$$

and the oscillation frequency $\lambda = 2\pi f = 4\pi\,\text{rad/s}$.

$$\boxed{T = 0.5\,\text{s}, \quad \lambda = 4\pi\,\text{rad/s}.}$$

2. The displacement $x(t)$ of the mass is given by

$$x(t) = A\cos(\lambda t - \phi).$$

Therefore the velocity (actually the speed) is

$$\dot{x}(t) = -A\lambda\sin(\lambda t - \phi)$$

At $t = 0$, we have

$$x(0) = A\cos(-\phi) = A\cos\phi \tag{9.35}$$

$$\dot{x}(0) = -A\lambda\sin(-\phi) = A\lambda\sin\phi \tag{9.36}$$

By squaring Eqn (9.35) and adding it to the square of [Eqn (9.36) divided by λ], we get

$$\begin{aligned} A^2\cos^2\phi + \frac{A^2\lambda^2\sin^2\phi}{\lambda^2} &= A^2 = x^2(0) + \frac{\dot{x}^2(0)}{\lambda^2} \\ \Rightarrow \qquad A &= \sqrt{(25\,\text{mm})^2 + \frac{(160\,\text{mm/s})^2}{(4\pi\,\text{rad/s})^2}} \\ &= 28.06\,\text{mm}. \end{aligned}$$

Substituting the value of A in Eqn (9.35), we get

$$\begin{aligned} \phi &= \cos^{-1}\frac{x(0)}{A} \\ &= \cos^{-1}\frac{25\,\text{mm}}{28.06\,\text{mm}} \\ &= 0.471\,\text{rad} \approx 27^\circ. \end{aligned}$$

$$\boxed{A = 28.06\,\text{mm}. \quad \phi = 0.471\,\text{rad}.}$$

3. The displacement, velocity, and acceleration of the mass at any time t can now be calculated as follows

$$\begin{aligned} x(t) &= A\cos(\lambda t - \phi) \\ \Rightarrow\quad x(1.5\,\mathrm{s}) &= 28.06\,\mathrm{mm}\cdot\cos(6\pi - 0.471) \\ &= 25\,\mathrm{mm}. \end{aligned}$$

$$\begin{aligned} \dot{x}(t) &= -A\lambda\sin(\lambda t - \phi) \\ \Rightarrow\quad \dot{x}(1.5\,\mathrm{s}) &= 28.06\,\mathrm{mm}\cdot(4\pi\,\mathrm{rad/s})\cdot\sin(6\pi - 0.471) \\ &= 160\,\mathrm{mm/s}. \end{aligned}$$

$$\begin{aligned} \ddot{x}(t) &= -A\lambda^2\cos(\lambda t - \phi) \\ \Rightarrow\quad \ddot{x}(1.5\,\mathrm{s}) &= 28.06\,\mathrm{mm}\cdot(4\pi\,\mathrm{rad/s})^2\cdot\cos(6\pi - 0.471) \\ &= -3.95\times 10^3\,\mathrm{mm/s^2} \\ &= -3.95\,\mathrm{m/s^2}. \end{aligned}$$

⑨

$$x(1.5\,\mathrm{s}) = 25\,\mathrm{mm}. \quad \dot{x}(1.5\,\mathrm{s}) = 160\,\mathrm{mm/s}. \quad \ddot{x}(1.5\,\mathrm{s}) = -3.93\,\mathrm{m/s^2}.$$

⑨ We can find the displacement and velocity at $t = 1.5\,\mathrm{s}$ without any differentiation. Note that the system completes 2 cycles in 1 second, implying that it will complete 3 cycles in 1.5 seconds. Therefore, at $t = 1.5\,\mathrm{s}$, it has the same displacement and velocity as it had at $t = 0\,\mathrm{s}$.

4. Maximum speed:

$$|\dot{x}_{\mathrm{max}}| = A\lambda = (28.06\,\mathrm{mm})\cdot(4\pi\,\mathrm{rad/s}) = 0.35\,\mathrm{m/s}.$$

Maximum acceleration:

$$|\ddot{x}_{\mathrm{max}}| = A\lambda^2 = (28.06\,\mathrm{mm})\cdot(4\pi\,\mathrm{rad/s})^2 = 4.43\,\mathrm{m/s^2}.$$

$$|\dot{x}_{\mathrm{max}}| = 0.35\,\mathrm{m/s}, \quad |\ddot{x}_{\mathrm{max}}| = 4.43\,\mathrm{m/s^2}.$$

5. The plot of $x(t)$ versus t is shown in Fig. 9.36. The phase angle ϕ represents the shift in $\cos(\lambda t)$ to the right by an amount $\frac{\phi}{\lambda}$.

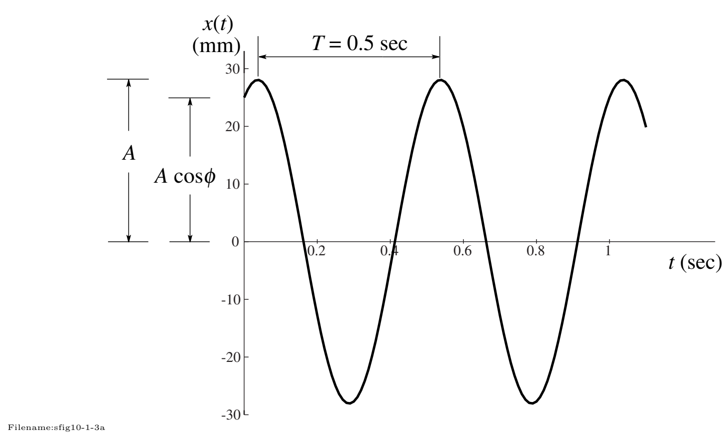


Figure 9.36:

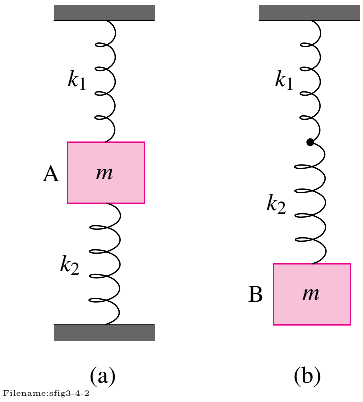


Figure 9.37:

SAMPLE 9.16 Springs in series versus springs in parallel: Two massless springs with spring constants k_1 and k_2 are attached to mass A *in parallel* (although they look superficially as if they are in series) as shown in Fig. 9.37. An identical pair of springs is attached to mass B *in series*. Taking $m_A = m_B = m$, find and compare the natural frequencies of the two systems. Ignore gravity.

Solution Let us pull each mass downwards by a small vertical distance y and then release. Measuring y to be positive downwards, we can derive the equations of motion for each mass by writing the balance of linear momentum for each as follows.

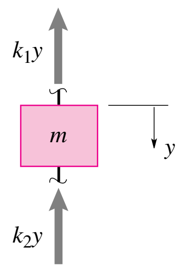


Figure 9.38: Free body diagram of the mass.

- **Mass A:** The free body diagram of mass A is shown in Fig. 9.38. As the mass is displaced downwards by y, spring 1 gets stretched by y whereas spring 2 gets compressed by y. Therefore, the forces applied by the two springs, $k_1 y$ and $k_2 y$, are in the same direction. The linear momentum balance of mass A in the vertical direction gives:

$$\begin{aligned} \sum F &= m a_y \\ \text{or} \qquad -k_1 y - k_2 y &= m\ddot{y} \\ \text{or} \qquad \ddot{y} + \left(\frac{k_1 + k_2}{m}\right) y &= 0. \end{aligned}$$

Let the natural frequency of this system be ω_p. Comparing with the standard simple harmonic equation $\ddot{x} + \lambda^2 x = 0$ (see box 9.3 on page 424), we get the natural frequency (λ) of the system:

$$\omega_p = \sqrt{\frac{k_1 + k_2}{m}} \tag{9.37}$$

$$\boxed{\omega_p = \sqrt{\frac{k_1 + k_2}{m}}}$$

- **Mass B:** The free body diagram of mass B and the two springs is shown in Fig. 9.39. In this case both springs stretch as the mass is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = y \tag{9.38}$$

Now, using the free body diagram of spring 2 and then writing linear momentum balance we get,

$$\begin{aligned} k_2 y_2 - k_1 y_1 &= \underbrace{m}_{0} a = 0 \\ y_1 &= \frac{k_2}{k_1} y_2 \end{aligned} \tag{9.39}$$

Solving (9.38) and (9.39) we get

$$y_2 = \frac{k_1}{k_1 + k_2} y.$$

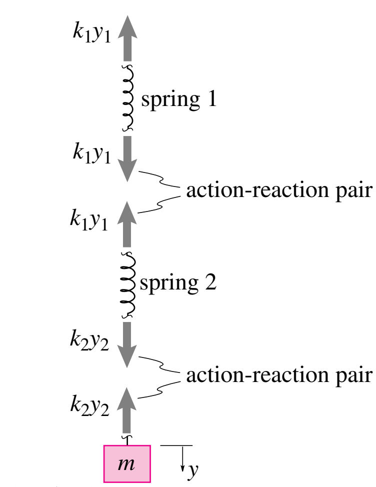


Figure 9.39: Free body diagrams

Now, linear momentum balance of mass B in the vertical direction gives:

$$
\begin{aligned}
-k_2 y_2 = m a_y &= m\ddot{y} \\
\text{or} \qquad m\ddot{y} + k_2 \overbrace{\frac{k_1}{k_1+k_2} y}^{y_2} &= 0 \\
\text{or} \qquad \ddot{y} + \frac{k_1 k_2}{m(k_1+k_2)} y &= 0.
\end{aligned} \tag{9.40}
$$

Let the natural frequency of this system be denoted by ω_s. Then, comparing with the standard simple harmonic equation as in the previous case, we get

$$
\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}. \tag{9.41}
$$

$$\boxed{\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}}$$

From (9.37) and (9.41)

$$
\frac{\omega_p}{\omega_s} = \frac{k_1+k_2}{\sqrt{k_1 k_2}}.
$$

Let $k_1 = k_2 = k$. Then, $\omega_p/\omega_s = 2$, *i.e.,* the natural frequency of the system with two identical springs in parallel is twice as much as that of the system with the same springs in series. Intuitively, the restoring force applied by two springs in parallel will be more than the force applied by identical springs in series. In one case the forces add and in the other they don't and each spring is stretched less. Therefore, we do expect mass A to oscillate at a faster rate (higher natural frequency) than mass B.

Comments:

1. Although the springs attached to mass A do not visually seem to be in parallel, from mechanics point of view they are parallel. You can easily check this result by putting the two springs visually in parallel and then deriving the equation of mass A. You will get the same equations. For springs in parallel, each spring has the *same displacement* but *different forces*. For springs in series, each has *different displacements* but the *same force*.
2. When many springs are connected to a mass in series or in parallel, sometimes we talk about their *effective* spring constant, *i.e.,* the spring constant of a single imaginary spring which could be used to replace all the springs attached in parallel or in series. Let the effective spring constant for springs in parallel and in series be represented by k_{pe} and k_{se} respectively. By comparing eqns. (9.37) and (9.41) with the expression for natural frequency of a simple spring mass system, we see that

   $$
   k_{pe} = k_1 + k_2 \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2}.
   $$

   These expressions can be easily extended for any arbitrary number of springs, say, N springs:

   $$
   k_{pe} = k_1 + k_2 + \ldots + k_N \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_N}.
   $$

SAMPLE 9.17 Figure 9.40 shows two responses obtained from experiments on two spring-mass systems. For each system

1. Find the natural frequency.
2. Find the initial conditions.

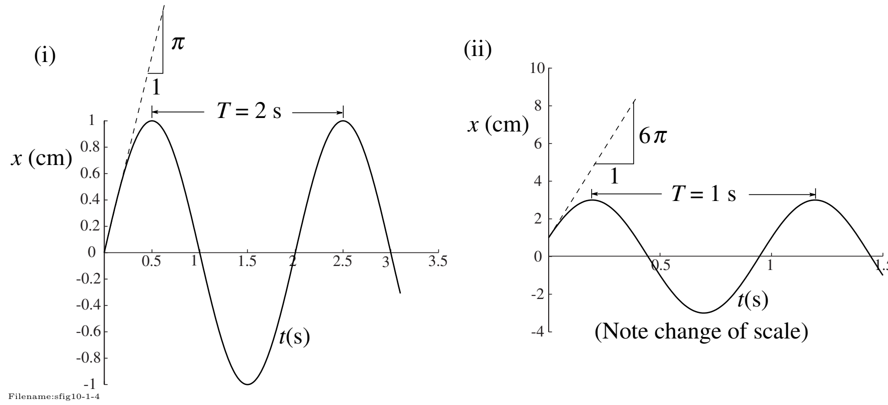


Figure 9.40:

Solution

1. **Natural frequency:** By definition, the natural frequency f is the number of cycles the system completes in one second. From the given responses we see that:

   Case(i): the system completes $\frac{1}{2}$ a cycle in 1 s.
   $$\Rightarrow \qquad f = \frac{1}{2}\ \text{Hz}.$$
   Case(ii): the system completes 1 cycle in 1 s.
   $$\Rightarrow \qquad f = 1\ \text{Hz}.$$
   It is usually hard to measure the fraction of cycle occurring in a short time. It is easier to first find the time period, *i.e.,* the time taken to complete 1 cycle. (10) Then the natural frequency can be found by the formula $f = \frac{1}{T}$. From the given responses, we find the time period by estimating the time between two successive peaks (or troughs): From Figure 9.40 we find that for

   Case (i):
   $$f = \frac{1}{T} = \frac{1}{2\,\text{s}} = \frac{1}{2}\ \text{Hz},$$
   Case (ii):
   $$f = \frac{1}{T} = \frac{1}{1\,\text{s}} = 1\ \text{Hz}$$
   $$\boxed{\text{case (i) } f = \tfrac{1}{2}\ \text{Hz.} \quad \text{case (ii) } f = 1\ \text{Hz.}}$$

(10) To estimate the frequency of some repeated motion in an experiment, it is best to measure the time for a large number of cycles, say 5, 10 or 20, and then divide that time by the total number of cycles to get an average value for the time period of oscillation.

2. **Initial conditions:** Now we are to find the displacement and velocity at $t = 0$ s for each case. Displacement is easy because we are given the displacement plot, so we just read the value at $t = 0$ from the plots:

   Case (i):
   $$x(0) = 0.$$
   Case (ii):
   $$x(0) = 1\ \text{cm}.$$

The velocity (actually the speed) is the time-derivative of the displacement. Therefore, we get the initial velocity from the slope of the displacement curve at $t = 0$.

Case (i):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{\pi\ \text{cm}}{1\ \text{s}} = 3.14\,\text{cm/s}.$$

Case (ii):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{6\pi\ \text{cm}}{1\ \text{s}} = 18.85\,\text{cm/s}.$$

Thus the initial conditions are

Case (i):	$x(0) = 0$	$\dot{x}(0) = 3.14\,\text{cm/s}$
Case (ii):	$x(0) = 1\,\text{cm}$	$\dot{x}(0) = 18.85\,\text{cm/s}$

Comments: Estimating the speed from the initial slope of the displacement curve at $t = 0$ is not a very good method because it is hard to draw an accurate tangent to the curve at $t = 0$. A slightly different line but still seemingly tangential to the curve at $t = 0$ can lead to significant error in the estimated value. A better method, perhaps, is to use the known values of displacement at different points and use the energy method to calculate the initial speed. We show sample calculations for the first system:

Case(i): We know that $x(0) = 0$. Therefore the entire energy at $t = 0$ is the kinetic energy $= \frac{1}{2}mv_0^2$. At $t = 0.5\,\text{s}$ we note that the displacement is maximum, *i.e.,* the speed is zero. Therefore, the entire energy is potential energy $= \frac{1}{2}kx^2$, where $x = x(t = 0.5\,\text{s}) = 1\,\text{cm}$.

Now, from the conservation of energy:

$$\begin{aligned}
\frac{1}{2}mv_0^2 &= \frac{1}{2}k(x_{t=0.5\,\text{s}})^2 \\
\Rightarrow \quad v_0 &= \sqrt{\frac{k}{m}}\cdot(x_{t=0.5\,\text{s}}) \\
&= \underbrace{\sqrt{\frac{k}{m}}}_{\lambda}\cdot(1\,\text{cm}) \\
&= 2\pi f\cdot(1\,\text{cm}) \\
&= 2\pi\cdot\frac{1}{2}\ \text{Hz}\cdot 1\,\text{cm} \\
&= 3.14\,\text{cm/s}.
\end{aligned}$$

Similar calculations can be done for the second system.

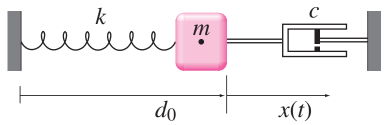


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Figure 9.41: Spring-mass dashpot.

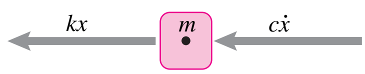

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Figure 9.42: Free body diagram of the mass at an instant t.

SAMPLE 9.18 A block of mass 10 kg is attached to a spring and a dashpot as shown in Figure 9.41. The spring constant $k = 1000\,\mathrm{N/m}$ and a damping rate $c = 50\,\mathrm{N{\cdot}s/m}$. When the block is at a distance d_0 from the left wall the spring is relaxed. The block is pulled to the right by 0.5 m and released. Assuming no initial velocity, find

1. the equation of motion of the block.
2. the position of the block at $t = 2\,\mathrm{s}$.

Solution

1. Let x be the position of the block, measured positive to the right of the static equilibrium position, at some time t. Let $\dot{x}$ be the corresponding speed. The free body diagram of the block at the instant t is shown in Figure 9.42.

   Since the motion is only horizontal, we can write the linear momentum balance in the x-direction ($\sum F_x = m\,a_x$):

$$\underbrace{-kx - c\dot{x}}_{\sum F_x} = m \underbrace{\ddot{x}}_{a_x}$$

$$\text{or} \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{9.42}$$

   which is the desired equation of motion of the block.

$$\boxed{\ddot{x} + \tfrac{c}{m}\dot{x} + \tfrac{k}{m}x = 0.}$$

2. To find the position and velocity of the block at any time t we need to solve Eqn (9.42). Since the solution depends on the relative values of m, k, and c, we first compute c^2 and compare with the *critical value* $4mk$.

$$\begin{aligned} c^2 &= 2500(\mathrm{N{\cdot}s/m})^2 \\ \text{and} \quad 4mk &= 4{\cdot}10\,\mathrm{kg}{\cdot}1000\,\mathrm{N/m} = 4000(\mathrm{N{\cdot}s/m})^2. \\ \Rightarrow \quad c^2 &< 4mk. \end{aligned}$$

   Therefore, the system is underdamped and we may write the general solution as (see box 9.7 on page 462)

$$x(t) = e^{-\frac{c}{2m}t}\,[A\cos\lambda_D t + B\sin\lambda_D t] \tag{9.43}$$

   where

$$\lambda_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 9.682\,\mathrm{rad/s}.$$

   Substituting the initial conditions $x(0) = 0.5\,\mathrm{m}$ and $\dot{x}(0) = 0\ \mathrm{m/s}$ in Eqn (9.43) (we need to differentiate Eqn (9.43) first to substitute $\dot{x}(0)$), we get

$$\begin{aligned} x(0) &= 0.5\,\mathrm{m} = A. \\ \dot{x}(0) &= 0 = -\frac{c}{2m}\cdot A + \lambda_D{\cdot}B \\ \Rightarrow \quad B &= \frac{Ac}{2m\lambda_D} = \frac{(0.5\,\mathrm{m}){\cdot}(50\,\mathrm{N{\cdot}s/m})}{2{\cdot}(10\,\mathrm{kg}){\cdot}(9.682\,\mathrm{rad/s})} = 0.13\,\mathrm{m}. \end{aligned}$$

   Thus, the solution is

$$x(t) = e^{(-2.5\frac{1}{\mathrm{s}})t}\,[0.50\cos(9.68\,\mathrm{rad/s}\ t) + 0.13\sin(9.68\,\mathrm{rad/s}\ t)]\ \mathrm{m}.$$

   Substituting $t = 2\,\mathrm{s}$ in the above expression we get $x(2\,\mathrm{s}) = 0.003\,\mathrm{m}$.

$$\boxed{x(2\,\mathrm{s}) = 0.003\,\mathrm{m}.}$$

SAMPLE 9.19 A structure, modeled as a single degree of freedom system, exhibits characteristics of an underdamped system under free oscillations. The response of the structure to some initial condition is determined to be $x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t)$ where $A = 0.3\,\text{m}$, $\xi \equiv$ damping ratio $= 0.02$, $\lambda \equiv$ undamped circular frequency $= 1\,\text{rad/s}$, and $\lambda_D \equiv$ damped circular frequency $= \lambda\sqrt{1-\xi^2} \approx \lambda$.

1. Find an expression for the ratio of energies of the system at the $(n+1)$th displacement peak and the nth displacement peak.
2. What percent of energy available at the first peak is lost after 5 cycles?

Solution

1. We are given that

$$x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t).$$

The structure attains its first displacement peak when $\sin\lambda_D t$ is maximum, *i.e.,*

$$\lambda_D t = \frac{\pi}{2} \quad \Rightarrow \quad t = \frac{\pi}{2\lambda_D}.$$

At this instant,

$$\begin{aligned} x(t) &= Ae^{-\xi\cdot\lambda\cdot\frac{\pi}{2\lambda_D}} \\ &= Ae^{-\frac{\pi}{2}\cdot\frac{\xi}{\sqrt{1-\xi^2}}} \\ &= (0.3\,\text{m})\cdot e^{-0.0314} \\ &= 0.29\,\text{m}. \end{aligned}$$

Let x_n and x_{n+1} be the values of the displacement at the nth and the $(n+1)$th peak, respectively. Since x_n and x_{n+1} are peak displacements, the respective velocities are zero at these points. Therefore, the energy of the system at these peaks is given by the potential energy stored in the spring. That is

$$E_n = \frac{1}{2}kx_n^2 \quad \text{and} \quad E_{n+1} = \frac{1}{2}kx_{n+1}^2. \tag{9.44}$$

Let t_n be the time at which the nth peak displacement x_n is attained, *i.e.,*

$$x_n = Ae^{-\xi\lambda t_n} \tag{9.45}$$

Since x_{n+1} is the next peak displacement, it must occur at $t = t_n + T_D$ where T_D is the time period of damped oscillations. Thus

$$x_{n+1} = Ae^{-\xi\lambda(t_n+T_D)} \tag{9.46}$$

From Eqns (9.44), (9.45), and (9.46)

$$\begin{aligned} \frac{E_{n+1}}{E_n} &= \frac{\frac{1}{2}k(Ae^{-\xi\lambda(t_n+T_D)})^2}{\frac{1}{2}k(Ae^{-\xi\lambda t_n})^2} \\ &= e^{-2\xi\lambda T_D}. \end{aligned}$$

$$\boxed{\frac{E_{n+1}}{E_n} = e^{-2\xi\lambda T_D}.}$$

2. Noting that $T_D = \frac{2\pi}{\lambda_D}$ and $\lambda_D = \lambda\sqrt{1-\xi^2}$, we get

$$\begin{aligned} E_{n+1} &= E_n\, e^{-2\xi\cancel{\lambda}\cdot\frac{2\pi}{\cancel{\lambda}\sqrt{1-\xi^2}}} \\ &= e^{-4\pi\frac{\xi}{\sqrt{1-\xi^2}}} \approx e^{-4\pi\xi} \\ \Rightarrow \quad E_{n+1} &= e^{-4\pi\xi}\, E_n. \end{aligned}$$

Applying this equation recursively for $n = n-1, n-2, \ldots, 1, 0,$, we get

$$\begin{aligned} E_n &= e^{-4\pi\xi}\cdot E_{n-1} \\ &= e^{-4\pi\xi}\cdot(e^{-4\pi\xi}\cdot E_{n-2}) \\ &= (e^{-4\pi\xi})^3\cdot E_{n-3} \\ &\vdots \\ &= (e^{-4\pi\xi})^n\cdot E_0. \end{aligned}$$

Now we use this equation to find the percentage of energy of the first peak ($n = 0$) lost after 5 cycles ($n = 5$):

$$\begin{aligned} \Delta E_5 &= \frac{E_0 - E_5}{E_0} \times 100 \\ &= \left(1 - e^{-4\pi\xi\cdot 5}\right) \times 100 \\ &= 71.5\%. \end{aligned}$$

$$\boxed{\Delta E_5 = 71.5\%.}$$

SAMPLE 9.20 **A SDOF spring-mass model from given data:** The following table is obtained for successive peaks of displacement from the simulation of free vibration of a mechanical system. Make a single degree of freedom mass-spring-dashpot model of the system choosing appropriate values for mass, spring stiffness, and damping rate.

Data:

peak no. n	0	1	2	3	4	5	6
time (s)	0.0000	0.6279	1.2558	1.8837	2.5116	3.1395	3.7674
peak x (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443

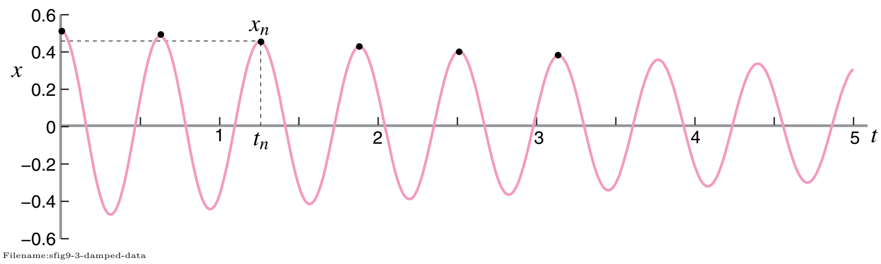

Figure 9.43: Oscillation data from the simulation of a mechanical system

Solution Since the data provided is for successive peak displacements, the time between any two successive peaks represents the period of oscillations. It is also clear that the system is underdamped because the successive peaks are decreasing. We can use the logarithmic decrement method to determine the damping in the system.

First, we find the time period T_D from which we can determine the damped circular frequency λ_D. From the given data we find that

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots = 0.6279\,\mathrm{s}$$

Therefore,

$$\begin{aligned} T_D &= 0.6279\,\mathrm{s}. \\ \Rightarrow \quad \lambda_D &= \frac{2\pi}{T_D} = 10\,\mathrm{rad/s}. \end{aligned} \tag{9.47}$$

Now we make a table for the logarithmic decrement of the peak displacements:

peak disp. x_n (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443
$\frac{x_n}{x_{n+1}}$	1.0658	1.0648	1.0647	1.0645	1,0637	1.0627	
$\ln\left(\frac{x_n}{x_{n+1}}\right)$	0.0637	0.0628	0.0627	0.0624	0.0618	0.0608	

Thus, we get several values of the logarithmic decrement $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$ (11)

(11) Theoretically, all of these values should be the same, but it is rarely the case in practice. When x_n's are measured from an experimental setup, the values of D may vary even more.

We take the average value of D:

$$D = \bar{D} = 0.0624. \tag{9.48}$$

Let the equivalent single degree of freedom model have mass m, spring stiffness k, and damping rate c. Then

$$\lambda_D = \lambda\sqrt{1-\xi^2} \approx \lambda = \sqrt{\frac{k}{m}}.$$

Thus, from Eqn (9.47),

$$\frac{k}{m} = \lambda^2 = 100(\,\mathrm{rad/s})^2, \tag{9.49}$$

and, since $D = \frac{cT_D}{2m}$, from Eqn (9.48) we get

$$\begin{aligned} c &= \frac{2mD}{T_D} \\ &= \frac{2m(0.0624)}{0.6279\,\mathrm{s}} \\ &= (0.1988\frac{1}{\mathrm{s}})m. \end{aligned} \tag{9.50}$$

Equations (9.49) and (9.50) have three unknowns: k, m, and c. We cannot determine all three uniquely from the given information. So, let us pick an arbitrary mass $m = 5\,\mathrm{kg}$. Then

$$\begin{aligned} k &= (100\frac{1}{\mathrm{s}^2})\cdot(5\,\mathrm{kg}) \\ &= 500\,\mathrm{N/\,m}, \end{aligned}$$

and

$$\begin{aligned} c &= (0.1988\frac{1}{\mathrm{s}})\cdot(5\,\mathrm{kg}) \\ &= 0.99\,\mathrm{N\cdot\,s/\,m}. \end{aligned}$$

$$\begin{aligned} m &= 5\,\mathrm{kg}, \\ k &= 500\,\mathrm{N/\,m}, \\ c &= 0.99\,\mathrm{N\cdot\,s/\,m}. \end{aligned}$$

Of course, we could choose many other sets of values for m, k, and c which would match the given response. In practice, there is usually a little more information available about the system, such as the mass of the system. In that case, we can determine k and c uniquely from the given response.

9.4 Coupled motions in 1D

Thinking of a car, a plane, a person on a bicycle or a satellite as a single particle is often edifying, and sufficient for many engineering purposes. However, the one-particle model is also often inadequate. That the parts of a machine or structure move relative to each other is obviously sometimes important; many important engineering systems have parts that move independently.

Here we begin the study of independent, but coupled, motions of parts. The independent motions are coupled in that the motion of each part may effects the motion of the others.

> **Example: Car suspension.**
> A model of a car suspension treats the wheel as one particle and the car as another. The wheel is coupled to the ground by a tire and to the car by the suspension. In a first analysis the only motion to consider would be vertical for both the wheel and the car. Think of the ground as moving up and down and 'forcing' the motion of the car and wheel system.

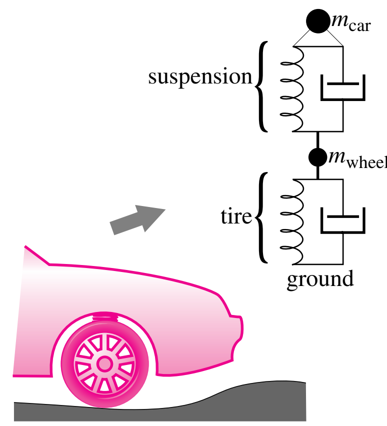

Figure 9.44: The vertical motions of a car and wheel might be considered as coupled motions in 1D.

Still using one-dimensional mechanics, we consider systems that can be modelled as two or more particles. Such one-dimensional coupled motion analysis is common in engineering practice in situations where there are connected parts that all move in about the same direction, but the parts do not move the same amount or necessarily at the same time. Many of the ideas generalize to systems where parts, each with one degree of freedom, are coupled together. Many generalizations apply even if each degree of freedom is quite different from the others. These generalizations to more general coupled motions come later in the book.

The primary goal in this section is to develop two skills:

- To write correct equations of motion for a line of particles connected to each other with springs and dashpots, and
- To simulate the motions of such systems on a computer.
- (the third of the two things, really implicit in the first two) To use the simulation results to find errors in the equations.

Further, we will introduce the concept of 'normal modes'. The simplest way of dealing with the coupled motion of two or more particles is

- to write $\vec{F} = m\vec{a}$ for each particle and then
- to use the forces on the free body diagrams to evaluate the forces.

Because the most common models for the interaction forces are springs and dashpots (see chapter 3), one needs to account for the relative positions and velocities of the particles.

Relative motion in one dimension

If the position of A is $\vec{r}_A$, and B's position is $\vec{r}_B$, then B's position *relative* to A is

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A.$$

Relative velocity and acceleration are similarly defined by subtraction, or by differentiating the above expression, as

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A \text{ and } \vec{\boldsymbol{a}}_{B/A} = \vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A.$$

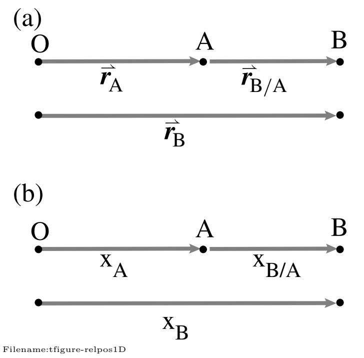

Figure 9.45: The relative position of points A and B in one dimension.

In one dimension, the relative position diagram of fig. 2.5 on page 44 becomes fig. 9.45. $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\imath}}$. So, we can write,

$$\begin{aligned} x_{B/A} &\equiv x_B - x_A, \\ v_{B/A} &\equiv v_B - v_A = \frac{d}{dt}x_{B/A}, \text{ and} \\ a_{B/A} &\equiv a_B - a_A = \frac{d}{dt}v_{B/A} = \frac{d^2}{dt^2}x_{B/A}. \end{aligned}$$

An alternative notation, discussed in Chapter 2, is x_{AB} where the directed line AB is equivalent to the position of B relative to A:

$$x_{AB} = x_{B/A}$$

Example: Two masses connected by a spring.
Consider the two masses on a frictionless support (fig. 9.46). Assume the spring is unstretched when $x_1 = x_2 = 0$. After drawing free body diagrams of the two masses we can write $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each mass:

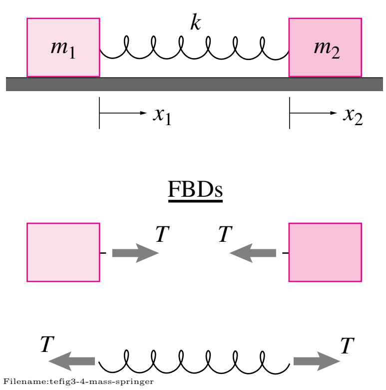

Figure 9.46:

$$\begin{aligned} &\textbf{mass 1:} \quad \vec{\boldsymbol{F}}_1 = m\vec{\boldsymbol{a}}_1 \quad \Rightarrow \quad T\hat{\boldsymbol{\imath}} = m_1\ddot{x}_1\hat{\boldsymbol{\imath}} \\ &\textbf{mass 2:} \quad \vec{\boldsymbol{F}}_2 = m\vec{\boldsymbol{a}}_2 \quad \Rightarrow \quad -T\hat{\boldsymbol{\imath}} = m_2\ddot{x}_2\hat{\boldsymbol{\imath}} \end{aligned} \tag{9.51}$$

The stretch of the spring is

$$\Delta\ell = x_2 - x_1$$

$$\text{so} \quad T = k\Delta\ell = k(x_2 - x_1). \tag{9.52}$$

Combining (9.51) and (9.52) we get

$$\begin{aligned} \ddot{x}_1 &= \left(\frac{1}{m_1}\right)k(x_2 - x_1) \\ \ddot{x}_2 &= \left(\frac{1}{m_2}\right)(-k(x_2 - x_1)) \end{aligned} \tag{9.53}$$

Note: Take care with signs when setting up this type of problem. You should check, for example, that if $x_2 > x_1$, mass 1 accelerates to the right ($\ddot{x}_1 > 0$) and mass 2 accelerates to the left($\ddot{x}_2 < 0$). It is easy to make sign errors. You've been warned!

The differential equations that result from writing $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for the separate particles are coupled second-order equations. The equations are 'coupled' in that the equation for m_1, say, includes the position x_2 or velocity v_2 of mass 2. Such systems of second order coupled equations are often solved on a computer by writing them as a system of first-order equations. You have two first-order equations for each of the second order equations because of the addition of equations like, for example, $\dot{x}_{17} = v_{17}$.

Example: Writing second-order ODEs as first-order ODEs.
Refer again to fig. 9.46 If we define $v_1 = \dot{x}_1$ and $v_2 = \dot{x}_2$ we can rewrite equation 9.53 as

$$\begin{aligned}
\dot{x}_1 &= v_1 \\
\dot{v}_1 &= \left(\frac{1}{m_1}\right) k(x_2 - x_1) \\
\dot{x}_2 &= v_2 \\
\dot{v}_2 &= \left(\frac{1}{m_2}\right) (-k)\,(x_2 - x_1)
\end{aligned}$$

or, defining $z_1 = x_1, z_2 = v_1, z_3 = x_2, z_4 = v_2$, we get

$$\begin{aligned}
\dot{z}_1 &= \quad z_2 \\
\dot{z}_2 &= -\left(\tfrac{k}{m_1}\right) z_1 + \left(\tfrac{k}{m_1}\right) z_3 \\
\dot{z}_3 &= \quad z_4 \\
\dot{z}_4 &= \tfrac{k}{m_2} z_1 - \tfrac{k}{m_2} z_3 \,.
\end{aligned}$$

Most numerical solutions depend on specifying numerical values for the various constants and initial conditions.

Example: computer solution
If we take, in consistent units, $m_1 = 1, k = 1, m_2 = 1, x_1(0) = 0, x_2(0) = 0, v_1(0) = 1$, and $v_2(0) = 0$, we can set up a well defined computer problem (please see the preface for a discussion of the computer notation). This problem corresponds to finding the motion just after the left mass was hit on the left side with a hammer.:

```
ODEs = {z1dot =  z2
        z2dot = -z1 + z3
        z3dot =  z4
        z4dot =  z1 - z3}
ICs  = {z1(0) =0, z2(0)=1, z3(0)=0, z4(0)=0}
solve ODEs with ICs from t=0 to t=10
plot z1 vs t.
```

This yields the plot shown in fig. 9.47.

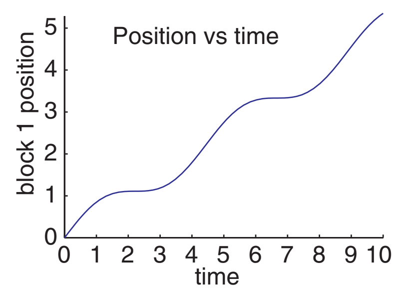

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Figure 9.47: Plot of the position of the left mass *vs.* time.

The same methods work for problems involving connections with dashpots.

Example: Multi-DOF system with a dashpot.
Consider m_B in fig. 9.48. Using the free body diagram shown linear momentum balance gives

$$\begin{aligned}
\sum \vec{F}_i &= m\vec{a}_B \\
\{(-T_{k_4} - T_{k_2} + T_{c_1} + T_{k_3})\hat{\imath} &= m a_B \hat{\imath}\} \\
\{\}\cdot\hat{\imath} \quad \Rightarrow \quad -\underbrace{T_{k_4}}_{k_4 x_B} - \underbrace{T_{k_2}}_{k_2(x_B - x_A)} + \underbrace{T_{c_1}}_{c_1(\dot{x}_D - \dot{x}_B)} + \underbrace{T_{k_3})}_{k_3(x_D - x_B)} &= m \underbrace{a_B}_{\ddot{x}_B} \\
k_2 x_A - (k_2 + k_4 + k_3)x_B + k_3 x_D - c_1\dot{x}_B + c_1\dot{x}_D &= m\ddot{x}_B
\end{aligned}$$

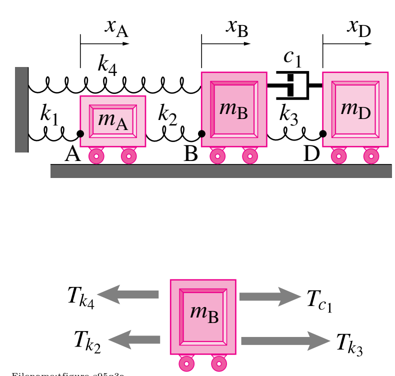

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Figure 9.48: A system with 3 masses, 4 springs and a dashpot. Also shown is free body diagram of mass B.

Similar equations could be written for masses A and C. Some things to note

- We assumed zeros for the displacements so that the system is in static equilibrium if $x_A = x_B = x_D = 0$.

- We have taken the sign convention that tension is positive for all springs and dashpots.
- All of the spring coefficients of x_B have a minus sign in front. That is because all springs, whether to the right or the left of mass B, provide a restoring force if mass B is displaced.
- All of the spring coefficients of x_A and x_D make a positive contribution because motion to the right of mass A or mass D causes a force to the right on mass B.

As for the example above, for any system of masses, linear springs and linear dashpots the set of momentum balance equations can be written in the form

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0} \tag{9.54}$$

where $\boldsymbol{x}$ is a list of positions of the masses. The mass matrix $[M]$ is diagonal because each equation corresponds to $F = ma$ for one mass. The damping and stiffness matrices $[C]$ and $[K]$ are symmetric because, as Jim Marley said, 'every reaction has a reaction'; if motion of mass 7 causes a stretch on the spring between it and mass 19 then motion of mass 19 causes a stretch on the same spring, similarly affecting mass 7. So row 17 column 9 has the same entry as row 9 column 17. As noted in the example below, the diagonal elements of $[M]$, $[C]$ and $[K]$ are positive (or zero).

Example: Matrix form

When the three momentum balance equations for fig. 9.48 are written, one for each mass, they can be assembled in matrix form as

$$\underbrace{\begin{bmatrix} m_A & 0 & 0 \\ 0 & m_B & 0 \\ 0 & 0 & m_C \end{bmatrix}}_{[M]} \underbrace{\begin{bmatrix} \ddot{x}_A \\ \ddot{x}_B \\ \ddot{x}_D \end{bmatrix}}_{\ddot{\boldsymbol{x}}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & c_1 & -c_1 \\ 0 & -c_1 & c_1 \end{bmatrix}}_{[C]} \underbrace{\begin{bmatrix} \dot{x}_A \\ \dot{x}_B \\ \dot{x}_D \end{bmatrix}}_{\dot{\boldsymbol{x}}} + \underbrace{\begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_4+k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}}_{[K]} \underbrace{\begin{bmatrix} x_A \\ x_B \\ x_D \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\boldsymbol{0}}.$$

The equation for m_B worked out at the start of this example corresponds to the second row of these matrices.

This form is convenient for numerical solution if it is written as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{v} \\ \dot{\boldsymbol{v}} &= -[M]^{-1}\big[[C]\boldsymbol{v} + [K]\boldsymbol{x}\big] \end{aligned}$$

For the three mass example this would represent 6 first-order differential equations.

Center of mass

For both theoretical and practical reasons it is often useful to pay attention to the motion of the average position of mass in the system. This average position is called the center-of-mass. For a collection of particles in one dimension the center-of-mass is

$$x_{\mathrm{CM}} = \frac{\sum x_i m_i}{m_{\mathrm{tot}}}, \tag{9.55}$$

where $m_{tot} = \sum m_i$ is the total mass of the system. The velocity and acceleration of the center-of-mass are found by differentiation to be

$$v_{CM} = \frac{\sum v_i m_i}{m_{tot}} \quad \text{and} \quad a_{CM} = \frac{\sum a_i m_i}{m_{tot}}. \tag{9.56}$$

If we imagine a system of interconnected masses and add the $\vec{F} = m\vec{a}$ equations from all the separate masses we can get on the left hand side only the forces from the outside; the interaction forces cancel because they come in equal and opposite (action and reaction) pairs. So we get:

$$\sum F_{external} = \sum a_i m_i = m_{tot} a_{CM}. \tag{9.57}$$

So the center-of-mass of a system (a system that may be deforming wildly) obeys the same simple governing equation as a single particle. Although our demonstration here was for particles in one dimension. The result holds for any bodies of any type in 1,2, or 3 dimensions.

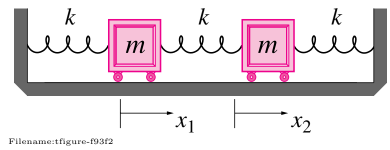


Figure 9.49: A two mass system. We define x_1 and x_2 so that the system is in equilibrium when $x_1 = x_2 = 0$.

Normal modes

Systems with many moving parts often move in complicated ways. Consider the two mass system shown in fig. 9.49. By drawing free body diagrams and writing linear momentum balance for the two masses we can write the equations of motion in matrix form (see eqn. (9.54)) as

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0}$$

where

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}.$$

Example: Complicated motion.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we get the motion shown in fig. 9.50a. Both masses move in a complicated way and not synchronously with each other.

On the other hand, all such systems, if started in just the right way, will move in a simple way.

Example: Simple motion: a normal mode.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in fig. 9.50b. Both masses move in a simple sine wave, synchronously and *in phase* with each other.

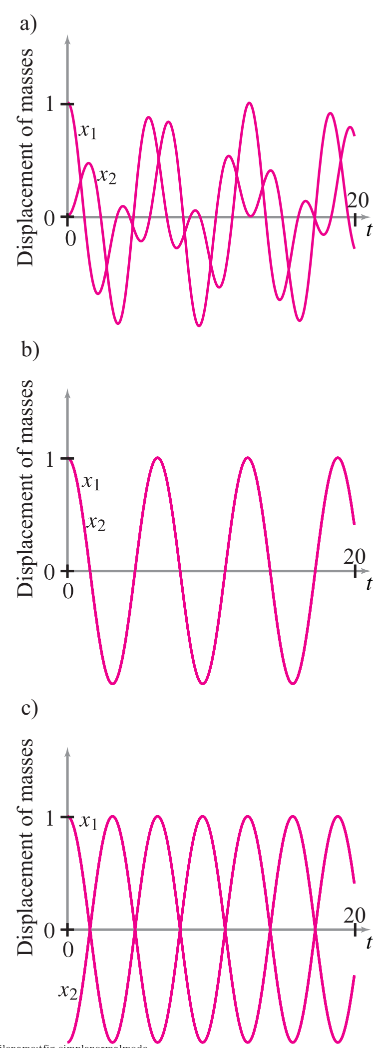


Figure 9.50: Motions of the masses from fig. 9.49 for three different initial conditions, all released from rest ($v_1 = v_2 = 0$)
a) $x_1 = 1,\ x_2 = 0$,
b) $x_1 = 1,\ x_2 = 1$, and
c) $x_1 = 1,\ x_2 = -1$.

That this system has this simple motion is intuitively apparent. If both of the equal masses are displaced equal amounts both have the same restoring force. So both move equal amounts in the ensuing motion. And nothing disturbs this symmetry as time progresses. In fact the frequency of vibration is exactly that of a single spring and mass (with the same k and m).

A given system can have more than one such simple motion.

Example: Another normal mode.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in fig. 9.50c. Both masses move in a simple sine wave, synchronously and exactly *out of phase* with each other. Being exactly out of phase is actually a form of being exactly in phase, but with a negative amplitude.

This motion is also intuitive. Each mass has restoring force of $3k\Delta x$. One k from a spring at the end and $2k$ because each mass experiences a spring with half the length (and thus twice the stiffness) in the middle (because the middle of the middle spring doesn't move in this symmetric motion).

The system above is about the simplest for demonstration of *normal mode* vibrations. But more complicated elastic systems always have such simple normal mode vibrations.

All elastic systems with mass have *normal mode* vibrations in which all masses

- have simple harmonic motion
- with the same frequency as all the other masses, and
- exactly in (or out) of phase with all of the other masses

Thus the first and second normal modes from fig. 9.50b,c can be written as

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_1 t \\ \cos\lambda_1 t \end{bmatrix}}_{\text{First normal mode}} \quad \text{and} \quad \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_2 t \\ -\cos\lambda_2 t \end{bmatrix}}_{\text{Second normal mode}}$$

where, by the physical reasoning in the examples we know that $\lambda_1 = \sqrt{k/m}$ and $\lambda_2 = \sqrt{3k/m}$. We could equally well have used the sine function instead of cosine.

Superposition of normal modes

Note that the governing equation (eqn. (9.4)) is 'linear' in that the sum of any two solutions is a solution. If we add the two solutions from fig. 9.50b,c we have a solution. And if we divide that sum by two we get a solution. And not just any solution, but the solution in fig. 9.50a. The top curve is the sum of the bottom two divided by two (The curves for $x_1(t)$ and $x_2(t)$ need to be added separately).

For more complicated systems it is not so easy to guess the normal modes. Most any initial condition will result in a complicated motion. Nonetheless the concept of normal modes applies to any system governed by the system of equations (eqn. (9.4)):

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0}.$$

Any collection of springs and masses connected any which way has normal mode vibrations. And because elastic solids are the continuum equivalent of a collection of springs and masses, the concept applies to all elastic structures. Here are the basic facts

- An elastic system with n degrees of freedom has n independent normal modes.
- In each normal mode i all the points move with the same angular frequency λ_i and exactly in phase.
- Any motion of the system is a superposition of normal modes (a sum of motions each of which is a normal mode).

Example: Musical instruments
The pitch of a bell is determined by that normal mode of the bell that has the lowest natural frequency. Similarly for violin and piano strings, marimba keys, kettle drums and the air-column in a tuba.

A recipe for finding the normal modes of more complex systems is given in box 9.8 on page 484.

Normal modes and single-degree-of-freedom systems

Any complex elastic system has simple normal mode motions. And all motions of the system can be represented as a superposition of normal modes. Hence sometimes we can think of every system as if it *is* a single degree of freedom system. For example, if a complex elastic system is forced, it will resononate if the frequency of forcing matches any of its normal mode (or natural) frequencies.

9.8 The math of, and how to find, normal modes

Consider a collection of n masses connected by springs whose motions are governed by eqn. (9.4)

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0},$$

where the positions of the masses are $\boldsymbol{x} = \boldsymbol{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]'$. The matrices $[M]$ and $[K]$ have to do with the masses and the network of springs, respectively. At this point in the book the examples are masses in a line, but the concepts are more general.

How to find a solution. The approach used by the professionals is to *guess* that there are "normal mode" solutions and then see if they are. A normal mode solution, with all masses moving sinusoidally and synchronously, is

$$\boldsymbol{x} = \begin{bmatrix} V_1 \cos\lambda t \\ V_2 \cos\lambda t \\ \vdots \end{bmatrix} = \boldsymbol{V} \cos\lambda t.$$

Upper case bold $\boldsymbol{V}$ (to distinguish it from lower case velocity) is a list of constants $[V_1, V_2, ...]'$. We could have used sin just as well as cos for our guess. Now we plug our guess into the governing equations to see if it is a good guess:

$$\begin{aligned} [M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} &= \boldsymbol{0} \\ [M]\frac{d^2}{dt^2}\{\boldsymbol{V}\cos\lambda t\} + [K]\{\boldsymbol{V}\cos\lambda t\} &= \boldsymbol{0} \\ -\lambda^2[M]\boldsymbol{V}\cos\lambda t + [K]\boldsymbol{V}\cos\lambda t &= \boldsymbol{0} \\ \left\{-\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V}\right\}\cos\lambda t &= \boldsymbol{0}. \end{aligned}$$

This equation has to hold true for all t therefore the constant column vector inside the brackets $\{\}$ must be zero:

$$\begin{aligned} -\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V} &= \boldsymbol{0} \\ \left[-\lambda^2[M] + [K]\right]\boldsymbol{V} &= \boldsymbol{0} \end{aligned}$$

The matrix $[M]$ is usually invertible. If $[M]$ is diagonal its inverse is $[M]$ with each element replaced by its reciprocal. Assuming $[M]^{-1}$ exists we can multiply through by $[M]^{-1}$ to get:

$$[M]^{-1}[K]\boldsymbol{V} = \lambda^2\boldsymbol{V},$$

where we used that $[M]^{-1}[M] = [1]$ = the identity matrix, and that $[1]\boldsymbol{V} = \boldsymbol{V}$. Defining the product $[B] = [M]^{-1}[K]$ and substituting we get the classic eigenvalue problem:

$$[B]\boldsymbol{V} = \lambda^2\boldsymbol{V}. \tag{9.58}$$

There is a lot to know about eqn. (9.58). Its a famous equation. eqn. (9.58) says that $\boldsymbol{V}$ is a vector that, when multiplied by $[B]$ gives itself back again, multiplied by a constant. For the special vector $\boldsymbol{V}$, being multiplied by the matrix $[B]$ is equivalent to being multiplied by the scalar λ^2.

Given $[B]$ there are generally n linear independent *eigen vectors* $\boldsymbol{V}^1, \boldsymbol{V}^2, ...\boldsymbol{V}^n$ with associated *eigen values* $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. Note, $[B]$ is generally not symmetric.

In the case of our vibration problem the eigen vectors are called *modes* or *eigen modes* or *mode shapes* or *normal modes*.

Recipe for finding normal modes

Given the matrices $[M]$ and $[K]$ proceed as follows.

- Calculate $[B] = [M]^{-1}K$
- Use a math computer program to find the eigenvalues and eigenvectors of $[B]$, call these $\boldsymbol{V}^i$ and λ_i^2. Usually this is a single command, like:

```
eig(B)
```

- For each i between 1 and n write each normal mode as $\boldsymbol{x}(t) = \boldsymbol{V}\cos(\lambda_i t)$ or as $\boldsymbol{x}(t) = \boldsymbol{V}\sin(\lambda_i t)$

For example, if

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}.$$

then, for any values of k and m, the computer will return for the eigen values and eigenvectors of $[B] = [M]^{-1}[K]$:

$$\boldsymbol{V}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{with } \lambda_1^2 = k/m \text{ and } \boldsymbol{V}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{with } \lambda_2^2 = 3k/m$$

Why are they called 'normal' modes?

The math here is relatively advanced, so trust or skip it if you don't have the needed linear algebra background. In math speak 'normal' sometimes means orthogonal. Here is the sense that 'normal' modes are orthogonal to each other. First, the matrix $[M]$ is generally both symmetric, non-singular and even positive definite, so $[M]$

- Has a symmetric inverse $[M]^{-1}$ with
$$[M]^{-1}[M] = [M][M]^{-1} = [\mathbf{1}] = [\text{identity matrix}],$$
- Has a unique positive definite square root $\sqrt{[M]}$ with
$$\sqrt{[M]}\sqrt{[M]} = [M],$$
- Has a unique inverse square root $[M]^{-1/2}$ with
$$[M]^{-1/2}[M][M]^{-1/2} = [\mathbf{1}].$$

First we use $[M]^{-1/2}$ to change coordinates from $\boldsymbol{x}$ to $\boldsymbol{y}$ as

$$\boldsymbol{x} = [M]^{-1/2}\boldsymbol{y} \quad \text{or} \quad \boldsymbol{y} = [M]^{1/2}\boldsymbol{x}.$$

Now we substitute this into the basic vibration equation ($[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0}$) and pre-multiply the whole equation by $[M]^{-1/2}$ to get

$$\underbrace{[M]^{-1/2}[M][M]^{-1/2}}_{[1]}\ddot{\boldsymbol{y}} + \underbrace{[M]^{-1/2}[K][M]^{-1/2}}_{[A]}\boldsymbol{y} = \boldsymbol{0}$$
$$\Rightarrow \quad \ddot{\boldsymbol{y}} + [A]\boldsymbol{y} = \boldsymbol{0}.$$

Now we look for solutions for $\boldsymbol{y}$ exactly as we did for $\boldsymbol{x}$ before. But, as opposed to $[B]$, $[A]$ is symmetric. So $[A]$ has n linear independent and mutually orthogonal *eigen vectors* $\boldsymbol{W}^1, \boldsymbol{W}^2, ...\boldsymbol{W}^n$ with associated *eigen values* $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. These $\boldsymbol{W}_i$ give the $\boldsymbol{V}_i$ by $\boldsymbol{V}_i = [M]^{-1/2}\boldsymbol{W}_i$. The λ_i are the same.

These coordinate-changed $\boldsymbol{W}_i = [M]^{1/2}\boldsymbol{V}_i$ are 'normal' (mutually orthogonal) but the more physical $\boldsymbol{V}_i$ are not, even though the $\boldsymbol{V}_i$ are called 'normal' modes. Or you can say that the $\boldsymbol{V}_i$ are mutually orthogonal with respect to the weighting $[M]$, *e.g.,* $\boldsymbol{V}_2'[M]\boldsymbol{V}_5 = 0$.

SAMPLE 9.21 For the given quantities and initial conditions, find $x_1(t)$ and $x_2(t)$. Assume the spring is unstretched when $x_1 = x_2$.

$$m_1 = 1\,\text{kg}, \quad m_2 = 2\,\text{kg}, \quad k = 3\,\text{N/m}, \quad c = 5\,\text{N/(m/s)}$$
$$x_1(0) = 1\,\text{m}, \quad \dot{x}_1(0) = 0, \quad x_2(0) = 2\,\text{m}, \quad \dot{x}_2(0) = 0.$$

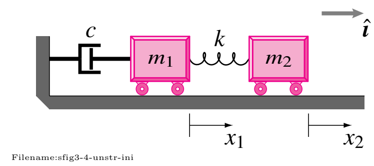


Figure 9.52:

Solution The free body diagrams of all components of the given system are shown below.

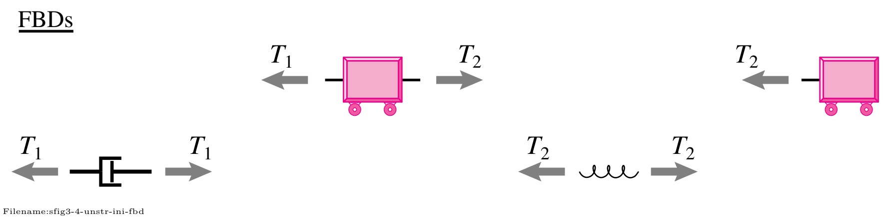


Figure 9.51:

The spring and dashpot laws give

$$T_1 = c\dot{x}_1 \qquad T_2 = k(x_2 - x_1). \tag{9.59}$$

The linear momentum balance for the two masses gives

$$\begin{aligned} & \sum \vec{F} = m\vec{a} \\ \text{mass 1:} \quad & -T_1\hat{\imath} + T_2\hat{\imath} = m_1\ddot{x}_1\hat{\imath} \\ \text{mass 2:} \quad & -T_2\hat{\imath} = m_2\ddot{x}_2\hat{\imath}. \end{aligned} \tag{9.60}$$

Applying the constitutive laws (9.59) to the momentum balance equations (9.60) gives

$$\begin{aligned} \ddot{x}_1 &= [k(x_2 - x_1) - c\dot{x}_1]/m_1 \\ \ddot{x}_2 &= [-k(x_2 - x_1)]/m_2. \end{aligned}$$

Defining $z_1 = x_1, z_2 = \dot{x}_1, z_3 = x_2, z_4 = \dot{x}_2$ gives

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= [k(z_3 - z_1) - cz_2]/m_1 \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= [-k(z_3 - z_1)]/m_2. \end{aligned}$$

The initial conditions are

$$z_1(0) = 1\,\text{m}, \quad z_2(0) = 0, \quad z_3(0) = 2\,\text{m}, \quad z_4(0) = 0.$$

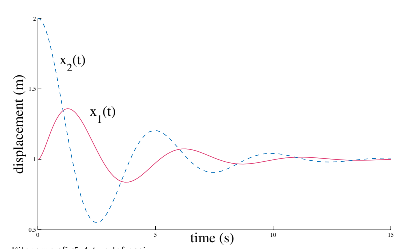


Figure 9.53: Plot of $x_1(t)$ and $x_2(t)$ obtained from the numerical solution of the equations of motion. Note that with $x_1(0) = 1\,\text{m}$ and $x_2(0) = 2\,\text{m}$, both masses settle down to $x_1 = x_2 = 1\,\text{m}$ after the oscillations die.

We are now set for numerical solution. Solving these equations numerically, we plot $x_1(t)$ and $x_2(t)$ as shown in fig. 9.53. From the solution, it is clear that both the masses settle down to the equilibrium position $x_1 = x_2 = 1\,\text{m}$ after the oscillations die down. In this position, the spring exerts no force as it is unstretched. Also note that the two masses move in the opposite direction immediately after set into motion as they must because of the opposite accelerations.

SAMPLE 9.22 A two mass vibratory MEMS gyroscope: A vibratory MEMS (microelectromechanical system) gyroscope employs two big plates as inertial masses, suspended by thin beams or 'springs' as shown in the figure. The two masses are made to vibrate (by electrical actuation) out of phase in the x-direction. Any rotation about the y-direction causes the masses to vibrate out of plane due to 'Coriolis acceleration' (you will learn about that in later chapters). We will restrict our attention to the planar motion of the gyroscope. A two degree of freedom spring-mass model is shown in the figure where $m = 34.5 \times 10^{-9}$ kg, $k_1 = 25\,\text{N/m}$, and $k_2 = 3\,\text{N/m}$.

1. Write the equations of motion for the two masses.
2. For the out of phase motion of the two masses, assume that $x_1(t) = -x_2(t) = x_0 \sin \lambda_n t$. Determine the natural frequency λ_n corresponding to this mode of vibration.

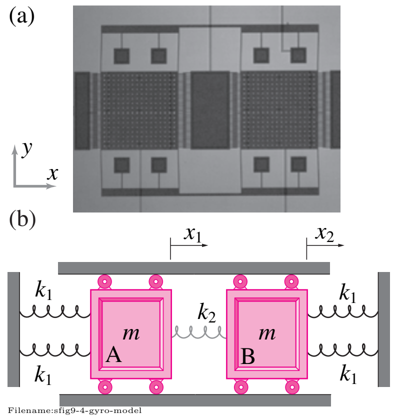


Figure 9.54: A MEMS vibratory gyroscope: (a) a micrograph of the two-mass structure. Each inertial mass (the plates with holes) is approximately 1 mm × 1 mm × 15μ m. The beams that hang the two masses act as springs. The comb drives on the left and right side of the two masses are used to drive the two masses to oscillate in the x-direction at their resonant frequencies. (b) a two degree of freedom spring-mass model of the driven gyroscope structure (without any damping).

Solution

1. The free body diagram of each mass is shown in shown in fig. 9.55. Assuming both x_1 and x_2 to be positive to in the x-direction, and $x_2 > x_1$ at the instant shown in the figure, we can write the equations of motion using the balance of linear momentum as

$$\begin{aligned}
\text{Mass A:} \quad m\ddot{x}_1 &= k_2(x_2 - x_1) - 2k_1 x_1 \\
&= -(2k_1 + k_2)x_1 + k_2 x_2 \\
\text{Mass B:} \quad m\ddot{x}_2 &= -k_2(x_2 - x_1) - 2k_1 x_2 \\
&= k_2 x_1 - (2k_1 + k_2)x_2.
\end{aligned}$$

These two equations can be also written in a convenient matrix form as

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1 + k_2) & k_2 \\ k_2 & -(2k_1 + k_2) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{9.61}$$

$$m\ddot{x}_1 = -(2k_1 + k_2)x_1 + k_2 x_2, \quad m\ddot{x}_2 = k_2 x_1 - (2k_1 + k_2)x_2$$

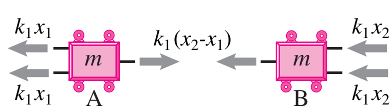


Figure 9.55: Partial free body diagram of the two masses. Only relevant forces (in the x-direction) are shown in the diagram.

2. The out of phase normal mode of vibration of the two masses is such that $x_1(t) = x_0 \sin \lambda_n t$ and $x_2 = -x_0 \sin \lambda_n t$, *i.e.,* the two masses have out of phase displacements ($x_1 = -x_2$). If we substitute these values of the displacements, we see that both equations turn out to be the same and they give,

$$-\lambda_n^2 x_0 \sin \lambda_n t = \frac{1}{m}(-2k_1 - k_2 - k_2)x_0 \sin \lambda_n t$$

from which it follows that,

$$\lambda_n = \sqrt{\frac{2(k_1 + k_2)}{m}}.$$

Substituting the given values of m, k_1, and k_2, we get,

$$\lambda_n = \sqrt{\frac{2(25 + 3)\,\text{N/m}}{34.5 \times 10^{-9}\,\text{kg}}} = 40.29 \times 10^3\,\text{rad/s}.$$

Thus the natural frequency corresponding to the out of phase vibration mode is 40.29×10^3 rad/s which corresponds to $f_n = \lambda_n / 2\pi = 6.4$ kHz.

$$f_n = 6.4\text{kHz}$$

SAMPLE 9.23 Normal modes from eigen analysis: Consider the two-mass MEMS gyroscope of Sample 9.22 again. Using the equations of motion derived in Sample 9.22,

1. Find the natural frequencies and the corresponding normal modes of vibration of the system.
2. Using initial conditions based on the normal modes, solve the equations of motion numerically and plot $x_1(t)$ and $x_2(t)$ together for each normal mode. From the plots, show that the time period of oscillation conforms to the natural frequencies found above.

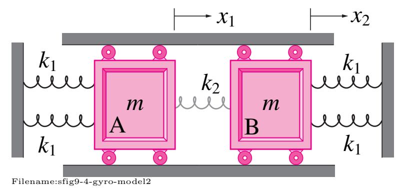


Figure 9.56: The two degree of freedom model of a two-mass MEMS vibratory gyroscope. Here, $m = 34.5 \times 10^9$ kg or 34.5μg, $k_1 = 25\,\mathrm{N/m}$, and $k_2 = 3\,\mathrm{N/m}$.

Solution

1. The equations of motion for the two degree of freedom model were obtained in eqn. (9.61) and are reproduced here:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1+k_2) & k_2 \\ k_2 & -(2k_1+k_2) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{9.62}$$

Let us assume a normal mode of vibration in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\sin\lambda_n t$$

where λ_n is the natural frequency of the system. Substituting this assumed motion in eqn. (9.62) and getting rid of $\sin\lambda_n t$ from both sides, we get,

$$-\lambda_n^2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1+k_2) & k_2 \\ k_2 & -(2k_1+k_2) \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Rearranging this equation a little bit, we can write it as **[A] V** = λ **V**, the standard eigenvalue problem, where $\lambda (= \lambda_n^2)$ is the eigenvalue of the matrix [**A**] and **V** is the corresponding eigenvector. Here,

$$\mathbf{A} = \begin{bmatrix} (2k_1+k_2)/m & -k_2/m \\ -k_2/m & (2k_1+k_2)/m \end{bmatrix}.$$

Now, we can go to a computer and find the eigenvalues and eigenvectors of [**A**]:

```
m = 34.5*10^(-9),  k1 = 25,  k2 = 3
A = [(2*k1+k2)/m     -k2/m;
     -k2/m           (2*k1+k2)/m]
lambda = eigenvalues(A)
v = eigenvectors(A)
```

By carrying out this computation, using appropriate commands in a computational package, we find the following two eigenvalues and the corresponding two eigenvectors:

$$\lambda^{(1)} = 1.449 \times 10^9, \qquad \mathbf{V}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

$$\text{and} \quad \lambda^{(2)} = 1.623 \times 10^9, \qquad \mathbf{V}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now, since we know that $\lambda = \lambda_n^2$, we can find the natural frequencies of our system by taking the square root of the eigenvalues just found. Thus,

$$\lambda_n^{(1)} = 3.807 \times 10^4\,\mathrm{rad/s} \quad \Rightarrow \quad f_n^{(1)} = \lambda_n^{(1)}/2\pi = 6.06\,\mathrm{kHz},$$

$$\text{and} \quad \lambda_n^{(2)} = 4.029 \times 10^4\,\mathrm{rad/s} \quad \Rightarrow \quad f_n^{(2)} = \lambda_n^{(2)}/2\pi = 6.41\,\mathrm{kHz}.$$

The corresponding normal modes or mode shapes are given by $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$. Please note that the components of an eigenvector are determined relative to each other, that is, the absolute numerical values are not unique, and any multiple of an eigenvector is also an eigenvector. For example, you could find $\mathbf{V}^{(1)} = [\sqrt{2} \quad \sqrt{2}]^T$, or $\mathbf{V}^{(1)} = [1/\sqrt{2} \quad 1/\sqrt{2}]^T$.

$$\lambda_n^{(1)} = 3.807 \times 10^4 \text{ rad/s}, \quad \mathbf{V}^{(1)} = [1 \quad 1]^T$$
$$\lambda_n^{(2)} = 4.029 \times 10^4 \text{ rad/s}, \quad \mathbf{V}^{(2)} = [1 \quad -1]^T$$

2. The normal modes thus found indicate that as long as we set the initial conditions for the two masses in the same proportion as one of the mode shapes (eigenvectors), the two masses will vibrate synchronously with the same frequency (corresponding to the chosen mode shape). So, we now simulate the motion of the two masses by solving the equations of motion numerically, using appropriate initial conditions.

We first write the equations of motion as a set of first order equations:

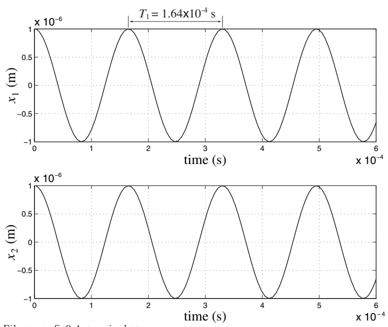
$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{u}_1 &= -\frac{(2k_1 + k_2)}{m} x_1 + \frac{k_2}{m} x_2 \\
\dot{x}_2 &= u_2 \\
\dot{u}_2 &= \frac{k_2}{m} x_2 - \frac{(2k_1 + k_2)}{m} x_2.
\end{aligned}$$

For the first mode, we set the initial conditions $x_1(0) = x_2(0) = 1\mu$m corresponding to the first eigenvector $\mathbf{V}^{(1)} = [1 \quad 1]^T$. Now, we are ready to solve the equations numerically.

```
ODEs = {x1dot = u1,
        u1dot = -(2*k1+k2)/m * x1 + k2/m * x2,
        x2dot = u2,
        u2dot = k2/m * x1 - (2*k1+k2)/m * x2}
ICs   = {x1(0)=1E-6, u1(0)=0, x2(0)=1E-6, u2(0)=0}
Set m = 34.5E-9, k1 = 25, k2 = 3,
Solve ODEs with ICs for t=0 to t=0.6E-3
Plot x1(t) and x2(t)
```

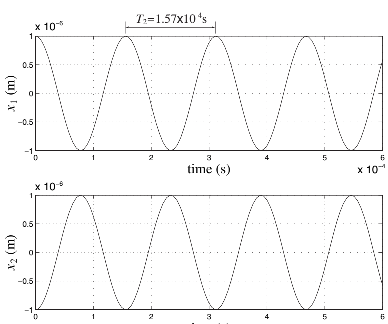
Note that we are solving the equations for only 0.6 milliseconds, that is, less than a millisecond. This is because we already know that the frequency is very high, roughly about 6 kHz, which means we can get six oscillations in one millisecond. The plot of $x_1(t)$ and $x_2(t)$ are shown in fig. 9.57. From this plot, we find that the time period of one oscillation is approximately 1.64×10^{-4} seconds, which gives a frequency of $\lambda_n = 2\pi/T = 3.8 \times 10^4$ rad/s.

Similarly, using the initial conditions $x_1(0) = 1\mu$m, and $x_2(0) = -1\mu$m (corresponding to the second eigenvector $\mathbf{V}^{(2)}$, we get the plot shown in fig. 9.58. From this plot, we find that the time period of one oscillation is approximately 1.57×10^{-4} seconds, which gives a frequency of $\lambda_n = 2\pi/T = 4 \times 10^4$ rad/s. Thus, the results of the numerical solution match the results obtained from the eigenvalue analysis.



Filename:sfig9-4-gyro-inphase

Figure 9.57: Motions of mass A and mass B, $x_1(t)$ and $x_2(t)$ in the first normal mode. Note that both masses move together (in phase) and oscillate with the same frequency, $\lambda_n^{(1)}$. In this mode, the middle spring, k_2 plays no role as it remains unstretched at all times $x_2(t) - x_1(t) = 0$ for all t.



Filename:sfig9-4-gyro-outofphase

Figure 9.58: Motions of mass A and mass B, $x_1(t)$ and $x_2(t)$ in the second normal mode. In this mode, each mass moves opposite to the other (out of phase) but both oscillate with the same frequency, $\lambda_n^{(2)}$.

SAMPLE 9.24 Flight of a toy hopper. A hopper model ① is made of two masses $m_1 = 0.4\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a spring with stiffness $k = 100\,\text{N/m}$ as shown in fig. 9.59. The unstretched length of the spring is $\ell_0 = 1\,\text{m}$. The model is released from rest from the configuration shown in the figure with $y_1 = 25.5\,\text{m}$ and $y_2 = 24\,\text{m}$.

1. Find and plot $y_1(t)$ and $y_2(t)$ for $t = 0$ to $2\,\text{s}$.
2. Plot the motion of m_1 and m_2 with respect to the center-of-mass of the hopper during the same time interval.
3. Plot the motion of the center-of-mass of the hopper from the solution obtained for $y_1(t)$ and $y_2(t)$ and compare it with analytical values obtained by integrating the center-of-mass motion directly.

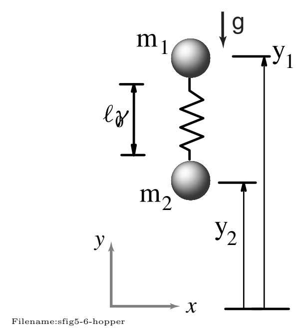


Figure 9.59:

① You can think of it as a model of a hopping robot or a hopping animal where the foot mass is modeled with m_2 and the body mass with m_1. The spring models the elasticity of the system. If you restrict this model to one dimensional vertical motion under gravity, it will hop, once released from some initial height. The number of hops depends on the intial conditions. Theoretically, it can hop forever with suitable initial conditions.

Solution The free-body diagrams of the two masses are shown in fig. 9.60. From the linear momentum balance in the y direction, we can write the equations of motion at once.

$$
\begin{aligned}
m_1\ddot{y}_1 &= -k(y_1 - y_2 - \ell_0) - m_1 g \\
\Rightarrow \quad \ddot{y}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \qquad (9.63) \\
m_2\ddot{y}_2 &= k(y_1 - y_2 - \ell_0) - m_2 g \\
\Rightarrow \quad \ddot{y}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g. \qquad (9.64)
\end{aligned}
$$

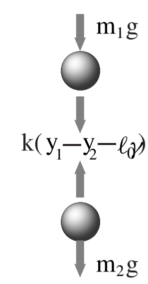


Figure 9.60: Free body diagram of the two masses m_1 and m_2

1. The equations of motion obtained above are coupled linear differential equations of second order. We can solve for $y_1(t)$ and $y_2(t)$ by numerical integration of these equations. As we have shown in previous examples, we first need to set up these equations as a set of first order equations.

   Letting $\dot{y}_1 = v_1$ and $\dot{y}_2 = v_2$, we get

$$
\begin{aligned}
\dot{y}_1 &= v_1 \\
\dot{v}_1 &= -\frac{k}{m_1}(y_1 - y_2) + \frac{k\ell_0}{m_1} - g \\
\dot{y}_2 &= v_2 \\
\dot{v}_2 &= \frac{k}{m_2}(y_1 - y_2) - \frac{k\ell_0}{m_2} - g.
\end{aligned}
$$

   Now we solve this set of equations numerically using some ODE solver and the following pseudocode.

```
ODEs = {y1dot = v1,
        v1dot = -k/m1*(y1-y2-l0) - g,
        y2dot = v2,
        v1dot = k/m1*(y1-y2-l0) - g}
IC   = {y1(0)=25.5, v1(0)=0, y2(0)=24, v2(0)=0}
Set  k=100, m1=0.4, m2=1, l0=1
Solve ODEs with IC for t=0 to t=2
Plot y1(t) and y2(t)
```

   The solution obtained thus is shown in fig. 9.61.

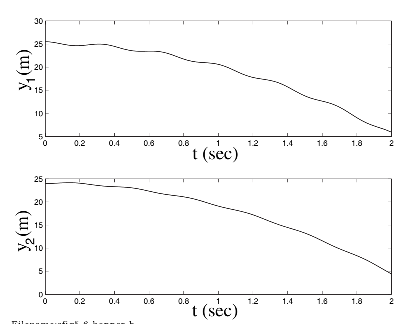


Figure 9.61: Numerically obtained solutions $y_1(t)$ and $y_2(t)$

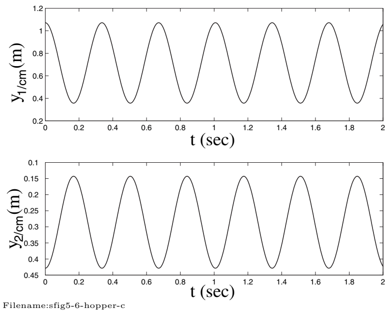


Figure 9.62: Numerically obtained solutions $y_{1/\mathrm{cm}}(t)$ and $y_{1/\mathrm{cm}}(t)$.

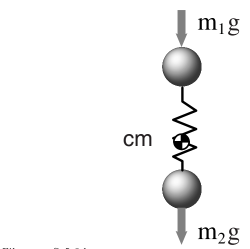


Figure 9.63: Free body diagram of the hopper as a single system. The spring force does not show up here since it becomes an internal force to the system

2. We can find the motion of m_1 and m_2 with respect to the center-of-mass by subtraction the motion of the center-of-mass, y_{cm} from y_1 and y_2. Since,

$$y_{\mathrm{cm}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} \tag{9.65}$$

we get,

$$\begin{aligned} y_{1/\mathrm{cm}} &= y_1 - y_{\mathrm{cm}} = \frac{m_2}{m_1 + m_2}(y_1 - y_2) \\ y_{2/\mathrm{cm}} &= y_2 - y_{\mathrm{cm}} = -\frac{m_1}{m_1 + m_2}(y_1 - y_2). \end{aligned}$$

The relative motions thus obtained are shown in fig. 9.62. We note that the motions of m_1 and m_2, as seen by an observer sitting at the center-of-mass, are simple harmonic oscillations.

3. We can find the center-of-mass motion $y_{\mathrm{cm}}(t)$ from y_1 and y_2 by using eqn. (9.65). The solution obtained thus is shown as a solid line in fig. 9.64. We can also solve for the center-of-mass motion analytically by first writing the equation of motion of the center-of-mass and then integrating it analytically.

The free-body diagram of the hopper as a single system is shown in fig. 9.63. The linear momentum balance for the system in the vertical direction gives

$$\begin{aligned} (m_1 + m_2)\ddot{y}_{\mathrm{cm}} &= -m_1 g - m_2 g \\ \Rightarrow \quad \ddot{y}_{\mathrm{cm}} &= -g. \end{aligned}$$

We recognize this equation as the equation of motion of a freely falling body under gravity. We can integrate this equation twice to get

$$y_{\mathrm{cm}}(t) = y_{\mathrm{cm}}(0) + \dot{y}_{\mathrm{cm}}(0)t - \frac{1}{2}gt^2.$$

Noting that $y_{\mathrm{cm}}(0) = 24.43\,\mathrm{m}$ (from eqn. (9.65)), and $\dot{y}_{\mathrm{cm}}(0) = 0$ (the system is released from rest), we get

$$y_{\mathrm{cm}}(t) = 24.43\,\mathrm{m} - \frac{1}{2}\cdot 9.81\,\mathrm{m/s^2}\cdot t^2.$$

The values obtained for the center-of-mass position from the above expression are shown in fig. 9.64 by small circles.

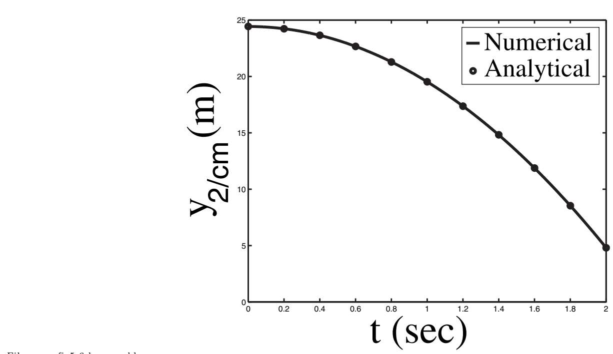


Figure 9.64: Numerically obtained solution for the position of the center-of-mass, $y_{\mathrm{cm}}(t)$.

SAMPLE 9.25 Conservation of linear momentum. Mr. P with mass $m_p =$ 200 lbm is standing on a cart with frictionless and massless wheels. The cart weighs half as much as Mr. P. Standing at one end of the cart, Mr. P spots an interesting object at the other end of the cart. Mr. P decides to walk to the other end of the cart to pick up the object. How far does he find himself from the object after he reaches the end of the cart?

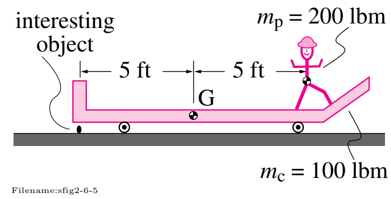


Figure 9.66: Mr. P spots an interesting object.

Solution From your own experience in small boats perhaps, you know that when Mr. P walks to the left the cart moves to the right. Here, we want to find how far the cart moves.

Consider the cart and Mr. P together to be the system of interest. The free-body diagram of the system is shown in Fig. 9.65(a).

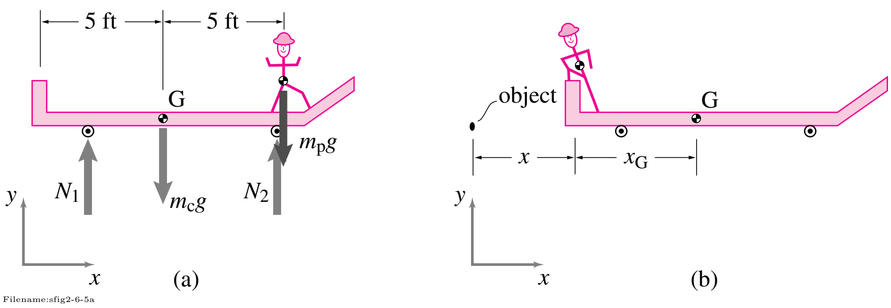


Figure 9.65: (a) Free-body diagram of Mr. P-and-the-cart system. (b) The cart has moved to the right by distance x when Mr. P reaches the other end.

From the diagram it is clear that there are no external forces in the x-direction. Therefore,

$$\dot{L}_x = \sum F_x = 0 \quad \Rightarrow \quad L_x = \text{constant}$$

that is, the linear momentum of the system in the x-direction is 'conserved'. But the initial linear momentum of the system is zero. Therefore,

$$L_x = m_{tot}(v_{cm})_x = 0 \text{ all the time} \quad \Rightarrow \quad (v_{cm})_x = 0 \text{ all the time.}$$

Because the horizontal velocity of the center-of-mass is always zero, the center-of-mass does not change its horizontal position. Now let x_{cm} and x'_{cm} be the x-coordinates of the center-of-mass of the system at the beginning and at the end, respectively. Then,

$$x'_{cm} = x_{cm}.$$

Now, from the given dimensions and the stipulated position at the end in Fig. 9.65(b),

$$x_{cm} = \frac{m_c x_G + m_p x_p}{m_c + m_p} \quad \text{and} \quad x'_{cm} = \frac{m_c(x_G + x) + m_p x}{m_c + m_p}.$$

Equating the two distances we get,

$$\begin{aligned} m_c x_G + m_p x_p &= m_c(x_G + x) + m_p x \\ &= m_c x_G + x(m_c + m_p) \\ \Rightarrow \quad x &= \frac{m_p x_p}{m_c + m_p} \\ &= \frac{200\,\text{lbm} \cdot 10\,\text{ft}}{300\,\text{lbm}} = 6\frac{2}{3}\text{ ft.} \end{aligned}$$

$\boxed{6.67\text{ ft}}$

[Note: if Mr. P and the cart have the same mass, the cart moves to the right the same distance Mr. P moves to the left.]

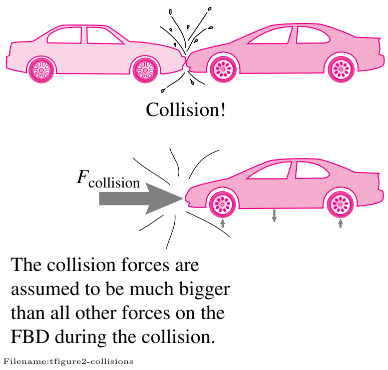


Figure 9.67: Here cars are shown colliding. A free body diagram of the right car shows the collision force and should not show other forces which are negligibly small. Here they are shown as negligibly small forces to give the idea that they may be much smaller than the collision force. The wheel reaction forces are neglected because of the spring compliance of the suspension and tires.

9.5 Collisions in 1D

Sometimes things interact in a sudden manner, like two cars in a head-on crash or a dropped cell-phone hitting the floor. Some sudden interactions are intentional, for example in sports the banging of racquets, bats, clubs, sticks, hands and legs with balls, pucks and bodies. And in machines there are sometimes intentionally sudden interactions like the clicking of a ratchet and the flip of an electric light switch. More esoteric 'sudden' interactions include those between subatomic particles in an accelerator and near passes of satellites with planets.

When two solids bump into each other a nearly discontinuous change in their velocities and/or angular velocities is needed to keep the bodies from interpenetrating. This sudden change in velocity demands large interaction. In the case of subatomic particles near nuclei and satellites near planets there might be no contact, but none-the-less there are large forces when the interaction distances get small. Estimating the effects of these large yet short-lived forces is the central problem in collision mechanics.

Two objects are said to *collide* when some interaction force or moment between them becomes so large that other forces acting on the bodies become negligible. For example, in a car collision the force of interaction at the bumpers may be many times the weight of the car or the reaction forces acting on the wheels. And so short acting that, although velocities change, positions change negligibly during the collision.

Collisional free body diagrams The analysis of collisions is a little different than the analysis of smooth motions, but still depends on free body diagrams (See fig. 9.67). Knowing which forces to include and which to ignore in a collisional free-body-diagram is a subtle issue. Some rules of thumb:

- ignore forces from gravity, springs, and at places where contact is broken in the collision, and

- include forces at places where new contact is made, or where contact is maintained.

The elementary analysis of rigid body collisions is based on these ideas:

I. Collision forces are big, so non-collisional forces are neglected in collisional free body diagrams.

II. Collision forces are of short duration, so the position and orientation of the colliding bodies do not change during the collision.

What happens during a collision

During a collision between what would generally be called "rigid" bodies *things get wild*. There are huge contact forces and stresses in the regions near the nominally① contacting points, there could be plastic deformation, fracture, and frictional slip. Elastic waves may travel all over the body, reflect and scatter this way and that. Altogether the contact interaction during the collision is the result of very complex deformations (see fig. 9.68).

Deformations (the lack of rigidity) give rise to the forces between colliding bodies. So what could the phrase "rigid-object collisions" mean? It is an oxymoron. Trying to understand the collision forces in detail, and how they are related to deformations, is way beyond this book. Actually, there is no unified theory of collisions so you can't read about it in any book. Loosely one might imagine that during part of the collision material is being squeezed, this is called the *compression* phase and later on it expands back in a *restitution* phase. But the realities of collisions are not necessarily so simple; the forces and deformations can vary in complex ways.

Soon after the collision, however, the vibrations often die out, each object may have negligible permanent change in shape, and the object returns to motions that are well described by rigid-object kinematics. To find out the net effect of the collision forces we use this one key idea:

III. The laws of mechanics apply *during* collisions even though rigid-object kinematics does not.

While the motions during a collision may be wildly complex, the general linear and angular momentum balance laws are still applicable. Rather than applying these laws to understand the details *during* a collision, we use them to summarize the overall result of the collision.

That is, in rigid-object collision analysis we do not pay attention to how the forces vary in time, or to the detailed trajectories, velocities or accelerations of any material points. Rather, we focus on the net change in the velocities of the colliding bodies that the collision forces cause. Thus, instead of using the differential-equation form of the linear momentum balance, angular-momentum balance and energy equations (Ia, IIa, and IIIa from the inside front cover) we use the time integrated forms (Ib, IIb, and IIIb). All that we note about a collisional force is its net impulse

$$\vec{\boldsymbol{P}}_{\text{coll}} = \int_{\text{collision time}} \vec{\boldsymbol{F}}_{\text{coll}}\, dt$$

in terms of which we have, for one object experiencing this impulse at point C

$$\begin{aligned} \vec{\boldsymbol{P}}_{\text{coll}} &= \Delta \vec{\boldsymbol{L}}, && (9.66)\\ \vec{\boldsymbol{r}}_{\text{C}/0} \times \vec{\boldsymbol{P}}_{\text{coll}} &= \Delta \vec{\boldsymbol{H}}_{/\text{O}}, \quad \text{and} && (9.67)\\ \text{Collisional dissipation} &= \Delta E_{\text{K}}. && (9.68) \end{aligned}$$

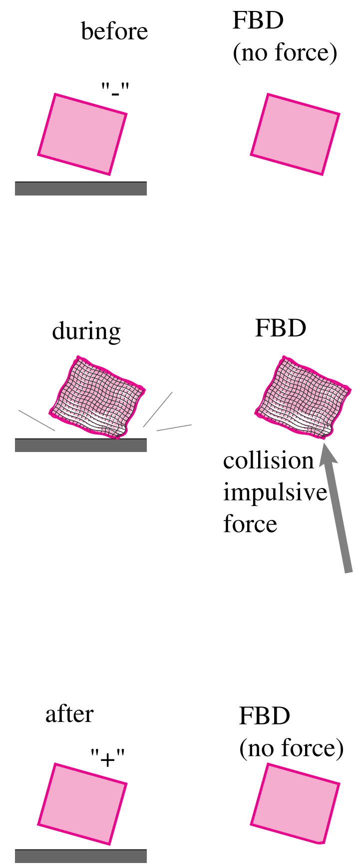


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Figure 9.68: Just before a collision is called "-", and just after is called "+". The only forces that show on a collisional free body diagram are those that are large and part of the impact. Either a force or an impulse may be shown. This figure exaggerates the difference between the before (-) and after (+) states. In analysis we assume that there is no change in the body's position or orientation from just before to just after the collision. The only net changes caused by the collision are the body's velocity and rotation rate.

① Nominally means "in name". That is, what one calls "contacting points" are not points at all, but regions of complex interaction.

Most often the first two of these, the impulse-momentum equations are used to find the motion after collision. The energy equation is just a check to make sure that the collisional dissipation is positive (otherwise the collision would be an energy source).

Extra assumptions are needed

The momentum balance equations, with the assumptions already discussed, are never enough in themselves to determine the outcome of a collision. The extra assumptions come in various forms. To minimize the algebra we discuss the issues first with one-dimensional collisions.

One dimensional collisions

Here we only consider collisions in the context of one-dimensional mechanics: all motion is constrained to one direction of motion by forces which we ignore. Only momentum and forces in, say, the $\hat{\imath}$ direction are included.

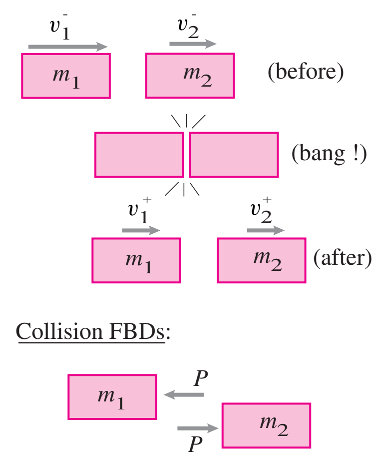

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Figure 9.69: Before the collisions the masses have velocities to the right of v_1^- and v_2^-. After the collision the velocities are v_1^+ and v_2^+. During the collision the impulse P acts to the right on mass 2 and to the left on mass 1.

Example: 1-D collisions

Consider two masses which collide along their common line of motion. All velocities and momenta are positive if to the right and P is the impulse *on* mass 2 from mass 1. The relevant impulse-momentum relations are

$$\begin{aligned}
\text{For mass 1} \quad -P &= m_1(v_1^+ - v_1^-),\\
\text{For mass 2} \quad P &= m_2(v_2^+ - v_2^-), \text{ and}\\
\text{For the system} \quad 0 &= (m_1 v_1^+ + m_2 v_2^+) - (m_1 v_1^- + m_2 v_2^-).
\end{aligned}$$

The third equation comes from a free body diagram of the system (ie, conservation of momentum) or by adding the first two equations. In any case, given the masses and initial velocities we have only two independent equations and we have three unknowns: v_1^+, v_2^+ and P. Momentum balance is not enough to determine the outcome of a collision.

To "close" (make solvable) the set of equations one needs to make extra assumptions.

Sticking collisions

The simplest assumption is that the masses *stick* together after the collision so

$$v_1^+ = v_2^+.$$

Such a collision is sometimes called a *perfectly plastic*, a *perfectly inelastic*, or a *dead* collision. Algebraic manipulations of the momentum equations and the "sticking" constitutive law give

$$\begin{aligned}
v_1^+ = v_2^+ &= (m_1 v_1^- + m_2 v_2^-)/m_{\text{tot}} \qquad (\text{where } m_{\text{tot}} = m_1 + m_2) \quad \text{and}\\
P &= (v_1^- - v_2^-) m_{\text{coll}} \qquad (\text{where } m_{\text{coll}} = \frac{m_1 m_2}{m_1 + m_2}).
\end{aligned}$$

The *collisional mass* or *contact mass* m_{coll}

$$m_{\text{coll}} = \frac{m_1 m_2}{m_1 + m_2} = \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}$$

is not the mass of anything. It is just a quantity that shows up repeatedly in collision calculations and theory. It is the reciprocal of the sum of the reciprocals of the two masses. If one mass is much bigger than the other, the contact mass is $m_{\text{coll}} \approx$ the smaller of the two masses. It is the proportionality constant relating the interaction force and the relative acceleration of the particles during the collision

$$m_{\text{coll}}(a_2 - a_1) = F \qquad \text{(with } F \text{ being the force of body 1 on body 2)}$$

and is thus related to the effective mass of box 12.1 on page 611.

More general 1-D collisions

The momentum equations can be re-arranged to better get at the essence of the situation which is that

- In the collision the system's center-of-mass velocity is unchanged, and
- The effect of the collision is to change the difference between the two mass velocities.

So we define the center-of-mass velocity v_{cm} and the velocity difference v_{rel} as

$$v_{\text{cm}} \equiv (m_1 v_1 + m_2 v_2)/m_{\text{tot}} \quad \text{and} \quad v_{\text{rel}} \equiv v_2 - v_1.$$

Note that before a collision the masses are approaching each other so $v_1^- > v_2^-$ and $v_{\text{rel}}^- < 0$. A little more algebra shows that for any P,

$$\begin{aligned} v_2^+ &= v_{\text{cm}} + \frac{m_1}{m_1 + m_2} v_{\text{rel}}^+, \\ v_1^+ &= v_{\text{cm}} - \frac{m_2}{m_1 + m_2} v_{\text{rel}}^+, \quad \text{and} \\ P &= (v_{\text{rel}}^+ - v_{\text{rel}}^-) m_{\text{coll}} \end{aligned}$$

That is, P acts on v_{rel} as if v_{rel} were the velocity of an object with mass m_{coll}. If $P = 0$ the equations above are a long winded way of saying that nothing happened, $v_1^+ = v_1^-$ and $v_2^+ = v_2^-$, and the masses pass right through each other.

If $P = -v_{\text{rel}}^- m_{\text{coll}}$ there is a sticking collision.

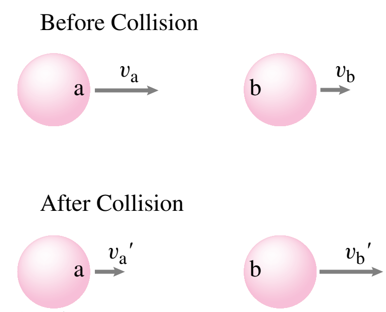


Figure 9.70: The rate of approach of two about-to-collide points is $v_a - v_b$. The rate of separation after collision is $v_b' - v_a'$. The simplest collision law says $v_b' - v_a' = e(v_a - v_b)$, where e is the coefficient of restitution.

Elastic collisions

Application of the above formulas will show that if

$$P = -2 v_{\text{rel}}^- m_{\text{coll}}$$

then the kinetic energy of the system after the collision is the same as the kinetic energy before. That is

$$\begin{aligned} {E_K}^+ &= {E_K}^- \\ \frac{m_1 {v_1^+}^2 + m_2 {v_2^+}^2}{2} &= \frac{m_1 {v_1^-}^2 + m_2 {v_2^-}^2}{2}. \end{aligned}$$

Also, $v_{\text{rel}}^+ = -v_{\text{rel}}^-$, the relative velocity maintains its magnitude and reverses its sign.

② A common mistake is to take e as a material property. It is not. e generally depends on the shapes and sizes of the contacting objects also (see box 9.9 on page 497).

③ **What did Newton say about collisions?** Newton swang spheres at the ends of string and banged them into each other and measured their bounce. He took account of air friction. At the point of this quote he has already discussed momentum conservation. Here is his statement of, and justification for, what we now call "Newton's law of collisions" (eqn. (9.69)):

"In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendula's bodies, and measuring their reflection, I determined the quantity of their elastic force; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity to which they met, as about 5 to 9. Balls of steel returned with almost the same velocity; those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. " (Newton's Principia Motte's translation revised, by Florian Cajori, Univ. of CA press, page 25, 1947)

The coefficient of restitution

We have that as P ranges from $-v_{\text{rel}}^- m_{\text{coll}}$ to $-2v_{\text{rel}}^- m_{\text{coll}}$, the collision ranges from sticking to an energy conserving reversal of relative velocities. The *coefficient of restitution e* is introduced as a way of interpolating between these cases. The most commonly used collision law can be summarized with this simple equation,

$$\overbrace{(v_b' - v_a')}^{\text{The speed with which colliding points are separating after the collision.}} = \underbrace{e}_{\text{The coefficient of restitution, assumed to be a constant for given materials.}} \overbrace{(v_a - v_b)}^{\text{The speed at which colliding points are approaching before collision.}}, \tag{9.69}$$

Or, more simply expressed, the collision law can be defined by either of the following two equations

$$\begin{aligned} v_{\text{rel}}^+ &= -e v_{\text{rel}}^- \quad \text{or} \\ P &= -(1+e) v_{\text{rel}}^- m_{\text{coll}}. \end{aligned}$$

If $e = 0$ we have a sticking collision. If $e = 1$ we have an energy conserving elastic collision. If e is between 0 and 1 the collision is somewhere between as dead and as alive as can be②. which can be summarized as, *the rate of separation is proportional to the rate of approach.* The coefficient e is called Newton's (see box 9.5) or Poisson's coefficient of restitution. Somewhat of a miracle is that a given pair of objects seems to have a coefficient of restitution that is roughly independent of the velocities. This is the result of a conspiracy by all kinds of deformation mechanisms that we don't really understand. But that e is a constant for a given pair of bodies is only an approximation that has roughly the same status (accuracy) as, say, the friction coefficient. Much lower status than the momentum balance equations.
③

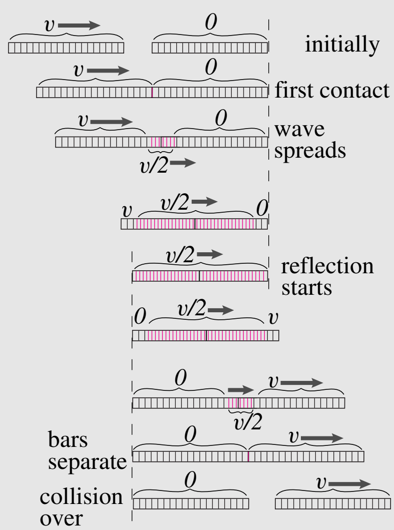
9.9 The axial collision of elastic rods: the unusual disappearance of vibrations

This box is not related to the skills covered in this book. It is an aside for those wondering how things work.

One approach to understanding collisions is to look at the stresses and deformations *during* the collision. This leads to the solution of partial differential equations. The material behavior needed to define those equations is usually not that well understood. So, hard as it is to solve such equations, even on a computer, the solution can be far from reality.

But to get a sense of things one can study an ideal system. The simple system we look at here was somewhat controversial amongst the great 19th century scientists Cauchy, Poisson and Saint-Venant (so said E.J. Routh in 1905).

Two identical linear elastic rods. Imagine two identical uniform linear elastic rods with length ℓ. The right one is stationary and the left one approaches it with speed v.



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No matter how the rods shake and vibrate, their elastic potential energy plus kinetic energy is constant.

Using reasoning beyond this book (see the paragraph for experts at the end of this box) one can explain this collision in detail, as illustrated in the sketches above. The pictures exaggerate the compression in the bar (For most materials the compression wouldn't be visible).

First the left rod moves like a rigid body towards the still rod at the right. Then contact is made and a compressional sound wave starts off spreading to the left and right. Behind the wave fronts is compressed material moving at speed $v/2$ to the right. To the right of the right wave front the material is still. To the left of the left-moving wave front the material is still moves at v. When the wave fronts meet the ends of their respective bars, the bars are compressed and all material is going to the right at $v/2$. Then both wave-fronts reflect off the ends of the bars and head back towards the contact point. To the left of the right-moving wave front (on the left bar) the material is still and uncompressed. To the right of the left-moving wave front (on the right bar) the material is uncompressed but moving to the right at speed v . Finally, the waves meet in the center and the bars separate. The right bar is now uniformly moving to the right at speed v and the left bar is still.

The result of this collision is that all of the momentum of the left bar is transferred to the right bar. The separation velocity is equal in magnitude to the approach velocity. The coefficient of restitution e is 1, and the kinetic energy of the system is the same after the collision as it was before.

Note that the collision itself was quick. The wave-fronts move at the speed of sound, about $1000\,\mathrm{m/s}$ for metals. So for 1 meter metal rods the collision takes a few thousandths of a second. But during that few thousandths of a second, the initial energy was partitioned into elastic strain energy and kinetic energy in different time-changing regions of the bar.

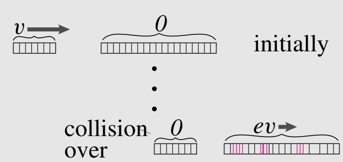
Despite all the complicated details, the elastic bars lead to the prediction of an 'elastic' collision. Maybe this is not surprising.

An elastic rod hits a rigid wall If you drop a 3 foot wooden dowel straight down on a thick concrete or stone floor it bounces quite well. Why? A wave analysis like that described above shows that a wave travelling from the first contact at the floor travels up the top and reflecting back to the bottom, leaving the rod moving uniformly up after the collisions just as fast as it was moving down before. Of course a wooden dowel is not perfectly described by the simple wave theory. And the ground is not perfectly rigid. So a real dowel's collision is not perfectly elastic.

But again we find that if we assume an elastic material that we predict an elastic collision. Again, no surprise. But the previous two examples are completely misleading!

Actually these are maybe *the only* examples where a detailed elastic theory predicts an elastic collision. More commonly the details are more like the next example.

Rods of different length If the rods have length ℓ_1 and $\ell_2 > \ell_1$ then the collision works out differently.



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When the reflection from the left end of the left rod comes back to the contact point, the rods separate. The left rod is stationary but the right rod has waves moving up and back. The average speed of the right rod is $(\ell_1/\ell_2)v$ so the effective coefficient of restitution is $e = \ell_1/\ell_2 < 1$. Later, after the vibrations have died out, the energy of the system will be less than initially. Or, even if the waves don't die out, the kinetic energy that can be accounted for in rigid-body mechanics is lost to remnant vibrations. Thus a totally elastic system leads to *inelastic* collisions. It is wrong to think that the restitution constant e depends on material; it also depends on the shapes and sizes of the objects. The amount of vibrational energy left after contact ends depends on shape and size.

For experts only: the wave equation In one-dimensional linear elasticity the displacement u to the right, of a point at location x on one or the other rod follows this partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}.$$

That is, the collision mechanics in detail is the finding of $u(x,t)$ that solves the wave equation above with the given initial conditions (one bar is moving the other isn't) and the boundary conditions (the ends of the bars have no stresses but when where they are in contact where they can have equal compressive stresses). The solution is most easily found by constructing right and left going waves that add to meet the initial conditions and boundary conditions (Routh).

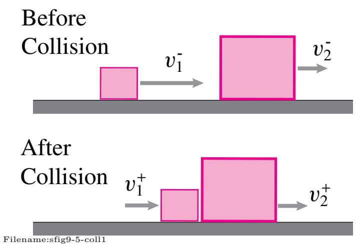

Figure 9.71:

SAMPLE 9.26 Collision without energy loss: A block of mass $m_1 = 2\,\text{kg}$ moves with speed $v_1 = 0.5\,\text{m/s}$ along the x-axis on a frictionless level ground behind another block of mass $m_2 = 10\,\text{kg}$ moving at a speed $v_2 = 0.2\,\text{m/s}$ in the same direction. The first block collides with the second block. Given that there is no loss of energy in this collision, find the speeds of the two blocks immediately after the collision.

Solution We are given the speeds of two blocks (of known masses) just before the collision. It is also given that there is no loss of energy in the collision. We have to find the speed of the two masses immediately after collision.

We know that the linear momentum of the system consisting of the two blocks is conserved during the collision. Thus, if v_1^- and v_2^- are the speeds of the two masses just before the collision and v_1^+ and v_2^+ are their respective speeds immediately after the collision, then we have

$$m_1 v_1^+ + m_2 v_2^+ = m_1 v_1^- + m_2 v_2^- \tag{9.70}$$

Since there is no loss of energy in the collision, the energy of the system is conserved. Thus, $E^- = E^+$, or

$$\frac{1}{2}m_1(v_1^+)^2 + \frac{1}{2}m_2(v_2^+)^2 = \frac{1}{2}m_1(v_1^-)^2 + \frac{1}{2}m_2(v_2^-)^2. \tag{9.71}$$

Thus, we have two equations (eqn. (9.70) and eqn. (9.71)) in two unknowns, v_1^+ and v_2^+, and hence we can solve for them. It is now only a question in algebra. From eqn. (9.71), we have

$$\begin{aligned} m_1\left[(v_1^+)^2 - (v_1^-)^2\right] &= m_2\left[(v_2^-)^2 - (v_2^+)^2\right] \\ \Rightarrow\quad m_1(v_1^+ + v_1^-)(v_1^+ - v_1^-) &= m_2(v_2^- + v_2^+)(v_2^- - v_2^+) \end{aligned} \tag{9.72}$$

But, from eqn. (9.70), $m_1(v_1^+ - v_1^-) = m_2(v_2^- - v_2^+)$. Hence,eqn. (9.72) simplifies to

$$\begin{aligned} v_1^+ + v_1^- &= v_2^+ + v_2^- \\ \Rightarrow\quad v_1^+ - v_2^+ &= v_2^- - v_1^-. \end{aligned} \tag{9.73}$$

Multiplying the above equation by m_1 and subtracting from eqn. (9.70), we get

$$\begin{aligned} (m_1 + m_2)v_2^+ &= 2m_1 v_1^- + v_2^-(m_2 - m_1) \\ \Rightarrow\quad v_2^+ &= \frac{2m_1}{m_1+m_2}v_1^- + \frac{m_2 - m_1}{m_1 + m_2}v_2^-. \end{aligned}$$

Now substituting the given values, $m_1 = 2\,\text{kg}$, $m_2 = 10\,\text{kg}$, $v_1^- = 0.5\,\text{m/s}$ and $v_2^- = 0.2\,\text{m/s}$ above, we get $v_2^+ = 0.3\,\text{m/s}$. Further, substituting the values of v_2^+ in eqn. (9.73), we get $v_1^+ = 0$, *i.e.,* the first mass comes to a halt!

$$\boxed{v_1^+ = 0 \text{ and } v_2^+ = 0.3\,\text{m/s}}$$

Comments: : Note that rather than using energy conservation equation directly as we did above, we could have used the given energy information t o set $e = 1$ (perfectly elastic collision) in eqn. (9.69) to get $v_2^+ - v_1^+ = -v_2^- + v_1^-$ (rather than deriving it as we did above). We can then solve this equation along with eqn. (9.70) to solve for v_1^+ and v_2^+.

SAMPLE 9.27 Estimating peak force in a collision: A metal ball of mass $m = 0.5\,\text{kg}$ strikes a stationary surface S_1 with velocity $\vec{v} = 10\,\text{m/s}\hat{\imath}$ and rebounds with velocity $\vec{v} = -9\,\text{m/s}\hat{\imath}$. In a different experiment the same ball strikes another stationary surface S_2 with the same initial velocity and has the same rebound velocity. The contact time during the two experiments were different: 0.1 s and 0.001 s respectively. Assuming that the collisional force between the ball and the two surfaces can be modeled as $F(t) = \frac{F_0}{2}(1 + \cos\frac{2\pi t}{T})$ (see fig. 9.72) where $-T/2 \le t \le T/2$ and T is the contact time, find the peak force F_0 in each case.

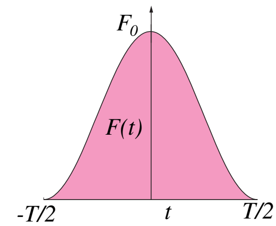

Figure 9.72:

Solution Let the collisional impulse acting on the ball be $\vec{P}$ (see fig. 9.73) given by

$$\vec{P} = \int_{-T/2}^{T/2} \vec{F}(t)dt.$$

From impulse-momentum relationship, we have

$$\vec{P} = \Delta\vec{L} = m\Delta\vec{v}.$$

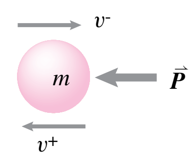

Figure 9.73: Free body diagram of the ball during collision with collisional impulse $\vec{P}$ acting on it.

Since in the case of each surface, $\Delta\vec{v}$ is the same ($\vec{v}^+ - \vec{v}^- = -19\,\text{m/s}\hat{\imath}$), the change in linear momentum $\Delta L = m\Delta\vec{v}$ is also the same. Hence, the impulse acting on the ball in each case has to be the same. Now, let $\vec{P}_1$ and $\vec{P}_2$ be the impulses acting on the ball during the collision with surface S_1 and S_2 respectively. Then,

$$\begin{aligned}
\vec{P}_1 &= -\int_{T_1/2}^{T_1/2} \vec{F}_1(t)dt\,\hat{\imath} \\
&= -\int_{-T_1/2}^{T_1/2} \frac{(F_0)_1}{2}\left(1 + \cos\frac{2\pi t}{T_1}\right) dt\hat{\imath} \\
&= -\frac{(F_0)_1 T_1}{2}\,\hat{\imath}
\end{aligned}$$

Similarly,

$$\vec{P}_2 = -\frac{(F_0)_2 T_2}{2}\,\hat{\imath}.$$

Now, setting $\vec{P}_1 = \Delta L$, we get

$$\begin{aligned}
-\frac{(F_0)_1 T_1}{2}\hat{\imath} &= -m\Delta v\,\hat{\imath} \\
\Rightarrow \quad (F_0)_1 &= \frac{2m\Delta v}{T_1} \\
&= \frac{2\cdot 0.5\,\text{kg}19\,\text{m/s}}{0.1\,\text{s}} \\
&= 1.9\,\text{N}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(F_0)_2 &= \frac{2m\Delta v}{T_2} \\
&= \frac{2\cdot 0.5\,\text{kg}19\,\text{m/s}}{0.001\,\text{s}} \\
&= 190\,\text{N}.
\end{aligned}$$

Clearly, the peak force is inversely proportional to the collision time. In fact, it is easy to see that for the given model of the impulsive force, the peak force $F_0 = \frac{2m\Delta v}{T}$. Thus if the change in momentum is constant, then the peak force varies as $1/T$.

$$(F_0)_1 = 1.9\,\text{N} \quad \text{and} \quad (F_0)_2 = 190\,\text{N}$$

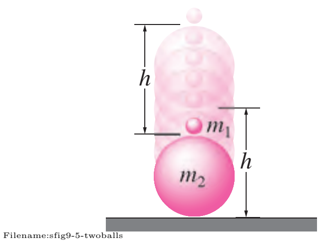

Figure 9.74:

SAMPLE 9.28 **A two-ball multiple collision experiment:** A tennis ball of approximate mass $m_1 = 60\,\text{gm}$ and a basketball of approximate mass $m_2 = 600\,\text{gm}$ are used in a fun collision experiment. The two balls are held in air, one on top of the other with a tiny gap between them, at a height h from the ground as shown in the figure. The two balls are released simultaneously from rest. The coefficient of restitution between the tennis ball and the basketball is $e_1 = 0.6$ and that between the basketball and the floor is $e_2 = 0.9$. Assume that the collision between the two balls takes place immediately after the basketball rebounds from the floor. Find the height of the tennis ball flight in terms of h as a result of the collision.

Solution We need to track two separate collisions here — one between the basketball and the floor, and second, between the tennis ball and the basketball. We can find the relevant vertical velocities before and after the collisions to determine the velocity of the tennis ball's flight which we can use to find the height of the flight. We will assume upward velocities to be positive.

Collision-1: Just before the basketball hits the floor, let its vertical velocity be v_2^- and let the tennis ball's speed at the same instant be v_1^-. Since both balls undergo free fall from height h before attaining these speeds, we have

$$v_1^- = v_2^- = -\sqrt{2gh}.$$

Now let v_2^+ be the speed of the basketball immediately after the collision with the ground (see fig. 9.75). Then,

$$v_2^+ = -e_2 v_2^- = e_2\sqrt{2gh}.$$

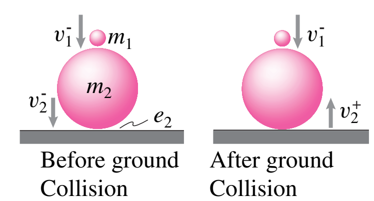

Figure 9.75: Collision of the basketball with the ground.

Collision-2: We assume that the second collision, the collision between the tennis ball and the basketball, takes place immediately after the first collision. Hence, the velocity of the tennis ball just before the collision with the basketball can be assumed to be $v_1^- = -\sqrt{2gh}$. The second collision is shown in fig. 9.76. The after impact velocities of the two balls are v_1^+ and v_2^{++}. Now, from collision law, we have

$$v_1^+ - v_2^{++} = -e_1(v_1^- - v_2^+) = -e_1(-\sqrt{2gh} - e_2\sqrt{2gh}) = \sqrt{2gh}e_1(1+e_2). \quad (9.74)$$

The conservation of linear momentum for the two-ball system gives

$$\begin{aligned} m_1 v_1^+ + m_2 v_2^{++} &= m_1 v_1^- + m_2 v_2^+) \\ \Rightarrow \quad v_1^+ + \frac{m_2}{m_1} v_2^{++} &= v_1^- + \frac{m_2}{m_1} v_2^+ \end{aligned}$$

Taking $M = m_2/m_1$, and substituting the values of v_1^- and v_2^+, we get

$$v_1^+ + M v_2^{++} = \sqrt{2gh}(Me_2 - 1). \quad (9.75)$$

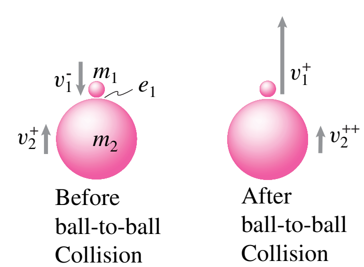

Figure 9.76: Collision of the tennis ball with the basketball.

Now solving eqn. (9.74) and eqn. (9.75) simultaneously, we get

$$v_1^+ = \frac{\sqrt{2gh}}{1+M}\left[M(e_1 + e_2 + e_1 e_2) - 1\right].$$

This is the velocity with which the tennis ball takes off on its vertical flight. Let the height of this flight be h_f. Then, from constant acceleration motion formula, we get $(v_1^+)^2 = 2gh_f$, or $h_f = (v_1^+)^2/2g$. Thus, from the derived expression for v_1^+ above, we get

$$h_f = \frac{h}{(1+M)^2}\left[M(e_1 + e_2 + e_1 e_2) - 1\right]^2.$$

Substituting $M = m_2/m_1 = 10$, $e_1 = 0.6$, and $e_2 = 0.9$ above, we get $h_f = 3.11h$. Thus the tennis ball flies off to three times its original height.

$$\boxed{h_f = 3.11h}$$

Note: From the expression obtained for v_1^+, we see that if M is very large then $v_1^+ = \sqrt{2gh}(e_1 + e_2 + e_1 e_2)$ and $h_f = (e_1 + e_2 + e_1 e_2)^2 h$.

9.6 Advanced vibrations: forcing and resonance

If the world of oscillators was as we have described them so far, especially in Section 9.3, there wouldn't be much to talk about. The undamped oscillators (of which there are none) would be oscillating away and the damped oscillators (all the real ones) would be damped out to no motion. The reason vibrations exist is because they are somehow excited. This excitement is also called *forcing* whether or not it is due to a literal mechanical force.

The most important idea of this section is the following

> If you shake something at about the same frequency at which it naturally oscillates you will eventually get large motions.

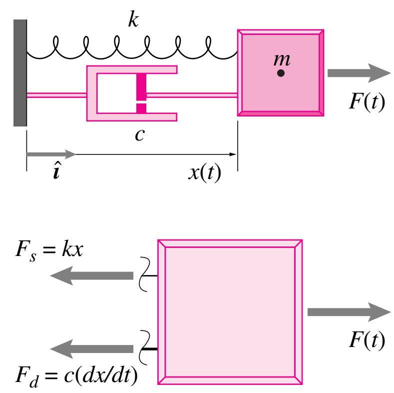

Figure 9.77: A forced mass-spring-dashpot is just a mass held in place by a spring and dashpot but pushed by a force $F(t)$ from some external source.

The rest of the section is largely a fleshing out of this idea.

The simplest example of a 'forced' harmonic oscillator is the mass-spring-dashpot system with an additional mechanical force applied to the mass. See fig. 9.77. Most of this section will be a study of this system. The governing equation for a forced damped oscillator can be derived from the free body diagram as follows, where vector notation helps keep the signs right:

$$\begin{aligned}
\sum \vec{F}_i &= m\vec{a} \\
-F_s\hat{\imath} - F_d\hat{\imath} + F(t)\hat{\imath} &= ma\hat{\imath} \\
\{\,(-kx - c\dot{x} + F(t))\,\hat{\imath} &= m\ddot{x}\hat{\imath}\,\} \\
\{\}\cdot\hat{\imath} \quad \Rightarrow \qquad -kx - c\dot{x} + F(t) &= m\ddot{x}
\end{aligned}$$

which is often re-arranged as

$$m\ddot{x} + c\dot{x} + kx = F(t). \qquad (9.76)$$

When $F(t) = 0$, there is no forcing and the governing equation reduces to that of the un-forced damped harmonic oscillator, eqn. (9.25).

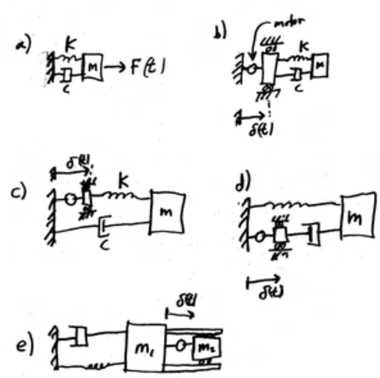

Figure 9.78: In all cases shown above the same forced oscillator eqn. (9.76) applies. In (a) a literal force is applied. In all the other cases the "forcing" is by a motor that moves something back and forth a distance δ. In (b) the support moves. In (c) and (d) just the spring or just the dashpot end is diplaced. In (e) an extra mass is moved relative to the main mass.

Equivalent ways to force an oscillator

There are many ways to "force" a system that all lead to the same forced-oscillator equation.

1. With a literal force as in fig. 9.77, shown again in fig. 9.78a.
2. By shaking the support, as in fig. 9.78b.
3. By displacing one end of the spring, but not the dashpot as in fig. 9.78c.
4. By displacing one end of the dashpot, but not the spring as in fig. 9.78d.

5. By displacing a second mass attached to the first with a motor that controls relative position, as in fig. 9.78e.

That these four systems all lead to the same governing equation follows from drawing free body diagrams, applying momentum balance, and collecting terms to match the form eqn. (9.76). Note that the meaning of some of the terms in the forced-oscillation equation is different for each system.

Types of forcing

In general this or that machine or structure could be forced in any number of complicated ways. But there are two special forcings of most common engineering interest:

- $F(t) = F_0$ (Constant force), and
- $F(t) = F \cos \lambda t$ (sinusoidal forcing).①

①Note again that we use λ (lambda) here instead of the more commonly used ω (omega) because we want to avoid confusion with the magnitude of angular velocity $\omega = |\vec{\boldsymbol{\omega}}|$ which itself could have oscillatory motion. It would be confusing (and bad math) to write $\omega = A \sin \omega t$ with ω having two different meanings in the same equation.

Constant force idealizes situations where the force doesn't vary much as due say, to gravity, a steady wind, or sliding dry friction. Sinusoidally varying forces are used to approximate oscillating forces as caused, say, by a vibrating support or earthquakes. Forces that are not sinusoidal can be thought of as sums of sine waves thus, in some sense, by knowing how a structure responds to sinusoidal forcing, at various frequencies, you know how it responds to all possible forcings②. Lets look at each of these two cases in detail.

②The best approximation of a function as a sum of sine waves is a Fourier series, a topic you learn in math, advanced physics or linear systems courses.

Forcing with a constant force

The case of constant forcing is both common and easy to analyze, so easy that it is often ignored (see fig. 9.27 on page 459). If $F(t) = F_0 =$ constant, then the general solution of eqnrefforcedODE for $x(t)$ is the same as the unforced case but with a constant added. The constant is F_0/k. The usual way of accommodating this case is to describe a new equilibrium point at $x = F_0/k$ and to pick a new deflection variable that is zero at that point. If we pick a new variable z defined as $z = x - F_0/k$, then substituting into eqn. (9.76) we get

$$m\ddot{z} + c\dot{z} + kz = 0, \tag{9.77}$$

which is the unforced oscillator equation. That is, constant forcing reduces to the case of no forcing if one merely changes what one calls zero to be the place where the mass is in equilibrium, taking account of the spring stretch (or compression) caused by the constant applied force. Thus the solution of the forced equation for x is equivalent to the unforced solution for z:

$$z(t) = x(t) - F_0/k = e^{(-\frac{c}{2m})t} \left(A \cos(\lambda_{\mathrm{d}} t) + B \sin(\lambda_{\mathrm{d}} t)\right) \tag{9.78}$$

where $\lambda_{\mathrm{d}} = \sqrt{\left(\frac{c}{2m}\right)^2 - k/m}$, as explained in box 9.7 on page 462.

An alternative approach is to use *superposition*. Here we say $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ satisfies $m\ddot{x} + c\dot{x} + kx = 0$ and $x_p(t)$ is any solution x_p of $m\ddot{x} + c\dot{x} + kx = F_0$. Any solution you like is called a "particular" solution. One easy solution is $x_p = F_0/k$. So the net solution is $x_p = F_0/k$ plus a solution x_h to the 'homogeneous' equation 9.77.

$$x(t) = \underbrace{e^{(-\frac{c}{2m})t}\left(A\cos(\lambda_d t) + B\sin(\lambda_d t)\right)}_{x_h} + \underbrace{F_0/k}_{x_{p1}} \qquad (9.79)$$

Example: Hanging mass.
The mass hanging from the support shown in fig. 9.79 obeys the equation

$$m\ddot{x} + c\dot{x} + kx = \underbrace{k\ell_0 + mg}_{F_0}$$

One particular solution x_p, the easiest one, has the mass hanging still. In this solution, the mass position is the unstretched length ℓ_0 of the spring plus the stretch of the spring due to gravity, $\Delta x = mg/k$. Because the mass is still in this solution, the dashpot constant c doesn't appear. So

$$x_p = \ell_0 + mg/k.$$

The homogeneous solution x_h is given by (9.78) and the general motion is the sum

$$\begin{aligned} x(t) &= x_p + x_h \\ &= (\ell_0 + mg/k) + e^{-\frac{ct}{2m}}\left(C\cos(\lambda_d t) + D\sin(\lambda_d t)\right) \end{aligned}$$

where C and D are constants determined by the initial conditions. For any initial condition and corresponding values of A and B, the motion eventually decays to the stationary particular solution with the mass hanging still (because the exponentials go to zero as $t \to \infty$).

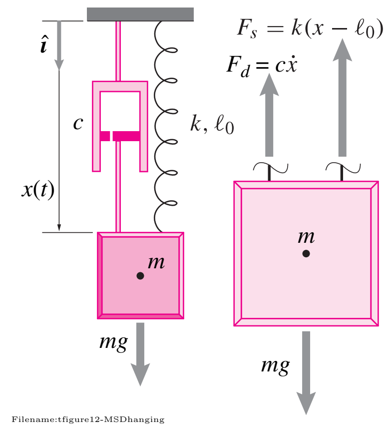

Figure 9.79: A mass hangs from a spring and dashpot. Its distance x from the support is due to the rest length ℓ_0 of the spring, and the stretch of the spring.

Forcing with a sinusoidally varying force

The motion resulting from sinusoidal forcing is of central interest in vibration analysis. In this case we imagine that $F(t) = F\cos pt$ where F is the amplitude of forcing and p is the angular frequency of the forcing. Note, we could just as well use $F(t) = F\sin pt$ for the forcing, sin and cos are both sinusoidal forcings.

The general solution of equation 9.76 is given by the sum of two parts. One is the general solution of equation 9.25, $x_h(t)$, and the other is *any* solution of equation 9.76, $x_p(t)$. The solution $x_h(t)$ of the damped oscillator equation 9.25 is called the 'homogeneous' or 'complementary' solution. Any solution $x_p(t)$ of the forced oscillator equation 9.76 is called a 'particular' solution.

We already know the solution $x_h(t)$ of the undamped governing differential equation 9.25. This solution is equation 9.26, 9.28, or 9.29, depending on the values of the mass, spring and damping constants. So the new problem is to find any solution to the forced equation 9.76. The easiest way to solve this (or any other) differential equation is to make a fortuitous guess (you may learn other methods in your math classes). In this case with

$$F(t) = F\cos(pt)$$

we make the guess that

$$x_p(t) = A\cos(pt) + B\sin(pt). \tag{9.80}$$

Basically this guess says "If you shake something with a sine wave it will probably move as a sine wave. But who knows the amplitude or phase?" Plugging this guess into the forced oscillator equation (9.76) we find values for A and B in box 9.11 on page 508.

Alternatively, a sum of sine waves can be written as a cosine wave (or sine wave) that has been shifted in phase as (see box 9.6 on page 456)

$$x_p(t) = A_0\cos(pt - \phi),$$

The value of forced amplitude is simply $A_0 = A^2 + B^2$ and is also given in terms of m, c, k, p and F in box 9.11. The forced amplitude A_0 is the central subject of this section. It answers the question 'How big are the oscillations when you shake something.?' Because the formula for A_0 is admittedly a mess, the answer is often given in a plot③. The general solution, therefore, is

$$x(t) = x_h(t) + x_p(t). \tag{9.81}$$

③Another more important reason that a plot is used is that often in a physical system one can measure the vibrations while never knowing a detailed accurate set of differential equations which would describe the system accurately.

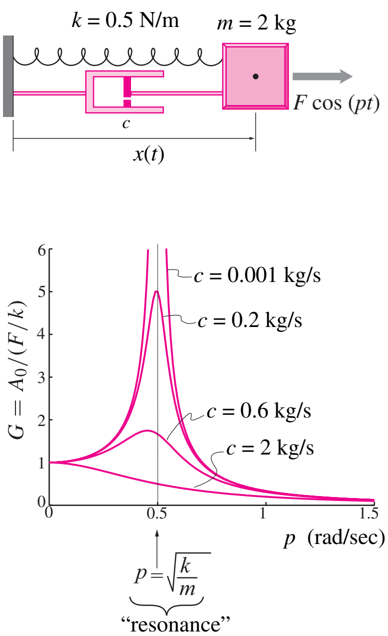

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Figure 9.80: Amplitude of oscillation vs forcing frequency for various dampings. Each curve shows the gain G vs the forcing frequency for a fixed damping. Note that when the damping is small ($c \ll 1$) and the forcing is close to the natural frequency of vibration $p \approx \lambda_n$ there is a 'resonant' peak in the amplitude of the response. The smaller the damping the higher and narrower is this peak. For very high damping the peak is at a slightly lower frequency. The mass-spring-dashpot system shown was used to generate the plots using the formulas from box 9.11 on page 508.

The homogeneous solution $x_h(t)$, the motion of the unforced system, is just decaying oscillations and is usually not of primary interest in vibrating systems. The particular solution $x_p(t)$ is steady oscillations. These oscillations are of central interest. In particular most often in engineering one wants these oscillations to be big or small.

Example: MEMs devices.
One general type of "**M**icro **E**lectronic **M**achine" consists of, basically, a vibrating beam. A beam with an effective mass 50μ gm and effective stiffness of $k = 500\,\mathrm{N/m} = 5\mu\,\mathrm{N}/\mu\,\mathrm{m}$ has

$$\lambda_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5\,\mathrm{N/m}}{50\cdot 10^{-9}\,\mathrm{kg}}} = \sqrt{\frac{\frac{500\mathrm{N}}{\mathrm{m}}\left(\frac{1\,\mathrm{kg\,m}/s^2}{1\mathrm{N}}\right)}{50\cdot 10^{-6}\,\mathrm{kg}}} = \sqrt{10^{10}\,\mathrm{s}^{-1}} = 10^5/\,\mathrm{s}$$

which corresponds to a frequency of $\lambda_n/2\pi \approx 15.7$k hz. That is, such a MEMs device would be a good receiver (or 'resonator') for 15.7k hz ultra-sonic vibrations. In this case resonance is useful to make the sensor sensitive.

The size of the oscillations scales with the size of the forcing F (this proportionality is known as 'linearity') and also depends on all the parameters m, c, k and p.

Frequency response and resonance

One way to show a structure's sensitivity to oscillatory loads is by a *frequency response* curve fig. 9.80. One curve shows the amplitude of vibrations vs the forcing frequency. The main idea of this section, resonance, shows as a peak in the frequency-response curve near the natural frequency $\lambda_n = \sqrt{k/m}$.

Recall that the natural frequency λ_n is the unforced frequency of undamped oscillation. The damped natural frequency λ_d, the frequency of decaying oscillations with damping present, is slightly slower (see, e.g., eqn. (9.26) on page 462.). The resonant frequency λ_{res}, the frequency of forcing for which the amplitude of motion is maximum (eqn. (9.11) on page 508), is slightly lower still. But, especially when the damping is low, there is only a small difference between the natural frequency, the damped frequency and the resonant frequency. So, in common language and engineering practice they are usually treated as one and the same.

In summary, the frequency response curve has a peak with forcing near to, but not exactly at, the natural frequency of unforced and undamped motion. But most engineers can reasonably assume, even though its not exactly true, that resonance occurs when the forcing frequency *is* the natural vibration frequency.

Resonance is good and bad

Sometimes an engineer studies vibrations with the hope of minimizing them, sometimes with the hope of maximizing them. Resonance is sometimes the problem and sometimes the solution.

Resonant vibrations are usually undesirable in machinery or cars. The vibrations can lead to large stresses, undesirable motions, or unpleasant sounds. A building resonating to earthquake vibrations may be more likely to fall down.

On the other hand, nuclear Magnetic Resonance imaging is used for medical diagnosis. In the old days, the resonant excitation of a clock pendulum was used to keep time. The resonance of quartz crystals is used to time most watches now-a-days. Self excited resonance is what makes musical instruments have such clear pitches. And resonant vibrations are used to give a larger signal in micro-mechanical sensors. In the electrical domain, radio tuners depend on resonance to pick out just one radio band.

Other systems

Most machines and structures are not exactly a point mass moving in one direction and constrained by a single spring and single dashpot. On the other hand, almost all machines have mass, elastic give, and some dissipation when they move. So most machines have natural oscillations after they are banged or disturbed somehow. And so most structures and machines can be shaken to large motions if the appropriate (or inappropriate, depending on your aims) frequency of force is applied④.

So the concepts introduced here for a single mass-spring-dashpot system apply to much more complex machines and structures. In particular, have natural vibration frequencies and they shake a lot (resonate) if forced at near those frequencies.

Experimental measurement

Because no real thing of interest is exactly a single mass-spring-dashpot the ideas of vibrations analysis are often not expressed in terms of (m, c and k). Rather, the more broad ideas of natural frequency, frequency response, and

④**Golden Gate bridge cables.** One Christmass day a few years ago one of us noted that the vertical cables on the Golden Gate bridge could be induced to oscillate quite visibly, maybe half a meter, if pushed by a person at the right frequency (about 0.5 hz). One cable, maybe 100 tons of steel, was shaking quite nicely when a police car pulled up and stopped. Through a megaphone and with an authoritative voice the officer sternly threatened "If you break it you have to pay for it."

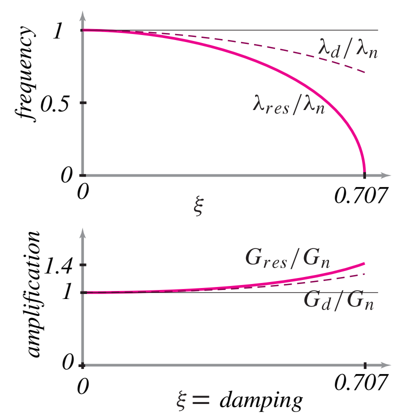


Figure 9.81: **Natural, damped and resonant frequencies.** The upper dashed plot shows the ratio of damped to natural frequency vs forcing frequency ratio ξ. The solid curve shows how the resonant frequency varies from the natural frequency as a function of frequency ratio. The lower plot shows how the amplification $G = A_0/(F/k)$ is nearly the same at the natural, damped and resonant frequencies. That the peak in the $c = 0.6\text{kg/s}$ curve of fig. 9.80 is slightly to the left of $p = \lambda_n = \sqrt{k/m}$ shows as $\lambda_{\text{res}}/\lambda_n$ being sligtly below 1 for $\xi \approx 0.3$. For low damping an engineer can treat the natural, damped and resonant frequencies as equal. Similarly the amplification when forced at the natural frequency is very close to the amplification when forcing is at the damped or at the resonant frequency.

resonance are considered on their own. Using either a large-scale computer model (say a 'finite-element' model) or measurement of the physical system itself, one can draw a frequency-response curve like fig. 9.80 on page 504.

Here's how. First, you a apply a sinusoidal force, say $F = F\cos(pt)$, to the structure at the point of interest. Then you measure the motion of a part of the structure of interest. You might instead measure a strain or rotation, but for definiteness let's assume you measure the displacement of some point on the structure δ.

If the structure is linear and has some damping, the eventual motion of the structure will eventually be a sinusoidal oscillation. In particular, you will measure that

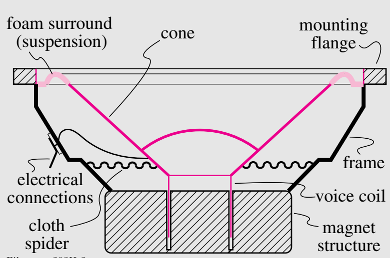
$$\delta = A_0 \cdot \cos(pt - \phi). \tag{9.83}$$

If you had applied half as big a force, you would have measured half the displacement, still assuming the structure is linear, so the ratio of the displacement to the force A_0/F is independent of the size of the force F. Let's define:

$$G = \frac{A_0}{F}$$

That is, the amplification gain G is the ratio of the amplitude of the displacement sine wave to the amplitude of the forcing sine wave. Plotting p on the x axis and G on the y axis, this experiment gives one point on the frequency response curve. Repeating for a range of forcing frequencies one can plot up the frequency response $G = G(p)$.

9.10 A Loudspeaker cone is a forced oscillator.



Cross-sectional view of a speaker.

A speaker, similar to the ones used in many home and auto speaker systems, is one of many devices which may be conveniently modeled as a one-degree-of-freedom mass-spring-dashpot system. A typical speaker has a paper or plastic cone, supported at the edges by a roll of plastic foam (the surround), and guided at the center by a cloth bellows (the spider). It has a large magnet structure, and (not visible from outside) a coil of wire attached to the point of the cone, which can slide up and down inside the magnet. (The device described above is, strictly speaking, the speaker driver. A complete speaker system includes an enclosure, one or more drivers, and various electronic components.) When you turn on your stereo, the amplifier forces a current through the coil in time with the music, causing the coil to alternately attract and repel the magnet. This rapid oscillation of attraction and repulsion results in the vibration of the cone which you hear as sound.

In the speaker, the primary mass is comprised of the coil and cone, though the air near the cone also contributes as 'added mass.' The 'spring' and 'dashpot' effects in the system are due to the foam and cloth supporting the cone, and perhaps to various magnetic effects. Speaker system design is greatly complicated by the fact that the air surrounding the speaker must also be taken into account. Changing the shape of the speaker enclosure can change the effective values of all three mass-spring-dashpot parameters. (You may be able to observe this dependence by cupping your hands over a speaker (gently, without touching the moving parts), and observing amplitude or tone changes.) Nevertheless, knowledge of the basic characteristics of a speaker (e.g., resonance frequency), is invaluable in speaker system design.

Our approximate equation of motion for the speaker is identical to that of the ideal mass-spring-dashpot above, even though the forcing is from an electromagnetic force, rather a than a direct mechanical force:

$$m\ddot{x} + c\dot{x} + kx = F(t) \text{ with } F(t) = \alpha i(t) \tag{9.82}$$

where $i(t)$ is the electrical current flow through the coil in amps, and α is the electro-mechanical coupling coefficient, in force per unit current.

Example: Shake table for earthquake response.
One way to get a frequency response curve for a building is to put a scale model on a "shake table". The base is then moved sinusoidally through a range of frequencies and the motion of the model is observed. This way one can find peaks in the frequency-response curve. These are frequencies that, to the extent they are prevalent in a feared earthquake, are likely to cause damage.

Transient response

As discussed, the full solution of eqn. (9.76) with forcing $F(t) = F_0 + F \cos pt$ is the sum of three terms

$$x(t) = x_h + x_{p1} + x_{p2}$$

The first of these has decaying oscillations, the second is a constant, and the third has steady oscillations. When added up the motion can look quite complicated, as seen in fig. 9.82. The main point is that after some initial complicated transient the motion eventually decays to steady oscillations ($x_{p2}(t) = A_0(\cos pt - \phi)$) plus an offset ($x_{p1} = F/k$).

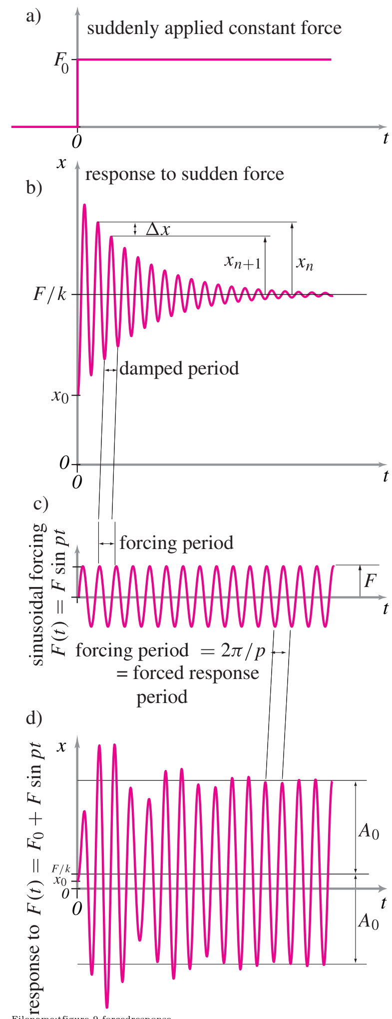


Figure 9.82: **Transient response.** (a) shows a suddenly applied force F_0. The response (b) to this force is a motion that starts at the initial position x_0 ($x_0 > 0$ in this illustration) and then oscillates about the new equilibrium F_0/k. The motion is identical to unforced motion, but offset. Thus it can be used to evaluate the rate of decay of oscillation and the damped period of oscillation. (c) A sinusoidal forcing causes. (d) the response if the mass is released at $x = x_0$ and suddenly both a constant force F and a sinusoidal force $F \sin pt$ are applied. The motion eventually settles into a sinusoidal oscillation at the forcing frequency (which is a little longer period than the damped oscillation in this illustration) with amplitude A_0.

9.11 Solution of the forced oscillator equation

The main equation for understanding forced oscillations is:

$$m\ddot{x} + c\dot{x} + kx = F_0 + F\cos pt.$$

Because the equation is linear we look for a solution which is the sum of three terms

$$x(t) = x_h + x_{p1} + x_{p2}$$

where x_h is the homogeneous solution from Eqns 9.26 - 9.29 on page 462, depending on whether the system is underdamped (oscillitory decay), critically damped or overdamped (non-oscillitory exponential decay). x_{p1} is a particular solution for the constant forcing F_0. $x_{p1}(t)$ was found in eqn. (9.79) on page 503 to be, simply, $x_{p1} = F_0/k$.

The last part of the solution, finding an x_{p2} for the forcing term $F\cos pt$ is found by guessing

$$x_{p2} = A\cos pt + B\sin pt.$$

When this guess is plugged into the equation

$$m\ddot{x} + c\dot{x} + kx = F\cos pt$$

every term is either a multiple of $\sin pt$ or $\cos pt$. Thus we get

$$\{\text{A collection of constants}\}\cos pt + \{\text{Another collection}\}\sin pt = 0$$

The only way a sum of a sine wave and cosine wave can be zero for all time is for both coefficients to be zero. Setting the two collections of constants above both to zero gives two simultaneous equations for the unknowns A and B in terms of m, c, k and p. These can be solved to give

$$A = \frac{(F/k)\left(1 - \frac{p^2}{(k/m)}\right)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2},$$

and

$$B = \frac{(F/k)(cp/k)}{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2}.$$

So we have found the particular solution for forcing with $F(t) = F\cos pt$, using A and B above, as

$$x_{p2} = A\cos pt + B\sin pt. \tag{9.84}$$

An alternative form for the solution is

$$x_{p2}(t) = A_0\cos(pt - \phi), \tag{9.85}$$

for which we can find the constants A_0 and ϕ using the trig identity $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$ described in box 9.6 on page 456. Applying this identity to the solution above we find the object of central interest, the forced amplitude

$$A_0 = \sqrt{(A^2 + B^2)} = \frac{F/k}{\sqrt{\left(\frac{c^2}{km}\right)\left(\frac{p^2}{k/m}\right) + \left(1 - \frac{p^2}{(k/m)}\right)^2}} \tag{9.86}$$

and also the phase angle

$$\phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{cp/k}{\left(1 - \frac{p^2}{(k/m)}\right)}\right). \tag{9.87}$$

All of the expressions above can be somewhat simplified if write them in terms of the *frequency ratio* $r = p/\lambda_{\text{n}} = p/\sqrt{k/m}$ and damping ratio $\xi = c/c_{\text{crit}} = c/2\sqrt{km}$ (The frequency ratio, damping ratio and some more specialized vibration words are defined on page 509.). Using these dimensionless quantities, the values of the constants in the solution $x_{p2}(t)$, namely eqn. (9.84) or eqn. (9.85), are:

$$A = \frac{(F/k)(1 - r^2)}{4\xi^2 r^2 + (1 - r^2)^2},$$

$$B = \frac{(F/k)(2\xi r)}{4\xi^2 r^2 + (1 - r^2)^2},$$

$$A_0 = \sqrt{(A^2 + B^2)} = \frac{F/k}{\sqrt{(2\xi r)^2 + (1 - r^2)^2}},$$

and

$$\phi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{2\xi r}{1 - r^2}\right).$$

These constants are for the particular forced solution $x_{p2}(t)$ of eqn. (9.84) or eqn. (9.85). Again, most important in all of this is the amplitude A_0 of the forced response. As you can see, the bottom of the fraction for A_0 gets quite small for small damping ($\xi \ll 1$) if the frequency ratio r is close to 1. That is,

> *the amplitude is big if the forcing is close to the natural frequency of vibration.*

Resonant frequency

In detail, the frequency at which the vibration amplitude A_0 is maximum is not exactly the unforced undamped natural frequency $\lambda_{\text{n}} = \sqrt{k/m}$. The resonant frequency λ_{res} is found by maximizing A_0 with respect to $r = \lambda/\lambda_{\text{n}}$. Setting $dA_0/dr = 0$ and solving for r we find

$$r_{\text{res}} = \sqrt{1 - 2\xi^2} \quad \Rightarrow \quad \lambda_{\text{res}} = \lambda_{\text{n}}\sqrt{1 - 2\xi^2} \tag{9.88}$$

The ratio of $\lambda_{\text{res}}/\lambda_{\text{n}}$ is plotted on fig. 9.80 on page 504. Also plotted is the ratio of A_0 at resonance to A_0 if forcing is at the natural frequency. The morals are that a) for small damping the natural frequency and resonant frequency are very close, and b) for all dampings, there is little error in calculating the amplitude of the maximum vibration response by approximating resonance as being at the natural frequency. Even when resonance is barely a viable concept, for systems that are critically damped, the error is only 40%.

Similarly one might think the damped natural frequency

$$\lambda_{\text{d}} = \lambda_{\text{n}}\sqrt{1 - \xi^2}$$

would be a better approximation to the resonant frequency. Actually, its about half way between the natural and resonant frequencies, as can be seen also on fig. 9.80.

9.12 The vocabulary of forced oscillations

Forced oscillations are so important and common that there is a specialized vocabulary for many of the terms and collections of commonly appearing terms. Here is a list, starting with the terms you know well.

m = the **mass** of the particle that is oscillating. For more complicated systems the mass m may represent an "effective" or "equivalent" mass.

c = the **damping coefficient**. c is used to describe the viscous drag, the resistence to motion $F_d = -c\dot{x}$.

k = the **spring constant**. k describes the elastic restoring "spring" force $F_s = -kx$.

F = the **forcing amplitude** for a sinusoidally varying applied force $F(t) = F \sin pt$ or $F(t) = F \cos pt$ or $F(t) = A \sin pt + B \cos pt$ with $F \equiv \sqrt{A^2 + B^2}$.

p = the **forcing frequency**. Some books will use the symbol ω for the forcing frequency.

The rest of the quantities below are completely determined by the quantities above (m, c, k, F and p).

$\lambda_\text{n} \equiv \sqrt{k/m}$ is the **natural frequency**. This is the frequency of oscillation if there is neither forcing nor damping. In that case $x(t) = A \cos \lambda_\text{n} t + B \sin \lambda_\text{n} t$. Many books use ω_n for the natural frequency.

$c_\text{crit} = 2 * \sqrt{km}$ is the **critical damping coefficient**. The relation of the actual damping c to the critical damping c_crit tells you whether a system is over-damped ($c > c_\text{crit}$ $\Rightarrow$ decay to equilibrium, when unforced, that is exponential) or under-damped ($c < c_\text{crit}$ $\Rightarrow$ decay to equilibrium, when unforced, that is oscillatory). See Fig. 9.30 on page 461. Sometimes c_crit is more simply written as c_c or c_{cr}.

$\xi \equiv c/c_\text{crit}$ is the **damping ratio**. The single number ξ ('ksee') tells you if a system is overdamped ($\xi > 1$) or underdamped ($\xi < 1$).

$r \equiv p/\lambda_\text{n} = p/\sqrt{k/m}$ is the **frequency ratio** . If $r > 1$ then the forcing is faster than the frequency of natural unforced vibrations. If $r < 1$ then the forcing is slower than the natural vibrations.

A_0 = the **response amplitude** . When a steady oscillatory force is applied the motion is eventually oscillatory. The amplitude of the motions is A_0, as in $x = A_0 cos(pt - \phi)$ with

$$A_0 = (F/k)/\sqrt{(2\xi r)^2 + (1 - r^2)^2)}.$$

$G \equiv A_0/(F/k)$ is the **gain** or **amplification**. G is the ratio of the eventual amplitude of the oscillator to the response that would occur if the same force was applied at zero frequency. It is the response amplitude scaled by the displacement that would occur if the same force was applied to a spring.

$\lambda_\text{res} = \lambda_\text{n}\sqrt{1 - 2\xi^2}$ is the **resonant frequency.** λ_res (also called λ_r or ω_r) is the frequency such that if $p = \lambda_\text{res}$ the amplification gain G is maximum. The resonant frequency is the frequency at which you force a system to get the biggest motions. The resonant frequency λ_res is rather close to the natural frequency λ_n in systems with small damping ratios. And these are also the systems that are prone to resonant vibrations.

λ_d is the **damped natural frequency** . If an underdamped system is released from rest it oscillates as the motions decay. The frequency of these oscillations is

$$\lambda_\text{d} = \lambda_\text{n}\sqrt{1 - \xi^2}.$$

The frequency λ_d of damped oscillations is a shade slower than the frequency λ_n of oscillation of the same system with no damping. When damping is small the natural frequency λ_n, the damped frequency λ_d and the resonant frequency λ_r are all close to each other (See fig. 9.80a).

G_n, G_res & G_d are the ampification **gains** (see G above) when forcing is at the natural, the resonant and the damped natural frequency respectively ($p = \lambda_\text{n}, \lambda_\text{res}$ & λ_d, see above). G_res is the biggest of these by definition. But it is not actually much bigger than G_n or G_d. These gains can be calculated using the formulas for G and A_0 above. They are plotted on fig. 9.80b.

D is the **logarithmic decrement**. D measures the rate of decay of unforced ($F = 0$) oscillations. The experimental definition, derivable from a graph of the motion, is

$$D = \ln(\frac{x_n}{x_{n+1}}).$$

In terms of m, c and k the logarithmic decrement is $D = \frac{cT}{2m} = 2\pi\xi$, as derived on page 462. If there is little damping, c is small ($\xi \ll 1$) and $D \approx (x_n - x_{n+1})/x_n$ is the fractional decrease in amplitude per oscillation. If $D = .1$ then each oscillation is about 10% smaller in amplitude than the previous one.

Q is the **quality factor**. For the mass-spring-dashpot system it is another way of describing the rate of decay of unforced oscillations.

$$\begin{aligned} Q &\equiv 2\pi(\text{energy of oscillator})/(\text{energy lost per cycle}) \\ &= 2\pi x_n^2/(x_n^2 - x_{n+1}^2) \\ &\approx \pi/D = 1/(2\xi) \quad \text{(for small damping)} \end{aligned}$$

The π in the definition of Q makes it so there is no π in the formula for the quality factor Q in terms of the damping ratio ξ. Note that, so long as damping is small, ξ, D and Q can each be found approximately from the other. A system with low damping ($\xi \ll 1$) has high quality ($Q \gg 1$) and slowly decaying osciallations and hence a small logarithmic decrement ($D \ll 1$).

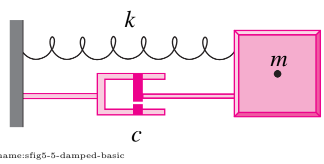


Figure 9.83:

SAMPLE 9.29 The mass-spring-dashpot system shown in the figure consists of a mass $m = 2\,\mathrm{kg}$, a spring with stiffness $k = 3200\,\mathrm{N/m}$ and a dashpot with damping coefficient $c = 10\,\mathrm{kg/s}$.

1. Is the system underdamped, critically damped or overdamped?
2. Find the damped natural frequency of the system.
3. What is the resonant frequency of the system.

Solution

1. The question about underdamped, critically damped, or overdamped can be answered conveniently by computing the damping ratio ξ. For an underdamped system, $\xi < 1$, for a crtically damped system, $\xi = 1$, and for an overdamped system $\xi > 1$. So, let us compute ξ. We know that
$$\xi = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}.$$
Thus, for the given system,
$$\xi = \frac{10\,\mathrm{kg/s}}{2\sqrt{3200\,\mathrm{N/m}\cdot 2\,\mathrm{kg}}} = \frac{10\,\mathrm{kg/s}}{160\,\mathrm{kg/s}} = 0.062.$$
Since $\xi < 1$, the system is underdamped.

$$\boxed{\text{Underdamped } (\xi = 0.062)}$$

2. The damped natural frequency, λ_d, is given by
$$\lambda_d = \lambda_n \sqrt{1-\xi^2}$$
where $\lambda_n = \sqrt{k/m}$ is the natural frequency of the system. Substituting the known values, we get
$$\lambda_d = \sqrt{\frac{3200\,\mathrm{N/m}}{2\,\mathrm{kg}}}\sqrt{1-(0.062)^2} = 39.92\,\mathrm{rad/s}$$
which is almost the same as the natural frequency $\lambda_n = 40\,\mathrm{rad/s}$.

$$\boxed{\lambda_d = 39.92\,\mathrm{rad/s}}$$

3. The resonant frequency of the system, λ_r, is given by
$$\lambda_r = \lambda_n \sqrt{1-2\xi^2}.$$
Substituting the known values of λ_n and ξ, we get
$$\lambda_r = 39.85\,\mathrm{rad/s}$$
which is the smallest among the three characteristic frequencies of the system — natural frequency, damped natural frequency, and the resonant frequency. For small values of ξ, however, the three frequencies are practically indistiguishable as is the case here.

$$\boxed{\lambda_r = 39.85\,\mathrm{rad/s}}$$

SAMPLE 9.30 **Response to a constant force:** A constant force $F = 50\,\text{N}$ acts on a mass-spring system as shown in the figure. Let $m = 5\,\text{kg}$ and $k = 10\,\text{kN/m}$.

1. Write the equation of motion of the system.
2. If the system starts from the initial displacement $x_0 = 0.01$ m with zero velocity, find the displacement of the mass as a function of time.
3. Plot the response (displacement) of the system against time and describe how it is different from the unforced response of the system.

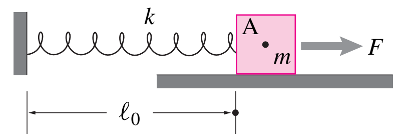


Figure 9.84:

Solution

1. The free-body diagram of the mass is shown in fig. 9.85 at a displacement x (assumed positive to the right). Applying linear momentum balance in the x-direction, *i.e.*, $(\sum \vec{F} = m\vec{a}) \cdot \hat{\imath}$, we get

$$\begin{aligned} F - kx &= m\ddot{x} \\ \Rightarrow \quad m\ddot{x} + kx &= F \end{aligned} \tag{9.89}$$

which is the equation of motion of the system.

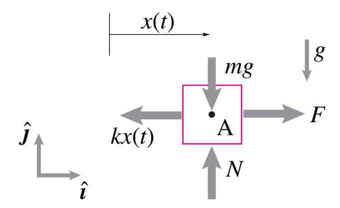


Figure 9.85: Free-body diagram of the mass.

2. The equation of motion has a non-zero right hand side. Thus, it is a nonhomogeneous differential equation. A general solution of this equation is made up of two parts — the homogeneous solution x_h which is the solution of the unforced system (eqn. (9.89) with $F = 0$), and a particular solution x_p that satisfies the nonhomogeneous equation. Thus,

$$x(t) = x_h(t) + x_p(t). \tag{9.90}$$

Now, let us find $x_h(t)$ and $x_p(t)$.

Homogeneous solution: $x_h(t)$ has to satisfy the homogeneous equation

$$m\ddot{x} + kx = 0.$$

Let $\lambda = \sqrt{k/m}$. Then, from the solution of unforced harmonic oscillator, we know that

$$x_h(t) = A\sin(\lambda t) + B\cos(\lambda t)$$

where A and B are constants to be determined later from initial conditions.

Particular solution: x_p must satisfy eqn. (9.89). Since the nonhomogeneous part of the equation is a constant (F), we guess that x_p must be a constant too (of the same form as F). Let $x_p = C$. Now we substitute $x_p = C$ in eqn. (9.89) and solve the resulting equation to determine C:

$$m\underbrace{\ddot{C}}_{0} + kC = F \quad \Rightarrow \quad C = F/k \quad \text{or} \quad x_p = F/k.$$

Substituting x_h and x_p in eqn. (9.90), we get

$$x(t) = A\sin(\lambda t) + B\cos(\lambda t) + F/k. \tag{9.91}$$

Now we use the given initial conditions to determine A and B.

$$\begin{aligned} x(t=0) &= B + F/k = x_0 \text{ (given)} \quad \Rightarrow \quad B = x_0 - F/k \\ \dot{x}(t) &= A\lambda\cos(\lambda t) - B\lambda\sin(\lambda t) \\ \Rightarrow \quad \dot{x}(t=0) &= A = 0 \text{ (given)} \quad \Rightarrow \quad A = 0. \end{aligned}$$

Thus,

$$x(t) = (x_0 - F/k)\cos(\lambda t) + F/k, \tag{9.92}$$

$$\text{and} \quad \dot{x}(t) = -\lambda(x_0 - F/k)\sin(\lambda t). \tag{9.93}$$

3. Let us plug the given numerical values, $k = 10\,\text{kN/m}, m = 5\,\text{kg}$, (which gives $\lambda = \sqrt{k/m} = 44.72\,\text{rad/s}$), $F = 50\,\text{N}$ and $x_0 = 0.01$ m in eqn. (9.92) and (9.93). The displacement and the velocity are now given as

$$
\begin{aligned}
x(t) &= (0.005\,\text{m})\cos(44.72\,\text{rad/s}\cdot t) + 0.005\,\text{m}, \\
\text{and} \quad \dot{x}(t) &= -(0.22\,\text{m/s})\sin(44.72\,\text{rad/s}\cdot t).
\end{aligned}
$$

This response is plotted in fig. 9.86 against time. Note that the oscillations of the mass are about a non-zero mean value, $x_{\text{eq}} = 0.005$ m. A little thought should reveal that this is what we should expect. When a mass hangs from a spring under gravity, the spring elongates a little, by mg/k to be precise, to balance the mass. Thus, the new static equilibrium position is not at the relaxed length ℓ_0 of the spring but at $\ell_0 + mg/k$. Any oscillations of the mass will be about this new equilibrium. The velocity, however, has a zero mean value which is what we expect from eqn. (9.93).

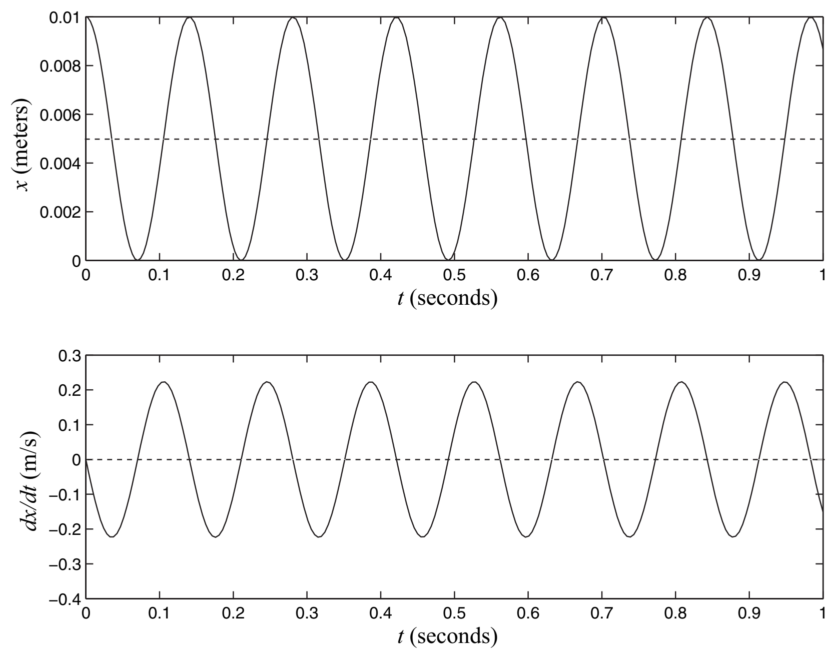


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Figure 9.86: Displacement of the mass as a function of time. Note that the mass oscillates about a nonzero value of x.

This problem is exactly like a mass hanging from a spring under gravity, a constant force, but just rotated by 90°. The new static equilibrium is at $x_{\text{eq}} = F/k$ and any oscillations of the mass have to be around this new equilibrium.

We can rewrite the response of the system by measuring the displacement of the mass from the new equilibrium. Let $\tilde{x} = x - F/k$. Then, eqn. (9.92) becomes

$$\tilde{x} = \tilde{x}_0 \cos(\lambda t)$$

where $\tilde{x}_0 = x_0 - F/k$ is the initial displacement. Clearly, this is the response of an unforced harmonic oscillator. Thus the effect of a constant force on a spring-mass system is just a shift in its static equilibrium position.

SAMPLE 9.31 A single degree of freedom damped oscillator has unknown mass, spring stiffness and damping coefficient. In order to find these quantities, the oscillator is subjected to a constant force $F_0 = 100\,\mathrm{N}$ and its transient response is recorded. The response is shown in fig. 9.87. The two peaks marked in the response plot correspond to $(t, x) = (0.2107\ \mathrm{s}, 0.01345\ \mathrm{m})$ and $(0.3525\ \mathrm{s}, 0.0117\ \mathrm{m})$ respectively. Find the system parameters m, k, and c.

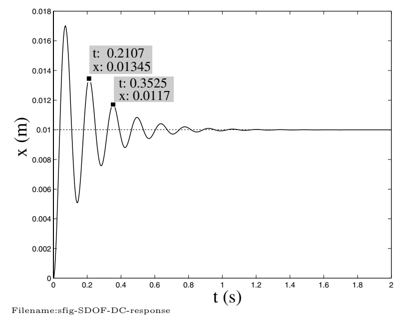


Figure 9.87:

Solution Let the mass, stiffness, and damping coefficient of the system be m, k, and c, respectively. Then the equation of motion of the system, subjected to a constant force F_0 is,

$$m\ddot{x} + c\dot{x} + kx = F_0$$

where $x(t)$ is the displacement at some instant t. From the solution of this equation, we know that the steady state solution (after the transient oscillations die) is merely a shift in the static equilibrium position, given by F_0/k. From the given response, we see that

$$\frac{F_0}{k} = 0.01\,\mathrm{m} \quad \Rightarrow \quad k = \frac{F_0}{0.01\,\mathrm{m}} = \frac{100\,\mathrm{N}}{0.01\,\mathrm{m}} = 10\,\mathrm{kN/m}.$$

Thus we have found one of the parameters, k. Now we need to find m and c.

Since two successive peaks are given in the transient response, we can use the logrithmic decrement to determine the damping ratio ξ from the relationship

$$\xi = \frac{1}{2\pi}\ln\left(\frac{x_n}{x_{n+1}}\right).$$

From the given data, $x_n = 0.01345\,\mathrm{m}$ and $x_{n+1} = 0.0117\,\mathrm{m}$. Therefore, ⑤

$$\xi = \frac{1}{2\pi}\ln\left(\frac{0.01345\,\mathrm{m}}{0.0117\,\mathrm{m}}\right) = 0.022.$$

⑤The calculation here is illustrative. In practice, you should not use just two data points for computing damping from the logarithmic decrement method. In general, we use several data points and determine the damping from taking an average of the decrements. See Sample 9.20 on page 475.

Since $\xi = c/c_c = c/(2\sqrt{km})$, we have

$$c = 2\xi\sqrt{km} = 0.044\sqrt{km} \tag{9.94}$$

This is just one equation in two unknowns, m and c (we already know k). So, we need another equation. From the peak to peak distance (in time), we can find the damped time period. That is $T_d = T_2 - T_1 = 0.3525\,\mathrm{s} - 0.2107\,\mathrm{s} = 0.1418\,\mathrm{s}$. But, $T_d = 2\pi/\lambda_d$, and $\lambda_d = \lambda_n\sqrt{1-\xi^2}$. Therefore,

$$\begin{aligned}
\lambda_n^2 \equiv \frac{k}{m} &= \frac{\lambda_d^2}{1-\xi^2} = \frac{4\pi^2}{T_d^2(1-\xi^2)} \\
\Rightarrow \quad m &= \frac{kT_d^2(1-\xi^2)}{4\pi^2} \\
&= \frac{10000\,\mathrm{N/m}\cdot(0.1418\,\mathrm{s})^2(1-0.022^2)}{4\pi^2} \\
&= 5.09\,\mathrm{kg}.
\end{aligned}$$

Now substituting the value of m and k in eqn. (9.94), we get

$$c = 0.044\sqrt{10000\,\mathrm{N/m}\cdot 5.09\,\mathrm{kg}} = 9.92\,\mathrm{kg/s}.$$

$$m = 5.09\,\mathrm{kg}, \quad k = 10\,\mathrm{kN/m}, \text{ and } c = 9.92\,\mathrm{kg/s}.$$

SAMPLE 9.32 Damping and forced response: When a single-degree-of-freedom damped oscillator (mass-spring-dashpot system) is subjected to a periodic forcing $F(t) = F_0 \sin(pt)$, then the response of the system is given by

$$x(t) = X \cos(pt - \phi)$$

where $X = \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}$, $\phi = \tan^{-1} \frac{2\xi r}{1-r^2}$, $r = \frac{p}{\lambda}$, $\lambda = \sqrt{k/m}$ and ξ is the damping ratio.

1. For $r \ll 1$, *i.e.,* the forcing frequency p much smaller than the natural frequency λ, how does the damping ratio ξ affect the response amplitude X and the phase ϕ?
2. For $r \gg 1$, *i.e.,* the forcing frequency p much larger than the natural frequency λ, how does the damping ratio ξ affect the response amplitude X and the phase ϕ?

Solution

1. If the frequency ratio $r \ll 1$, then r^2 will be even smaller; so we can ignore r^2 terms with respect to 1 in the expressions for X and ϕ. Thus, for $r \ll 1$,

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{1} = \frac{F_0}{k} \\ \phi &= \tan^{-1}(2\xi r) \approx \tan^{-1} 0 = 0 \end{aligned}$$

that is, the response amplitude does not vary with the damping ratio ξ, and the phase also remains constant at zero. As an example, we use the full expressions for X and ϕ for plotting them against ξ for $r = 0.01$ in fig. 9.88

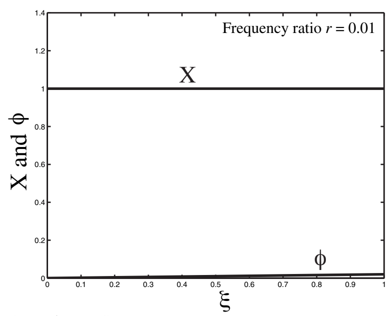


Figure 9.88:

For $r \ll 1$, $X \approx F_0/k$, and $\phi \approx 0$

2. If $r \gg 1$, then the denominator in the expression for X, $4\xi^2 r^2 + (1-r^2)^2 \approx r^4$ (because we can ignore all other terms with respect to r^4. Similarly, we can ignore 1 with respect to r^2 in the expression for ϕ. Thus, for $r \gg 1$,

$$\begin{aligned} X &= \frac{F_0/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}} \approx \frac{F_0/k}{r^2} = 0 \\ \phi &= \tan^{-1} \frac{2\xi r}{-r^2} \approx \tan^{-1} \frac{2\xi}{-r} \approx \tan^{-1}(-0) = \pi. \end{aligned}$$

Once again, we see that the response amplitude and phase do not vary with ξ. This is also evident from fig. 9.89 where we plot X and ϕ using their full expressions for $r = 10$. The slight variation in ϕ around π goes away as we take higher values of r.

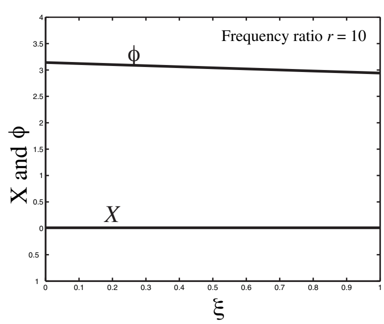
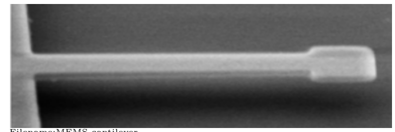


Figure 9.89:

For $r \gg 1$, $X \approx 0$, and $\phi \approx \pi$

Thus, we see that the damping in a system does not affect the response of the system much if the forcing frequency is far away from the natural frequency.

SAMPLE 9.33 A MEMS (microelectromechanical system) cantilever resonator (shown in the figure) is modeled as a single degree of freedom oscillator (SDOF) oscillator. Using load deflection measurements, the stiffness of the beam (equivalent to the spring stiffness) is found to be 90 N/m. The beam is excited using electrical actuation and its resonant frequency is determined under two different conditions: (i) the beam vibrating in vacuum where the viscous damping is negligible, and (ii) the beam vibrating in ambient conditions where the airflow around it causes viscous damping. If the two frequencies are found to be 30 kHz and 28.4 kHz, respectively, find the equivalent mass m and the damping ratio for the SDOF model. If the beam is subjected to a periodic actuation at the free end by a force $F(t) = F\sin(2\pi f t)$ where $F = 50\mu$ N and $f = 25$ kHz, find the steady state displacement amplitude and the phase of the free end of the resonator.



Filename:MEMS-cantilever

Figure 9.90: A MEMS cantilever resonator: Such resonators are typically silicon cantilever beams fabricated using micromachining process. These beams have geometric dimensions in micrometers and can be as small as a couple of micrometers long and a few nanometers in width and thickness.

Solution First we need to find m and c for the equivalent mass-spring-dashpot model. In the first case, where the resonant frequency is found in vacuum, we neglect damping, *i.e.,* $c = 0$. Therefore, the given frequency is the natural frequency. However, it is f_n, not the circular natural frequency λ_n. Now, $\lambda_n = 2\pi f_n$, hence

$$\sqrt{\frac{k}{m}} = 2\pi f_n \quad\Rightarrow\quad m = \frac{k}{4\pi^2 f_n^2} = \frac{90\,\mathrm{N/m}}{4\pi^2(30000\,\mathrm{s}^{-1})^2} = 2.533\times 10^{-9}\,\mathrm{kg}.$$

We now use the damped natural frequency to find the damping ratio ξ. Since, we are given $f_d = 28.4$ kHz, and we know that $\lambda_d = \lambda_n\sqrt{(1-\xi^2)}$, we have

$$\begin{aligned} 2\pi f_d &= 2\pi f_n\sqrt{1-\xi^2} \\ \xi &= \sqrt{1-\left(\frac{f_d}{f_n}\right)^2} = \sqrt{1-\left(\frac{28.4\,\mathrm{kHz}}{30\,\mathrm{kHz}}\right)^2} \\ &= 0.32. \end{aligned}$$

Now, we know the values of all system parameters for our SDOF model of the MEMS resonator — m, k and ξ (can find c if required from ξ, k and m). For the given sinusoidal forcing, the equation of motion of the SDOF oscillator is:

$$m\ddot{x} + c\dot{x} + kx = F\sin(2\pi f t).$$

We can write the steady state solution as the particular solution $x(t) = A_0\sin(pt - \phi)$ where $p = 2\pi f$, and the displacement amplitude A_0 and the phase ϕ are given by the following expressions:

$$A_0 = \frac{F/k}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}, \quad\text{and}\quad \phi = \tan^{-1}\left(\frac{2\xi r}{1-r^2}\right).$$

Since, $r = \frac{p}{\lambda_n} = \frac{2\pi f}{2\pi f_n} = \frac{25}{30} = 0.833$, we have,

$$\begin{aligned} A_0 &= \frac{(50^{-6}\,\mathrm{N})/(90\,\mathrm{N/m})}{\sqrt{(2\cdot 0.32\cdot 0.833)^2 + (1-0.833^2)^2}} \\ &= 9.04\times 10^{-7}\,\mathrm{m} = 0.904\mu\mathrm{m}. \end{aligned}$$

Similarly, we find the phase as,

$$\begin{aligned} \phi &= \tan^{-1}\frac{2\cdot 0.32\cdot 0.833}{1-0.833^2} \\ &= 1.05\,\mathrm{rad}. \end{aligned}$$

$$m = 2.533\times 10^{-9}\,\mathrm{kg},\ \xi = 0.32,\ A_0 = 0.904\mu\mathrm{m},\ \text{and}\ \phi = 1.05\,\mathrm{rad}$$

SAMPLE 9.34 Energetics of resonance: Consider the response of a damped harmonic oscillator to a periodic forcing. Find the work done on the system by the periodic force during a single cycle of the force and show how this work varies with the forcing frequency and the damping ratio.

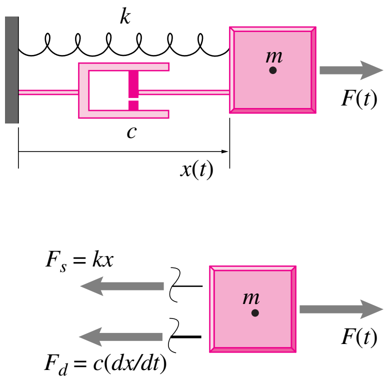


Figure 9.91:

Solution Let us consider the damped harmonic oscillator shown in fig. 9.91 with $F(t) = F\sin(pt)$. The equation of motion of the system is $m\ddot{x} + c\dot{x} + kx = F\sin(pt)$ and the response of the system may be expressed as $X\sin(pt-\phi)$ where $X = (F/k)/\sqrt{(2\xi r)^2 + (1-r^2)^2}$ and $\phi = \tan^{-1}(2\xi r/(1-r^2))$, with $r = p/\lambda_n$, $\lambda_n = \sqrt{k/m}$ and $\xi = c/(2\sqrt{km})$.

We can compute the work done by the applied force on the system in one cycle by evaluating the integral

$$W = \int_{\text{onecycle}} F(t)\,dx$$

But, $x = X\sin(pt-\phi) \quad\Rightarrow\quad dx = Xp\cos(pt-\phi)dt$. Therefore,

$$\begin{aligned}
W &= \int_0^{2\pi/p} F\sin(pt)\cdot Xp\cos(pt-\phi)\,dt \\
&= FXp\int_0^{2\pi/p} \sin(pt)\cos(pt-\phi)\,dt \\
&= FXp\int_0^{2\pi/p} \sin(pt)\left(\cos(pt)\cos\phi + \sin(pt)\sin\phi\right)\,dt \\
&= FXp\left[\cos\phi\cdot\frac{1}{2}\int_0^{2\pi/p}\sin(2pt)\,dt + \sin\phi\cdot\frac{1}{2}\int_0^{2\pi/p}(1-\cos(2pt))\,dt\right] \\
&= \frac{FXp}{2}\left[\cos\phi\left(-\frac{\cos(2pt)}{2p}\right)\Big|_0^{2\pi/p} + \sin\phi\left(t-\frac{\sin(2pt)}{2p}\right)\Big|_0^{2\pi/p}\right] \\
&= \frac{FXp}{2}\left[\frac{\cos\phi}{2p}(-1+1) + \frac{2\pi}{p}\sin\phi + 0\right] \\
&= \frac{FXp}{2}\cdot\frac{2\pi}{p}\sin\phi \\
&= F\pi X\sin\phi
\end{aligned}$$

Although the expression obtained above for W looks simple, we must substitute for X and ϕ to see the dependence of W on the damping ratio ξ and the frequency ratio r.

$$W = \frac{(F\pi)\cdot(F/k)}{\sqrt{(2\xi r)^2 + (1-r^2)^2}}\cdot\frac{2\xi r}{\sqrt{(2\xi r)^2+(1-r^2)^2}} = \frac{(2\pi F^2\xi r)/k}{(2\xi r)^2 + (1-r^2)^2} \qquad (9.95)$$

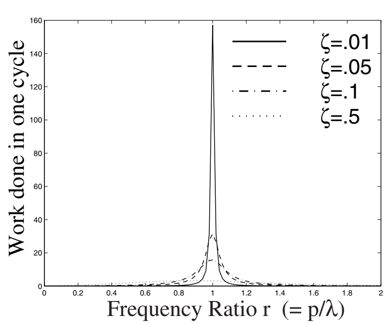


Figure 9.92:

Unfortunately, this expression is too complicated to see the dependence of W on ξ and r. However, we know that for small $r(\ll 1)$, $\phi \approx 0$ and for large $r(\gg 1)$, $\phi \approx \pi$, implying that W is almost zero in both these cases. On the other hand, for r close to one, that is, close to resonance, $\phi \approx \pi/2 \Rightarrow \sin\phi \approx 1$, but the response amplitude X is large (for small ξ), which makes W to be big near the resonance. Figure 9.92 shows a plot of W against r, using eqn. (9.95), for different values of ξ. It is clear from the plot that the work done on the system in a single cycle is much larger close to the resonance for lightly damped systems. This explains why the response amplitude keeps on growing near resonance.

$$\boxed{W = \pi FX\sin\phi}$$

Problems for Chapter 9

Unconstrained 1D dynamics

9.1 Force and motion in 1D

Preparatory Problems

9.1.1 Give three examples of real life objects where you might use the idealization, for dynamic calculations, that the object is a particle in unconstrained 1D motion. *

9.1.2 A car is going downhill on a constant slope straight road. for finding out the car's speed at the end of the road you model it as a particle. For specifying initial velocity, which point on the car would you consider? *

9.1.3 The acceleration of a particle is given as a function of time, $a(t)$. Is this information sufficient to find the speed of the particle at the end of, say, T seconds? *

9.1.4 If a particle has constant acceleration, its linear momentum (a) remains constant, (b) changes linearly with time, or (c) changes quadratically with time. Which one is true? *

9.1.5 In a motorcycle race on a straight track, the speed of a motorcyclist at the 200 m mark is recorded. Given that the rider started from rest, find the acceleration of the motorcycle from the given information, provided the acceleration (a) is constant or (b) varies linearly with time. *

9.1.6 The force acting on a particle is a given function of time. If you plot the force vs time and find the area under the graph, from that area can you determine (a) the net displacement of the particle, (b) the average velocity of the particle, or (c) the change in linear momentum of the particle? *

9.1.7 If the linear momentum of a body remains constant in time, it must have (a) a constant force acting on it, (b) no net force acting on it, or (c) a sinusoidal force acting on it. *

9.1.8 The distance between two points in a bicycle race is 10 km. How many minutes does a bicyclist take to cover this distance if he/she maintains a constant speed of 15 mph. *

9.1.9 A 5 kN constant force acts on a 1 kg object for 5 seconds that was initially at rest. Find the speed of the object at the end of (a) 5 seconds, and (b) 10 seconds.

9.1.10 Given that $\dot{x} = k_1 + k_2 t$, $k_1 = 1\,\text{ft/s}$, $k_2 = 1\,\text{ft/s}^2$, and $x(0) = 1\,\text{ft}$, what is the displacement at the end of 10 seconds?

9.1.11 Find $x(3\,\text{s})$ given that

$$\dot{x} = x/(1\,\text{s}) \quad \text{and} \quad x(0\,\text{s}) = 1\,\text{m}$$

or, expressed slightly differently,

$$\dot{x} = cx \quad \text{and} \quad x(0\,\text{s}) = x_0,$$

where $c = 1\,\text{s}^{-1}$ and $x_0 = 1\,\text{m}$. Make a sketch of x versus t. *

9.1.12 A ball of mass m is dropped from rest at a height h above the ground. Find the position and velocity as a function of time (as well as m and g, if needed). Neglect air friction. When does the ball hit the ground? What is the velocity of the ball just before it hits?

9.1.13 The speed of a particle varies sinusoidally as $v = A\sin[ct]$, where $A = 0.5\,\text{m/s}$ and $c = 3\,\text{rad/s}$. Let the initial position of the particle be $x(0) = 0$. Find the position of the particle at $t = \pi/2\,\text{s}$.

9.1.14 The speed of a particle is directly proportional to its position and is given as $\dot{x} = x/\text{s}$. If the initial position, $x(0\,\text{s}) = 1\,\text{m}$, how far would the particle be from the origin in 5 seconds?

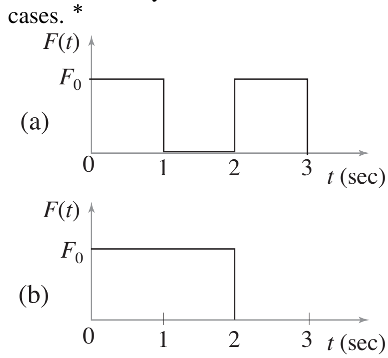
More-Involved Problems

9.1.15 Consider a force $F(t)$ acting on a cart over a 3 second span. In case (a), the force acts in two impulses of one second duration each as shown in fig. 9.1.15. In case (b), the force acts continuously for two seconds and then goes to zero. Given that the mass of the cart is 10 kg, $v(0\,\text{s}) = 0$, and $F_0 = 10\,\text{N}$, for each force profile,

a) Find the speed of the cart at the end of 3 seconds, and

b) Find the distance travelled by the cart in 3 seconds.

Comment on your answers for the two cases. *



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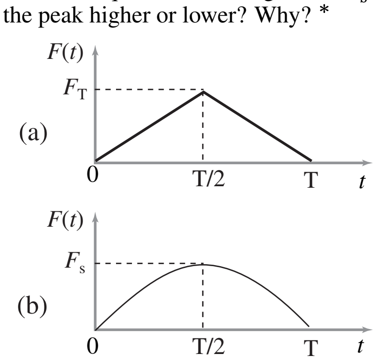
Problem 9.1.15

9.1.16 A car of mass m is accelerated by applying a triangular force profile shown in fig. 9.1.16(a). Find the speed of the car at $t = T$ seconds. If the same speed is to be achieved at $t = T$ seconds with a sinusoidal force profile, $F(t) = F_s \sin\frac{\pi t}{T}$, find the required force magnitude F_s. Is the peak higher or lower? Why? *



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Problem 9.1.16

9.1.17 A particle of mass $m = 1\,\text{kg}$ is acted upon by a short duration force given by

$$F(t) = \begin{cases} F_0 t & 0 \le t \le 1\,\text{s} \\ F_0(2-t) & 1\,\text{s} < t \le 2\,\text{s} \end{cases}$$

where $F_0 = 5\,\text{N}$. If the particle starts from rest, find the speed of the particle as a function time. Sketch the given force profile as a function of time and draw the corresponding speed $v(t)$ as a function of time. What is the speed of the particle at $t = 2\,\text{s}$?

9.1.18 A ball of mass m is dropped vertically from rest at a height h above the ground. Air resistance causes a drag force on the ball directly proportional to the speed v of the ball, $F_d = bv$. Find the velocity and position of the ball as a function of time. Find the velocity as a function of position. Gravity is non-negligible, of course.

9.1.19 A sinusoidal force acts on a 1 kg mass as shown in the figure and graph below. The mass is initially still; i. e.,

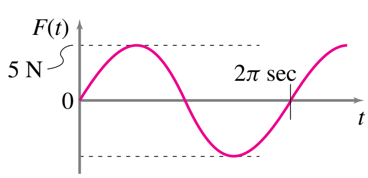
$$x(0) = v(0) = 0$$

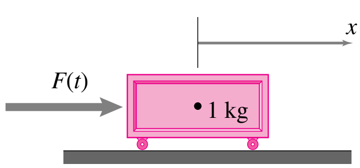
.

a) What is the velocity of the mass after 2π seconds?

b) What is the position of the mass after 2π seconds?

c) Plot position x versus time t for the motion.





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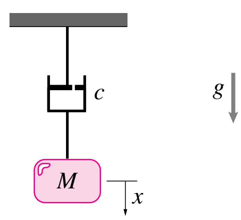
Problem 9.1.19

9.1.20 A motorcycle accelerates from 0 mph to 60 mph in 5 seconds. Find the average acceleration in m/s^2. How does this acceleration compare with g, the acceleration of an object falling near the earth's surface?

9.1.21 A car moves on a straight road with an initial velocity $v_0 = 30\,\text{m/s}$. Let its position at $t = 0$ be $x = 0$. For the first 5 s it has no acceleration, and thereafter it brakes with a retarding force that gives it a constant acceleration $a_x = -10 m/s^2$. Calculate the velocity and the x-coordinate of the car when $t = 8\,\text{s}$ and when $t = 12\,\text{s}$, and find the distance travelled by the car from start until it comes to a final stop.

9.1.22 A grain of sugar falling through honey has a negative acceleration proportional to the difference between its velocity and its 'terminal' velocity, which is a known constant v_t. Write this sentence as a differential equation, defining any constants you need. Solve the equation assuming some given initial velocity v_0.

9.1.23 The mass-dashpot system shown below is released from rest at $x = 0$. Determine an equation of motion for the particle of mass m that involves only $\dot{x}$ and x (a first-order ordinary differential equation). The damping coefficient of the dashpot is c.



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Problem 9.1.23

9.1.24 Due to gravity, a particle falls in air with a drag force proportional to the speed squared.

1. Write $\sum \vec{F} = m\vec{a}$ in terms of variables you clearly define,
2. find a constant speed motion that satisfies your differential equation,
3. pick numerical values for your constants and for the initial height. Assume the initial speed is zero
   a) set up the equation for numerical solution,
   b) solve the equation on the computer,
   c) make a plot with your computer solution and show how that plot supports your answer to (2).

9.1.25 In quadratic drag problems, the deceleration is proportional to the square of velocity, *i.e.,* $a = \frac{dv}{dt} = -kv^2$. Assume that a particle with initial velocity $v(0) = v_0$ experiences quadratic drag.

a) How long does it take for the particle to reduce its speed to half of its initial speed (*i.e.,* find t such that $v(t) = \frac{1}{2}v_0$)?

b) Find the position of the particle as a function of velocity. How far does the particle move from its initial position when its velocity drops to half its initial value?

9.1.26 A bullet penetrating flesh slows approximately as it would if penetrating water. The drag on the bullet is about $F_D = c\rho_w v^2 A/2$ where ρ_w is the density of water, v is the instantaneous speed of the bullet, A is the cross sectional area of the bullet, and c is a drag coefficient which is about $c \approx 1$. Assume that the bullet has mass $m = \rho_l AL$ where ρ_l is the density of lead, A is the cross sectional area of the bullet and L is the length of the bullet (approximated as cylindrical). Assume $m = 2$ grams, entering velocity $v_0 = 400\,\text{m/s}$, $\rho_l/\rho_w = 11.3$, and bullet diameter $d = 5.7\,\text{mm}$.

a) Plot the bullet position vs time.

b) Assume the bullet has effectively stopped when its speed has dropped to 5 m/s, what is its total penetration distance?

c) According to the equations implied above, what is the penetration distance in the limit $t \to \infty$?

d) How would you change the model to make it more reasonable in its predictions for long time?

9.1.27 A force pulls a particle of mass m towards the origin according to the law (assume same equation works for $x > 0, x < 0$)

$$F = Ax + Bx^2 + C\dot{x}$$

Assume $\dot{x}(0) = 0$.
Using numerical solution, find values of A, B, C, m, and x_0 so that

1. the mass never crosses the origin,
2. the mass crosses the origin once,
3. the mass crosses the origin many times.

[Hint: Vary one parameter at a time and choose a different set of parameter values for each case.]

9.2 Energy methods in 1D

Preparatory Problems

9.2.1 A mass m is at position x moving at velocity v and being acted upon by force F. For each of the quantities below:

i give the symbol used for the quantity

ii describe the quantity in words

iii give a formula to evaluate the quantity in terms of some or all of m, x, v and F and any other variables you may need.

iv Give the standard units for the quantity in the SI system.

v Give the standard units for the quantity in the English system.

a) Power

b) Kinetic energy

c) Work

d) Potential energy

9.2.2 Write an equation relating the two words in each of these pairs. If any conditions or descriptions of the situation are needed, give them. If you know more than one equation (or form for a given equation), give all that you know. All should be given in the context of this section: 1D motion.

a) work and power

b) work and kinetic energy

c) power and kinetic energy

d) work and potential energy

e) potential energy and kinetic energy

9.2.3 A force $F = F_0 \sin(ct)$ acts on a particle with mass $m = 3\,\text{kg}$ which has position $x = 3\,\text{m}$, velocity $v = 5\,\text{m/s}$ at $t = 2\,\text{s}$. $F_0 = 4\,\text{N}$ and $c = 2/\text{s}$. At $t = 2\,\text{s}$ evaluate (give numbers and units):

a) a,

b) E_K,

c) P,

d) $\dot{E}_\text{K}$,

e) the rate at which the force is doing work.

9.2.4 A force only depends on position according to $F = C_0 + C_1 x$ where C_0 and C_1 are constants. What is the work done by this force when the point to which it is applied moves from x_1 to x_2? Answer in terms of some or all of C_0, C_1, x_1 and x_2.

9.2.5 Find the potential E_P associated with each of these force fields.

a) $F = 0$.

b) $F = F_0$ (=constant).

c) $F = kx$.

d) $F = A\sin(x/x_0)$.

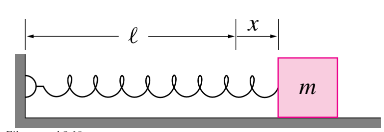
e) $F = c/x^2$.

9.2.6 Consider a spring-mass system with $m = 2\,\text{kg}$ and $k = 5\,\text{N/m}$. The mass is pulled to the right a distance $x = x_0 = 0.5\,\text{m}$ from the unstretched position and released from rest. No external forces act on the mass.

a) What are the initial potential and kinetic energy of the system?

b) What is the potential and kinetic energy of the system as the mass passes through the static equilibrium (unstretched spring) position?

c) What is the speed of the mass when it passes through the static equilibrium position?



Problem 9.2.6

9.2.7 A mass m is held in place by a spring whose restoring force is $T(x) = kx$. Derive the equation of motion of the system (that is, find the acceleration a in terms of x).

9.2.8 The peak propulsion force on a 4-wheel-drive car is about μmg where $\mu \approx 1$ for rubber on road (a bit more for fancy racing tires). Assume a car starts from rest at position zero. Answer the following questions with symbols and with numbers (using $\mu = 1$, $m = 1000\,\text{kg}$, and $g = 10\,\text{m/s}^2$).

a) What is the minimum distance required to reach $v_1 = 60\,\text{mph}$?

b) What is the extra distance required to get from $v_1 = 60\,\text{mph}$ up to $v_2 = 70\,\text{mph}$?

c) What is the peak power used by the engine in getting up to $v_1 = 60\,\text{mph}$ (assuming no dissipation and no air friction)?

9.2.9 A car (mass $m = 1000\,\text{kg}$) traveling at speed $v_0 = 30\,\text{m/s}$ crashes into a brick wall and comes to a stop as the front end of the car compresses a distance $d = 1\,\text{m}$. Answer with symbols and numbers. Assume constant deceleration during the crash. Neglect the mass of crushing region of the car.

a) What is the total energy dissipated in the crash?

b) What is the force of the car on the wall?

c) What is the force of the wall on the car?

d) What is the deceleration of the car passengers (assuming they are strapped in and move with the bulk of the car). Answer in g's?

e) Assuming an $m_p = 50\,\text{kg}$ person, what is the force of the seat belts on the person (answer in body weight).

f) If a parent was holding a 15 kg child on his lap, what force would he need to hold on to the child through the crash (answer in N and in number of child body weights).

More-Involved Problems

9.2.10 A kid ($m = 90\,\text{lbm}$) stands on a $h = 10\,\text{ft}$ wall and jumps down, accelerating with $g = 32\,\text{ft/s}^2$. Upon hitting the ground with straight legs, she bends them so her body slows to a stop over a distance $d = 1\,\text{ft}$. Neglect the mass of her legs. Assume constant deceleration as she brakes the fall.

a) What is the total distance her body falls? *

b) What is the potential energy lost?

c) How much work must be absorbed by her legs?

d) What is the force of her legs on her body? Answer in symbols, numbers and numbers of body weight (*i.e.,* find F/mg).

9.2.11 In traditional archery, when pulling an arrow back the force increases approximately linearly up to the peak 'draw force' F_{draw} that varies from about $F_{draw} = 25\,\mathrm{lbf}$ for a bow made for a small person to about $F_{draw} = 75\,\mathrm{lbf}$ for a bow made for a big strong person. The distance the arrow is pulled back, the draw length ℓ_{draw}, varies from about $\ell_{draw} = 2\,\mathrm{ft}$ for a small adult to about 30 inch for a big adult. An arrow has mass of about 300 grain (1 grain ≈ 64.8milli gm, so an arrow has mass of about $19.44 \approx 20\,\mathrm{gm} \approx 3/4\,\mathrm{ounce}$). Give all answers in symbols and numbers.

a) What is the range of speeds you can expect an arrow to fly?

b) What is the range of heights an arrow might go if shot straight up (it's a bad approximation, but for this problem neglect air friction)?

9.2.12 A big person ($m = 100\,\mathrm{kg}$) jumps on a trampoline which we model as a linear spring with stiffness k. You know that the trampoline deflects $d_0 = 20\,\mathrm{cm}$ under the stationary weight mg of the person (use $g = 10\,\mathrm{m/s^2}$). Assume there is no dissipation and the person is jumping repeatedly a height $h = 1\,\mathrm{m}$ above the unloaded surface of the trampoline. Give all answers with symbols and numbers.

a) What is the stiffness k of the spring (answer in terms of some or all of m, g and d_0).

b) What is the maximum deflection of the trampoline during these jumps?

c) What is the peak force of the trampoline on the jumper? (answer in symbols, Newtons, and numbers of body weights).

9.2.13 For the car of problem 9.2.8 what is the average power required to reach speed v_1? There are two plausible ways to calculate this power:

$$\bar{P}_1 = \int_0^x P(x')\,dx'/x \qquad \text{and}$$
$$\bar{P}_2 = \int_0^t P(t')\,dt'/t\ .$$

Use both. Do the two methods give the same answer? If so, why, and will the answers be the same for all problems? If not, why not, in what cases will the answers agree, and, when they differ, which one is right?

9.2.14 For problem 9.2.9 which answers would change, and in which way, if the deceleration was not exactly constant during the crash? That is, for which quantities would be bigger, which smaller, which the same, for which would the answer depend on the nature of the non-constant acceleration?

9.2.15 The earth's gravitational pull on a mass m is $F = -\frac{mgR^2}{r^2}$, where mg is the pull at the surface of the earth and R is the radius of the earth. Assume a ballistic rocket is shot straight up with a launch velocity of v_0 (measured in a 'fixed' not-rotating-with-the-earth frame). Assume the rocket goes in a straight radial line as the earth turns underneath it (relative to the surface of the earth this rocket would be launched somewhat to the West to cancel the earths rotation). Assume the period of active thrust is negligibly short (hence the word *ballistic*: "relating to or characteristic of the motion of objects moving under their own momentum and the force of gravity").

a) Solve for v as a function of r (and some or all of m, g, R and v_0).

b) Find the maximum height the rocket reaches.

c) Find the 'escape velocity' v_{escape}, the minimum launch speed needed for the rocket to never return.

d) On one graph plot height (r or $r - R$) vs t for a v_0 just below v_{escape} and for v_0 just greater than v_{escape}. If you use numerical methods to make this plot use $g = 10\,\mathrm{m/s^2}$, $R = 6400\,\mathrm{km}$, and $m = 1\,\mathrm{kg}$. Make sure your axes are such that you can see a clear qualitative difference between the two cases.

9.2.16 The power available to a very strong accelerating cyclist over short periods of time (up to, say, about 1 minute) is about 1 horsepower. Assume a rider starts from rest and uses this constant power. Assume a mass (bike + rider) of 150 lbm, a realistic drag force of $.006\,\mathrm{lbf/(ft/s)^2}v^2$. Neglect other drag forces.

a) What is the peak (steady state) speed of the cyclist?

b) Using analytic or numerical methods make an accurate plot of speed vs. time.

c) What is the acceleration as $t \to \infty$ in this solution?

d) What is the acceleration as $t \to 0$ in your solution?

e) How would you improve the model to fix the problem with the answer above?

9.3 Elementary vibration analysis

Preparatory Problems

9.3.1 The basic model.

a) Draw a spring (k) mass (m) system in a configuration where the spring is stretched.

b) On the drawing indicate the variable x.

c) Draw a free body diagram of the mass.

d) Write the equation of linear momentum balance for the mass.

e) Rearrange the momentum balance equation to get the harmonic-oscillator equation in standard form.

f) Write the general solution to the harmonic oscillator equation in two different ways (one as a sum of a sine and cosine function and one as a phase shifted sine or cosine function).

g) What is the natural frequency of this system?

h) What is the period?

i) What is the frequency (or circular frequency)?

j) Find the solution for the special case that the mass is released from rest at $x(0) = x_0$.

- give the analytic expression.
- plot the position vs time for at least one whole cycle of motion.
- with the same time scale, plot velocity vs time (what is the peak velocity).
- with the same time scale, plot both the potential and kinetic energies vs time.

k) Find the solution for the special case that the mass is launched at v_0 from the rest position (just the analytic form, no need to repeat all the parts just above).

9.3.2 Does the function $x = C_1 e^{\lambda t} + C_2 e^{-\lambda t}$ satisfy the harmonic oscillator equation $\ddot{x} + \lambda^2 x = 0$ for any, possibly special, values of C_1 and C_2? Show that it does or does not.

9.3.3 Given that $\ddot{x} = -cx$, with $c = 1/\text{s}^2$, $x(0) = 1$ m, and $\dot{x}(0) = 0$ find:

a) $x(\pi \text{ s}) = ?$

b) $\dot{x}(\pi \text{ s}) = ?$

9.3.4 Given that $\ddot{x} + \lambda^2 x = C_0$, $x(0) = x_0$, and $\dot{x}(0) = 0$, find the value of x at $t = \pi/\lambda$ s.

9.3.5 A mass m is connected to a spring k and released from rest with the spring stretched a distance d from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? Neglect gravity and friction. Answer in terms of some or all of m, k, and d. *

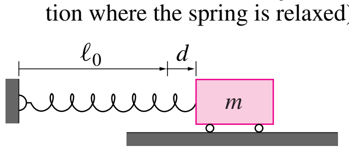
9.3.6 A spring k with rest length ℓ_0 is attached to a mass m which slides frictionlessly on a horizontal ground as shown. At time $t = 0$ the mass is released from rest with the spring stretched a distance d. Measure the mass position x relative to the wall.

a) What is the acceleration of the mass just after release?

b) Find a differential equation which describes the horizontal motion x of the mass.

c) What is the position of the mass at an arbitrary time t?

d) What is the speed of the mass when it passes through $x = \ell_0$ (the position where the spring is relaxed)?



Filename:s97f1
Problem 9.3.6

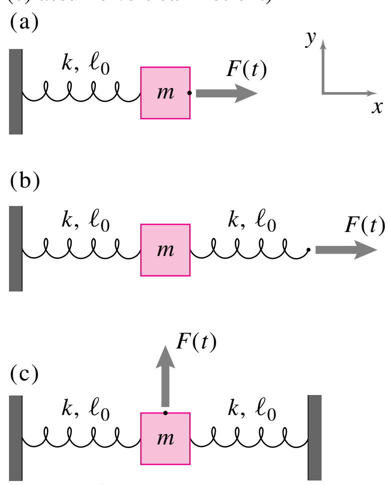
9.3.7 Reconsider the spring-mass system from problem 9.3.6.

a) Find the potential and kinetic energy of the spring mass system as functions of time.

b) Assigning numerical values to the various variables, use a computer to make a plot of the potential and kinetic energy as a function of time for several periods of oscillation. Are the potential and kinetic energy ever equal at the same time? If so, at what position $x(t)$?

c) Make a plot of kinetic energy versus potential energy. What is the phase relationship between the kinetic and potential energy?

9.3.8 For the three spring-mass systems shown in the figure, find the equation of motion of the mass in each case. All springs are massless and are shown in their relaxed states. Ignore gravity. (In problem (c) assume vertical motion.) *



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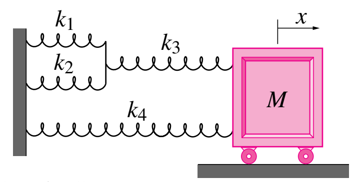
Problem 9.3.8

More-Involved Problems

9.3.9 A spring and mass system is shown in the figure.

a) First, as a review, let k_1, k_2, and k_3 equal zero and k_4 be nonzero. What is the natural frequency of this system?

b) Now, let all the springs have nonzero stiffness. What is the stiffness of a single spring equivalent to the combination of k_1, k_2, k_3, k_4? What is the frequency of oscillation of mass M?



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Problem 9.3.9

9.3.10 **Mass** m **hangs from a spring** with constant k and which has the length l_0 when it is relaxed (i.e., when no mass is attached). It only moves vertically.

a) Draw a Free Body Diagram of the mass.

b) Write the equation of linear momentum balance. *

c) Reduce this equation to a standard differential equation in x, the position x of the mass. *

d) Verify that one solution is that $x(t)$ is constant at $x = l_0 + mg/k$.

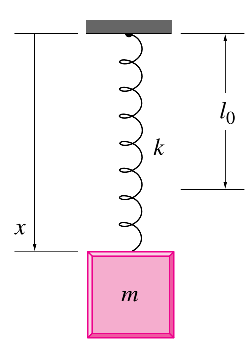
e) What is the meaning of that solution? (That is, describe in words what is going on.) *

f) Define a new variable $\hat{x} = x - (l_0 + mg/k)$. Substitute $x = \hat{x} + (l_0 + mg/k)$ into your differential equation and note that the equation is simpler in terms of the variable $\hat{x}$. *

g) Assume that the mass is released from an an initial position of $x = D$. What is the motion of the mass? *

h) What is the period of oscillation of this oscillating mass? *

i) Why might this solution not make physical sense for a long, soft spring if the initial stretch is large. In other words, what is wrong with this solution if $D > \ell_0 + 2mg/k$? *

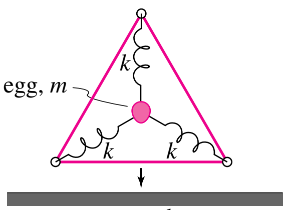


Filename:pg141-1
Problem 9.3.10

9.3.11 One of the winners in an egg-drop contest was a structure in which rubber bands held the egg at the center of it. Here is a model. Consider the egg to be a particle of mass m and the springs to be linear with spring constants k. Consider only a two-dimensional version of the winning design as shown in the figure. Assume the frame hits the ground on one of the straight sections. Assume small motions (deflection $\ll$ side-length) and that the springs do not buckle.

a) what will be the frequency of vibration of the egg after impact?

b) What is the maximum vertical deflection of the egg (relative to its equilibrium position)?



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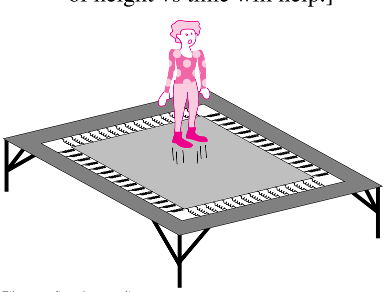
Problem 9.3.11

9.3.12 A person jumps on a trampoline. The trampoline is modeled as having an effective vertical undamped linear spring with stiffness $k = 200\,\text{lbf/ ft}$. The person is modeled as a rigid mass $m = 150\,\text{lbm}$. $g = 32.2\,\text{ft/s}^2$.

a) What is the period of motion if the person's motion is so small that her feet never leave the trampoline? *

b) What is the maximum amplitude of motion (amplitude of the sine wave) for which her feet never leave the trampoline? *

c) (harder) If she repeatedly jumps so that her feet clear the trampoline by a height $h = 5\,\text{ft}$, what is the period of this motion (note, the contact time is *not* exactly half of a vibration period)? [Hint, a neat graph of height vs time will help.] *



Filename:pfigure3-trampoline

Problem 9.3.12: A person jumps on a trampoline.

9.3.13 A mass moves on a frictionless surface. It is connected to a dashpot with damping coefficient b to its right and a spring with constant k and rest length ℓ to its left. At the instant of interest, the mass is moving to the right and the spring is stretched a distance x from its position where the spring is unstretched. There is gravity.

a) Draw a free body diagram of the mass at the instant of interest.

b) Derive the equation of motion of the mass. *

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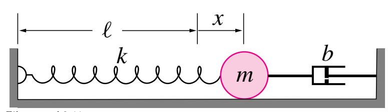

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Problem 9.3.13

9.3.14 The equation of motion of an unforced mass-spring-dashpot system is, $m\ddot{x} + c\dot{x} + kx = 0$, as discussed in the text. For a system with $m = 0.4\,\text{kg}$, $c = 10\,\text{kg/ s}$, and $k = 5\,\text{N/m}$,

a) Find whether the system is underdamped, critically damped, or overdamped.

b) Sketch a typical solution of the system.

c) Make an accurate plot of the response of the system (displacement vs time) for the initial conditions $x(0) = 0.1\,\text{m}$ and $\dot{x}(0) = 0$.

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9.3.15 Experiments conducted on free oscillations of a damped oscillator reveal that the amplitude of oscillations drops to 25% of its peak value in just 3 periods of oscillations. The period os oscillation is measured to be 0.6 s and the mass of the system is known to be 1.2 kg. Find the damping coefficient and the spring stiffness of the system.

9.3.16 You are required to design a mass-spring-dashpot system that, if disturbed, returns to its equilibrium position the quickest. You are given a mass, $m = 1\,\text{kg}$, and a damper with $c = 10\,\text{kg/ s}$. What should be the stiffness of the spring? Your solution needs to include your definition of "'quickest".

9.4 Coupled motion in 1D

The primary emphasis of this section is setting up correct differential equations (without sign errors) and solving these equations on the computer.

Preparatory Problems

9.4.1 Write the following set of coupled second order ODE's as a system of first order ODE's.

$$\begin{aligned}\ddot{x}_1 &= k_2(x_2 - x_1) - k_1 x_1\\ \ddot{x}_2 &= k_3 x_2 - k_2(x_2 - x_1)\end{aligned}$$

9.4.2 The solution of a set of a second order differential equations is:

$$\begin{aligned}\xi(t) &= A\sin\omega t + B\cos\omega t + \xi^*\\ \dot{\xi}(t) &= A\omega\cos\omega t - B\omega\sin\omega t,\end{aligned}$$

where A and B are constants to be determined from initial conditions and ξ^* is a known constant. Assume A and B are the only unknowns.

a) Write the equations in matrix form which you would need to solve in order to find A and B in terms of $\xi(0)$ and $\dot{\xi}(0)$.

b) Solve the equations in symbols.

c) Solve for the numerical constants A and B using the matrix form, if $\xi(0) = 0$, $\dot{\xi}(0) = 0.5$, $\omega = 0.5\,\text{rad/s}$ and $\xi^* = 0.2$.

9.4.3 A set of first order linear differential equations is given:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 + kx_1 + cx_2 &= 0.\end{aligned}$$

Write these equations in the form $\dot{\vec{\boldsymbol{x}}} = [\mathbf{A}]\vec{\boldsymbol{x}}$, where $\vec{\boldsymbol{x}} = \left\{\begin{array}{c} x_1\\ x_2\end{array}\right\}$.

9.4.4 Write the following pair of coupled ODE's as a set of first order ODE's.

$$\begin{aligned}\ddot{x}_1 + x_1 &= \dot{x}_2\sin t\\ \ddot{x}_2 + x_2 &= \dot{x}_1\cos t\end{aligned}$$

9.4.5 The following set of differential equations can be written in first order form, and in particular, in matrix form $\dot{\vec{x}} = [\mathbf{A}]\vec{x} + \vec{c}$. In general equations of motion are not so simple, but *linear* cases like this are prevalent in the analytic study of dynamical systems.

$$\begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 + 5\Omega^2 x_1 - 4\Omega^2 x_2 &= 2\Omega^2 v_1^* \\ \dot{x}_4 - 4\Omega^2 x_1 + 5\Omega^2 x_2 &= -\Omega^2 v_1^* \end{aligned}$$

9.4.6 Write each of the following equations as a system of first order ODE's.

a) $\ddot{\theta} + \lambda^2\theta = \cos t$,

b) $\ddot{x} + 2p\dot{x} + kx = 0$,

c) $\ddot{x} + 2c\dot{x} + k\sin x = 0$.

9.4.7 A train moves at a constant absolute velocity $v\hat{\imath}$. A passenger, idealized as a point mass, walks at an absolute absolute velocity $u\hat{\imath}$, where $u > v$. What is the velocity of the passenger relative to the train?

9.4.8 Two equal masses, each denoted by the letter m, are on an air track. One mass is connected by a spring to the end of the track. The other mass is connected by a spring to the first mass. The two spring constants are equal and represented by the letter k. In the rest configuration (springs are relaxed) the masses are a distance ℓ apart. Motion of the two masses x_1 and x_2 is measured relative to this configuration.

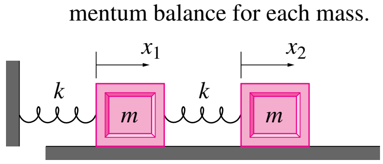
a) Write the potential energy of the system for arbitrary displacements x_1 and x_2 at some time t.

b) Write the kinetic energy of the system at the same time t in terms of $\dot{x}_1$, $\dot{x}_2$, m, and k.

c) Write the total energy of the system.

d) Draw a free body diagram for each mass.

e) Write the equation of linear momentum balance for each mass.



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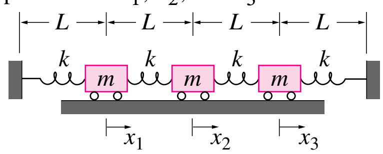
Problem 9.4.8

9.4.9 For the three-mass system shown, draw a free body diagram of each mass. Write the spring forces in terms of the displacements x_1, x_2, and x_3.



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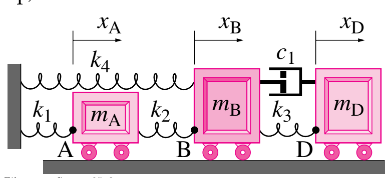
Problem 9.4.9

9.4.10 The springs shown are relaxed when $x_A = x_B = x_D = 0$. In terms of some or all of m_A, m_B, m_D, x_A, x_B, x_D, $\dot{x}_A$, $\dot{x}_B$, $\dot{x}_C$, and k_1, k_2, k_3, k_4 and c_1, find the acceleration of block B. *



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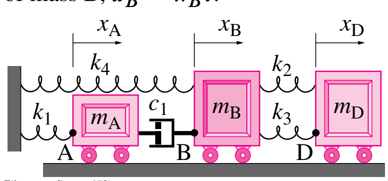
Problem 9.4.10

9.4.11 A system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$. Given $k_1, k_2, k_3, k_4, c_1, m_A, m_B, m_D, x_A, x_B, x_D, \dot{x}_A, \dot{x}_B$, and $\dot{x}_D$, find the acceleration of mass B, $\vec{a}_B = \ddot{x}_B\hat{\imath}$. *



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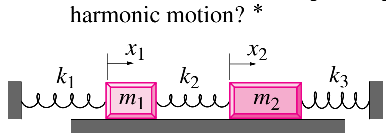
Problem 9.4.11

9.4.12 A two degree of freedom mass-spring system, made up of two unequal masses m_1 and m_2 and three springs with unequal stiffnesses k_1, k_2 and k_3, is shown in the figure. All three springs are relaxed in the configuration shown. Neglect friction.

a) Derive the equations of motion for the two masses.

b) Does each mass undergo simple harmonic motion? *



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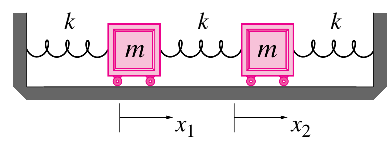
Problem 9.4.12

9.4.13 Normal Modes. Three equal springs (k) hold two equal masses (m) in place. There is no friction. x_1 and x_2 are the displacements of the masses from their equilibrium positions.

a) How many independent normal modes of vibration are there for this system? *

b) Assume the system is in a normal mode of vibration and it is observed that $x_1 = A\sin(ct) + B\cos(ct)$ where A, B, and c are constants. What is $x_2(t)$? (The answer is not unique. You may express your answer in terms of any of A, B, c, m and k.) *

c) Find all of the frequencies of normal-mode-vibration for this system in terms of m and k. *



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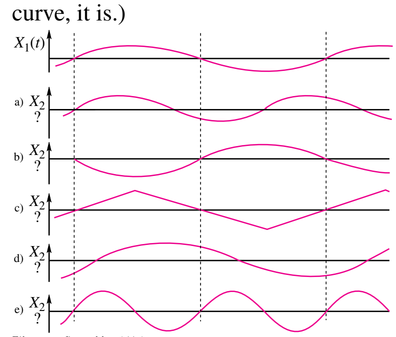
Problem 9.4.13

9.4.14 $x_1(t)$ and $x_2(t)$ are measured positions on two points of a vibrating structure. $x_1(t)$ is shown. Some candidates for $x_2(t)$ are shown. Which of the $x_2(t)$ could possibly be associated with a normal mode vibration of the structure? Answer "could" or "could not" next to each choice and briefly explain your answer (If a curve looks like it is meant to be a sine/cosine curve, it is.)



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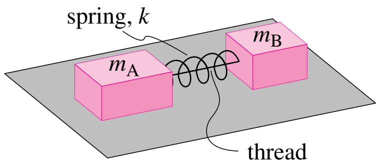
Problem 9.4.14

More-Involved Problems

9.4.15 A massless spring with constant k is held compressed a distance δ from its relaxed length by a thread connecting blocks A and B which are still on a frictionless table. The blocks have mass m_A and m_B, respectively. The thread is suddenly but gently cut, the blocks fly apart and the spring falls to the ground. Find the speed of block A as it slides away. *



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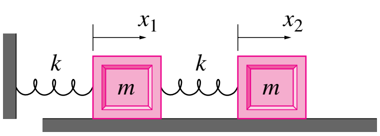
Problem 9.4.15

9.4.16 In the system below the masses are in equilibrium with the springs when $x_1 = x_2 = 0$.

a) First do problem 9.4.8.

b) Pick parameter values and initial conditions of your choice and simulate a motion of this system. Make a plot of the motion of, say, one of the masses vs time,

c) Explain how your plot does or does not make sense in terms of your understanding of this system. Is the initial motion in the right direction? Are the solutions periodic? Bounded? etc.



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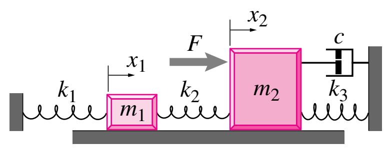
Problem 9.4.16

9.4.17 Two masses are connected to fixed supports and each other with the three springs and dashpot shown. The force F acts on mass 2. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when the springs are unstretched. The ground is frictionless. The governing equations for the system shown can be written in first order form if we define $v_1 \equiv \dot{x}_1$ and $v_2 \equiv \dot{x}_2$.

a) Write the governing equations in a neat first order form. Your equations should be in terms of any or all of the constants m_1, m_2, k_1, k_2, k_3, C, the constant force F, and t. Getting the signs right is important.

b) Write computer commands to find and plot $v_1(t)$ for 10 units of time. Make up appropriate initial conditions.

c) For constants and initial conditions of your choosing, plot x_1 vs t for enough time so that decaying erratic oscillations can be observed.



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Problem 9.4.17

9.4.18 The three beads of masses m, $2m$, and m connected by massless linear springs of constant k slide freely on a straight rod. Let x_i denote the displacement of the i^{th} bead from its equilibrium position at rest.

a) Write expressions for the total kinetic and potential energies.

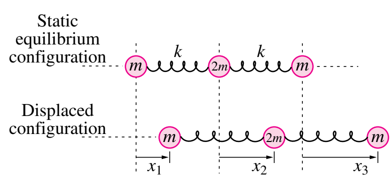
b) Write an expression for the total linear momentum.

c) Draw free body diagrams for the beads and use Newton's second law to derive the equations for motion for the system.

d) Verify that total energy and linear momentum are both conserved.

e) Show that the center of mass must either remain at rest or move at constant velocity.

f) What can you say about vibratory (sinusoidal) motions of the system?



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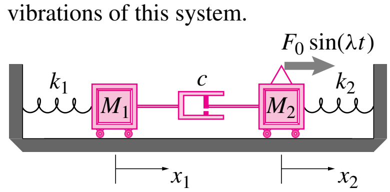
Problem 9.4.18

9.4.19 Two masses are connected to fixed supports and each other with the two springs and dashpot shown. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when both springs are unstretched.

For the special case that $C = 0$ and $F_0 = 0$ clearly define two different set of initial conditions that lead to normal mode vibrations of this system.



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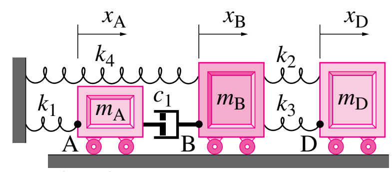
Problem 9.4.19

9.4.20 As in problem 9.4.11, a system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$.

a) In the special case when $k_1 = k_2 = k_3 = k_4 = k$, $c_1 = 0$, and $m_A = m_B = m_D = m$, find a normal mode of vibration. Define it in any clear way and explain or show why it is a normal mode in any clear way. *

b) In the same special case as in (a) above, find another normal mode of vibration. *



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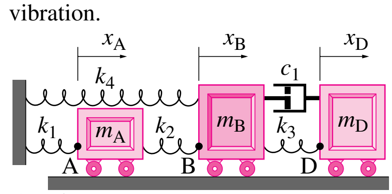
Problem 9.4.20

9.4.21 As in problem 9.4.10, a system of three masses, four springs, and one damper are connected as shown. In the special case when $c_1 = 0$, find the normal modes of vibration.



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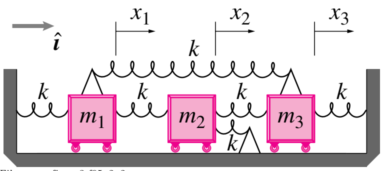
Problem 9.4.21

9.4.22 Normal modes. All three masses have $m = 1\,\mathrm{kg}$ and all 6 springs are $k = 1\,\mathrm{N/m}$. The system is at rest when $x_1 = x_2 = x_3 = 0$.

a) Find as many different initial conditions as you can for which normal mode vibrations result. In each case, find the associated natural frequency. (we will call two initial conditions $[v]$ and $[w]$ different if there is no constant c so that $[v_1\ v_2\ v_3] = c[w_1\ w_2\ w_3]$. Assume the initial velocities are zero.)

b) For the initial condition
$$[x_0] = [\,0.1\,\mathrm{m}\ \ 0\ \ 0\,],$$
$$[\dot{x}_0] = [\,0\ \ 2\,\mathrm{m/s}\ \ 0\,]$$
what is the initial (immediately after the start) acceleration of mass 2?



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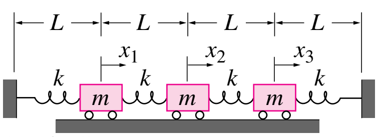
Problem 9.4.22

9.4.23 For the three-mass system shown, assume $x_1 = x_2 = x_3 = 0$ when all the springs are fully relaxed. One of the normal modes is described with the initial condition $(x_{10}, x_2, x_3) = (1, 0, -1)$.

a) What is the angular frequency ω for this mode? Answer in terms of $L, m, k,$ and g. (Hint: Note that in this mode of vibration the middle mass does not move.) *

b) Make a neat plot of x_2 versus x_1 for one cycle of vibration with this mode.



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Problem 9.4.23

9.4.24 Two blocks with masses M and m are connected by a spring with constant k and free length ℓ_0 that can sustain compression. Mass M is resting on the ground at the start. There is gravity. The upwards vertical displacement of mass m is x, which is zero when the spring is at its rest length and M is on the ground.

a) For what value of x is the system in static equilibrium?

b) Find a differential equation governing the motion of the m assuming M remains on the ground.

c) Draw a free body diagram of M.

d) For what value of x is M on the verge of lifting off the ground.

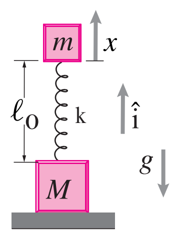
e) Defining y as the height of the lower mass, write two coupled differential equations for the motion of m and M if both masses are in the air.

f) Find the value of $x < 0$ so that if the system is started from rest with that x and $y = 0$ that the ground reaction force on M just goes to zero.

g) Starting here, this problem is more of a project than a typical homework problem. Assume $x(t = 0)$ is less than the value computed above. Write a computer program that integrates the equations of motion until M lifts off and then switches to integrating the equations for the two masses in the air.

h) modify your program so that if M hits the ground again, it sticks until the ground reaction force goes to zero again.

i) By playing around, this way or that, see if you can find a special value for $x(t = 0)$ so that the bouncing continues indefinitely. (This is a perhaps surprising result, that a system with plastic collisions can continue to bounce indefinitely.)



Filename:pfigure-2blocksandspring
Problem 9.4.24

9.5 1D Collisions

Preparatory Problems

9.5.1 Before a collision two particles, $m_A = 1\,\mathrm{kg}$ and $m_B = 2\,\mathrm{kg}$, have velocities of $v_A^- = 10\,\mathrm{m/s}$ and $v_B^- = 5\,\mathrm{m/s}$. After the collision the velocity of A is $v_A^+ = 6\,\mathrm{m/s}$.

a) What is the momentum of A before the collision?

b) What is the momentum of B before the collision?

c) What is the system momentum before the collision?

d) What is the momentum of A after the collision?

e) What is the system momentum after the collision?

f) What is the momentum of B after the collision?

g) What is the impulse that A applies to B during the collision?

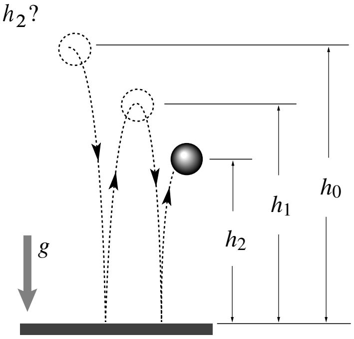
h) What is the impulse that B applies to A during the collision?

i) What is the kinetic energy of the system before the collision?

j) What is the kinetic energy of the system after the collision?

k) What is the coefficient of restitution?

9.5.2 A ball is dropped from a height of $h_0 = 10\,\mathrm{m}$ onto a hard stationary surface. After the first bounce, it reaches a height of $h_1 = 6.4\,\mathrm{m}$. What is the coefficient of restitution between the ball and ground? What is the height of the second bounce, h_2?



Filename:Danef94s1q7
Problem 9.5.2

9.5.3 A 20 gram, 500 m/s bullet embeds in an initially-stationary 50 kg rigid block. What is the coefficient of restitution? What is the velocity of the block after this collision? *

9.5.4 A ball of mass m is dropped vertically from a height h. The only force acting on the ball in its flight is gravity. The ball strikes the ground with speed v^- and after collision it rebounds vertically with reduced speed v^+ directly proportional to the incoming speed, $v^+ = ev^-$, where $0 < e < 1$. What is the maximum height the ball reaches after one bounce, in terms of h, e, and g. *

9.5.5 Set up the following equations in matrix form and solve for v_A and v_B, if $v_0 = 2.6\,\mathrm{m/s}$, $e = 0.8$, $m_A = 2\,\mathrm{kg}$, and $m_B = 500\mathrm{g}$:

$$\begin{aligned} m_A v_0 &= m_A v_A + m_B v_B \\ -e v_0 &= v_A - v_B. \end{aligned}$$

More-Involved Problems

9.5.6 Before a collision two particles, $m_A = 7\,\mathrm{kg}$ and $m_B = 9\,\mathrm{kg}$, have velocities of $v_A^- = 6\,\mathrm{m/s}$ and $v_B^- = 2\,\mathrm{m/s}$. The coefficient of restitution is $e = .5$. Find the impulse of mass A on mass B and the velocities of the two masses after the collision.

9.5.7 Two frictionless masses $m_A = 2\,\mathrm{kg}$ and mass $m_B = 5\,\mathrm{kg}$ travel on straight collinear paths with speeds $V_A = 5\,\mathrm{m/s}$ and $V_B = 1\,\mathrm{m/s}$, respectively. The masses collide since $V_A > V_B$. Find the amount of energy lost in the collision. The coefficient of restitution is $e = 0.5$.

Filename:Danef94s3q7
Problem 9.5.7

9.5.8 A ball of mass m is dropped from height h onto the solid hard ground where its coefficient of restitution is $e < 1$. The gravitational constant is g.

a) How many times does the ball bounce before it comes to a stop?

b) How long does it take from first release until it comes to a stop?

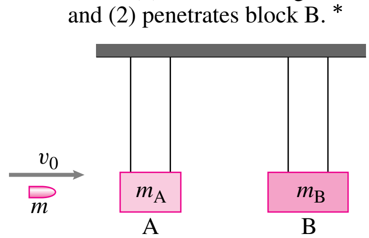
c) What is the total distance the ball travels before coming to a stop (add up and down distances)?

9.5.9 A bullet of mass m with initial speed v_0 is fired in the horizontal direction through block A of mass m_A and becomes embedded in block B of mass m_B. Each block is suspended by thin wires. The bullet causes A and B to start moving with speed of v_A and v_B respectively. Determine

a) the initial speed v_0 of the bullet in terms of v_A and v_B, *

b) the velocity of the bullet as it travels from block A to block B,and *

c) the energy loss due to friction as the bullet (1) moves through block A and (2) penetrates block B. *



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Problem 9.5.9

9.5.10 A basketball with mass m_b is dropped from height h onto the hard solid ground on which it has coefficient of restitution e_b. Just on top of the basketball, falling with it and then bouncing against it after the basketball hits the ground, is a small rubber ball with mass m_r that has a coefficient of restitution e_r with the basketball.

a) In terms of some or all of m_b, m_r, h, g, e_b and e_r how high does the rubber ball bounce (measure height relative to the collision point)?

b) Assuming the coefficients of restitution are less than or equal to one, for given h, what mass and restitution parameters maximize the height of the bounce of the rubber ball and what is that height?

9.5.11 Show that it is necessary that $|e| \leq 1$ for the net kinetic energy (sum of the two kinetic energies of the colliding particles) to not increase.

9.5.12 According to the problem above, unless energy is created in the collision (as in an explosion), $-1 \leq e \leq 1$. Show that, for given masses and given initial velocities, that the loss of system kinetic energy is maximized by $e = 0$.

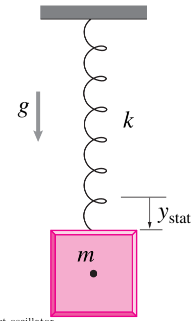
9.6 Advanced vibration

Preparatory Problems

9.6.1 Given that $\ddot{\theta} + k^2\theta = \beta \sin \omega t$, $\theta(0) = 0$, and $\dot{\theta}(0) = \dot{\theta}_0$, find $\theta(t)$.

9.6.2 A mass-spring-dashpot system has $m = 1\,\mathrm{kg}$, $k = 10\,\mathrm{kN/m}$, and $c = 5\,\mathrm{kg/\,s}$. Find the natural frequency, damping ratio, and the damped frequency of the system. Specify whether the system is underdamped, critically damped or overdamped.

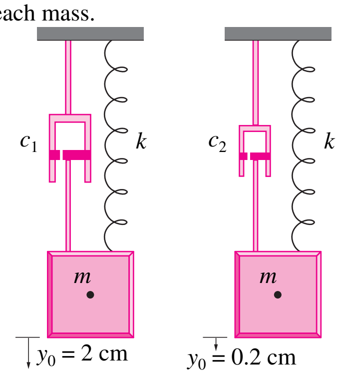
9.6.3 A mass-spring oscillator hangs vertically under gravity. The mass rests in static equilibrium by stretching the spring by an amount $y_{\text{static}} = 0.025\,\mathrm{m}$. Take your favorite value of g and find the natural frequency of the oscillator. How much time does the oscillator take to complete one oscillation?



Filename:pfig9-6-vert-oscillator
Problem 9.6.3

9.6.4 You are given to design a SDOF damped oscillator that should show no oscillations at all when disturbed from the equilibrium (*i.e.,* , it should return to equilibrium without overshooting on the other side). You are given a spring with stiffness $k = 500\,\mathrm{N/m}$, a hydraulic damper with $c = 10\,\mathrm{kg/\,s}$, and you have a choice of masses from $m = 1\,\mathrm{kg}$ to $m = 10\,\mathrm{kg}$ in the increments of half kg. Find the appropriate mass.

9.6.5 Two SDOF oscillators with the same k and m but different c's are hung from the ceiling as shown in the figure. The one on the left is pulled down 2 cm and let go. The other is pulled down by 0.2 cm and let go. Which oscillator undergoes more number of oscillations before reaching the steady state. Find the steady state displacement of each mass.



Filename:pfig9-6-2oscillators
Problem 9.6.5

9.6.6 The natural frequency, λ_n, of a SDOF system is 150 rad/s. Find the minimum damping (ξ) that the system must have for the resonant frequency to occur below 100 rad/s?

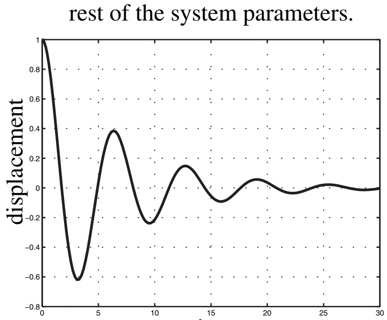
9.6.7 A machine that can be modeled as a SDOF system is put under vibration test for estimating the system parameters m, k, and c. First, a transient test is conducted by disturbing the machine from its equilibrium and letting its settle down to equilibrium again. The transient response is recorded as a displacement versus time plot and is shown in fig. 9.6.7(a). Next, a sinusoidal forcing is of amplitude F_0 and angular frequency p is applied on the machine and its steady state response is recoded along with the forcing function. This response is shown in fig. 9.6.7(b).

a) Mark the relevant points on the transient response plot and explain, with equations, which systems parameters can be determined using what information from this plot.

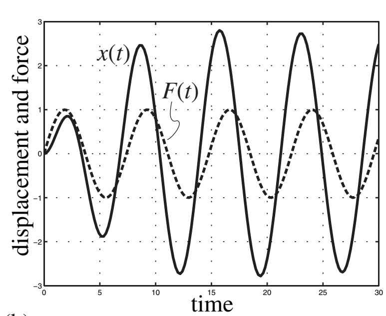
b) On the steady state plot, mark the phase difference between the response and the forcing function. From the given phase, can you find out whether $p > \lambda_n$ or $p < \lambda_n$?

c) From the phase difference of the steady state response and the information obtained from the transient response, can you determine the frequency ratio r? Explain with appropriate equations.

d) From the amplitude of the steady state response, and the rest of the information obtained above, find the rest of the system parameters.



(a)



(b)

Filename:pfig9-6-response-plots

Problem 9.6.7

9.6.8 Three SDOF systems, each with the same mass but different stiffnesses, $k_1 = k$, $k_2 = 2k$, and $k_3 = 4k$, and different damping, $c_1 = c$, $c_2 = 2c$, and $c_3 = c/2$, are subjected to the same periodic forcing, $F = F_0 \sin pt$ where p is less than the resonant frequency of each of the systems. Sketch approximately the response of each of the three oscillators assuming all of them to be underdamped. Clearly mark the transient and steady state part of the response, and indicate the relative values of the response amplitudes.

More-Involved Problems

9.6.9 A 3 kg mass is suspended by a spring ($k = 10\,\mathrm{N/m}$) and forced by a 5 N sinusoidally oscillating force with a period of 1 s. What is the amplitude of the steady-state oscillations (ignore the "homogeneous" solution)

9.6.10 A machine produces a steady-state vibration due to a forcing function described by $Q(t) = Q_0 \sin \omega t$, where $Q_0 = 5000N$. The machine rests on a circular concrete foundation. The foundation rests on an isotropic, elastic half-space. The equivalent spring constant of the half-space is $k = 2{,}000{,}000\,\mathrm{N{\cdot}m}$ and has a damping ratio $d = c/c_c = 0.125$. The machine operates at a frequency of $\omega = 4$ Hz.

1. What is the natural frequency of the system?
2. If the system were undamped, what would the steady-state displacement be?
3. What is the steady-state displacement given that $d = 0.125$?
4. How much additional thickness of concrete should be added to the footing to reduce the damped steady-state amplitude by 50%? (The diameter must be held constant.)

9.6.11 The transient response of an oscillatory system shows exponential decay of the peak displacements at each cycle. The second peak is found to be twice as big as the fifth peak. Find the damping ratio xi for the system. How many cycles does it take for the peak displacement to drop below 5% of the first peak displacement?

9.6.12 A 50 kg engine is mounted on springs with an equivalent single spring stiffness of 1200 N/m. Using various means, enough damping needs to be provided so that any unwanted vibration dies quickly. Assume that this objective is met by dissipating 80% of the available energy in a single cycle of vibration. Find the damping coefficient of the system.

9.6.13 Consider the system shown in the figure. You are given that $m = 10\,\mathrm{kg}$, $k = 50\,\mathrm{N/m}$, and $c = 5\,\mathrm{kg/s}$. A periodic force $F = F_0 \cos pt$ acts on the system as shown where $F_0 = 25\,\mathrm{N}$ and $p = 2.5\,\mathrm{rad/s}$.

a) Find the resonant frequency of the system.

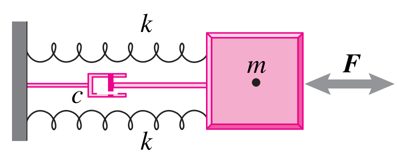
b) Find the steady state response of the system, specifying the amplitude and phase of the motion.

c) What is the displacement amplification ($G = A/(F_0/k)$)?

d) Find the work done by the force on the system in one cycle.

e) Find the energy lost to the damper in one cycle.

f) Find the quality factor, Q, of the system using the energy calculations.

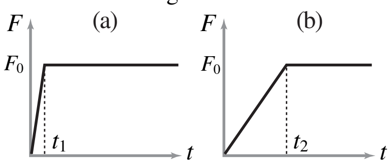


Filename:pfig9-6-damped-forced

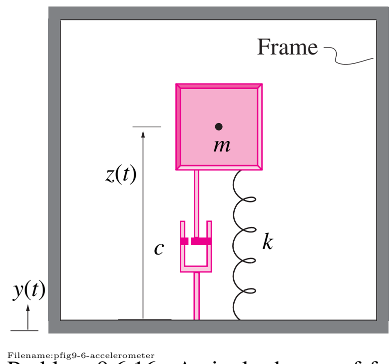
Problem 9.6.13

9.6.14 A MEMS cantilever beam resonator is used for mass measurement of biological molecules by comparing the shift in the resonant frequency of the beam after the test molecule is attached to the free end of the beam. In a SDOF model of the resonator, it is equivalent to finding the difference in the resonant frequency of the system with mass m and $m + \Delta m$. If the 'effective mass' of the beam (mass to be used in the SDOF model) is 2.05×10^{-15} kg, the stiffness is $0.625\,\mathrm{N/m}$, and the Q of the resonator is 900, find the shift in the resonant peak in Hz when a biological molecule of mass 1.36×10^{-21} kg is attached to end of the beam (equivalently to the mass m).

9.6.15 A damped mass-spring system is subjected to a constant load $F_0 = 50\,\mathrm{N}$ by ramping the load to the constant level in (a) $t_1 = 2\,\mathrm{s}$ and (b) $t_2 = 10\,\mathrm{s}$. If the mass of the system $m = 1\,\mathrm{kg}$, the natural frequency $\lambda_n = 62\,\mathrm{rad/s}$, and the damping ratio $\xi = 0.2$, find the difference in the settling time of the system to the steady state between the two given cases.



Filename:pfig9-6-ramped-forcing
Problem 9.6.15



Filename:pfig9-6-accelerometer
Problem 9.6.16: A single degree of freedom spring-mass model of an accelerometer.

9.6.16 **An accelerometer** is a sensor that is used to measure acceleration of a body. It can be modeled as a single degree of freedom spring-mass-dashpot system that is attached to a body frame as shown in the figure. Assume that the body undergoes vertical motion denoted by $y(t)$ and, as a result, the mass of the accelerometer undergoes vertical motion $z(t)$ relative to the frame. From a measurement of $z(t)$ we want to know if we can determine the acceleration $\ddot{y}$ of the frame. Neglect gravity.

a) What is the absolute or inertial acceleration of the mass in terms of $z(t)$ and $y(t)$ and their derivatives?

b) Write the equation of motion of the mass in terms of z and y.

c) Assume that $y(t) = y_0 \sin pt$. What is the magnitude of acceleration of the frame (that is, the peak acceleration of this sine wave)?

d) Find the steady state response $z(t)$ of the accelerometer when $y(t) = y_0 \sin pt$ (a big mess). Plot the amplitude of the response vs the amplitude of the frame acceleration as the frequency p is varied.

e) One would like a signal (z) that is proportional to acceleration $\ddot{y}$ independent of the frequency of shaking. Show that this requires that the frequency of frame motion p must be much smaller than the natural frequency λ_n.

f) What is the amplitude of the response (the magnitude of z) of the accelerometer when $p \gg \lambda_n$?

9.6.17 Consider the accelerometer described in Problem 9.6.16. Assume that the frame undergoes a sinusoidal motion given by $y(t) = y_0 \sin pt$.

a) Find the response $z(t)$ of the accelerometer.

b) Given that $m = 0.5\,\mathrm{kg}$, $k = 5\ \mathrm{kN/m}$, and $c = 10\,\mathrm{kg/s}$, find the maximum acceleration that the accelerometer can sense, assuming the accelerometer to work in the frequency range much below its natural frequency (*i.e.*, $p/\lambda_n \ll 1$). Express your answer in terms of the gravitational acceleration g (it is customary to talk about acceleration of various things in terms of 'so many g's').

CHAPTER 10

Particle dynamics in space (unconstrained)

This chapter is about the vector equation $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for one particle. Concepts and applications include ballistics and planetary motion. The differential equations of motion are set-up in cartesian coordinates and integrated either numerically, or for special simple cases, by hand. Constraints, forces from ropes, rods, chains, floors, rails and guides that can only be found once one knows the acceleration, are not considered.

Contents

The previous chapter was about particles that move in a straight line. Now we will consider particles that move in more complicated ways. More specifically, in this chapter we will consider the curving motion of a single particle using cartesian coordinates. We will be able to calculate the path of a hit baseball (perhaps taking account of air friction), a satellite, or a bungy jumper.

The key tool is, in Newton's words,

> *"Any change of motion is proportional to the force that acts, and it is made in the direction of the straight line in which that force is acting."*

Realizing that the quantification of motion is the product of mass and velocity, and that the rate of change of velocity is acceleration, in modern language we could rephrase Newton's as:

> *'the net force on a particle is its mass times its acceleration.'*

Informally we think 'force causes motion in the direction of the force'. Then, thinking more carefully we fill in the details that in this context 'motion' means acceleration and that the amount of force needed for a given acceleration is also proportional to the mass.

You can also think of $\vec{F} = m\vec{a}$ as a special case of the more general principle of linear momentum balance (LMB) for a system, where the system of interest is just a single particle. If we start with the general form of LMB given in the front cover, and discussed in general terms in chapter 1, we get:

$$\begin{aligned} \sum \vec{F}_\mathrm{i} &= \dot{\vec{L}} && \text{Linear momentum balance for any system} \\ &= \sum m_i \vec{a}_\mathrm{i} && \text{for a system of particles} \\ &= m\vec{a} && \text{for one particle} \end{aligned}$$

If we define $\vec{F}$ to be the net force on the particle ($\vec{F} = \sum \vec{F}_\mathrm{i}$) then linear momentum balance becomes 'Newton's second law',

$$\vec{F} = m\vec{a}. \tag{10.1}$$

Does force cause acceleration or is it the other way around? Whether force causes acceleration or acceleration necessitates force, the issue of *causality*, is a philosophical question of no import. All that shows up in the math, and in any problem solution, is that when there is a net force there

Filename:FountainDetroit5037

Figure 10.1: Some small blobs of water fly in nearly (neglecting air friction) parabolic arcs. This fountain is in the Detroit airport.

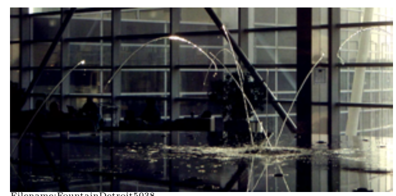


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Figure 10.2: Small streams start to show the arcs in a still photo.

Filename:FountainDetroit5041

Figure 10.3: A full parabola shows.

Filename:FountainDetroit5040

Figure 10.4: The water runs in continuous streams. Some fancy valve work (under the visible part of the fountain) is required to get such a laminar stream that holds together for the whole flight. This water draws its own graph of the trajectory of its particles. You can do the same kind of thing with a squirt gun next to a blackboard.

is acceleration of mass, and when there is acceleration of mass there is a net force. When a car crashes into a pole there is a big force and a big deceleration of the car. You could think of the force on the bumper as causing the car to slow down rapidly. Or you could think of the rapid car deceleration as necessitating a force. It is only a matter of personal taste because in both cases the same eqn. (10.1) applies. Equations don't have a 'cause' side and a 'result' side (If $A = B$ does A cause B or does B cause A?).

Acceleration is the second derivative of position

What is acceleration? If $\vec{r}(t)$ is the position of a particle relative to some origin, the particle's acceleration is

$$\vec{a} \equiv \ddot{\vec{r}}.$$

As for scalars, one or two dots over a vector is a short hand notation for the first or second time derivative. In the next section we'll explain how to take the derivative of a vector. As explained in box 10.1 the vector diferentiation has to be done using an appropriate coordinate system.

10.1 Newton's laws are accurate in a Newtonian reference frame

Acceleration is calculated from position using a particular coordinate system. For our purposes here, a coordinate system is also a *reference frame*. The calculation of acceleration of a particle depends on how the coordinate system itself is moving. So the simple equation

$$\vec{F} = m\vec{a}$$

has as many different interpretations as there are differently moving coordinate systems (and there are an infinite number of those). In each different coordinate system, the coordinates of a given particle are different from the coordinates in another system. And the calculated accelerations are also different. Sir Isaac Newton was sitting on earth contemplating position relative to the ground at his feet when he noticed that his second law accurately described things like falling apples. So the equation $\vec{F} = m\vec{a}$ is valid using coordinate systems that are fixed to the earth. Well, not quite. Isaac noticed that the motion of the planets around the sun only followed his law if the acceleration was calculated using a coordinate system that was still relative to 'the fixed stars.' With a fixed-star coordinate system you calculate slightly (about 0.25%) different accelerations for things like falling apples than you do using a coordinate system that is stuck to the earth. And nowadays when astrophysicists try to figure out how the laws of mechanics explain the shapes of spiral galaxies, they realize that none of the so-called 'fixed stars' are so totally fixed. They need even more care to pick a coordinate system where eqn. (10.1) is accurate.

Despite all this confusion, it is generally agreed that no matter where you are there exists some coordinate system for which Newton's laws are incredibly accurate.

Further, once you know one 'good' coordinate system you know many others. Any system which translates (has no relative rotation) with constant velocity relative to a 'good' system is also a 'good' system. Why? Because the difference between the accelerations measured in the two frames is the relative acceleration of the frames, which is zero. Mechanics is the same on a constant velocity train or plane as on a stationary plane or train. Any reference frame in which Newton's laws are accurate is called a $\mathcal{N}$*ewtonian reference frame*. Sometimes people also call such a frame a $\mathcal{F}$*ixed frame*, as in 'fixed to the earth' or 'fixed to the stars'. But a Newtonian frame could also be 'fixed' to a constant velocity train or plane.

For most engineering purposes a coordinate system attached to the ground under your feet is a good approximation to a $\mathcal{N}$ewtonian frame. Fortunately. Or else apples would fall differently. Imagine Newton's apple having fallen on some crazy curved path leaving Newton confounded and the subject of mechanics still a mystery. The fall of apples, both in Newton's day and now, is well predicted using Newton's laws and treating the ground as a $\mathcal{N}$ewtonian frame. However, if you are interested in trajectory control of sattelites, you need to use something more like the 'fixed stars' as your (even more accurate) Newtonian reference frame in order to make accurate predictions using Newton's laws.

10.1 Dynamics of a particle in space

Time derivative of a vector: position, velocity and acceleration

From here to the end of the book most of our calculations will involve vector-valued functions of time. For example, the vectors linear momentum $\vec{\boldsymbol{L}}$ and angular momentum $\vec{\boldsymbol{H}}$ have a central place in mechanics. Evaluating them depends, in turn, on understanding the relation between position $\vec{\boldsymbol{r}}$, and its rate of change, called velocity $\vec{\boldsymbol{v}}$. We also need to know the relation between velocity $\vec{\boldsymbol{v}}$ and its rate of change, the acceleration $\vec{\boldsymbol{a}}$.

What do we mean by the rate of change of a vector? The rate of change of any quantity, including a vector, is the ratio of the change of that quantity to the amount of time that passes, for very small amounts of time. ①

$$\text{rate of change of any } (thing) \equiv \frac{\text{amount } thing \text{ changes}}{\text{amount of time for that change}}$$

The notation for the rate of change of a vector $\vec{\boldsymbol{r}}$ is

$$\frac{d\vec{\boldsymbol{r}}}{dt}.$$

Or, in the short hand 'dot' notation invented by Newton for just this purpose, $\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}}$. The definition of the derivative $\frac{d\vec{\boldsymbol{r}}}{dt}$ or $\dot{\vec{\boldsymbol{r}}}$ is the same as for any*thing* else,

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\boldsymbol{r}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{r}}(t + \Delta t) - \vec{\boldsymbol{r}}(t)}{\Delta t}$$

where the top of the fraction is the change in $\vec{\boldsymbol{r}}$ and the bottom is the change in t. Underlying most dynamics calculations are derivatives of $\vec{\boldsymbol{r}}(t)$. But we also sometimes need to take derivatives of linear momentum $\vec{\boldsymbol{L}}$, angular momentum $\vec{\boldsymbol{H}}$ and some other quantities (e.g., the angular velocity $\vec{\boldsymbol{\omega}}$ of a rigid object). But all of these quantities somehow depend on the derivatives of $\vec{\boldsymbol{r}}(t)$.

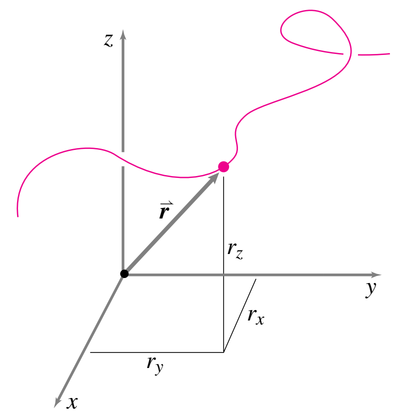

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Figure 10.5: The path of a particle and its position at one time projected onto Cartesian coordinates. The path, or trajectory, of the particle is the sequence of locations of the tip of the $\vec{\boldsymbol{r}}$ vector, if the vector is drawn with its tail at the origin.

①As you should remember from Calculus, these words really describe the average rate of change over the time interval. Only in the mathematical limit, as the time interval approaches zero, is the ratio of "amount of change over the time interval" not just approximately, but exactly, the instantaneous rate of change.

Cartesian coordinates

A simple way to think about vector derivatives is with cartesian coordinates. A moving point has a location $\vec{\boldsymbol{r}}$, relative to the origin of a 'good' (i.e., $\mathcal{N}$ewtonian) reference frame as shown in fig. 10.5, which can be written as:

$$\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}} \qquad \text{or} \qquad \vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}.$$

So velocity is the derivative of $\vec{\boldsymbol{r}}$. Since the base vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are constant, differentiation to get velocity and acceleration is simple:

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} \qquad \text{and} \qquad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}.$$

The idea is illustrated in fig. 10.6. Let's take

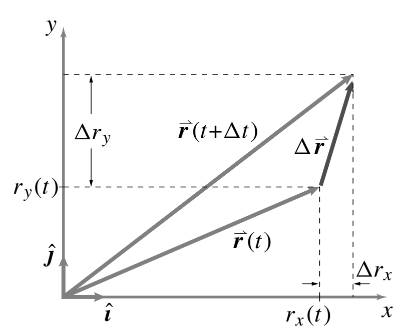

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Figure 10.6: Change of position in Δt broken into components in 2-D . $\Delta\vec{\boldsymbol{r}}$ is $\vec{\boldsymbol{r}}(t + \Delta t) - \vec{\boldsymbol{r}}(t)$. $\Delta\vec{\boldsymbol{r}}$ has components Δr_x and Δr_y. So $\Delta\vec{\boldsymbol{r}} = \Delta r_x\hat{\boldsymbol{\imath}} + \Delta r_y\hat{\boldsymbol{\jmath}}$. In the limit as Δt goes to zero, $\dot{\vec{\boldsymbol{r}}}$ is the ratio of $\Delta\vec{\boldsymbol{r}}$ to Δt.

$$\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath} \quad \text{or} \quad \vec{r}(t) = r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}.$$

We can apply the definition of derivative and find

$$\begin{aligned}
\dot{\vec{r}}(t) &= \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} \\
&= \frac{\overbrace{\left(r_x(t+\Delta t)\hat{\imath} + r_y(t+\Delta t)\hat{\jmath}\right)}^{\vec{r}(t+\Delta t)} - \overbrace{\left(r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}\right)}^{\vec{r}(t)}}{\Delta t} \\
&= \frac{r_x(t+\Delta t) - r_x(t)}{\Delta t}\hat{\imath} + \frac{r_y(t+\Delta t) - r_y(t)}{\Delta t}\hat{\jmath} \\
&= \dot{r}_x(t)\hat{\imath} + \dot{r}_y(t)\hat{\jmath}
\end{aligned}$$

thus showing the palatable result that

> the components of the velocity vector are the time derivatives of the components of the position vector.

(**Beware.** Later in the book we will use base vectors that change in time, such as polar coordinate base vectors, path basis vectors, or basis vectors attached to a rotating frame. For these vectors the components of the vector's derivative will *not* be the derivatives of its components. See box 10.2.)

Figure 10.7 shows a particle P's path, its position at a sequence of times. The position vector $\vec{r}_{P/O}$ is the arrow from the origin to a point on the curve, a different point on the curve at each instant of time. The velocity $\vec{v}$ at time t is the rate of change of position at that time, $\vec{v} \equiv \dot{\vec{r}}$.

Example: Given position as a function of time, find the velocity.
Given that the position of a point is:

$$\vec{r}(t) = C_1\cos(\omega\, t)\hat{\imath} + C_2\sin(\omega\, t)\hat{\jmath}$$

with C_1, C_2 and ω given constants what is the velocity (a vector) at a given time t?

First we note that the components of $\vec{r}(t)$ have been given implicitly as

$$r_x(t) = C_1\cos(\omega\, t) \qquad \text{and} \qquad r_y(t) = C_2\sin(\omega\, t).$$

Then we find the velocity by differentiating each of the components with respect to time and re-assembling as a vector to get

$$\vec{v}(t) = \dot{\vec{r}} = -C_1\omega\sin(\omega\, t)\hat{\imath} + C_2\omega\cos(\omega\, t)\hat{\jmath}$$

Now we evaluate this expression with the given values of C_1, C_2, ω and t .

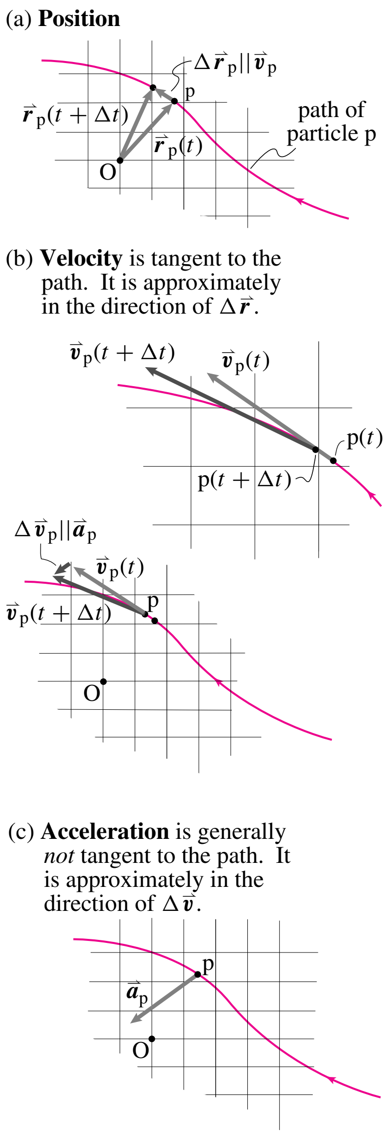

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Figure 10.7: A particle moving on a curve. (a) shows the position vector is an arrow from the origin to the point on the curve. On the position curve the particle is shown at two times: t and $t+\Delta t$. The velocity at time t is roughly parallel to the difference between these two positions. The velocity is then shown at these two times in (b). The acceleration is roughly parallel to the difference between these two velocities. In (c) the acceleration is drawn on the path roughly parallel to the difference in velocities.

Are position, velocity and acceleration all parallel? Sometimes this is a right intuition. For example, after some time has passed the change in position is exactly the average velocity. And the change in velocity is exactly the average acceleration. So in the long run, if something accelerates in some more-or-less constant direction then the position will change in that same direction. But actually, at any instant in time, position, velocity and acceleration are basically unrelated.

Example: Position, velocity and acceleration can be mutually orthogonal.
Here is a motion where, at least at one instant in time, the position, velocity, and acceleration are mutually orthogonal as in fig. 10.8. For example, look at the path in fig. 10.9. At the point where the path intersects the y axis the position relative to the origin is in the $\hat{\jmath}$ direction, the velocity is tangent to the path in the $\hat{\imath}$ direction and the acceleration is at least partially up, in the $\hat{k}$ direction. Working this out with equations, if we take the position as a function of time to be

$$\vec{r}(t) = A\hat{\jmath} - Bt\hat{\imath} + Ct^2\hat{k}$$

we can calculate the velocity and acceleration by differentiation as

$$\vec{v} = \frac{d\vec{r}}{dt} \quad = \quad -B\hat{\imath} + 2Ct\hat{k}, \quad \vec{a} = \frac{d\vec{v}}{dt} \quad = \quad 2C\hat{k}.$$

So, at $t = 0$,

$$\vec{r} = A\hat{\jmath}, \quad \vec{v} = -B\hat{\imath}, \qquad \text{and} \qquad \vec{a} = 2C\hat{k}.$$

The dot products between $\vec{r}$, $\vec{v}$ and $\vec{a}$ are: $\vec{r}\cdot\vec{v} = 0$, $\vec{v}\cdot\vec{a} = 0$, and $\vec{r}\cdot\vec{a} = 0$, so these vectors are mutually orthogonal at the instant marked. (**Aside:** Why is there a $-B$ in this example? Answer: no reason, we could have used $+B$ just as well.)

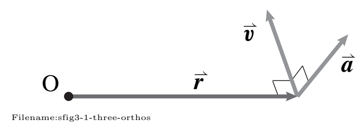

Figure 10.8: It is possible for position, velocity and acceleration to be mutually orthogonal.

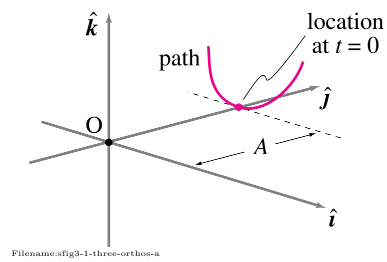

Figure 10.9: A path where, at the marked dot, the position (relative to the origin), velocity and acceleration can be mutually orthogonal: Position is in the $\hat{\jmath}$ direction, velocity is in the $\hat{\imath}$ direction and (if the particle moves at constant speed), the acceleration is in the $\hat{k}$ direction.

In constant rate circular motion position (relative to the circle's center) and velocity remain perpendicular for all time, and so do velocity and acceleration. However, the directions of position, velocity and acceleration are not arbitrary. For example, there is no motion where position, velocity and acceleration are exactly mutually orthogonal for an extended time. Imagine a slender circular cone. If position is measured relative to the apex of the cone then constant-rate circular motion about the base of the cone *almost* has position, velocity and acceleration mutually orthogonal for all time. But position and acceleration are only exactly orthogonal in the limit as the cone becomes infinitely slender.

The product rule of differentiation

We know three ways to multiply vectors: multiplying a vector by a scalar, taking the dot product of two vectors, and taking the cross product of two vectors (please review Chapter 2). Because all of these quantities might be functions of time we need to know how to differentiate products. It's simple. All three kinds of vector multiplication obey 'the product rule' that you learned in freshmen calculus.

$$\begin{aligned}
\frac{d}{dt}(a\vec{A}) &= \dot{a}\vec{A} + a\dot{\vec{A}} \\
\frac{d}{dt}(\vec{A}\cdot\vec{B}) &= \dot{\vec{A}}\cdot\vec{B} + \vec{A}\cdot\dot{\vec{B}} \\
\frac{d}{dt}(\vec{A}\times\vec{B}) &= \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}.
\end{aligned}$$

The proofs of these identities is the same as the proof used for scalar multiplication, it follows from the definition of derivative (above) and the elimination of terms with Δ^2 as negligible compared to terms with just Δ in the limit $\Delta \to 0$.

Example: Derivative of a vector of constant length.
Assume a vector $\vec{C}$ has constant length so

$$|\vec{C}| = \text{constant} \qquad \text{and} \qquad |\vec{C}|^2 = \text{constant}$$

so, differentiating and using the product rule, working from left to right:

$$\frac{d}{dt}|\vec{C}|^2 = \frac{d}{dt}(\vec{C}\cdot\vec{C}) = \vec{C}\cdot\dot{\vec{C}} + \dot{\vec{C}}\cdot\vec{C} = 2\vec{C}\cdot\dot{\vec{C}} = 0.$$

Because $\vec{C}\cdot\dot{\vec{C}} = 0$ we then know that $\vec{C} \perp \dot{\vec{C}}$. That is, for any vector $\vec{C}$ that has constant length, its rate of change is perpendicular to itself. This is a useful fact to remember about time-varying constant-magnitude vectors, especially time-varying unit vectors.

To make this more intuitive, imagine a dog on a taut fixed-length leash anchored to the ground. The length of the leash is the magnitude $|\vec{C}|$ of the position vector $\vec{C}$, from ground-to-neck, and is constant. So our result is obvious, the neck can only move with a velocity $\dot{\vec{C}}$ that is tangent to the circle that the neck moves on because the tangent of a circle is orthogonal to the radius.

In 3D, space-dogs on taut leashes can only move tangent to the sphere they are stuck on $|\vec{C}| = \text{constant} \quad \Rightarrow \quad \vec{C}\cdot\dot{\vec{C}} = 0 \quad \Rightarrow \quad \vec{C} \perp \dot{\vec{C}}$. And, intuitively again, all tangents to the surface of a sphere are orthogonal to the radius of the sphere at that point.

Dynamics in space

Isaac Newton wondered how the planets move around the sun. By applying his equation $\vec{F} = m\vec{a}$, his law of gravitation, his calculus, and his inimitable geometric reasoning, he learned a lot about the moon and the planets. After you learn the material in this section you will know enough to reproduce many of Newton's calculations. You won't need to be a Newton-like genius to solve Newton's differential equations. You can solve them on a computer. And you can use the same computer approach to find motions that Newton could never find, say the trajectory of projectile with a realistic model of air friction. In this chapter, the the basic recipe is this: ①

① Eventually you may gain the math skills to shortcut this brute-force numerical approach, at least for some simple problems. But for most problems, even math geniuses use the numerical approach here.

$$\text{Write} \quad \vec{F} = m\vec{a} \quad \text{and solve the equations.}$$

In some sense it's that simple.

A sure-fire recipe. Here's how to find the motion of a particle:

1. Draw a free body diagram of the particle,
2. Find the forces on the particle in terms of its position, velocity and time. External forces (external forces might come, for example, from a spring, dashpot, gravity, or air friction),
3. Write the linear momentum balance equation for the particle (translation: write $\vec{F} = m\vec{a}$).
4. Break the vector equation into components to make 2 or 3 2nd order scalar ODEs, in 2 or 3 dimensions, respectively.

5. Write the 2 or 3 2nd order ODEs in first order form. You now have 4 or 6 first order ordinary differential equations (for a 2 or 3 dimensional problem, respectively).
6. Write these first order equations in standard form, with all the time derivatives on the left hand side.
7. Feed these equations to the computer, substituting values for the various parameters and appropriate initial conditions.
8. Plot some aspect(s) of the solution and
   a) Use the solution to help you find errors in your formulation, and
   b) Interpret the solution so that it makes sense to you and increases your understanding of the system of study.

Instantaneous dynamics. Some problems are even easier, problems of the *instantaneous dynamics* type. They use the equations of dynamics but do not track the motion over time.

Example: Knowing the forces find the acceleration.
Say you know the forces on a particle at some instant in time, say $\vec{F}_1$ and $\vec{F}_2$, and you just want to know the acceleration at that instant. The answer is given directly by linear momentum balance as

$$\sum \vec{F}_i = m\vec{a} \quad \Rightarrow \quad \vec{a} = \frac{\vec{F}_1 + \vec{F}_2}{m}$$

Sometimes this 'instantaneous' dynamics, with the motion given and the forces to be determined, is called *'inverse dynamics'*. The inside back cover of the book compares the solution methods for instantaneous dynamics to those where differential equations need be solved.

Analytic solution. Some problems involving motion are simple and you can determine almost all you want to know with pencil and paper. You can bypass the whole computer recipe above.

Example: Parabolic trajectory of a projectile
If we assume a constant gravitational field, neglect air drag, and take the y direction as up the only force acting on a projectile is $\vec{F} - mg\hat{\jmath}$. Thus the "equations of motion" (linear momentum balance) are

$$-mg\hat{\jmath} = m\vec{a}.$$

Taking the dot product of this equation with $\hat{\imath}$ and $\hat{\jmath}$ (equivalent to taking the x and y components) we get the following two differential equations,

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

which are decoupled and have the general solution

$$\vec{r} = (A + Bt)\hat{\imath} + (C + Dt - gt^2/2)\hat{\jmath}$$

which is a parametric description of all possible trajectories. By making plots or by using simple algebra you could convince yourself that these trajectories are parabolas for all possible A, B, C, and D. That is, neglecting air drag, the predicted trajectory of a thrown ball is a parabola.

Some other special problems turn out to be easy, although you might not recognize such problems at first glance.

Example: Mass tethered by a zero-length spring
Imagine a massless spring whose unstretched length is zero (See page 323 in section 6.1 for a discussion of zero length springs). Assume one end is connected to a pivot at the origin and the other to a particle. Neglect gravity and air drag. The force on the mass is thus proportional to its distance from the pivot and the spring constant and pointed towards the origin: $\vec{F} = -k\vec{r}$. Thus linear momentum balance yields

$$-k\vec{r} = m\vec{a}.$$

Breaking into components we get

$$\ddot{x} = (-k/m)x \quad \text{and} \quad \ddot{y} = (-k/m)y.$$

Thus the motion can be thought of as two independent harmonic oscillators, one in the x direction and one in the y direction. The general solution is

$$\vec{r} = \left(A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\imath} + \left(C\cos\sqrt{\frac{k}{m}}\,t + D\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\jmath}$$

which is always an ellipse (special cases of which are a circle and a straight line).

Even with gravity and springs together some (only some) problems are easy.

Example: Mass hanging from a zero-length spring
If gravity in the $-\hat{\jmath}$ direction is included in the above problem the solution is only changed by the addition of a constant to be:

$$\vec{r}_{\text{with gravity}} = \vec{r}_{\text{previous example}} - (mg/k)\hat{\jmath}$$

Analytic methods sometimes just can't do the job. Some problems are hard and can't be solved without a computer.

Example: Trajectory with quadratic air drag.
For motions of things you can see with your bare eyes moving in air, the drag force is roughly proportional to the speed squared and opposes the motion. Thus the total force on a particle is $\vec{F} = -mg\hat{\jmath} - Cv^2(\vec{v}/v)$, where $\vec{v}/v$ is a unit vector in the direction of motion. So linear momentum balance gives

$$-mg\hat{\jmath} - Cv\vec{v} = m\vec{a}.$$

If we dot this equation with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\ddot{x} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{x} \quad \text{and} \quad \ddot{y} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{y} - g.$$

These are two coupled second order equations that are probably not solvable with pencil and paper. But they are easily put in the form of a set of four first order equations for direct numerical solution.

On the edge. Some problems are within the reach of advanced analytic methods, but might be easier to solve with a computer.

Example: Path of the earth around the sun.
Assume the sun is big and unmovable with mass M and the earth has mass m. Take the origin to be at the sun. The force on the earth is $\vec{F} = -(mMG/r^2)(\vec{r}/r)$ where $\vec{r}/r$ is a unit vector pointing from the sun to the earth. So linear momentum balance gives

$$-\frac{mMG\vec{r}}{r^3} = m\vec{a}.$$

This equation *can* be solved with pencil and paper, Newton did it but many of us find it too tricky. On the other hand the equations of motion for planetary trajectories are easily broken into components and then into a set of 4 ODEs which can be easily solved on the computer. Either by pencil and paper, or by investigation of numerical solutions, you will find that all solutions are conic sections (straight lines, parabolas, hyperbolas, and ellipses). The special case of circular motion is not far from what the earth does around the sun, what the moon does around the earth, and what most artificial satellites do around the earth.

Summary

If, given the time, the particle's position and the particles velocity, you know the force on a particle, then you know $\vec{F}(t, \vec{r}, \vec{v})$②. That means you can write $\vec{F} = m\vec{a}$ as

$$\vec{a} = \vec{F}(t, \vec{r}, \vec{v})/m$$

where $\vec{F}$ is known. This can, in turn, be written as two vector first order equations

$$\begin{aligned} \dot{\vec{r}} &= \vec{v} \\ \dot{\vec{v}} &= \vec{F}(t, \vec{r}, \vec{v})/m. \end{aligned} \tag{10.2}$$

which are equivalent, written out long hand, to the 6 first order equations

$$\begin{aligned} \dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \\ \dot{v}_x &= F_x(t, x, y, z, v_x, v_y, v_z)/m \\ \dot{v}_y &= F_y(t, x, y, z, v_x, v_y, v_z)/m \\ \dot{v}_z &= F_z(t, x, y, z, v_x, v_y, v_z)/m. \end{aligned} \tag{10.3}$$

Given the position and velocity at some starting time, these equations can be integrated, sometimes by hand but generally on the computer, to give position and velocity as a function of time.

② Typically you would know this because an applied force would vary in time in a known way (the dependence on t), gravity and spring forces would vary with position in a known way (dependence on r), and you would know forces due to various friction (dependence on $\vec{v}$).

Example: Simple ballistics.
This is an example that can be solved with pencil and paper. A computer is not needed. It is the classic from high school and freshman physics. A particle has only one force on it, gravity. A free body diagram is shown in fig. 10.10. Linear momentum balance gives

$$\begin{aligned} \vec{F} &= m\vec{a} \\ -mg\hat{\jmath} &= m\vec{a} \\ \vec{a} &= -g\hat{\jmath} \end{aligned}$$

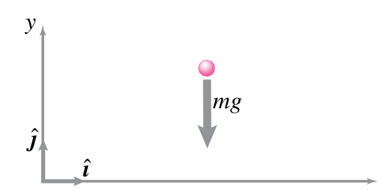

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Figure 10.10: Free body diagram of a particle in flight, neglecting air friction.

So

$$\begin{aligned}\dot{x} &= v_x\\ \dot{y} &= v_y\\ \dot{v}_x &= 0\\ \dot{v}_y &= -g\end{aligned}$$

Integrating the last two of these equations and plugging the result into the first two we get:

$$\begin{aligned}\vec{\boldsymbol{v}} &= v_{0x}\hat{\boldsymbol{\imath}} + (v_{0y} - gt)\hat{\boldsymbol{\jmath}}\\ \vec{\boldsymbol{r}} &= (x_0 + v_{0x}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{0y}t - gt^2/2)\hat{\boldsymbol{\jmath}}\end{aligned}$$

This solution is plotted various ways in fig. 10.22 on page 549.

More complicated examples are given in the samples on the following pages.

10.2 The rate of change of a vector depends on reference frame

The time derivative of a vector can be found by differentiating each of its components. This calculation depended on having a reference frame, an imaginary piece of big graph paper, and a corresponding set of base (or basis) vectors, say $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. But there can be more than one piece of imaginary graph paper. You could be holding one, Jo another, and Tanya a third. Each could be moving their graph paper around and on each paper the same given vector would change in a different way.

As noted earlier, but for the special case of one frame moving at constant velocity (without rotation) with respect to another, the rate of change of a given vector is different if calculated in different reference frames.

For mechanics we have to differentiate vectors with respect to a Newtonian frame.

Because most often we use the "fixed" ground under us as a practical approximation of a Newtonian frame, we label a Newtonian frame with a curly script $\mathcal{F}$, for fixed. So, when being fussy about notation we will sometimes write

$$\overset{\mathcal{F}\cdot}{\vec{\boldsymbol{r}}}_{\mathrm{B/O}} = \text{The velocity of point B as calculated in frame } \mathcal{F}.$$

Non-Newtonian frames

Even though the laws of mechanics are not valid in non-Newtonian frames, non-Newtonian frames are useful help with the understanding of the motion of and forces on systems composed of objects with complex relative motion. So eventually we need to understand frames that accelerate and rotate with respect to each other and with reference to Newtonian frames. Such non-Newtonian frames will be discussed in later chapters.

SAMPLE 10.1 Velocity and acceleration from derivative of position: The position vector of a particle is given as a functions time:

$$\vec{r}(t) = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}$$

where $C_1 = 1\,\text{m}$, $C_2 = 3\,\text{m/s}$, $C_3 = 1\,\text{m/s}^2$, and $C_4 = 2\,\text{m/s}$.

1. Find the position, velocity, and acceleration of the particle at $t = 2\,\text{s}$.
2. Find the change in the position of the particle between $t = 2\,\text{s}$ and $t = 3\,\text{s}$.

Solution We are given,

$$\vec{r} = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}.$$

Therefore,

$$\begin{aligned}\vec{v} &\equiv \dot{\vec{r}} = \frac{d\vec{r}}{dt} = (C_2 + 2C_3 t)\hat{\imath} + C_4\hat{\jmath}\\ \vec{a} &\equiv \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = 2C_3\hat{\imath}.\end{aligned}$$

1. Substituting the given values of the constants and $t = 2\,\text{s}$ in the equations above we get,

$$\begin{aligned}\vec{r}(t=2\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 2\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 4\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 2\,\text{s})\hat{\jmath}\\ &= 11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath}\\ \vec{v}(t=2\,\text{s}) &= (3\frac{\text{m}}{\text{s}} + 2\cdot 1\frac{\text{m}}{\text{s}^2}\cdot 2\,\text{s})\hat{\imath} + (2\frac{\text{m}}{\text{s}})\hat{\jmath}\\ &= 7\,\text{m/s}\hat{\imath} + 2\,\text{m/s}\hat{\jmath}\\ \vec{a}(t=2\,\text{s}) &= (2\cdot 1\frac{\text{m}}{\text{s}^2})\hat{\imath} = 2\,\text{m/s}^2\hat{\imath}.\end{aligned}$$

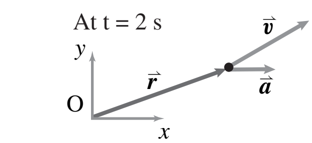

Figure 10.11:

$$\boxed{\vec{r} = (11\hat{\imath} + 4\hat{\jmath})\,\text{m}, \quad \vec{v} = (7\hat{\imath} + 2\hat{\jmath})\,\text{m/s}, \quad \vec{a} = 2\,\text{m/s}^2\hat{\imath}}$$

2. The change in the position of the particle between the two time instants is,

$$\Delta\vec{r} = \vec{r}(t=3\,\text{s}) - \vec{r}(t=2\,\text{s}).$$

We already have $\vec{r}$ at $t = 2\,\text{s}$. We need to calculate $\vec{r}$ at $t = 3\,\text{s}$.

$$\begin{aligned}\vec{r}(t=3\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 3\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 9\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 3\,\text{s})\hat{\jmath}\\ &= 19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta\vec{r} &= (19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}) - (11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath})\\ &= 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}.\end{aligned}$$

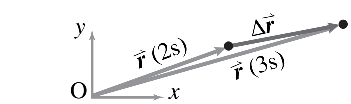

Figure 10.12:

$$\boxed{\Delta\vec{r} = 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}}$$

SAMPLE 10.2 Velocity and acceleration from position on a helix. Given that the position of a particle is

$$\vec{r} = A\cos(\lambda t)\hat{\imath} + B\sin(\lambda t)\hat{\jmath} + Ct\hat{k},$$

where A, B, C, and λ are constants, find

1. the velocity as a function of time,
2. the acceleration as a function of time,
3. a condition under which the acceleration vector is normal to the velocity vector.

Solution

1. The velocity:

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} \\ &= \frac{d}{dt}[A\cos(\lambda t)\hat{\imath} + B\sin(\lambda t)\hat{\jmath} + Ct\hat{k}] \\ &= -A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k}.\end{aligned}$$

$$\vec{v} = -A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k}$$

2. The acceleration:

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} \\ &= \frac{d}{dt}[-A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} \\ &= -A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}\end{aligned}$$

$$\vec{a} = -A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}$$

3. The velocity vector and the acceleration vector will be orthogonal to each other if $\vec{v}\cdot\vec{a} = 0$. Taking the dot product of the two vectors, we find,

$$\begin{aligned}\vec{v}\cdot\vec{a} &= (-A\lambda\sin(\lambda t)\hat{\imath} + B\lambda\cos(\lambda t)\hat{\jmath} + C\hat{k})\cdot(-A\lambda^2\cos(\lambda t)\hat{\imath} - B\lambda^2\sin(\lambda t)\hat{\jmath}) \\ &= A^2\lambda^3\sin(\lambda t)\cos(\lambda t) - B^2\lambda^3\sin(\lambda t)\cos(\lambda t) \\ &= (A^2 - B^2)\lambda^3\sin(\lambda t)\cos(\lambda t).\end{aligned}$$

Now, this dot product must be zero for all t if $\vec{a}$ is normal to $\vec{v}$. This is indeed the case if $A = B$. Thus, the condition for orthogonality o $\vec{v}$ and $\vec{a}$ is $A = B$.

$$A = B \quad \Rightarrow \quad \vec{v}\cdot\vec{a} = 0$$

Note: The path is an elliptical helix with axis in the z direction. The z-component of velocity is constant so the acceleration is entirely in the xy plane. In fact, the acceleration vector points from the particle towards the axis of the helix.

SAMPLE 10.3 Position from velocity. Assume the expression for velocity $\vec{v}$ of a particle is given: $\vec{v} = v_0\hat{\imath} - gt\,\hat{\jmath}$. Find the expressions for the x and y coordinates of the particle at a general time t, if the initial coordinates at $t = 0$ are (x_0, y_0). Plot the path of the particle taking $x0 = 0$, $y_0 = 80\,\mathrm{m}$, $v_0 = 2\,\mathrm{m/s}$, $g = 10\,\mathrm{m/s^2}$, and $t = 1 \ldots 4\,\mathrm{s}$.

Solution The position vector of the particle at any time t is

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}.$$

We are given that

$$\vec{r}(t=0) = x_0\hat{\imath} + y_0\hat{\jmath}.$$

Now

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= v_0\hat{\imath} - gt\,\hat{\jmath} \\ \text{or} \qquad dx\hat{\imath} + dy\,\hat{\jmath} &= (v_0\hat{\imath} - gt\,\hat{\jmath})\,dt. \end{aligned}$$

Integrating both sides of the equation with appropriate limits, we get

$$\begin{aligned} \int_{x_0\hat{\imath}+y_0\hat{\jmath}}^{x\hat{\imath}+y\hat{\jmath}} (dx\hat{\imath} + dy\,\hat{\jmath}) &= \int_0^t (v_0\hat{\imath} - gt\,\hat{\jmath})dt \\ \int_{x_0}^{x} dx\,\hat{\imath} + \int_{y_0}^{y} dy\,\hat{\jmath} &= v_0\hat{\imath}\int_0^t dt - g\,\hat{\jmath}\int_0^t t\,dt \\ (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} &= v_0 t\hat{\imath} - \frac{1}{2}gt^2\hat{\jmath} \\ x\hat{\imath} + y\hat{\jmath} &= (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}. \end{aligned}$$

Therefore,

$$\vec{r}(t) = (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}$$

and the (x, y) coordinates are

$$\begin{aligned} x(t) &= x_0 + v_0 t \\ y(t) &= y_0 - \frac{1}{2}gt^2. \end{aligned}$$

$$\boxed{(x_0 + v_0 t,\ y_0 - \tfrac{1}{2}gt^2)}$$

Plugging in $x0 = 0$, $y_0 = 80\,\mathrm{m}$, $v_0 = 2\,\mathrm{m/s}$, $g = 10\,\mathrm{m/s^2}$, and taking 20 points between $t = 0$ to $t = 4$, we compute the values of x and y and plot them to get the path of the particle. The plot is shown in fig. 10.14 with a few intermediate positions marked on the path.

Comments: From the x and y coordinates, it is possible to get the equation of the path of the particle by eliminating the time from the two equations. From the expression for $x(t)$, we get $t = (x - x_0)/v_0$. Substituting this expression for t in the equation for $y(t)$, we get,

$$y - y_0 = \frac{g}{2v_0^2}(x - x_0)^2$$

which is the equation of the path. From this equation it should be clear that the path is parabolic. It is easier to see this if you shift the origin to (x_0, y_0) and use the new coordinates $\hat{x} = x - x_0$ and $\hat{y} = y - y_0$. Then, in terms of the new coordinates, the path becomes,

$$\hat{y} = \frac{g}{2v_0^2}\hat{x}^2.$$

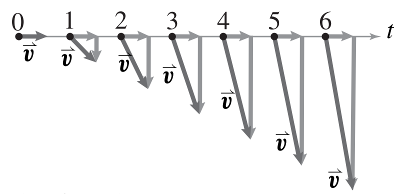

Figure 10.13: The given velocity vector at different time instants ($t = 0,\ 1\ 2$, etc.). Note that the x-component of the velocity remains constant (v_0) while the y-component grows linearly with time ($-gt$).

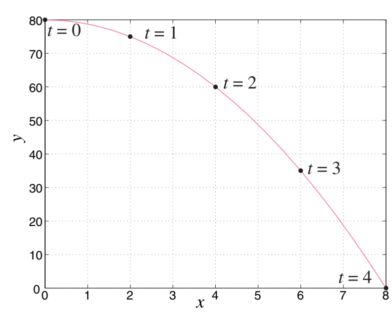

Figure 10.14: Path of the particle plotted by computing the coordinates $x(t)$ and $y(t)$ at 20 equal time intervals between $t = 0$ to 4 seconds. Five positions are marked on the path corresponding to $t = 0$, 1, 2, 3, and 4 seconds.

SAMPLE 10.4 Acceleration of a point mass in 3-D. A ball of mass $m = 13\,\mathrm{kg}$ is being pulled by three strings as shown in Fig. 10.15. The tension in each string is $T = 13\,\mathrm{N}$. Find the acceleration of the ball.

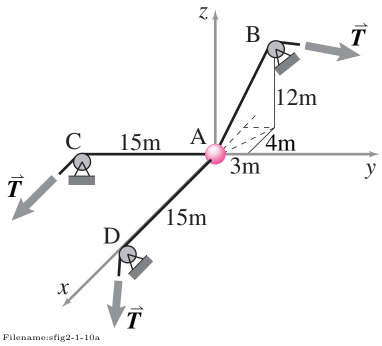

Figure 10.15: A ball in 3-D

Solution The forces acting on the body are shown in the free-body diagram in Fig.10.16. From geometry:

$$
\begin{aligned}
\hat{\boldsymbol{\lambda}} &= \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2 + 3^2 + 12^2}} \\
&= \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}.
\end{aligned}
$$

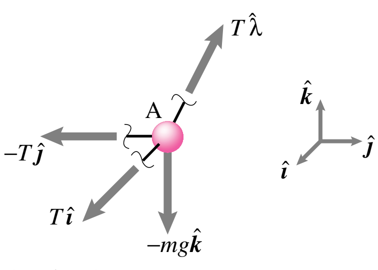

Figure 10.16: FBD of the ball

The balance of linear momentum for the ball gives

$$
\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \tag{10.4}
$$

where

$$
\begin{aligned}
\sum \vec{\boldsymbol{F}} &= T\hat{\boldsymbol{\imath}} - T\hat{\boldsymbol{\jmath}} + T\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{k}} \\
&= T\left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}\right) - mg\hat{\boldsymbol{k}} \\
&= \frac{T}{13}(9\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - mg\hat{\boldsymbol{k}}.
\end{aligned}
$$

Substituting $\sum \vec{\boldsymbol{F}}$ in eqn. (10.4):

$$
\vec{\boldsymbol{a}} = \frac{T}{13m}(9\hat{\boldsymbol{\imath}} + 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - g\hat{\boldsymbol{k}}.
$$

Now plugging in the given values: $T = 13\,\mathrm{N}$, $m = 13\,\mathrm{kg}$, and $g = 10\,\mathrm{m/s^2}$, we get

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= \frac{\cancel{13}\,\mathrm{N}}{\cancel{13}\cdot 13\,\mathrm{kg}}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - 10\,\mathrm{m/s^2}\,\hat{\boldsymbol{k}} \\
&= (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\mathrm{m/s^2}.
\end{aligned}
$$

$$
\boxed{\vec{\boldsymbol{a}} = (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\mathrm{m/s^2}}
$$

SAMPLE 10.5 **Projectile motion with air drag.** A projectile is fired into the air at an initial angle θ_0 and with initial speed v_0. The air resistance to the motion is proportional to the square of the speed of the projectile. Take the constant of proportionality to be k. Find the equations of motion of the projectile in the horizontal and vertical directions assuming the air resistance to be in the opposite direction of the velocity.

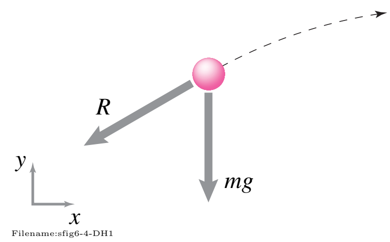

Figure 10.17: FBD of the projectile.

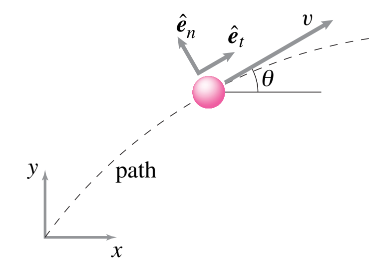

Figure 10.18:

Solution The free body diagram of the projectile is shown in the figure at some constant t during motion. At the instant shown, let the velocity of the projectile be $\vec{v} = v\hat{e}_t$ where

$$\hat{e}_t = \vec{v}/|\vec{v}| = (v_x/|\vec{v}|)\hat{\imath} + (v_y/|\vec{v}|)\hat{\jmath}.$$

Then the force due to air resistance is

$$\vec{R} = -kv^2\hat{e}_t.$$

Now applying the linear momentum balance on the projectile, we get

$$\begin{aligned} \vec{R} + m\vec{g} &= m\vec{a} \\ \text{or} \qquad -kv^2\hat{e}_t - mg\hat{\jmath} &= m\overbrace{(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})}^{\vec{a}}. \end{aligned} \tag{10.5}$$

Noting that $v = |\vec{v}| = |\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, and dotting both sides of eqn. (10.5) with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\begin{aligned} -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\imath}) &= m\ddot{x} \\ -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\jmath}) - mg &= m\ddot{y}. \end{aligned}$$

Rearranging terms and carrying out the dot products, we get

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_x/|\vec{v}|) \\ \ddot{y} &= -g - \frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_y/|\vec{v}|). \end{aligned}$$

Substituting these expressions in to the equations for $\ddot{x}$ and $\ddot{y}$ we get

$$\boxed{\ddot{x} = -\tfrac{k}{m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad \ddot{y} = -\tfrac{k}{m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g}$$

For initial conditions we could use

$$x_0 = 0, y_0 = 0 \qquad \text{and} \qquad \dot{x}_0 = v_{x0} = v_0\cos\theta_0, \qquad \text{and} \qquad \dot{y}_0 = v_{y0} = v_0\sin\theta_0.$$

Note that θ changes with time. We can express θ in terms of $\dot{x}$ and $\dot{y}$. We can calculate the slope of the trajectory from

$$\tan\theta = \frac{\dot{y}}{\dot{x}}.$$

SAMPLE 10.6 Trajectory of a food-bag. In a flood hit area relief supplies are dropped in a 20 kg bag from a helicopter. The helicopter is flying parallel to the ground at 200 km/h and is 80 m above the ground when the package is dropped. How much horizontal distance does the bag travel before it hits the ground? Take the value of g, the gravitational acceleration, to be $10\,\mathrm{m/s^2}$. Ignore air drag.

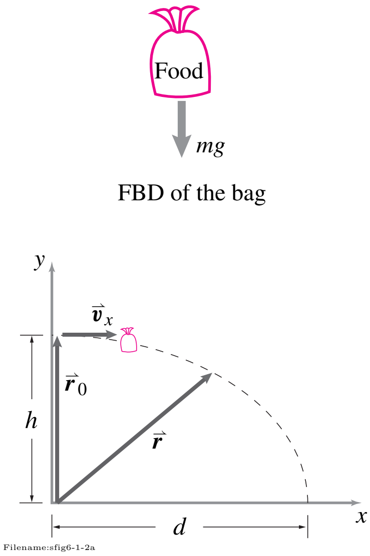

Figure 10.19: Free-body diagram of the bag and the geometry of its motion.

Solution You must have solved such problems in elementary physics courses. Usually, in all projectile motion problems the equations of motion are written separately in the x and y directions, realizing that there is no force in the x direction, and then the equations are solved. Here we show you how to write and keep the equations in vector form all the way through.

The free-body diagram of the bag during its free flight is shown in Fig. 10.19. The only force acting on the bag is its weight. Therefore, from the linear momentum balance for the bag we get

$$m\vec{\boldsymbol{a}} = -mg\hat{\boldsymbol{j}}.$$

Let us choose the origin of our coordinate system on the ground exactly below the point at which the bag is dropped from the helicopter. Then, the initial position of the bag $\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{j}} = 80\,\mathrm{m}\hat{\boldsymbol{j}}$. The fact that the bag is dropped from a helicopter flying horizontally gives us the initial velocity of the bag:

$$\vec{\boldsymbol{v}}(0) \equiv \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}} = 200\,\mathrm{km/h}\hat{\boldsymbol{\imath}}.$$

So now we have a 2nd order differential equation (from linear momentum balance):

$$\ddot{\vec{\boldsymbol{r}}} = -g\hat{\boldsymbol{j}}$$

with two initial conditions:

$$\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{j}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}}$$

which we can solve to get the position vector of the bag at any time. Since the basis vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{j}}$ do not change with time, solving the differential equation is a matter of simple integration:

$$\begin{aligned} \ddot{\vec{\boldsymbol{r}}} &\equiv \frac{d\dot{\vec{\boldsymbol{r}}}}{dt} = -g\hat{\boldsymbol{j}} \\ \int d\dot{\vec{\boldsymbol{r}}} &= -\hat{\boldsymbol{j}}\int g\,dt \\ \text{or} \qquad \dot{\vec{\boldsymbol{r}}} &= -gt\hat{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1 \end{aligned} \tag{10.6}$$

and integrating once again, we get

$$\begin{aligned} \vec{\boldsymbol{r}} &= \int(-gt\hat{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1)\,dt \\ &= -\frac{1}{2}gt^2\hat{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1 t + \vec{\boldsymbol{c}}_2 \end{aligned} \tag{10.7}$$

where $\vec{\boldsymbol{c}}_1$ and $\vec{\boldsymbol{c}}_2$ are constants of integration and are vector quantities. Now substituting the initial conditions in eqn. (10.6) and eqn. (10.7) we get

$$\begin{aligned} \dot{\vec{\boldsymbol{r}}}(0) &= v_x\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{c}}_1, \quad \text{and} \\ \vec{\boldsymbol{r}}(0) &= h\hat{\boldsymbol{j}} = \vec{\boldsymbol{c}}_2. \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} \vec{\boldsymbol{r}}(t) &= -\frac{1}{2}gt^2\hat{\boldsymbol{j}} + v_x t\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{j}} \\ &= v_x t\hat{\boldsymbol{\imath}} + (h - \frac{1}{2}gt^2)\hat{\boldsymbol{j}}. \end{aligned}$$

So how do we find the horizontal distance traveled by the bag from our solution? The distance we are interested in is the x-component of $\vec{r}$, *i.e.,* $v_x t$. But we do not know t. However, when the bag hits the ground, its position vector has no y-component, *i.e.,* we can write $\vec{r} = d\hat{\imath} + 0\hat{\jmath}$ where d is the distance we are interested in. Now equating the components of $\vec{r}$ with the obtained solution, we get

$$d = v_x t \quad \text{and} \quad 0 = h - \frac{1}{2} g t^2.$$

Solving for t from the second equation and substituting in the first equation we get

$$d = v_x \sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

$$\boxed{d = 222\,\text{m}}$$

Comments: Here we have tried to show you that solving for position from the given acceleration in vector form is not really any different than solving in scalar form provided the unit vectors involved are fixed in time. As long as the right hand side of the differential equation is integrable, the solution can be obtained. If the method shown above seems too "*mathy*" or intimidating to you then follow the usual scalar way of doing this problem.

The scalar method:

From the linear momentum balance, $-mg\hat{\jmath} = m\vec{a}$, writing the acceleration as $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$ and equating the x and y components from both sides, we get

$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

Now using the formula for distance under uniform acceleration from Chapter 3, $x = x_0 + v_0 t + \frac{1}{2} a t^2$, in both x and y directions, we get

$$\begin{aligned}
d &= \overbrace{x_0}^{0} + v_x t + \frac{1}{2} \overbrace{a_x}^{0} t^2 \\
&= v_x t \\
0 &= \overbrace{y_0}^{h} + \overbrace{v_y}^{0} t + \frac{1}{2} \overbrace{a_y}^{-g} t^2 \\
&= h - \frac{1}{2} g t^2 \\
\Rightarrow \quad t &= \sqrt{\frac{2h}{g}}.
\end{aligned}$$

Substituting for t in the equation for d we get

$$d = v_x \sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

as above.

SAMPLE 10.7 Cartoon mechanics: The cannon. It is sometimes claimed that students have trouble with dynamics because they built their intuition by watching cartoons. This claim could be rebutted on many grounds.

1) Students don't have trouble with dynamics! They love the subject.
2) Nowadays many cartoons are made using 'correct' mechanics, and
3) the cartoons are sometimes more accurate than the pedagogues anyway.

Problem: What is the path of a cannon ball? In the cartoon world the cannon ball goes in a straight line out the cannon then comes to a stop and then starts falling. Of course a good physicist knows the path is a parabola. Or is it?

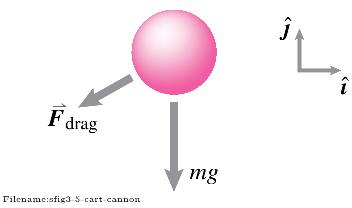

Figure 10.20:

Solution The drag force on a cannon ball moving through air is approximately proportional to the speed squared and resists motion. Gravity is approximately constant. Then

$$\begin{aligned}
\vec{F}_{\text{drag}} &= cv^2 \cdot (\text{unit vector opposing motion}) \\
&= cv^2 \cdot \left(\frac{-\vec{v}}{|\vec{v}|}\right) \\
&= -c|\vec{v}|\vec{v} \\
&= -c\sqrt{\dot{x}^2 + \dot{y}^2}\,(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})
\end{aligned}$$

So the linear momentum balance gives

$$\sum \vec{F} = \dot{\vec{L}}$$

$$\left\{-mg\hat{\jmath} - c\sqrt{\dot{x}^2 + \dot{y}^2}(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})\right\}$$

$$\begin{aligned}
\{\}\cdot\hat{\imath} \quad &\Rightarrow \quad \ddot{x} = \left[-c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x}/m\right] \\
\{\}\cdot\hat{\jmath} \quad &\Rightarrow \quad \ddot{y} = -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y}/m - g
\end{aligned}$$

Solving these equations numerically with reasonable values ③ of $\dot{x}_0$, $\dot{y}_0$, m and c gives which is closer to a cartoon's triangle than to a naive physicist's parabola.

③ To be precise, if the launch speed is much faster than the 'terminal velocity' of the falling ball.

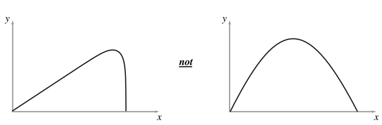

Figure 10.21:

10.2 Linear momentum, angular momentum, work and energy

If you know a particle's starting position and velocity, and you know the force on it as it moves, then you can use $\vec{F} = m\vec{a}$ to predict its path. That is the central idea of the previous section. We had no need for ideas related to momentum, angular momentum, work and energy. For one particle the one equation $\vec{F} = m\vec{a}$ tells the whole story. So, before we go on to discuss them further, let us be clear:

> The concepts of linear and angular momentum, work and energy are not needed to study particle mechanics. $\vec{F} = m\vec{a}$ is enough.

So, why do we bother to devote a section to these topics? Because

- These concepts will sometimes be needed when we discuss more complex systems;
- These concepts sometimes provide a shorter route for answering some dynamics questions;
- The simplest place to introduce the concepts is in the context of one particle;
- The concepts give a way to check the consistency of solutions of $\vec{F} = m\vec{a}$; and
- The concepts can be an aid to physical intuition.

For more complex systems, principles of momentum and energy transcend $\vec{F} = m\vec{a}$ and can generally not be derived from $\vec{F} = m\vec{a}$. But for a single particle, all of these are derived concepts, as worked out in box 10.4 on page 557. Note that

> All of the facts and theorems below apply to any motion of a particle that is consistent with $\vec{F} = m\vec{a}$.

Example: Simple ballistics solution.
Consider a ball thrown up at 45°: $\vec{F} = -mg\hat{\jmath}$, $\vec{r}(0) = \vec{0}$ and $\vec{v}(0) = v_0\hat{\imath} + v_0\hat{\jmath}$. We claimed (page 539) that a solution is

$$\vec{r} = v_0 t\hat{\imath} + (v_0 t - gt^2/2)\hat{\jmath} \qquad \text{and} \qquad \vec{v} = v_0\hat{\imath} + (v_0 - gt)\hat{\jmath}.$$

This solution is plotted various ways in fig. 10.22. These functions of time are consistent with the initial conditions. Further they are consistent with the governing equations, the so called 'equations of motion', $\dot{\vec{v}} = \vec{F}/m$ and $\dot{\vec{r}} = \vec{v}$. All of the momentum and energy principles below must therefore apply.

Some of the ideas apply even if $\vec{F} \neq m\vec{a}$. For example, the work of a force is defined for imagined motions that might never occur.

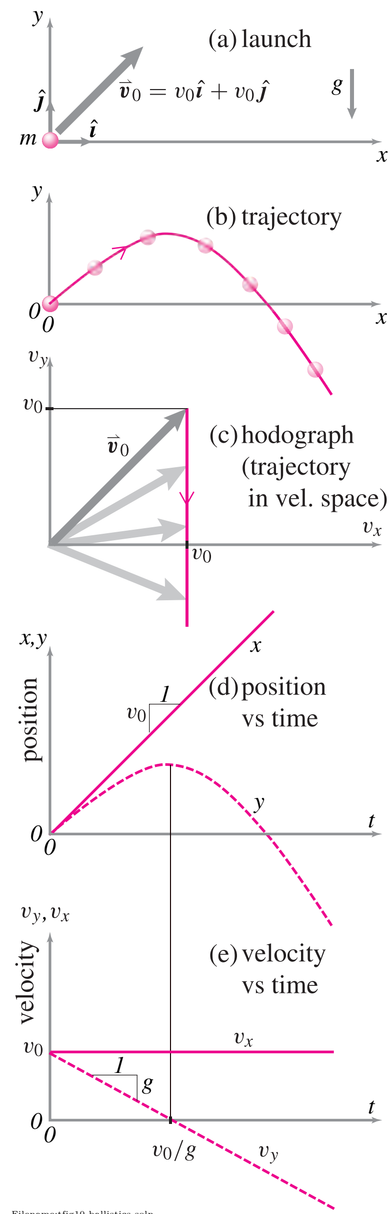

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Figure 10.22: **(a) A simple ballistics problem.** (b) The well known parabolic trajectory. (c) The velocity trajectory, called a *hodograph*, shows the sequence of positions of the tip of the velocity vector. The velocity starts with $v_x = v_y$ and then v_y decreases. (d) Plot of x and y components vs time. The y component has a parabolic shape similar to the particle trajectory because x is proportional to t in this problem. (e) The velocity components versus time. Note, (b) is a 'cross plot' of the two plots in (d), and (c) is a cross plot of (e).

Linear momentum

Linear momentum for a particle is defined as $\vec{L} = m\vec{v}$. The particle momentum-balance theorems (facts) are

$$\vec{F} = \frac{d}{dt}\vec{L} \qquad \text{and} \qquad \underbrace{\int_{t_1}^{t_2} \vec{F}\,dt}_{\text{Linear impulse}} = \vec{L}_2 - \vec{L}_1$$

These are so trivially related to $\vec{F} = m\vec{a}$ that it is hard to see any content in them. And, indeed, if we were only studying the mechanics of single particles we probably would not have introduced the concept of linear momentum. Nonetheless, the general result does apply:

> The net force $\vec{F}$ on a particle is the rate of change of its linear momentum, $\dot{\vec{L}}$.

A special important case is when there is no force and linear momentum is conserved (doesn't change). For a single particle momentum conservation means constant velocity motion.

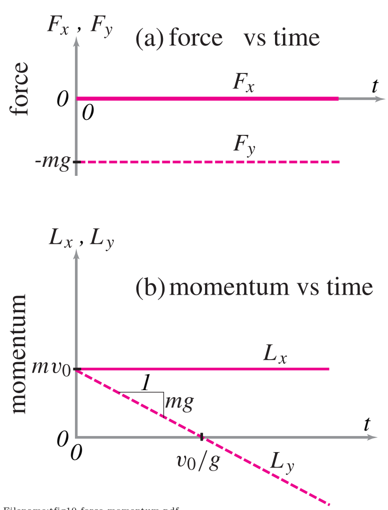

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Figure 10.23: The force and linear momentum from the problem in fig. 10.22. Note that the momentum curves are the integrals of the corresponding force curves.

Example: Linear momentum check

For the simple ballistics solution above we evaluate the left side of the momentum balance equation

$$\int_0^t \vec{F}\,dt' = -mgt\,\hat{\jmath}.$$

Then evaluate the right side:

$$\Delta\vec{L} = \vec{L}_2 - \vec{L}_1 = [m(v_0\hat{\imath} + (v_0 - gt)\hat{\jmath})] - [m(v_0\hat{\imath} + v_0\hat{\jmath})] = -mgt\,\hat{\jmath}$$

and check for equality: $-mgt\,\hat{\jmath} = -mgt\,\hat{\jmath}$. This force and momentum is plotted in fig. 10.23. The solution *is* consistent with linear momentum balance. Note that in this example there is no change in the component of linear momentum in the $\hat{\imath}$ direction; there is no force in the x direction so L_x is conserved.

Angular momentum

In dynamics angular momentum is one of the most important ideas, important theoretically and for solving problems. For one particle

> Angular momentum relative to point C is $\vec{H}_{/C} = \vec{r}_{/C} \times (m\vec{v})$,

where $\vec{r}_{/C}$ is the position of the particle relative to fixed point C and $\vec{v}$ is the velocity of the particle. Angular momentum can be calculated relative to any point C. Which point you pick affects the value of the angular momentum. Sometimes $\vec{H}_{/C}$ it is written without the "/" as $\vec{H}_C$. The key angular momentum theorems (facts) are:

$$\underbrace{\vec{r}_{/C} \times \vec{F}}_{\vec{M}_C} = \dot{\vec{H}}_{/C} \qquad \text{and} \qquad \underbrace{\int_{t_1}^{t_2} \vec{r}_{/C} \times \vec{F}\, dt}_{\text{Angular impulse}} = (\vec{H}_{/C})_2 - (\vec{H}_{/C})_1.$$

The torque $\vec{M}_C$ of *all* the external forces acting on a particle about point C is the rate of change of its angular momentum $\dot{\vec{H}}_{/C}$ about point C.

The intuitive notion is that angular momentum represents how much a particle is 'going around' point C. A particle gets more credit for going faster, for being more massive, and for being farther away①.

If the force on the particle is zero or passes through the point C, the torque (moment) of the force is zero and its angular momentum is conserved.

①**Angular momentum contradicts common intuition.** The notion that angular momentum is bigger if a given mass is further away is counterintuitive to most of us. Imagine a one kilogram is going around your head once per second at a distance of a meter. Now imagine a second equal mass going around at a radius of 10 meters and only going around once every 100 seconds. The first mass whirls around your head 100 times for each slow orbit of the second mass. Even though the second mass certainly doesn't make its rotation feel so present its angular momentum is as big. Intuitive or not, this is how angular momentum is defined. Its a useful concept so its worth adjusting your intuition to match it.

Example: Angular momentum check.
Using the same ballistics example we check the solution for consistency with angular momentum balance. For no good reason lets use the origin for the angular momentum reference point. We could use any point. Again we compare the left and right sides and check for equality.

$$\begin{aligned}
\underbrace{\int_0^t \vec{r}_{/0} \times \vec{F}\, dt}_{\text{Left side}} &= \int_0^t \left(v_0 t'\hat{\imath} + (v_0 - g t'^2/2)\hat{\jmath}\right) \times (-mg\hat{\jmath})\; dt' \\
&= \int_0^t -v_0 m g t' \hat{k}\, dt' \quad = \quad -v_0 m g \hat{k} t^2/2 \\
\underbrace{\vec{H}_{/02} - \vec{H}_{/01}}_{\text{Right side}} &= \vec{r}(t) \times (m\vec{v}(t)) \; - \; \vec{r}(0) \times (m\vec{v}(0)) \\
&= m\left(v_0 t\hat{\imath} + (v_0 t - g t^2/2)\hat{\jmath}\right) \times (v_0\hat{\imath} + (v_0 - g t)\hat{\jmath}) - m\left(\vec{0} \times \vec{v}(0)\right) \\
&= \left(v_0 t - g t^2 - v_0 t + g t^2/2\right) m v_0 \hat{k} \quad = \quad -v_0 m g t^2 \hat{k}/2.
\end{aligned}$$

This torque and angular momentum are plotted in fig. 10.24. The 'left side' agrees with the 'right side' and angular momentum balance is satisfied, as it must be. In 2D problems like this there is only one non-trivial component to angular momentum, that in the out-of-plane (z) direction. In this case the angular momentum H_z is not conserved because the gravity force does have a torque about the z axis.

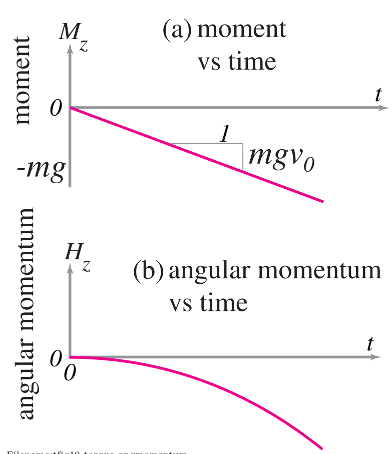

Figure 10.24: The torque and angular momentum about the origin for the problem in fig. 10.22. For this 2D problem only the z component is non-trivial. Note that the angular momentum is the integral of the torque.

Power balance

The power P of a force $\vec{F}$ on a particle with velocity $\vec{v}$ is $P = \vec{F} \cdot \vec{v}$. The kinetic energy of a particle is $E_K = mv^2/2$. The power balance equations are

$$P = \dot{E}_K \qquad \text{and} \qquad \int_{t_1}^{t_2} P\, dt = E_{K2} - E_{K1}.$$

The power P of *all* the external forces acting on a particle is the rate of change of its kinetic energy $\dot{E}_K$. Or, integrating in time, the power added up over time is the net change in kinetic energy.

The situation is similar to that for 1D motion (section 9.2).

Power and work

The integral of power with respect to time can be replaced with a path integral for the work of a force. The key idea is in the differential expressions for an increment of work:

$$dW = P\,dt = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}dt = \vec{\boldsymbol{F}} \cdot \frac{d\vec{\boldsymbol{r}}}{dt}dt = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}.$$

So

$$\begin{aligned} W_{12} &= W_{12} \\ \text{Power integrated in time} &= \text{Sum of work increments} \\ \int_{t_1}^{t_2} P\,dt &= \int_1^2 dW \\ \int_{t_1}^{t_2} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\,dt &= \int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \end{aligned} \tag{10.8}$$

Thus the power balance equation, integrated in time is equivalent to the "work energy" equation:

$$\begin{aligned} \text{Work on particle} &= \text{Change in its kinetic energy} \\ W_{12} &= \Delta E_{\mathrm{K}} \\ &= E_{\mathrm{K}2} - E_{\mathrm{K}1} \end{aligned} \tag{10.9}$$

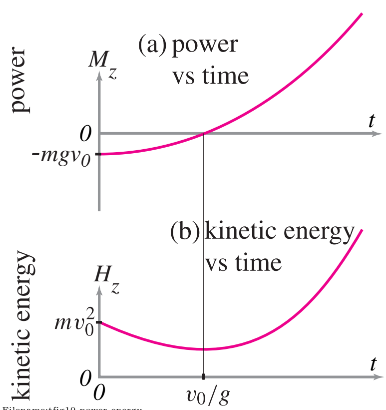

Figure 10.25: The power of the gravitational force and the total kinetic energy are plotted vs time for the ballistics problem of fig. 10.22. Note that the kinetic energy is the integral of the power. The gravitational power starts out negative, goes to zero when the particle reaches the apex of its trajectory and then becomes positive evermore as the downwards gravitational force acts on a downwards moving particle. The kinetic energy starts at mv_0^2 and drops to $mv_0^2/2$ at the particle apex when v_y goes to zero. Then the kinetic energy increases forever more as the gravitational force does more and more work.

Example: Energy check.
We can check the simple ballistics solution for consistency with energy balance. First lets compare the work to the change of kinetic energy.

$$\underbrace{\int_0^t \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\,dt}_{\text{Work}} = \int_0^t (-mg\hat{\boldsymbol{\jmath}}) \cdot \left(v_0\hat{\boldsymbol{\imath}} + (v_0 - gt')\hat{\boldsymbol{\jmath}}\right) dt'$$

$$= \int_0^t -mg(v_0 - gt')\,dt' = -mgv_0t + mg^2t^2/2$$

$$\underbrace{E_{\mathrm{K}2} - E_{\mathrm{K}1}}_{\text{Change in kinetic energy}} = m(|\vec{\boldsymbol{v}}|^2 - |\vec{\boldsymbol{v}}_0|^2)/2$$

$$= m\left((v_0^2 + (v_0 - gt)^2 - 2v_0^2\right)/2 = \underbrace{-mgv_0t + mg^2t^2/2}_{\text{agrees}}$$

This power and kinetic energy are plotted in fig. 10.25. In this check we have not taken advantage of the fact that this particular force is conservative.

The work of a force $\vec{F}$: W_{12}

Previously in Physics, and more recently in one dimensional dynamics here, you learned that

Work is force times distance.

This is actually a special case of the formula

$$P = \vec{F} \cdot \vec{v}.$$

How is that? If $\vec{F}$ is constant and parallel to the displacement $\Delta\vec{x}$, then

$$\begin{aligned} W_{12} &= \int \dot{W}\, dt = \int P\, dt = \int \vec{F} \cdot \underbrace{\vec{v}\, dt}_{d\vec{x}} = \int \vec{F} \cdot d\vec{x} = \vec{F} \cdot \int d\vec{x} \\ &= \vec{F} \cdot \Delta\vec{x} = F\Delta x = \text{Force} \cdot \text{distance}. \end{aligned}$$

In 2 and 3 dimensions there are subtleties involved with the concept of work because of its dependence on which path in space the force works on. These 'path dependence' subtleties are often covered in some detail in calculus courses in the sections on vector calculus, path integrals, gradient and curl. We discuss the relevant highlights below.

Potential energy of a force

Some forces (read force fields $\vec{F} = \vec{F}(\vec{r})$) have the property that the work they do is independent of the path followed by the material point as the force acts. If the work of a force is path independent in this way (see box 10.3 on page 555, then a potential energy can be defined so that the work done by the force is the decrease in the Potential Energy

$$-\Delta E_{\mathrm{P}} = W_{12} = E_{\mathrm{P}1} - E_{\mathrm{P}2}$$

The common examples are listed below:

- **linear spring:** $E_{\mathrm{P}} = (1/2)k(\text{stretch})^2$.
- **gravity near earth's surface:** $E_{\mathrm{P}} = mgh$
- **gravity between spheres or points:** $E_{\mathrm{P}} = -MmG/r$
- **constant force $\vec{F}$ acting on a point:** $E_{\mathrm{P}} = -\vec{F} \cdot \vec{r}$

In all cases a constant could be added to the potential energy and it would still be a legitimate potential energy for the force.

In the cases of the spring and gravity between spheres, the change in potential energy is the net work done by the spring or gravity on the pair of objects between which the force acts. If both ends of a spring are moving, the net work of the spring on the two objects to which it is connected is the decrease in potential energy of the spring.

There is a possible source of confusion in our using the same symbol E_{P} to represent the potential work of an external force and for internal potential energy. In practice, however, they are used identically, so we use the same symbol for both. The potential energy in a stretched spring is the same whether it is the cause of force on a system or it is internal to the system.

Example: Checking conservation of energy
Because the gravity force is conservative we can also check our simple ballistics solution for consistency with conservation of energy. Taking the potential energy as $E_{\mathrm{P}} = mgh = mgy$ we find, as expected, that the solution does have the property that

$$
\begin{aligned}
E_{tot1} &\stackrel{?}{=} E_{tot2} \\
E_{\mathrm{K}1} + E_{\mathrm{P}1} &\stackrel{?}{=} E_{\mathrm{K}2} + E_{\mathrm{P}2} \\
mv_0^2 + 0 &\stackrel{?}{=} \left(v_0^2 + (v_0 - gt)^2\right) m/2 + mg \overbrace{(v_0 t - gt^2/2)}^{y} \\
mv_0^2 &\stackrel{\checkmark}{=} mv_0^2 \quad \text{(Checks)}
\end{aligned}
$$

Using momentum and energy as a check of a numerical solution

You obtain a numerical solution to $\vec{F} = m\vec{a}$ by setting up the set of first order differential equations 10.2 on page 539. In turn, these can be written in explicit scalar form as eqn. (10.3).

While you solve these equations you can add further first order equations that you can use in your energy and momentum checks. These evaluate the integrals for linear impulse, angular impulse and work.

$$
\begin{aligned}
\frac{d}{dt}(\text{linear impulse}) &= \vec{F}, \\
\frac{d}{dt}(\text{angular impulse}) &= \vec{r}_{/C} \times \vec{F}, \text{ and} \\
\dot{W} &= \vec{F} \cdot \vec{v}.
\end{aligned}
$$

The first two equations are short hand for 2 (or 3) first order scalar equations for motion in 2 (or 3) spatial dimensions. If these are added to the system of ordinary differential equations that you solve, they can be used to check the solution.

Summary on using energy and momentum to check a solution

Because the momentum and energy facts and theorems apply to any motion consistent with $\vec{F} = m\vec{a}$ they can be used as a consistency check on any solutions you find to the differential equations of motion ($\vec{F} = m\vec{a}$).

Here is the general situation. You are given $\vec{F}(t, \vec{r}, \vec{v})$. You are given initial conditions $\vec{r}_0$ and $\vec{v}_0$ at, say, $t = 0$. Using computer integration or pencil and paper methods, you solve the differential equation $\vec{a} = \vec{F}/m$ to get $\vec{r}(t)$ and $\vec{v}(t)$. Now your solution can be checked for consistency with the energy and momentum theorems. In particular, your solution, if it is correct, must satisfy

- Linear momentum balance: $\int_0^t \vec{F}\,dt = \vec{L}_2 - \vec{L}_1$;
- Angular momentum balance: $\int_0^t \vec{r} \times \vec{F}\,dt = \vec{H}_2 - \vec{H}_1$;
- Work-energy: $\int_0^t \vec{F} \cdot \vec{v}\,dt = E_{K2} - E_{K1}$.

These have been used in the simple ballistics example above. That linear momentum balance, angular momentum balance and energy balance, all are consistent to an assumed solution lends credence to its correctness. For simple problems with such simple analytical solutions, using this consistency is not the most efficient way of checking a candidate solution's veracity. We would be better off just plugging the proposed solution back into the differential equation to see if it was satisfied. But in more complex problems and in numerical solutions, checks like those here are sometimes simpler to make. Some more comments about these checks:

- The angular momentum check can be used relative to any fixed point you choose. If you can find a point where, say, the applied force has no moment, then the change of angular momentum should be zero about that point.
- If the applied force is conservative, the work integral can be replaced by the change in potential energy and the work-energy check is a check of the conservation of energy.
- If you try to make the checks with pencil and paper the checks can sometimes be harder to implement than it was to find the original solution.

10.3 Conservative forces and non-conservative forces

Imagine that the force $\vec{F}$ on a particle is known to depend on the position $\vec{r}$ of a particle as it moves. This dependence of $\vec{F}$ on $\vec{r}$ is called a *force field*:

$$\vec{F} = \vec{F}(\vec{r}).$$

As the particle moves from one point $\vec{r}_1$ to another $\vec{r}_2$ we can evaluate the work of this force field as

$$W_{12} = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\,dt = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot d\vec{r}.$$

But what if the particle moves between the same two points but along a different path, is the work W_{12} the same? If that is true then the work going from $\vec{r}_1$ to $\vec{r}_2$ and back would be zero. Which means the work of the force when the particle moves on any closed path would be zero. Here is an example of a force field $\vec{F}(\vec{r})$ in the xy plane where the work in going on a closed path, from $\vec{r}_1$ to $\vec{r}_1$, from home to home, is not zero:

$$\vec{F} = C[\hat{k} \times \vec{r}] = C[-y\hat{\imath} + x\hat{\jmath}]$$

where C is a constant. This force pushes the particle around in circles. So, if the particle moves on the circular path

$$\vec{r} = r_0[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}] \qquad (0 \le \theta \le 2\pi)$$

then the work is the force magnitude times the arc-length (the force is parallel to the velocity for this path) and so,

$$\int_{\theta_1}^{\theta_2} \vec{F} \cdot d\vec{r} = 2\pi C r_0 \neq 0.$$

This force field gives non-zero work for some closed paths, thus is path dependent for open paths and therefor is *non-conservative*. How can you tell if a force field is conservative or not. This, you learn in vector calculus, holds if the curl of $\vec{F}$ is zero, $\vec{\nabla} \times \vec{F} = \vec{0}$, everywhere.

Forces from any combination of springs and gravity are always conservative.

- These checks are often *very useful*, and this is perhaps an understatement, for checking the validity of numerical solutions of dynamics equations. Basically you shouldn't trust yours or any body else's code unless such checks have been made. It is hard to write correct code without making such checks. And such checks are a strong sign of code reliability because an error in computer code will usually lead to an error in momentum balance, angular momentum balance or energy balance.

10.4 Derivation of momentum, angular momentum and energy theorems for a point mass

For a point-mass particle the principles of linear momentum, angular momentum and energy are theorems that can be derived simply from

$$\vec{F} = m\vec{a}$$

as follows.

Linear momentum

Define linear momentum as $\vec{L} = m\vec{v}$ then differentiating we have the equation $\vec{F} = m\vec{a}$. It is not so much a derivation but a restatement to write:

$$\vec{F} = \dot{\vec{L}}.$$

Integrating both sides in time we get

$$\underbrace{\int_{t_1}^{t_2} \vec{F}\, dt}_{\text{impulse}} = \underbrace{\vec{L}_2 - \vec{L}_1}_{\text{change of momentum}} .$$

This is the principle of impulse and momentum.

Angular momentum

Start with $\vec{F} = m\vec{a}$ and take the cross product of both sides with the position relative to a fixed point C and you get

$$\vec{r}_{/C} \times \vec{F} = \vec{r}_{/C} \times (m\vec{a}) .$$

Now if we define $\vec{H}_{/C} = \vec{r}_{/C} \times (m\vec{v})$ we can differentiate to find that, writing out all details,

$$\begin{aligned}
\dot{\vec{H}}_{/C} &= \frac{d}{dt}\left(\vec{r}_{/C} \times (m\vec{v})\right) \\
&= \frac{d}{dt}\left((\vec{r} - \vec{r}_C) \times m\vec{v}\right) \\
&= m\left(\dot{\vec{r}} - \dot{\vec{r}}_C\right) \times \vec{v} + m\left(\vec{r} - \vec{r}_C\right) \times \dot{\vec{v}} \\
&= m\left(\vec{v} - \vec{0}\right) \times \vec{v} + m\left(\vec{r} - \vec{r}_C\right) \times \dot{\vec{v}} \\
&= m\underbrace{\vec{v} \times \vec{v}}_{\vec{0}} + m\left(\vec{r} - \vec{r}_C\right) \times \dot{\vec{v}} \\
&= m\vec{r}_{/C} \times (m\dot{\vec{v}}) \\
&= \vec{r}_{/C} \times (m\vec{a})
\end{aligned}$$

Putting these together we have

$$\vec{r}_{/C} \times \vec{F} = \dot{\vec{H}}_{/C}.$$

Integrating both sides with respect to time we get that the net angular impulse is the change in angular momentum.

$$\int_{t_1}^{t_2} \vec{r}_{/C} \times \vec{F}\, dt = (\vec{H}_{/C})_2 - (\vec{H}_{/C})_1 .$$

Power and kinetic energy

The power equation is found with a shade more difficulty. We take the equation $\vec{F} = m\vec{a}$ and dot both sides with the velocity $\vec{v}$ of the particle:

$$\vec{F} \cdot \vec{v} = m\vec{a} \cdot \vec{v}. \tag{10.10}$$

Evaluating $\vec{v} \cdot \vec{a}$ is most easily done with the benefit of hindsight. So we cheat and look at the time derivative of the speed squared:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}v^2\right) &= \frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v}) \\
&= \frac{1}{2}\left(\dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}}\right) \\
&= \vec{v} \cdot \dot{\vec{v}} \\
&= \vec{v} \cdot \vec{a}
\end{aligned}$$

Applying this result to eqn. (10.10) we get

$$\underbrace{\vec{F} \cdot \vec{v}}_{P} = \frac{d}{dt}\underbrace{\left(\frac{1}{2}mv^2\right)}_{E_K},$$

the energy (or power balance) equation for a particle.

Integrating in time we get

$$\underbrace{\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt}_{\text{Work}} = \underbrace{E_{K2} - E_{K1}}_{\text{Change in kinetic energy}} ,$$

Power and work and energy

Because $\vec{F} \cdot \vec{v}\, dt = \vec{F} \cdot d\vec{r}$ the time integral of power can be replaced with a path integral, the standard work integral:

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}.$$

If $\vec{F}$ is a conservative force field, meaning a function of position, then $E_P(\vec{r})$ exists, so that

$$-\vec{\nabla} E_P = \vec{F} \quad \text{then} \quad \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = E_P(\vec{r}_2) - E_P(\vec{r}_1)$$

and the work-energy equation becomes

$$E_2 = E_1$$

Where $E = E_K + E_P$ is defined as the total energy.

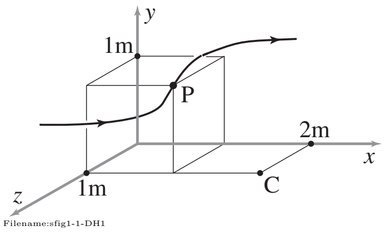

Figure 10.26:

SAMPLE 10.8 Basic calculations: Find $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{/\mathrm{C}}$, $\dot{\vec{H}}_{/\mathrm{C}}$, E_K, $\dot{E}_K$ for a given particle P with mass $m_P = 1\,\mathrm{kg}$, given position, velocity, acceleration, and a point C. Specifically, we are given $\vec{r}_P = (\hat{\imath}+\hat{\jmath}+\hat{k})\,\mathrm{m}$, $\vec{v}_P = 3\,\mathrm{m/s}(\hat{\imath}+\hat{\jmath})$, $\vec{a}_P = 2\,\mathrm{m/s^2}(\hat{\imath}-\hat{\jmath}-\hat{k})$, and $\vec{r}_C = (2\hat{\imath}+\hat{k})\,\mathrm{m}$.

Solution Since $\vec{r}_P = (\hat{\imath}+\hat{\jmath}+\hat{k})\,\mathrm{m}$ and $\vec{r}_C = (2\hat{\imath}+\hat{k})\,\mathrm{m}$,

$$\vec{r}_{P/C} = \vec{r}_P - \vec{r}_C = (-\hat{\imath}+\hat{\jmath})\,\mathrm{m}.$$

So we have the motion quantities

$$\begin{aligned}
\vec{L} &= m\vec{v}_P \\
&= (1\,\mathrm{kg})\cdot[(3\,\mathrm{m/s})(\hat{\imath}+\hat{\jmath})] \\
&= 3(\hat{\imath}+\hat{\jmath})\frac{\mathrm{kg\cdot m}}{\mathrm{s}} \\
&= 3\,\mathrm{N\cdot s}(\hat{\imath}+\hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{L}} &= m\vec{a}_P \\
&= (1\,\mathrm{kg})[(2\,\mathrm{m/s^2})(\hat{\imath}-\hat{\jmath}-\hat{k})] \\
&= 2(\hat{\imath}-\hat{\jmath}-\hat{k})\frac{\mathrm{kg\cdot m}}{\mathrm{s^2}} \\
&= 2\,\mathrm{N}(\hat{\imath}-\hat{\jmath}-\hat{k})
\end{aligned}$$

$$\begin{aligned}
\vec{H}_{/\mathrm{C}} &= \vec{r}_{P/C}\times m\vec{v}_P \\
&= [(-\hat{\imath}+\hat{\jmath})\,\mathrm{m}]\times[(1\,\mathrm{kg})3\,\mathrm{m/s}(\hat{\imath}+\hat{\jmath})] \\
&= -(6\,\mathrm{kg\cdot m^2/\,s})\hat{k}
\end{aligned} \tag{10.11}$$

$$\begin{aligned}
\dot{\vec{H}}_{/\mathrm{C}} &= \vec{r}_{P/C}\times m\vec{a} \\
&= [(-\hat{\imath}+\hat{\jmath})m]\times[(1\,\mathrm{kg})2\,\mathrm{m/s^2}(\hat{\imath}-\hat{\jmath}-\hat{k})] \\
&= -(2\,\mathrm{kg\cdot m^2/\,s^2})(\hat{\imath}+\hat{\jmath})
\end{aligned}$$

$$\begin{aligned}
E_\mathrm{K} &= \frac{1}{2}m|\vec{v}_P|^2 \\
&= \frac{1}{2}(1\,\mathrm{kg})(3\sqrt{2}\,\mathrm{m/s})^2 \\
&= 9\frac{\mathrm{kg\cdot m^2}}{\mathrm{s^2}} \\
&= 9\,\mathrm{N\cdot m}
\end{aligned}$$

$$\begin{aligned}
\dot{E}_\mathrm{K} &= \frac{d}{dt}(\frac{m}{2}\vec{v}_P\cdot\vec{v}_P) \\
&= \frac{m}{2}[\vec{v}_P\cdot\dot{\vec{v}}_P + \dot{\vec{v}}_P\cdot\vec{v}_P] \\
&= m\vec{v}_P\cdot\vec{a}_P \\
&= 1\,\mathrm{kg}[(3\,\mathrm{m/s})(\hat{\imath}+\hat{\jmath})]\cdot[(2\,\mathrm{m/s^2})(\hat{\imath}-\hat{\jmath}-\hat{k})] \\
&= 0.
\end{aligned}$$

Note: $\frac{d}{dt}(\frac{1}{2}v^2) \neq |\vec{v}||\vec{a}|$.

SAMPLE 10.9 Direct application of the formulas: A 2 kg block is moving with a velocity $\vec{v}(t) = u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}$, where $u_0 = 5\,\text{m/s}$, $v_0 = 10\,\text{m/s}$, and $c = 0.5/\,\text{s}$. Consider the time interval between $t_1 = 1\,\text{s}$ to $t_2 = 3\,\text{s}$.

1. Find the net change in the linear momentum of the block, $\Delta\vec{L} = \vec{L}(t_2) - \vec{L}(t_1)$.
2. Find the force $\vec{F}(t)$ on the block and compute the impulse $\int_{t_1}^{t_2} \vec{F}\,dt$ and show that it is the same as $\Delta\vec{L}$ computed above.
3. Find the change in kinetic energy from direct computation of energy and compare with work done by computing $\int_{t_1}^{t_2} P\,dt$.

Solution

1. For the given block we have, $\vec{L} = m\vec{v} = m(u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath})$. Therefore,

$$\Delta L = \vec{L}(t_2) - \vec{L}(t_1) = m u_0\left(e^{-ct_2} - e^{-ct_1}\right)\hat{\imath}.$$

Substituting the given values, $m = 2\,\text{kg}$, $u_0 = 5\,\text{m/s}$, $v_0 = 10\,\text{m/s}$, $c = 0.5/\,\text{s}$, $t_1 = 1\,\text{s}$, and $t_2 = 3\,\text{s}$ we get

$$\Delta\vec{L} = 2\,\text{kg}\cdot 5\,\text{m/s}(e^{-0.5/\,\text{s}\cdot 3\,\text{s}} - e^{-0.5/\,\text{s}\cdot 1\,\text{s}})\hat{\imath} = -(3.83\,\text{kg}\cdot\text{m/s})\hat{\imath}.$$

$$\boxed{\Delta\vec{L} = -(3.83\,\text{kg}\cdot\text{m/s})\hat{\imath}}$$

2. To calculate the impulse, $\int \vec{F}\,dt$, we need to find the force first. SInce $\vec{F} = m\vec{a} = m\dot{\vec{v}}$, we get

$$\vec{F}(t) = m\frac{d}{dt}(u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}) = -mcu_0 e^{-ct}\hat{\imath}.$$

Hence, the impulse is

$$\begin{aligned}
\int_{t_1}^{t_2}\vec{F}\,dt &= -\int_{t_1}^{t_2} mcu_0 e^{-ct}\,dt\hat{\imath} = mu_0(e^{-ct_2} - e^{-ct_1}) \\
&= 2\,\text{kg}\cdot 5\,\text{m/s}(e^{-0.5/\,\text{s}\cdot 3\,\text{s}} - e^{-0.5/\,\text{s}\cdot 1\,\text{s}})\hat{\imath} \\
&= -3.83\ \text{kg}\cdot\text{m/s}\hat{\imath}
\end{aligned}$$

which is, expectedly, the same answer as obtained above for ΔL.

3. To find the kinetic energy, we need the speed of the particle, $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$. Now, the change in kinetic energy is

$$\begin{aligned}
\Delta E_K &= E_{K2} - E_{K1} = \frac{1}{2}m\left(v_2^2 - v_1^2\right) \\
&= \frac{1}{2}m\left(u_0^2 e^{-2ct_2} + v_0^2 - u_0^2 e^{-2ct_1} - v_0^2\right) \\
&= \frac{1}{2}mu_0^2\left(e^{-2ct_2} - e^{-2ct_1}\right) = -7.95\,\text{N·m}.
\end{aligned}$$

Now, we can compare this value by computing the work done $\int P\,dt$, since $\Delta E_K = \int P\,dt$. To compute the power $P = \vec{F}\cdot\vec{v}$, we need to find the dot product between the force and the velocity. Since $\vec{F} = -mcu_0 e^{-ct}\hat{\imath}$, and $\vec{v} = u_0 e^{-ct}\hat{\imath} + v_0\hat{\jmath}$, we get, $\vec{F}\cdot\vec{v} = -mcu_0^2 e^{-2ct}$. Therefore, the work done is,

$$\begin{aligned}
W &= \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} -mcu_0^2 e^{-2ct}\,dt \\
&= -mcu_0^2(e^{-2ct_2} - e^{-2ct_1}) = -7.95\,\text{N·m}.
\end{aligned}$$

$$\boxed{\Delta E_K = -7.95\,\text{N·m}, \quad W = -7.95\,\text{N·m}}$$

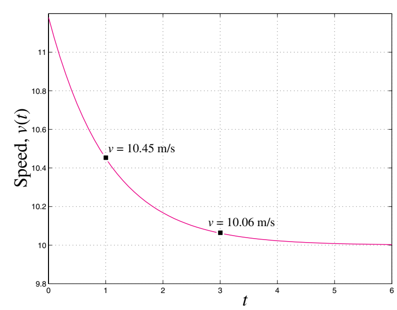

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Figure 10.27: The plot of speed $v = |\vec{v}| = \sqrt{v_x^2 + v_y^2}$ vs time. The speeds at t_1 and t_2 are of interest for computing the kinetic energy at the two instants.

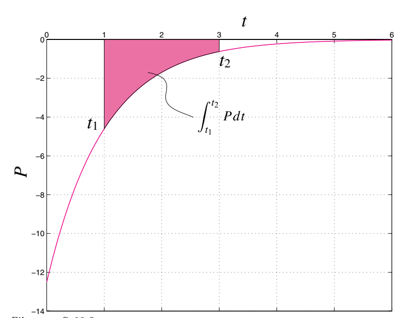

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Figure 10.28: The area under the $P(t)$ curve between t_1 and t_2 is the work done $W = \int_{t_1}^{t_2} P\,dt$.

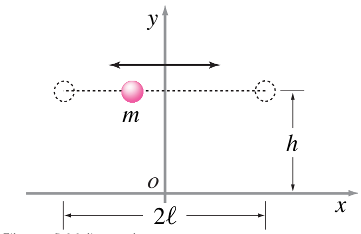


Figure 10.29:

SAMPLE 10.10 Angular momentum: direct application of the formula. The position of a particle of mass $m = 0.5\,\text{kg}$ is $\vec{r}(t) = \ell\,\sin(\lambda t)\hat{\imath} + h\hat{\jmath}$; where $\lambda = \pi/2\,\text{rad/s}$, $h = 2\,\text{m}$, $\ell = 2\,\text{m}$, and $\vec{r}$ is measured from the origin.

1. Find the net change in linear momentum $\Delta\vec{L}$ of the particle between $t = 1\,\text{s}$ and $t = 3\,\text{s}$.
2. Find the net change in angular momentum $\Delta\vec{H}_{/O}$ of the particle about the origin between $t = 1\,\text{s}$ and $t = 3\,\text{s}$.
3. Find the angular impulse $\int_{1\,\text{s}}^{3\,\text{s}} M dt$ about the origin and compare the result with $\Delta\vec{H}_{/O}$ found above.

Solution

1. Linear momentum: Let the two instants of interest be $t_1 (= 1\,\text{s})$ and $t_2 (= 3\,\text{s})$. The net change in linear momentum, $\Delta\vec{L} = \vec{L}_2 - \vec{L}_1 = m(\vec{v}_2 - \vec{v}_1)$. Since $\vec{v} = \dot{\vec{r}} == \ell\,\lambda\,\cos(\lambda t)\hat{\imath}$, we get

$$\begin{aligned}\Delta\vec{L} &= m(\vec{v}_2 - \vec{v}_1) = m\,\ell\,\lambda(\cos\lambda t_2 - \cos\lambda t_1)\hat{\imath}\\ &= (0.5\,\text{kg})(2\,\text{m})\left(\frac{\pi}{2}\,\text{rad/s}\right)\left(\cos\frac{\pi}{2} - \cos\frac{3\pi}{2}\right)\hat{\imath}\\ &= \vec{0}.\end{aligned}$$

The answer makes sense because both $\vec{v}_1 = \vec{0}$ and $\vec{v}_2 = \vec{0}$. In fact, finding the velocity at $t_1 = 1\,\text{s}$ and $t_2 = 3\,\text{s}$ would have made the calculation much simpler.

$$\boxed{\Delta\vec{L} = \vec{0}}$$

2. Angular momentum: The net change in angular momentum between t_1 and t_2 is,

$$\Delta\vec{H}_{/O} = (\vec{H}_{/O})_2 - (\vec{H}_{/O})_1 = \vec{r}_{/O} \times m\underbrace{\vec{v}_2}_{\vec{0}} - \vec{r}_{/O} \times m\underbrace{\vec{v}_1}_{\vec{0}} = \vec{0}.$$

$$\boxed{\Delta\vec{H}_{/O} = \vec{0}}$$

Note that it so happens that velocities at the two instants are zero and hence, both $(\vec{H}_{/O})_1$ and $(\vec{H}_{/O})_2$ are zero, making $\Delta\vec{H}_{/O}$ also zero. It is, however, possible that we could get $\Delta\vec{H}_{/O}$ to be zero even if the $(\vec{H}_{/O})_1$ and $(\vec{H}_{/O})_2$ were non-zero (when they are equal).

3. Moment impulse: Now, let us find the impulse due to the moment, $\int M dt$ between the two given time instants and see if that matches with the net zero change in angular momentum. We first need to compute the moment $\vec{M}_O = \vec{r}_{/O} \times \vec{F} = \vec{r}_{/O} \times m\vec{a}$:

$$\vec{M}_{/o}(t) = \vec{r}_{/O} \times m\vec{a} = (\ell\,\sin(\lambda t)\hat{\imath} + h\hat{\jmath}) \times m(-\ell\,\lambda^2\sin\lambda t\hat{\imath}) = m\,\ell\,h\,\lambda^2\sin\lambda t\hat{k}.$$

Therefore, the impulse due to this moment is

$$\begin{aligned}\int_{t_1}^{t_2}\vec{M}_{/O}dt &= \int_{1\,\text{s}}^{3\,\text{s}}(m\,\ell\,h\,\lambda^2\sin\lambda t\hat{k})dt = m\,\ell\,h\,\lambda^2\hat{k}\int_{1\,\text{s}}^{3\,\text{s}}\sin(\lambda t)dt\\ &= m\,\ell\,h\,\lambda^2\hat{k}\left[-\frac{\cos\lambda t}{\lambda}\right]_{1\,\text{s}}^{3\,\text{s}} = m\,\ell\,h\,\lambda\hat{k}\left[\cos\frac{3\pi}{2} - \cos\frac{\pi}{2}\right]\\ &= \vec{0}\end{aligned}$$

as expected. It can also be seen from a plot of $|\vec{M}_{/O}|$ vs t, as shown in fig. 10.30, that the net area under the moment between t_1 and t_2 is zero, giving a zero moment impulse.

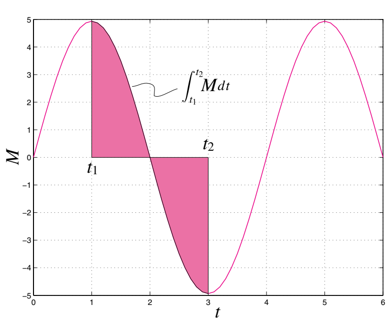


Figure 10.30: Plot of $M(t)$ where $\vec{M}_{/O}(t) = M(t)\hat{k}$. The area under the moment curve between t_1 and t_2 is the magnitude of the moment impulse.

10.3 Central-force motion and celestial mechanics

One of Isaac Newton's greatest achievements was the explanation of Kepler's laws of planetary motion. Kepler, using the meticulous observations of Tycho Brahe characterized the orbits of the planets about the sun with his 3 famous laws:

- Each planet travels on an ellipse with the sun at one focus.
- Each planet goes faster when it is close to the sun and slower when it is further. It speeds and slows so that the line segment connecting the planet to the sun sweeps out area at a constant rate.
- Planets that are further from the sun take longer to go around. More exactly, the periods are proportional to the lengths of the ellipses to the 3/2 power.

Newton, using his equation $\vec{F} = m\vec{a}$ and his law of universal gravitational attraction, was able to formulate a differential equation governing planetary motion. He was also able to solve this equation and found that it exactly predicts all three of Kepler's laws.

The Newtonian description of planetary motion is the most historically significant example of *central-force motion* where,

- the only force acting on a particle is directed towards the origin of a given coordinate system, and
- the magnitude of the force depends only on distance between attracting points.

If we define the position of the particle as $\vec{r}$ with magnitude r, linear momentum balance for central-force motion is

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ \Rightarrow \quad \vec{F} &= m\vec{a} \\ \Rightarrow \quad F(r)\left(\frac{-\vec{r}}{r}\right) &= m\ddot{\vec{r}} \end{aligned} \tag{10.12}$$

where $-\vec{r}/r$ is a unit vector pointed toward the origin and $F(r)$ is the magnitude of the origin-attracting force.

For the rest of this section we consider some of the consequences of eqn. (10.12). We start with the most historically important example.

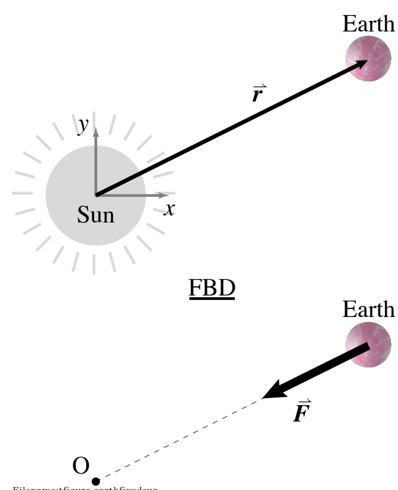


Figure 10.31: The earth moving around a fixed sun. The attraction force $\vec{F}$ is directed "centrally" towards the sun and has magnitude proportional to both masses and inversely proportional to the distance squared.

Motion of the earth around a fixed sun

For simplicity let's assume that the sun does not move and that the motion of the earth lies in a plane. Newton's law of gravitation says that the attractive force of the sun on the earth is proportional to the masses of the sun and earth and inversely proportional to the distance between them squared (fig. 10.31). Thus we have ①

$$F = \frac{G m_e m_s}{r^2}$$

where m_e and m_s are the masses of the earth and sun, r is the distance between the earth and sun. 'Big G' is a universal constant $G \approx 6.67 \cdot 10^{-11}\,\mathrm{N\,m^2/\,kg^2}$. What is the vector-valued force on the earth? It is its magnitude times a unit vector in the appropriate direction.

$$\begin{aligned} \vec{F} &= \left(\frac{G m_e m_s}{r^2}\right)\left(\frac{-\vec{r}}{|\vec{r}|}\right) \\ \Rightarrow\quad \vec{F} &= -G m_e m_s \left(\frac{\vec{r}}{r^3}\right) \\ \Rightarrow\quad \vec{F} &= -G m_e m_s \left(\frac{x\hat{\imath} + y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) \end{aligned} \tag{10.13}$$

where we have used that $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, $r = |\vec{r}| = \sqrt{x^2+y^2}$, and $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$. Now we can write the linear momentum balance equation for the earth in great detail.

$$\begin{aligned} \vec{F} &= m\vec{a} \\ \Rightarrow\quad -G m_e m_s \left(\frac{x\hat{\imath} + y\hat{\jmath}}{(x^2+y^2)^{3/2}}\right) &= m_e\,(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \end{aligned} \tag{10.14}$$

Taking the dot product of equation 10.14 with $\hat{\imath}$ and $\hat{\jmath}$ (*i.e.,* taking x and y components) gives two scalar second order ordinary differential equations:

$$\ddot{x} = \frac{-G m_s x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-G m_s y}{(x^2+y^2)^{3/2}}. \tag{10.15}$$

This pair of coupled second order differential equations describes the motion of the earth.② Pencil and paper solution is possible, Newton did it, but is a little too hard for this book. So we resort to computer solution. To set this up we put equations eqn. (10.15) in the form of a set of coupled first order ordinary differential equations. If we define $z_1 = x$, $z_2 = \dot{x}$, $z_3 = y$, and $z_4 = \dot{y}$. We can now write equations 10.15 as

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -G m_s z_1/(z_1^2 + z_3^2)^{3/2} \\ \dot{z}_3 &= z_4 \\ \dot{z}_4 &= -G m_s z_3/(z_1^2 + z_3^2)^{3/2}. \end{aligned} \tag{10.16}$$

To actually solve these numerically we need a value for $G m_s$ and initial conditions. The solutions of these equations on the computer are all, within numerical error, consistent with Kepler's laws.

Without a full solution, there are some things we can figure out relatively easily.

① Soon after Newton, Cavendish found G in his lab by delicately measuring the *small* attractive force between two balls. The gravitational attraction between two 1 kg balls a meter apart is about a ten-millionth of a billionth of a Newton (a Newton is about a fifth of a pound).

② Note that G appears in the product $G m_s$. Newton didn't know the value of big G, but he could do a lot of figuring without it. All he needed was the product $G m_s$ which he could find from the period and radius of the earth's orbit. The entanglement of G with the mass of the sun is why some people call Cavendish's measurement of big G, "weighing the sun". From Newton's calculation of $G m_s$ and Cavendish's measurement of G you can find m_s. Naturally, the real history is a bit more complicated; Cavendish called his experiment 'weighing the earth'.

Circular orbits

We generally think of the motions of the planets as being roughly circular orbits. In fact, for any attractive central force one of the possible motions is a circular orbit. Rather than trying to derive this, let's assume a circular solution and see if it solves the equations of motion. A constant speed circular orbit with angular frequency ω and radius r_o obeys the parametric equation

$$\begin{aligned}
& & \vec{\boldsymbol{r}} &= r_o\,(\cos(\omega t)\hat{\boldsymbol{\imath}} + \sin(\omega t)\hat{\boldsymbol{\jmath}}) \\
\text{differentiating twice} & \Rightarrow & \ddot{\vec{\boldsymbol{r}}} &= -\omega^2 r_o\,(\cos(\omega t)\hat{\boldsymbol{\imath}} + \sin(\omega t)\hat{\boldsymbol{\jmath}}) \\
& & &= -\omega^2\vec{\boldsymbol{r}}. \qquad (10.17)
\end{aligned}$$

Comparing eqn. (10.17) with eqn. (10.12) we see we have an identity (a solution to the equation) if

$$\omega^2 = \frac{F(r)}{mr}.$$

In the case of gravitational attraction where $m = m_e$ we have $F(r) = Gm_sm_e/r^2$ so we get circular motion with

$$\omega^2 = \frac{Gm_s}{r^3} \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{1}{Gm_s}} \quad r^{\frac{3}{2}} \qquad (10.18)$$

because angular frequency is inversely proportional to the period ($\omega = 2\pi/T$). We have, for the special case of circular orbits, derived Kepler's third law. The orbital period is proportional to the orbital size to the 3/2 power.

Conservation of energy

Any force of the form

$$\vec{\boldsymbol{F}} = -F(r)\frac{\vec{\boldsymbol{r}}}{r}$$

is conservative and is associated with a potential energy given by the indefinite integral

$$E_{\text{P}} = \int_{-\infty}^{r} F(r)dr.$$

For the case of gravitational attraction, the potential energy is

$$E_{\text{P}} = \frac{-Gm_sm_e}{r}$$

where we could add an arbitrary constant. Thus, one of the features of planetary motion is that for a given orbit the energy is constant in time:

$$\begin{aligned}
\text{Constant} &= E_{\text{K}} + E_{\text{P}} \\
&= \frac{1}{2}mv^2 + \frac{-Gm_sm_e}{r} \\
&= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{-Gm_sm_e}{\sqrt{x^2 + y^2}}. \qquad (10.19)
\end{aligned}$$

If that constant is bigger than zero then the orbit has enough energy to have positive kinetic energy even when infinitely far from the sun. Such orbits are said to have more than “escape velocity” and they do indeed have open hyperbola-shaped orbits, and only pass close to the sun at most once.

Motion of rockets and artificial satellites

Rockets and the like move around the earth much like planets, comets and asteroids move around the sun. All of the equations for planetary motion apply. But you need to substitute the mass of the earth for m_s and the mass of the satellite for m_e. Thus we can write the governing equation eqn. (10.14) as

$$-GMm\left(\frac{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) \quad = \quad m\,(\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}}) \tag{10.20}$$

where now M is the mass of the earth and m is the mass of the satellite. At the surface of the earth $r = R$, the earth's radius, and $GM/R^2 = g$ so we can rewrite the governing equation for rockets and the like as

$$-gR^2\left(\frac{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) \quad = \quad (\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}})\,. \tag{10.21}$$

Another central-force example: force proportional to radius

A less famous, but also useful, example of central force is where the attraction force is proportional to the radius. In this case the governing equations are:

$$\begin{aligned} \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -k\vec{\boldsymbol{r}} &= m\ddot{\vec{\boldsymbol{r}}} \\ -k(x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}) &= m\,(\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}})\,. \end{aligned} \tag{10.22}$$

Dotting both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get two uncoupled linear homogeneous constant coefficient differential equations:

$$\ddot{x}+\frac{k}{m}x=0 \quad \text{and} \quad \ddot{y}+\frac{k}{m}y=0.$$

These you recognize as the harmonic oscillator equations so we can pick off the general solutions immediately as:

$$x = A\cos(\lambda t)+B\sin(\lambda t) \quad \text{and} \quad y = C\cos(\lambda t)+D\sin(\lambda t) \tag{10.23}$$

where $A, B, C,$ and D are arbitrary constants which are determined by initial conditions. For all $A, B, C,$ and D eqn. (10.23) describes an ellipse (or a special case of an ellipse, like a circle or a straight line). In the case of planetary motion we also had ellipses. In this case, however, the center of attraction is at the center of the ellipse and not at one of the foci.

Conservation of angular momentum and Kepler's second law

If we take the linear momentum balance equation eqn. (10.12) and take the cross product of both sides with $\vec{\boldsymbol{r}}$ we get the following.

$$
\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right) &= m\ddot{\vec{\boldsymbol{r}}} \\
\Rightarrow \quad \vec{\boldsymbol{r}} \times \left(F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right)\right) &= \vec{\boldsymbol{r}} \times \left(m\ddot{\vec{\boldsymbol{r}}}\right) \\
\Rightarrow \quad \vec{\boldsymbol{0}} &= \frac{d}{dt}\left(m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}\right) \qquad \text{(because } \dot{\vec{\boldsymbol{r}}} \times \dot{\vec{\boldsymbol{r}}} = \vec{\boldsymbol{0}}\text{)} \\
\Rightarrow \quad \text{constant} &= m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}. \qquad (10.24)
\end{aligned}
$$

But this last quantity is exactly the rate at which area is swept out by a moving particle. Thus Kepler's third law has been derived for all central-force motions (not just inverse square attractions). The last quantity is also the angular momentum of the particle. Thus for a particle in central force motion we have derived conservation of angular momentum from $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.

SAMPLE 10.11 Circular orbits of planets: Refer to eqn. (10.15) in the text that governs the motion of planets around a fixed sun.

1. Let $x = A\cos(\lambda t)$ and $y = A\sin(\lambda t)$. Show that x and y satisfy the equations of planetary motion and that they describe a circular orbit.
2. Show that the solution assumed above satisfies Kepler's third law by showing that the orbital period $T = 2\pi/\lambda$ is proportional to the 3/2 power of the size of the orbit (which can be characterized by its radius).

Solution

1. The governing equation of planetary motion can be written as

$$\begin{aligned} \frac{\ddot{x}}{x} = \frac{-Gm_s}{(x^2+y^2)^{3/2}} &= \frac{\ddot{y}}{y} \\ \Rightarrow \quad \ddot{x}y - \ddot{y}x &= 0 \end{aligned} \tag{10.25}$$

Now,

$$\begin{aligned} x &= A\cos(\lambda t) \quad \Rightarrow \quad \ddot{x} = -\lambda^2 A\cos(\lambda t) \\ y &= A\sin(\lambda t) \quad \Rightarrow \quad \ddot{y} = -\lambda^2 A\cos(\lambda t) \end{aligned}$$

Substituting these values in eqn. (10.25), we get

$$-\lambda^2 A^2 \cos(\lambda t)\cdot\sin(\lambda t) + \lambda^2 A\sin(\lambda t)\cdot\cos(\lambda t) \stackrel{\checkmark}{=} 0$$

Thus the assumed form of x and y satisfy the governing equations of planetary motion, *i.e.*, $x(t) = A\cos(\lambda t)$ and $y(t) = A\sin(\lambda t)$ form a solution of planetary motion. Now, it is easy to show that

$$x^2 + y^2 = A^2\cos^2(\lambda t) + A^2\sin^2(\lambda t) = A^2,$$

i.e., x and y satisfy the equation of a circle with radius A. Thus, the assumed solution gives a circular orbit.

2. Substituting $x = A\cos(\lambda t)$ in eqn. (10.15), and noting that square of the radius of the orbit is $r^2 = x^2 + y^2 = A^2$, we get

$$\begin{aligned} -\lambda^2 A\cos(\lambda t) &= -Gm_s\frac{A\cos(\lambda t)}{r^3} \\ \Rightarrow \quad \lambda^2 &= \frac{Gm_s}{A^3} \\ \text{or} \quad \left(\frac{2\pi}{T}\right)^2 &= \frac{Gm_s}{A^3} \\ \Rightarrow \quad T^2 &= \frac{4\pi^2}{Gm_s}A^3 \\ \text{or} \quad T &= KA^{3/2} \end{aligned}$$

where $K = 2\pi/\sqrt{Gm_s}$ is a constant. Thus the orbital period T is proportional to the 3/2 power of the radius, or the size, of the circular orbit.

Of, course, the same holds true for elliptic orbits too, but it is harder to show that analytically using cartesian coordinates, x and y.

SAMPLE 10.12 Numerical computation of satellite orbits: The following data is known for an earth satellite: mass = 2000 kg, the distance to the closest point, the perigee, on its orbit from the earth's surface = 1100 km, and its velocity at perigee, which is purely tangential, is 9500 m/s. The radius of the earth is 6400 km and the acceleration due to gravity $g = 9.81\,\mathrm{m/s^2}$.

1. Solve the equations of motion of the satellite numerically with the given data and show that the orbit of the satellite is elliptical. Find the apogee of the orbit and the speed of the satellite at the apogee.
2. From the data at apogee and perigee show that the angular momentum and the energy of the satellite are conserved.
3. Find the orbital period of the satellite and show that it satisfies Kepler's third law (in equality form).

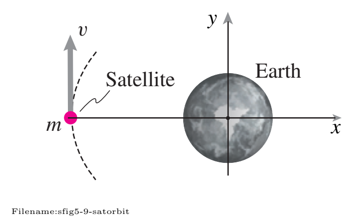


Figure 10.32:

Solution

1. The equations of motion of a satellite around a fixed earth are

$$\ddot{x} = \frac{-gR^2 x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-gR^2 y}{(x^2+y^2)^{3/2}}.$$

where g is the acceleration due to gravity and R is the radius of the earth (see eqn. (10.20) in the text). From the given data at perigee, the initial conditions are

$$x(0) = -7500\,\mathrm{km}, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 9500\,\mathrm{m/s}.$$

In order to solve the equations of motion by numerical integration, we first rewrite these equations as four first order equations:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -gR^2 z_1/(z_1^2+z_3^2)^{3/2} \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -gR^2 z_3/(z_1^2+z_3^2)^{3/2}.
\end{aligned}$$

Now the given initial conditions in terms of the new variables are

$$z1(0) = -7.5\times 10^6\,\mathrm{m}, \quad z2(0) = 0, \quad z3(0) = 0, \quad z4(0) = 9500\,\mathrm{m/s}.$$

We are now ready to go to a computer. We implement the following pseudocode on the computer to solve the problem.

```
ODEs ={z1dot=z2,  z2dot=-g*R^2*z_1/(z_1^2+z_3^2)^{3/2},
   z3dot=z4,  z4dot=-g*R^2*z_3/(z_1^2+z_3^2)^{3/2}}
IC   ={z1(0)=-7.5E06, z2(0)=0, z3(0)=0, z4(0)=9500}
Set  g = 9.81, R = 6.4E06
Solve ODEs with IC for t=0 to t=4E04
Plot z1 vs z3
```

Results obtained from implementing the code above with a Runge-Kutta method based integrator is shown in fig. 10.33 where we have also plotted the earth centered at the origin to put the orbit in perspective. The orbit is clearly elliptical. From the computer output, we find the following data for the apogee.

$$x = 4.0049\times 10^7\,\mathrm{m}, \quad \dot{x} = 0, \quad y = 0, \quad \dot{y} = -1.7791\times 10^3\,\mathrm{m/s}$$

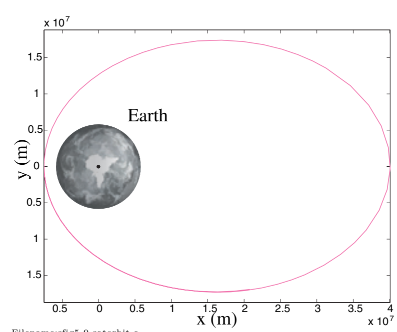


Figure 10.33: The elliptical orbit of the satellite, obtained from numerical integration of the equations of motion.

2. The expressions for energy E and angular momentum H for a satellite are,

$$\begin{aligned}
E &= E_K + E_P = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{GMm}{r} \\
\vec{H}_o &= \vec{r}\times m\vec{v} = (x\hat{\imath}+y\hat{\jmath})\times m(\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}) = m(x\dot{y}-y\dot{x})\hat{k}
\end{aligned}$$

At both apogee and perigee, $y = 0$ and the velocity (which is tangential) is in the y direction, *i.e.,* $\dot{x} = 0$. Therefore, the expressions for energy and angular momentum become simpler:

$$\begin{aligned} E &= \frac{1}{2}m\dot{y}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{y}^2 - \frac{gR^2m}{|x|}, \\ \text{and} \quad H &= mx\dot{y}. \end{aligned}$$

Let E_1 and H_1 be the energy and the angular momentum of the satellite at the perigee, respectively, and E_2 and H_2 be the respective quantities at the apogee. Then, from the given data,

$$\begin{aligned} E_1 = \frac{1}{2}m\dot{y}_1^2 - \frac{gR^2m}{|x_1|} &= \frac{1}{2}2000\,\text{kg}\cdot(9500\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{7.5\times10^6\,\text{m}} \\ &= -1.6901\times10^{10}\ \text{Joules} \\ H_1 = mx_1\dot{y}_1 &= 2000\,\text{kg}\cdot(-7.5\times10^6\,\text{m})\cdot(9500\,\text{m/s}) \\ &= -1.4250\times10^{14}\ \text{N·m}\cdot\text{s} \\ E_2 = \frac{1}{2}m\dot{y}_2^2 - \frac{gR^2m}{|x_2|} &= \frac{1}{2}2000\,\text{kg}\cdot(-1779\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{4.0049\times10^7\,\text{m}} \\ &= -1.6901\times10^{10}\text{Joules} \\ H_2 = mx_2\dot{y}_2 &= 2000\,\text{kg}\cdot(4.0049\times10^7\,\text{m})\cdot(-1779\,\text{m/s}) \\ &= -1.4250\times10^{14}\ \text{N·m}\cdot\text{s} \end{aligned}$$

Clearly, the energy and the angular momentum are conserved.

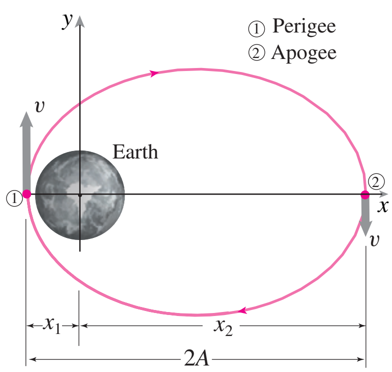


Figure 10.34: The elliptical orbit of the satellite. The perigee and apogee are marked as points 1 and 2 on the orbit.

3. From the computer output, we find the time at which the satellite returns to the perigee for the first time. This is the orbital period. From the output data, we get the orbital period to be 3.6335×10^4 s $=$ 10.09 hrs. Now let us compare this result with the analytical value of the orbital period.

   Let A be the semimajor axis of the elliptic orbit. Then the square of the orbital time period T is given by

$$T^2 = \frac{4\pi^2A^3}{gR^2}.$$

For the orbit we have obtained by numerical integration,

$$\begin{aligned} 2A &= |x_1| + |x_2| \\ &= 7.5\times10^6\,\text{m} + 4.0049\times10^7\,\text{m} \\ &= 4.7549\times10^7\,\text{m} \\ \Rightarrow \quad A &= 2.3774\times10^7\,\text{m} \end{aligned}$$

Hence,

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2\cdot(2.3774\times10^7\,\text{m})^3}{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}} \\ &= 3.6335\times10^4\,\text{s}. \end{aligned}$$

which is the same value as obtained from numerical solution.

$$T = 3.6335\times10^4\,\text{s} = 10.09\,\text{hrs}$$

SAMPLE 10.13 Zero-length spring and central force motion: A zero-length spring ③ (the relaxed length is zero) is tied to a mass $m = 1\,\text{kg}$ on one end and fixed on the other end. The spring stiffness is $k = 1\,\text{N}/m$.

1. Find appropriate initial conditions for the mass so that its trajectory is a straight line along the y-axis.
2. Find appropriate initial conditions for the mass so that its trajectory is a circle.
3. Can you find any condition on initial conditions that guarantees elliptic orbits of the mass?
4. Let $\vec{r}(0) = 0.5\,\text{m}\hat{\imath}$ and $\dot{\vec{r}}(0) = (0.5\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s}$. Describe the motion of the mass by plotting its trajectory for 12 s.

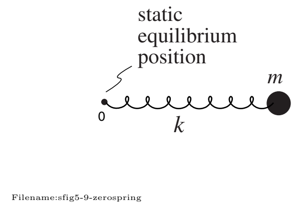


Figure 10.35:

③ No spring can have zero relaxed length, however, a spring can be configured in various ways to make it behave as if it has zero relaxed length. See box 6.1 on page 323

Solution Let the position of the mass be $\vec{r}$ at some instant t. Since the relaxed length of the spring is zero, the stretch in the spring is $|\vec{r}|$ and the spring force on the mass is $-k\vec{r}$. Then the equation of motion of the mass is

$$\begin{aligned}
-k\vec{r} &= m\ddot{\vec{r}} \\
-k(x\hat{\imath} + y\hat{\jmath}) &= m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \\
\Rightarrow \quad \ddot{x} + \frac{k}{m}x &= 0 \\
\text{and} \quad \ddot{y} + \frac{k}{m}y &= 0.
\end{aligned}$$

Thus the equations of motion are decoupled in the x and y directions. The solutions, as discussed in the text (see eqn. (10.23)), are

$$\begin{aligned}
x &= A\cos(\lambda t) + B\sin(\lambda t) \\
\text{and} \quad y &= C\cos(\lambda t) + D\sin(\lambda t)
\end{aligned} \tag{10.26}$$

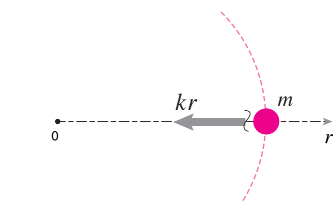


Figure 10.36: Free-body diagram of the mass.

where the constants A, B, C, D are determined from initial conditions. Let us take the most general initial conditions $x(0) = x_0, \dot{x}(0) = \dot{x}_0, y(0) = y_0$, and $\dot{y}(0) = \dot{y}_0$. By substituting these values in x and y equations above and their derivatives, we get

$$A = x_0, \quad B = \dot{x}_0/\lambda, \quad C = y_0, \quad D = \dot{y}_0/\lambda.$$

Substituting these values we get

$$\begin{aligned}
x &= x_0\cos(\lambda t) + \dot{x}_0/\lambda\sin(\lambda t) \\
\text{and} \quad y &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t).
\end{aligned} \tag{10.27}$$

1. For a straight line motion along the y-axis, we should have the x-component of motion identically zero. We can, therefore, set $x_0 = 0, \dot{x}_0 = 0$ and take any value for y_0 and $\dot{y}_0$ to give

$$\begin{aligned}
x(t) &= 0 \\
\text{and} \quad y(t) &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t).
\end{aligned}$$

2. For a circular trajectory, we must pick initial conditions such that we get $x^2 + y^2 = (\text{a constant})^2$. We can easily achieve this by choosing, say, $x(0) = x_0, \dot{x}(0) = 0, y(0) = 0$, and $\dot{y}(0) = x_0\lambda$. Substituting these values in eqn. (10.27), we get

$$x^2 + y^2 = x_0^2\cos^2(\lambda t) + \left(\frac{x_0\lambda}{\lambda}\right)^2\sin^2(\lambda t) = x_0^2$$

which is a circular orbit of radius x_0. Note that the initial position of the mass for this orbit is $\vec{r}(0) = x_0\hat{\imath}$), and the initial velocity is ($\vec{v}(0) = x_0\lambda\hat{\jmath}$), *i.e.,* the velocity is normal to the position vector ($\vec{r}\cdot\vec{v} = 0$), and the magnitude of the velocity is dependent on the magnitude of the position vector, in fact , it must be exactly equal to the product of the distance from the center and the orbital frequency λ.

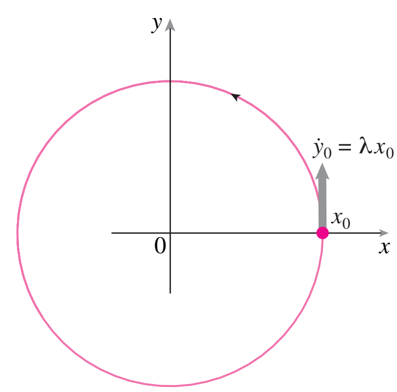


Figure 10.37: Circular trajectory of the mass.

3. In order to have elliptic orbits, the initial conditions should be selected such that x and y satisfy the equation of an ellipse. By examining the solutions in eqn. (10.27), we see that if we set $\dot{x}_0 = 0$ and $y_0 = 0$ and let the other two initial conditions have any arbitrary value, x_0 and $\dot{y}_0$, we get

$$\begin{aligned} x(t) &= x_0 \cos(\lambda t), \\ \text{and} \quad y(t) &= (\dot{y}_0/\lambda)\sin(\lambda t), \\ \Rightarrow \quad \frac{x^2}{x_0^2} + \frac{y^2}{(\dot{y}_0/\lambda)^2} &= \cos^2(\lambda t) + \sin^2(\lambda t) \\ &= 1 \end{aligned}$$

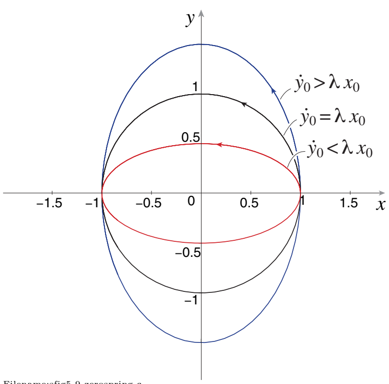
which is the equation of an ellipse with semimajor axis x_0 and semiminor axis $\dot{y}/\lambda$. Of course, the symmetry of the equations implies that we could also get elliptic orbits by setting $x_0 = 0$ and $\dot{y}_0 = 0$, and letting the other two initial conditions be arbitrary. Thus the condition for elliptic orbits is to have the initial velocity normal to the position vector, *e.g.*,

$$\begin{aligned} \vec{r}(0) = x_0\hat{\imath} \quad &\text{and} \quad \dot{\vec{r}}(0) = \dot{y}_0\hat{\jmath}, \\ \text{or} \quad \vec{r}(0) = y_0\hat{\jmath} \quad &\text{and} \quad \dot{\vec{r}}(0) = \dot{x}_0\hat{\imath}, \end{aligned}$$

or, more generally,

$$\vec{r}(0) = r_0\hat{\boldsymbol{\lambda}} \quad \text{and} \quad \dot{\vec{r}}(0) = v\hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector along the position vector of the mass and $\hat{\boldsymbol{n}}$ is normal to $\hat{\boldsymbol{\lambda}}$. Note that the condition obtained in (b) for circular orbits is just a special case of the condition for elliptic orbits (well, a circle is just a special case of an ellipse). Therefore, if we keep x_0 fixed and vary $\dot{y}_0$ we can get different elliptic orbits, including a circular one, based on the same major axis. Taking $x_0 = 1\,\mathrm{m}$, we show different orbits obtained for the mass by varying $\dot{y}_0$ in fig. 10.38.



Filename:sfig5-9-zerospring-c

Figure 10.38: Elliptic orbits of the mass obtained from the initial conditions $x_0 = 1\,\mathrm{m}, \dot{x}_0 = 0, y_0 = 0$, and various values of $\dot{y}_0$.

4. By substituting the given initial values $x_0 = 0.5\,\mathrm{m}, \dot{x}(0) = 0.5\,\mathrm{m/s}, y(0) = 0$ and $\dot{y} = 0.6\,\mathrm{m/s}$ in eqn. (10.27) and and noting that $\lambda \equiv \sqrt{k/m} = \sqrt{(1\,\mathrm{N/m})/(1\,\mathrm{kg})} = (1/\,\mathrm{s})$, we get

$$\begin{aligned} x(t) &= (0.5\,\mathrm{m}) \cdot \cos\left(\frac{1}{s} \cdot t\right) + \left(\frac{0.5\,\mathrm{m/s}}{s}\right) \cdot \sin\left(\frac{1}{s} \cdot t\right) \\ y(t) &= \left(\frac{0.6\,\mathrm{m/s}}{s}\right) \cdot \sin\left(\frac{1}{s} \cdot t\right) \end{aligned}$$

The functions $x(t)$ and $y(t)$ do not seem to describe any simple geometric path immediately. We could, perhaps, do some mathematical manipulations and try to get a relationship between x and y that we can recognize. In stead, let us plot the orbit on a computer to see the path that the mass takes during its motion with these initial conditions. To plot this orbit, we evaluate x and y at, say, 100 values of t between 0 and 10 s and then plot x vs y.

```
t = [0 0.1 0.2 ... 9.9 10]
x = 0.5 * cos(t) + 0.6 * sin(t)
y = 0.6 * sin(t)
plot x vs y
```

The plot obtained by performing these operations on a computer is shown in fig. 10.39.

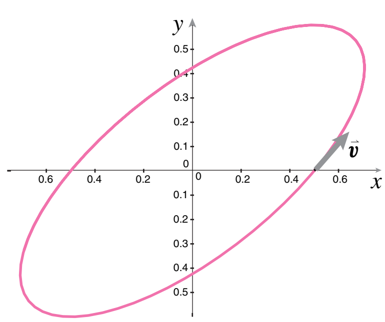


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Figure 10.39: The orbit of the mass obtained from the initial conditions $x_0 = 0.5\,\mathrm{m}, \dot{x}_0 = 0.5\,\mathrm{m/s}, y_0 = 0$, and $\dot{y}_0 = 0.6\,\mathrm{m/s}$.

Problems for Chapter 10

Particle dynamics in space

10.1 Dynamics of a particle in space

Preparatory Problems

10.1.1 Given $\vec{r}(t) = A\sin(\omega t)\hat{\imath} + Bt\hat{\jmath} + C\hat{k}$, find

1. $\vec{v}(t)$
2. $\vec{a}(t)$
3. $\vec{r}(t) \times \vec{a}(t)$.

10.1.2 A particle of mass =3 kg travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\text{s}})\,\text{m}\hat{\imath} + (\cos\frac{t}{\text{s}})\,\text{m}\hat{\jmath} + te^{\frac{t}{\text{s}}}\,\text{m/s}\hat{k}$. At $t = 3$ s, find the particle's

a) velocity and
b) acceleration.

10.1.3 A particle of mass $m = 2$ kg travels in the xy-plane with its position known as a function of time, $\vec{r} = 3t^2\,\text{m/s}^2\hat{\imath} + 4t^3\frac{\text{m}}{s^3}\hat{\jmath}$. At $t = 5$ s, find the particle's

a) velocity and *
b) acceleration, and *
c) draw the three vectors.

10.1.4 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\text{m/s})\hat{\imath} - (1.5\,\text{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

10.1.5 If $\dot{\vec{r}} = (u_0\sin\Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω as constants, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. *

10.1.6 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\text{m/s}^2\hat{\imath} - (3\,\text{m/s}^2 - 1\,\text{m/s}^3\,t)\hat{\jmath} - 5\,\text{m/s}^4\,t^2\hat{k}$, write the vector equation $\vec{v} = \int \vec{a}\,dt$ as three scalar equations (*i.e.*, find $v_x(t)$, $v_y(t)$, and $v_z(t)$).

10.1.7 Find $\vec{r}(5\,\text{s})$ given that $\dot{\vec{r}} = v_1\sin(ct)\hat{\imath} + v_2\hat{\jmath}$ and $\vec{r}(0) = 2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}$, and that v_1 is a constant 4 m/s, v_2 is a constant 5 m/s, and c is a constant $4\,\text{s}^{-1}$.

10.1.8 Let $\dot{\vec{r}} = v_0\cos\alpha\hat{\imath} + v_0\sin\alpha\hat{\jmath} + (v_0\tan\theta - gt)\hat{k}$, where v_0, α, θ, and g are constants. If $\vec{r}(0) = \vec{0}$, find $\vec{r}(t)$.

10.1.9 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\hat{\imath} + R\cos t\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\text{s})$.

More-Involved Problems

10.1.10 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\text{m}})$ m. where $x(t) = t^3(\frac{\text{m}}{s^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}$ s?

10.1.11 The position of a particle is given by $\vec{r}(t) = (t^2\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{\text{s}}}\,\text{m}\hat{\jmath})$. What are the velocity and acceleration of the particle as functions of time? Draw the path of the particle and show the vectors $\vec{v}$ and $\vec{a}$ at $t = 1$ s. *

10.1.12 A particle travels on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

10.1.13 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\text{m}}})$ m. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\text{m/s}^2$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3$ s?

10.1.14 A particle starts at the origin in the xy-plane, $(x_0 = 0, y_0 = 0)$ and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{s^2})t^2}$ m/s and $x(t) = t$ m/s, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What are the distance of the particle from the origin and its velocity and acceleration when $x = 3$ m?

10.1.15 For a particle, $\sum\vec{F} = m\vec{a}$. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{a} = -2\,\text{ft/s}^2\hat{\imath} + 5\,\text{ft/s}^2\hat{\jmath}$.

a) Write the equation

$$\sum\vec{F} = m\vec{a}$$

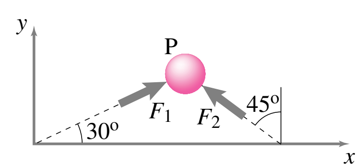
in vector form (evaluating each side as much as possible).

b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).

c) Write the equation in matrix form.

d) Find $F_1 = |\vec{F}_1|$ and $F_2 = |\vec{F}_2|$ by the following methods:

1. from the scalar equations using hand algebra,
2. from the matrix equation using a computer, and
3. from the vector equation using a cross product.



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Problem 10.1.15

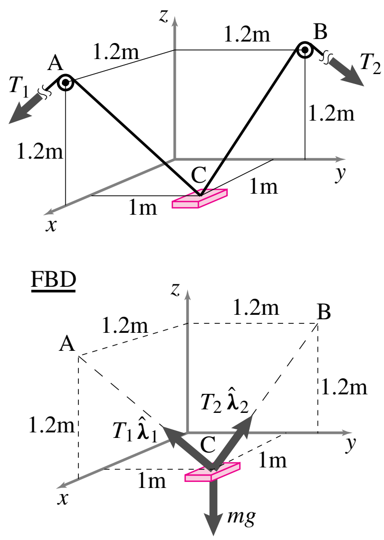
10.1.16 Three forces, $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{k}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\text{m/s}^2\hat{\imath} + 2.2\,\text{m/s}^2\hat{\jmath} + 1.7\,\text{m/s}^2\hat{k}$. Write the equation $\sum\vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

10.1.17 In three-dimensional space with no gravity a particle with $m = 3\,\mathrm{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\vec{a} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})\ \mathrm{m/s^2}$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the pseudo-code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]

```
% Incomplete PSEUDO-CODE file
m   = 3;
a   = [ 1 2 3]';
~
rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';
uAB = rAB/(magnitude of rAB);
      .
      .
      .
T   =
```

10.1.18 The rate of change of linear momentum of a particle is known in two directions: $\dot{L}_x = 20\,\mathrm{kg\,m/s^2}$, $\dot{L}_y = -18\,\mathrm{kg\,m/s^2}$ and unknown in the z direction. The forces acting on the particle are $\vec{F}_1 = 25\,\mathrm{N}\hat{\imath} + 32\,\mathrm{N}\hat{\jmath} + 75\,\mathrm{N}\hat{k}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2y}\hat{\jmath}$ and $\vec{F}_3 = -F_3\hat{k}$. Using $\sum \vec{F} = \dot{\vec{L}}$, separate the vector equation into scalar equations in the x, y, and z directions. Solve these equations (maybe with the help of a computer) to find F_{2x}, F_{2y}, and F_3.

10.1.19 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200\,\mathrm{N}$ and $T_2 = 1500\,\mathrm{N}$, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]



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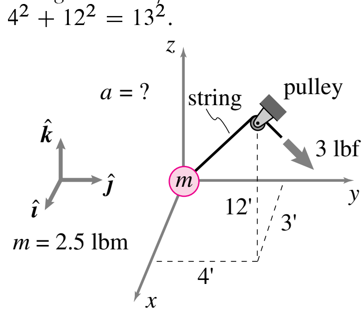
Problem 10.1.19

10.1.20 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and lbf/lbm $= g$. Use $g = 32\,\mathrm{ft/s^2}$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.



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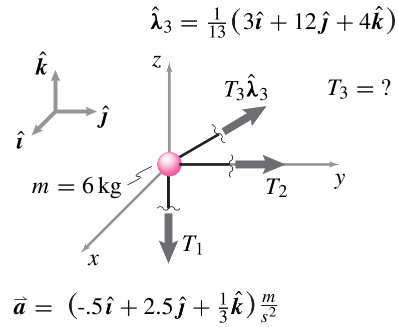
Problem 10.1.20

10.1.21 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{a} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\mathrm{m/s^2}$.

a) Given the free body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3. *



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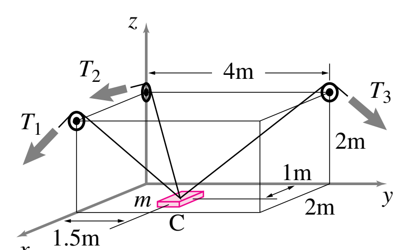
Problem 10.1.21

10.1.22 An object C of mass 2 kg is pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = \left(-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}\right)\,\mathrm{m/s^2}$.

a) Draw a free body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use $\boldsymbol{\lambda}$'s as unit vectors along the strings.

c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.



Filename:pfigure-s94h2p9
Problem 10.1.22

10.1.23 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\begin{aligned}\vec{r} = \quad & (5\,\mathrm{m})\cos^2(t^2/\,\mathrm{s}^2)\hat{\imath} \\ + \quad & (5\,\mathrm{m})\sin(t^2/\,\mathrm{s}^2)\cos(t^2/\,\mathrm{s}^2)\hat{\jmath}\end{aligned}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

10.1.24 *Computer question:* **What's the plot? What's the mechanics question?** Shown are shown some pseudo computer commands that are not commented adequately, unfortunately, and no computer is available at the moment.

a) Draw as accurately as you can, assigning numbers etc, the plot that results from running these commands.

b) See if you can guess a mechanical situation that is described by this program. Sketch the system and define the variables to make the script file agree with the problem stated.

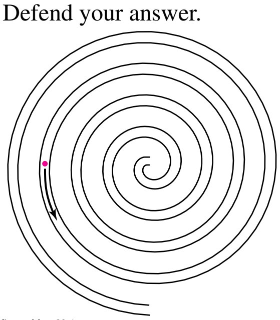
```
ODEs = {z1dot = z2
        z2dot = 0}
ICs = {z1 =1, z2=1}

Solve ODEs with ICs from t=0 to t=5

plot z2 and z1 vs t on the same plot
```

Filename:pfigure-s94f1p3
Problem 10.1.24

10.1.25 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



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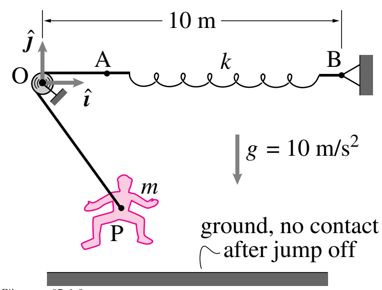
Problem 10.1.25

10.1.26 Bungy Jumping. In a relatively safe bungy jumping system, people jump up from the ground while being pulled up by a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The ideal pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2$ m and a stiffness of $k = 200$ N/m. The inextensible massless rope from A to P has length $\ell_r = 8$ m, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$

a) Assume you are given the position of the person $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Then use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\vec{r}_0 = 1\,\text{m}\hat{\imath} - 5\,\text{m}\hat{\jmath}$ and $\boldsymbol{v}_0 = \boldsymbol{0}$ write computer commands to find her position at $t = \pi/\sqrt{2}$ s.

c) Find the answer to part (b) with pencil and paper (that is, find an analytic solution to the differential equations, a final numerical answer is desired).



Filename:s97p1-3
Problem 10.1.26: Conceptual setup for a bungy jumping system.

10.1.27 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

10.1.28 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10$ m/s, the drag proportionality constant is $C = 5$ N/(m/s). The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.
- Compare the two plots and comment on the differences, if any.

10.1.29 A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\vec{F}_d = -bv^2\hat{e}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates. *

10.1.30 The equations of motion from problem 10.1.29 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations. *

b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plots its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball? Show this with an accurate plot of the trajectory. (Make sure to integrate long enough for the ball to get back to the ground.)

10.1.31 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{\boldsymbol{v}}|$, where $\vec{\boldsymbol{v}}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{r} = \vec{0}$ with $\vec{v} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

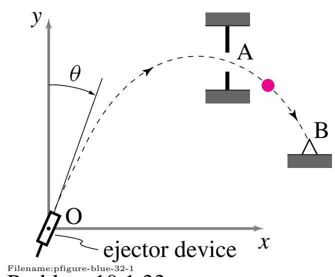
d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

10.1.32 Someone in a violent part of the world shot a projectile at someone else. The basic facts:
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag force $\propto cv^2$ with $c = .01$ kg / m.
Gravity $g = 10$ m/s.

a) Write and execute computer code to find the height at $t = 1$ s. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, numerical setup, find height at 1 s].

b) Estimate the height at $t = 1$ s using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the duration of one second of your solution.]

10.1.33 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that is passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_o. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell, 2\ell)$ and $(3\ell, \ell)$, respectively, where ℓ is your favorite length unit.



Problem 10.1.33

10.2 Momentum and energy for particle motion

Preparatory Problems

10.2.1 What symbols do we use for the following quantities? What are the definitions of these quantities? Which are vectors and which are scalars? What are the SI and US standard units for the following quantities?

a) linear momentum
b) rate of change of linear momentum
c) angular momentum
d) rate of change of angular momentum
e) kinetic energy
f) rate of change of kinetic energy
g) moment
h) work
i) power

10.2.2 Does angular momentum depend on reference point? (Assume that all candidate points are fixed in the same Newtonian reference frame.)

10.2.3 Does kinetic energy depend on reference point? (Assume that all candidate points are fixed in the same Newtonian reference frame.)

10.2.4 What is the relation between the dynamics 'Linear Momentum Balance' equation and the statics 'Force Balance' equation?

10.2.5 What is the relation between the dynamics 'Angular Momentum Balance' equation and the statics 'Moment Balance' equation?

10.2.6 A ball of mass $m = 0.1$ kg is thrown from a height of $h = 10$ m above the ground with velocity $\vec{v} = 120$ km/h$\hat{\imath} - 120$ km/h$\hat{\jmath}$. What is the kinetic energy of the ball at its release?

10.2.7 A ball of mass $m = 0.2$ kg is thrown from a height of $h = 20$ m above the ground with velocity $\vec{v} = 120$ km/h$\hat{\imath} - 120$ km/h$\hat{\jmath} - 10$ km/h$\hat{k}$. What is the kinetic energy of the ball at its release?

10.2.8 How do you calculate P, the power of all external forces acting on a particle, from the forces $\vec{F}_i$ and the velocity $\vec{v}$ of the particle?

10.2.9 A particle A has velocity $\vec{v}_A$ and mass m_A. A particle B has velocity $\vec{v}_B = 2\,\vec{v}_A$ and mass equal to the other $m_B = m_A$. What is the relationship between:

a) $\vec{L}_{\text{A}}$ and $\vec{L}_{\text{B}}$,
b) $\vec{H}_{\text{A/C}}$ and $\vec{H}_{\text{B/C}}$, and
c) E_{KA} and E_{KB}?

10.2.10 A bullet of mass 50 g travels with a velocity $\vec{v} = 0.8$ km/s$\hat{\imath} + 0.6$ km/s$\hat{\jmath}$. (a) What is the linear momentum of the bullet? (Answer in consistent units.)

10.2.11 A particle has position $\vec{r} = 4$ m$\hat{\imath} + 7$ m$\hat{\jmath}$, velocity $\vec{v} = 6$ m/s$\hat{\imath} - 3$ m/s$\hat{\jmath}$, and acceleration $\vec{a} = -2$ m/s$^2\hat{\imath} + 9$ m/s$^2\hat{\jmath}$. For each position of a point P defined below, find $\vec{H}_P$, the angular momentum of the particle with respect to the point P.

a) $\vec{r}_P = 4\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$,
b) $\vec{r}_P = -2\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$, and
c) $\vec{r}_P = 0\,\text{m}\hat{\imath} + 7\,\text{m}\hat{\jmath}$,
d) $\vec{r}_P = \vec{0}$

10.2.12 The position vector of a particle of mass 1 kg at an instant t is $\vec{r} = 2\,\text{m}\hat{\imath} - 0.5\,\text{m}\hat{\jmath}$. If the velocity of the particle at this instant is $\vec{v} = -4\,\text{m/s}\hat{\imath} + 3\,\text{m/s}\hat{\jmath}$, compute (a) the linear momentum $\vec{L} = m\vec{v}$ and (b) the angular momentum ($\vec{H}_{/\text{O}} = \vec{r}_{/\text{O}} \times (m\vec{v})$).

10.2.13 The position of a particle of mass $m = 0.5\,\text{kg}$ is $\vec{r}(t) = \ell\,\sin(\omega t)\hat{\imath} + h\hat{\jmath}$; where $\omega = 2\,\text{rad/s}, h = 2\,\text{m}, \ell = 2\,\text{m}$, and $\vec{r}$ is measured from the origin.

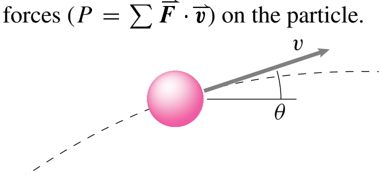
a) Find the kinetic energy of the particle at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.
b) Find the rate of change of kinetic energy at $t = 0\,\text{s}$ and $t = 5\,\text{s}$.

10.2.14 For a particle

$$E_\text{K} = \frac{1}{2}\,m\,v^2.$$

Why does it follow that $\dot{E}_\text{K} = m\,\vec{v}\cdot\vec{a}$? [hint: write v^2 as $\vec{v}\cdot\vec{v}$ and then use the product rule of differentiation.]

10.2.15 Consider a projectile of mass m at some instant in time t during its flight. Let $\vec{v}$ be the velocity of the projectile at this instant (see the figure). In addition to the force of gravity, a drag force acts on the projectile. The drag force is proportional to the square of the speed (speed$\equiv |\vec{v}| = v$) and acts in the opposite direction. Find an expression for the net power of these forces ($P = \sum \vec{F}\cdot\vec{v}$) on the particle.



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Problem 10.2.15

10.2.16 A 10 gm wad of paper is tossed into the air. At a some instant, the position, velocity, and acceleration of its center of mass are $\vec{r} = 3\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath} + 6\,\text{m}\hat{k}$, $\vec{v} = -9\,\text{m/s}\hat{\imath} + 24\,\text{m/s}\hat{\jmath} + 30\,\text{m/s}\hat{k}$, and $\vec{a} = -10\,\text{m/s}^2\hat{\imath} + 24\,\text{m/s}^2\hat{\jmath} + 32\,\text{m/s}^2\hat{k}$, respectively. What is the translational kinetic energy of the wad at the instant of interest?

10.2.17 A 2 kg particle moves so that its position $\vec{r}$ is given by

$$\vec{r}(t) = [5\sin(at)\hat{\imath} + bt^2\hat{\jmath} + ct\hat{k}]\,\text{meters}$$

where a $= \pi/\,\text{sec}, b = .25/\,\text{sec}^2, c = 2/\,\text{sec}$.

a) What is the linear momentum of the particle at $t = 1$ sec?
b) What is the force acting on the particle at $t = 1$ sec?

10.2.18 A particle A has mass m_A and velocity $\vec{v}_\text{A}$. A particle B at the same location has mass $m_B = 2\,m_A$ and velocity equal to the other $\vec{v}_\text{B} = \vec{v}_\text{A}$. Point C is a reference point. What is the relationship between:

a) $\vec{L}_\text{A}$ and $\vec{L}_\text{B}$,
b) $\vec{H}_{\text{A/C}}$ and $\vec{H}_{\text{B/C}}$, and
c) E_{KA} and E_{KB}?

10.2.19 A particle of mass $m = 3\,\text{kg}$ moves in space. Its position, velocity, and acceleration at a given time are $\vec{r} = 2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath} + 5\,\text{m}\hat{k}$, $\vec{v} = -3\,\text{m/s}\hat{\imath} + 8\,\text{m/s}\hat{\jmath} + 10\,\text{m/s}\hat{k}$, and $\vec{a} = -5\,\text{m/s}^2\hat{\imath} + 12\,\text{m/s}^2\hat{\jmath} + 16\,\text{m/s}^2\hat{k}$, respectively. For this particle at the instant of interest, find its:

a) linear momentum $\vec{L}$,
b) rate of change of linear momentum $\dot{\vec{L}}$,
c) angular momentum about the origin $\vec{H}_{/\text{O}}$,
d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/\text{O}}$,
e) kinetic energy E_K, and
f) rate of change of kinetic energy $\dot{E}_\text{K}$.

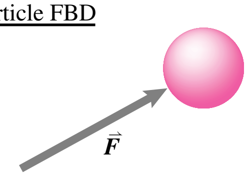
10.2.20 A particle has position $\vec{r} = 3\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath} + 4\,\text{m}\hat{k}$, velocity $\vec{v} = 2\,\text{m/s}\hat{\imath} - 3\,\text{m/s}\hat{\jmath} + 7\,\text{m/s}\hat{k}$, and acceleration $\vec{a} = 1\,\text{m/s}^2\hat{\imath} - 8\,\text{m/s}^2\hat{\jmath} + 3\,\text{m/s}^2\hat{k}$. For each position of a point P defined below, find the rate of change of angular momentum, $\dot{\vec{H}}_P$, of the particle with respect to the point P.

a) $\vec{r}_P = 3\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath} + 4\,\text{m}\hat{k}$,
b) $\vec{r}_P = 6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath} + 8\,\text{m}\hat{k}$,
c) $\vec{r}_P = -9\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath} - 12\,\text{m}\hat{k}$, and
d) $\vec{r}_P = \vec{0}$

More-Involved Problems

10.2.21 A particle of mass $m = 6\,\text{kg}$ is moving in space. Its position, velocity, and acceleration at some instant are $\vec{r} = 1\,\text{m}\hat{\imath} - 2\,\text{m}\hat{\jmath} + 4\,\text{m}\hat{k}$, $\vec{v} = 3\,\text{m/s}\hat{\imath} + 4\,\text{m/s}\hat{\jmath} - 7\,\text{m/s}\hat{k}$, and $\vec{a} = 5\,\text{m/s}^2\hat{\imath} + 11\,\text{m/s}^2\hat{\jmath} - 9\,\text{m/s}^2\hat{k}$, respectively. At this instant, find:

a) the net force $\sum\vec{F}$ on the particle,
b) the net moment on the particle about the origin $\sum\vec{M}_\text{O}$ due to the applied forces, and
c) the power P of the applied forces.



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Problem 10.2.21: FBD of the particle

10.2.22 At a time of interest, a particle with mass $m_1 = 5\,\text{kg}$ has position, velocity, and acceleration $\vec{r}_1 = 3\,\text{m}\hat{\imath}$, $\vec{v}_1 = -4\,\text{m/s}\hat{\jmath}$, and $\vec{a}_1 = 6\,\text{m/s}^2\hat{\jmath}$, respectively. Another particle with mass $m_2 = 5\,\text{kg}$ has position, velocity, and acceleration $\vec{r}_2 = -6\,\text{m}\hat{\imath}$, $\vec{v}_2 = 5\,\text{m/s}\hat{\jmath}$, and $\vec{a}_2 = -4\,\text{m/s}^2\hat{\jmath}$, respectively. For this system of two particles, and at this time, find its

a) linear momentum $\vec{L}$,
b) rate of change of linear momentum $\dot{\vec{L}}$
c) angular momentum about the origin $\vec{H}_{/\text{O}}$,
d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/\text{O}}$,

e) kinetic energy E_K, and
f) rate of change of kinetic energy $\dot{E}_K$.

10.2.23 A particle of mass $m = 250$ gm is shot straight up (parallel to the y-axis) from the x-axis at a distance $d = 2$ m from the origin. The velocity of the particle is given by $\overline{\boldsymbol{v}} = v\hat{\boldsymbol{\jmath}}$ where $v^2 = v_0^2 - 2ah$, $v_0 = 100$ m/s, $a = 10\,\mathrm{m/s^2}$ and h is the height of the particle from the x-axis.

a) Find the linear momentum of the particle at the outset of motion ($h = 0$).
b) Find the angular momentum of the particle about the origin at the outset of motion ($h = 0$).
c) Find the linear momentum of the particle when the particle is 20 m above the x-axis.
d) Find the angular momentum of the particle about the origin when the particle is 20 m above the x-axis.

10.3 Central force motion

Experts note that these problems do not use polar coordinates or any other fancy coordinate systems. Such descriptions come later in the text. At this point we want to lay out the basic equations and the qualitative features that can be found by numerical integration of the equations using Cartesian (xyz) coordinates.

Preparatory Problems

10.3.1 What exactly is meant by "central force motion"?

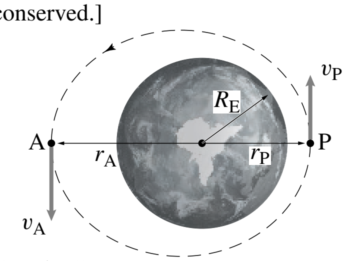
10.3.2 Under what circumstances is the angular momentum of a system, calculated relative to a point C which is fixed in a Newtonian frame, conserved?

10.3.3 The mass of the earth is M, the mass of a satellite orbiting the earth is m, the radius of the earth is R, the force of gravity at the earth's surface is mg, the universal gravitational constant is G.

a) If the sattelite is at distance r what is the force of the earth's gravity in terms of r, M, m and G?
b) If the sattelite is at distance r what is the force of the earth's gravity in terms of r, R, m and g? (hint: evaluate the formula from the first part at $r = R$).

More-Involved Problems

10.3.4 A satellite is put into an elliptical orbit around the earth and has a speed v_P at position P. Find an expression for the speed v_A at position A (in terms of R_E, r_p, r_A, g, and v_P. The radii to A and P are, respectively, r_A and r_P. [Hint: both total energy and angular momentum are conserved.]



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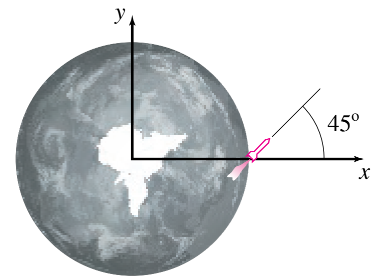
Problem 10.3.4

10.3.5 An intercontinental misile, modelled as a particle, is launched on a ballistic trajectory from the surface of the earth. The force on the missile from the earth's gravity is $F = mgR^2/r^2$ and is directed towards the center of the earth. When it is launched from the equator it has speed v_0 and in the direction shown, 45° from horizontal (both measured relative to a Newtonian reference frame). For the purposes of this calculation ignore the earth's rotation. You can think of this problem as two-dimensional in the plane shown. If you need numbers, use the following values:

$m = 1000$ kg = missile mass
$g = 10\,\mathrm{m/s^2}$ at the earth's surface,
$R = 6,400,000$ m = earth's radius, and
$v_0 = 9000$ m/s.

The distance of the missile from the center of the earth is $r(t)$.

a) Draw a free body diagram of the missile. Write the linear momentum balance equation. Break this equation into x and y components. Rewrite these equations as a system of-4 first order ODE's suitable for computer solution. Write appropriate initial conditions for the ODE's.
b) Using the computer (or any other means) plot the trajectory of the rocket after it is launched for a time of 6670 seconds. [Hint: use a much shorter time when debugging your program.] On the same plot draw a (round) circle for the earth.



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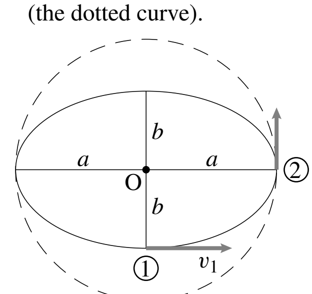
Problem 10.3.5: In intercontinental ballistic missile launch.

10.3.6 A particle of mass 2 kg moves in the horizontal xy-plane under the influence of a central force $\overline{\boldsymbol{F}} = -k\overline{\boldsymbol{r}}$ (attraction force proportional to distance from the origin), where $k = 200$ N/m and $\overline{\boldsymbol{r}}$ is the position of the particle relative to the force center. Neglect all other forces.

a) Show that circular trajectories are possible, and determine the relation between speed v and circular radius r_o which must hold on a circular trajectory. [hint: Write $\overline{\boldsymbol{F}} = m\overline{\boldsymbol{a}}$, break into x and y components, solve the separate scalar equations, pick fortuitous values for the free constants in your solutions.]
b) It turns out that trajectories are in general elliptical, as depicted in the diagram.
For a particular elliptical trajectory with $a = 1$ m and $b = 0.8$ m, the velocity of the particle at point 1 is observed to be perpendicular to the radial direction, with magnitude v_1, as shown. When the particle reaches point 2, its velocity is again perpendicular to the radial direction.
Determine the speed increment Δv which would have to be added (instantaneously) to the particle's speed at point 2 to transfer it to the circular trajectory through point 2 (the dotted curve).



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Problem 10.3.6

10.3.7 Circular motion. Generally when people talk about central force motion they not only mean that the only force is directed at the origin but that the magnitude of the force only depends on the distance from the origin. Thus in 2D

$$\vec{F} = \underbrace{-\frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}}}_{\text{Unit vector}} \cdot \underbrace{F\left(\sqrt{x^2+y^2}\right)}_{\text{Magnitude of force}}$$

where the scalar function $F(r)$ expresses the dependence of the central attractive force on distance $r = \sqrt{x^2+y^2}$. Consider a particle with mass m on a candidate circular orbit

$$\vec{r} = R\cos\lambda t\hat{\imath} + R\sin\lambda t\hat{\jmath}$$

with constant speed $v = |\vec{v}| = |\dot{\vec{r}}| = R\lambda$. For each of the cases below find the speed v for circular motion at radius R. Find this by plugging the circular motion equation into $\vec{F} = m\vec{a}$ using the form of $F(r)$ given. Answer in terms of other constants given (e.g., k, m, M, G)

a) $F(r) = kr$ (zero-rest-length attractive spring)

b) $F(r) = GMm/r^2$ (inverse-square gravitational attraction)

c) $F(r) = r^n$ (arbitrary power law attraction)

d) $F(r) = F(r)$ (arbitrary function). In this case you need to find how the speed depends on F in general.

e) Can you find a function $F(r)$ for which there are two or more circular orbits at the same speed v?

10.3.8 Circular motion, numerical solution. For each of the cases in problem 10.3.7 pick values for the physical constants. Then pick initial conditions which, according to theory, should give circular orbits. Then numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits. How big is the disrepancy between your numerical solution and an exact circle?

10.3.9 Two equal mass satellites have circular orbits at two different radii. The one that is closer to the earth has smaller potential energy and bigger kinetic energy. Which satellite has bigger total energy?

10.3.10 Find initial conditions corresponding to circular motion for a central force problem and simulate this motion on the computer. Use any central force attraction law you like (e.g., zero-length spring, inverse square,...) Check that you get closed circular orbits by plotting several revolutions. Now, in your simulation, apply a slight drag force opposing motion $\vec{F} = -c\vec{v}$. Pick a value for c so that the orbit slowly spirals in (say, less than 10% per orbit).

a) Make a plot of the spiraling orbit.

b) Plot the speed $|\vec{v}|$ vs time as it spirals in.

c) How is it that a drag force causes the satellite to speed up? Is that numerical error? An approximation in our formulation of the governing equations? A relativistic effect? What?

10.3.11 Circular motion, numerical solution. For each of the cases in problem 10.3.7 pick values for the physical constants.

a) Pick initial conditions which, according to theory, should give circular orbits.

b) Numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits.

c) How big is the disrepancy between your numerical solution and an exact circle? Between the theoretically predicted period and the actual period?

10.3.12 Conic sections, numerical solution. Newton discovered that with $\vec{F} = m\vec{a}$ and a central attractive force of $F = C/r^2$ that all motions were conic sections. In particular, consider this problem, all in consistent units: $m = 1, C = 1, x_0 = 1, y_0 = 0, \dot{x}_0 = 0, \dot{y}_0 = v_0$. Newton claimed that there is a special values for v_0, lets call them v_0^c and v_0^p with $v_0^c < v_0^p$ so that

- for $v_0 < v_0^c$ all orbits are ellipses with maximum distance from the origin being 1;
- for $v_0 = v_0^c$ the orbit is a circle of radius 1;
- for $v_0^c < v_0 < v_0^p$ the orbit is an ellipse with minimum distance from the origin being 1;
- for $v_0 = v_0^p$ the orbit is a left-opening parabola with base at the initial condition. (to draw the complete parabola you need also to use al the same initial conditions but with $\dot{y}_0 = -v_0$).
- for $v_0^p < v_0$ the orbit is a left-opening hyperbola, asymptoting to a straight line. (again you need to use $\dot{y}_0 = -v_0$ to draw the complete hyperbola.

a) By a sequence of more or less systematic numerical guesses find as accurately, as you can, v_0^c and v_0^p.

b) Numerically solve the 4 coupled first order ODEs for initial conditions that correspond to each of the 5 cases above.

c) Plot all 5 cases on one plot showing the 5 shapes clearly.

CHAPTER 11

Two or more particles in space (unconstrained)

This more advanced chapter concerns the motion of two or more particles in space. We will use $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each particle. We will use Cartesian coordinates only. The start is the set up of "two-body" type problems which are easily generalized to 3 or more particles. The first section concerns smooth motions due to forces from gravity, springs, smoothly applied forces and friction. The second section concerns the sudden change in velocities when impulsive forces are applied.

Contents

In the previous chapter you saw that once you know the forces on a particle, or how to find those forces given a particle's position, velocity and time, you can easily set up the equations of motion. That is, the linear momentum balance equation for a particle

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}},$$

with initial conditions, gives a well defined mathematical problem. The solution of this math problem gives the position and velocity of the particle as a function of time. The solution may be hard or impossible to find with pencil and paper, but can usually be found quite directly using numerical integration.

Now we generalize this idea to two, three or more particles. In one model of the universe every one of its parts is made of particles, and each particle obeys Newton's laws. We could think of all materials as made of atoms, and of all the atoms moving in deterministic ways governed by Newton's laws and known force laws. If we knew the initial positions and velocities accurately enough, then we could accurately predict the motions of all things for all time. ①

① The mathematician and mechanician Laplace (1749-1827) imagined a 'vast intellect' that could solve the differential equations that describe the universe. "Laplace's demon" was a hyper mega super computer with access to perfect data: *"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes."*

To put it in other words, given a simple atomic view of the world and a big computer, we could end a course on dynamics here. You know how to use $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each atom, so you could then simulate anything by simulating the motions of the atoms which make it up.

Of course there are some serious limitations to this point of view, so before proceeding, we list some serious caveats:

- there are no computers big enough to keep track of the 10^{23} or so atoms needed to describe macroscopic objects or the 10^{79} or so atoms in the universe;
- the laws of interaction between the most fundamental particles are not given by Newton's laws but by quantum chromodynamics, or whatever;
- one feature of the rules of the world, as physicists now understand them, is that they are not deterministic, quantum mechanics says that you *cannot* know the state of the world perfectly;
- the state of the world (the positions and velocities of all the bits is not that well known);
- the solutions of dynamics equations are often unstable in that the smallest of errors in the initial conditions propagates into a large error in the predicted motion (so called "chaos theory");

- some common descriptions of mechanical interactions, particularly those for contact between nominally rigid objects, are genuinely non-deterministic in that the governing equations do not have unique solutions; and finally
- massive simulations, even if accurate, are not always the best way to understand how things work.

Despite these limitations, in this chapter we look at the nature of systems of interacting particles. Using this particle model we can, for example, derive some results about angular momentum that turn out to be reliable, despite the questionable microscopic physics. Also, the multi-particle model of the systems is good for intuition and is also useful for modeling machines with many parts as well as of galaxies.

11.1 Coupled motions of particles in space

Assume you know enough about a system so that you know the forces on each particle if someone tells you the time and the positions and velocities of all the particles. This means you can write the governing equations for the system of particles like this:

$$\begin{aligned} \vec{\boldsymbol{a}}_1 &= \frac{1}{m_1}\vec{\boldsymbol{F}}_1 \\ \vec{\boldsymbol{a}}_2 &= \frac{1}{m_2}\vec{\boldsymbol{F}}_2 \\ \vec{\boldsymbol{a}}_3 &= \frac{1}{m_3}\vec{\boldsymbol{F}}_3 \\ &\text{etc.} \end{aligned} \tag{11.1}$$

where $\vec{\boldsymbol{F}}_1$, $\vec{\boldsymbol{F}}_2$ *etc.* are the total of the forces on the corresponding particles. The force on each particle may come from air-friction, from springs or dashpots connected here and there, or from gravity interactions with other particles, from known applied loads, *etc.*. One way or another, all the forces on all the particles are known given the time, the positions and velocities of the particles. Thus eqn. (11.1) can be written as a system of first order differential equations in standard form, ready for computer simulation. Given accurate initial conditions and a good computer then the motions of all the particles can be found accurately.

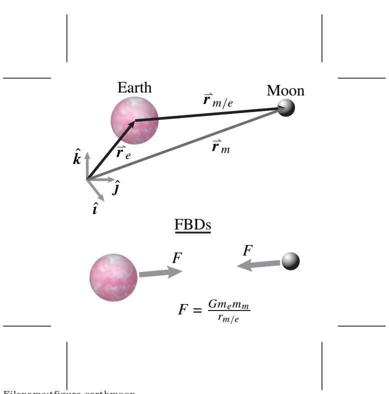

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Figure 11.1: The earth and moon. Position is measured relative to some "fixed" point C.

Example: Coupled motion of the earth and moon in three dimensions.
Let's neglect the sun and just look at the coupled motions of the earth and moon. They attract each other by the same law of gravity that we used for the sun and earth. The difference between this problem and a "central-force" problem is that we now need to look at the 'absolute' positions of the earth and the moon ($\vec{\boldsymbol{r}}_e$ and $\vec{\boldsymbol{r}}_m$), as well as the 'relative' position $\vec{\boldsymbol{r}}_{m/e} \equiv \vec{\boldsymbol{r}}_m - \vec{\boldsymbol{r}}_e$ (fig. 11.1).

The linear momentum balance equations are now

$$m_e \ddot{\vec{r}}_e = \frac{-G m_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3} \quad \text{and} \tag{11.2}$$

$$m_m \ddot{\vec{r}}_m = \frac{+G m_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3}, \tag{11.3}$$

which, when broken into x, y, and z components give 6 second order ordinary differential equations. These equations can be written as 12 first order equations by defining a list of 12 z variables: $z_1 = x_e, z_2 = \dot{x}_e, z_3 = \dot{y}_e, z_4 = y_e$, *etc.*

After you find solutions, using various initial conditions you can check if the computer finds such truths (that is, features of the exact solution of the differential equations) as:

1. that the line between the earth and moon always lies on one fixed plane,
2. the center-of-mass moves at constant speed on a straight line,
3. relative to the center-of-mass both the earth and moon travel on paths that are conic sections (circle, ellipse, parabola, hyperbola or a straight line).
4. the total energy ($E_K + E_P$) of the system is constant,
5. and that the angular momentum of the system about the center-of-mass is a constant.

These facts are discussed further below in the subsection on 'Two-particle central force motion'.

Momentum and energy of systems

There are a plethora of theorems about the momentum and energy of systems of particles. These are discussed in section C. The simplest of these are just the ones that you get from adding up the results for a single particle from section 10.2:

Linear momentum balance. $\displaystyle\sum_{\text{all forces}} \vec{F}_j = \sum m_i \vec{a}_i.$

Either because the forces between particles in a system are usually assumed to come in equal and opposite pairs or because it is an independent postulate of mechanics for general systems, the force sum can be replaced with a sum over all the external forces.

Angular momentum balance. $\displaystyle\sum_{\text{all forces}} \vec{r}_{j/C} \times \vec{F}_j = \sum \vec{r}_{i/C} \times (m_i \vec{a}_i).$

As for linear momentum, the force sum can be replaced with only the forces that act externally on the system.

Power balance. $\displaystyle\sum_{\text{all forces}} \vec{F}_j \cdot \vec{v}_j = \frac{d}{dt} \sum m_i \vec{v}_i^2/2.$

In this case the sum is over *all* the forces, internal and external. The simplification to just external forces doesn't apply to system kinetic energy like it does for momentum and angular momentum.

The one-body problem

Lets review one special problem from the previous section. The 'one-body' problem should properly be about the mechanics of a single particle interacting with nothing else. Such a particle moves at constant velocity and is too boring to get a name. Instead, when people refer to the 'one-body' problem they are talking about a particle flying around a stationary point mass to which it is attracted. That stationary point mass is held in place by, well, who knows what. Its just an idealized thing anchored by a massless structure. The 'one-body' problem is to find the motion of the particle flying around.

As we discussed in the previous sections, if the gravitational attraction follows an inverse square law then the particle moves on a plane on a curve which is either an elipse, a circle, a parabola or a hyperbola. These are, quite accurately, the trajectories of the planets and comets around the sun.

The two-body problem: two mutually attracting particles

If two particles are attracted equally to each other by mutually central forces, and no other forces act, this is called 'the two-body problem'①. Assume the two particles are m_1 and m_2 with positions $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$ (relative to the origin of a coordinate system fixed in a Newtonian frame). The force on particle 1 from particle 2 is

①This is such a famous problem in the history of science that people use it for word play to describe certain social situations. For example if two people in a couple are having trouble finding jobs in the same city they are said to have a 'two-body problem'.

$$\vec{\boldsymbol{F}}_{12} = F(r_{12})\frac{\vec{\boldsymbol{r}}_{12}}{r_{12}}$$

where $\vec{\boldsymbol{r}}_{12} = \vec{\boldsymbol{r}}_{2/1}$ is the position of particle 2 relative to particle 1, r_{12} is the distance $|\vec{\boldsymbol{r}}_{12}|$ between the particles and F is the magnitude of the attractive force. We assume the force on particle 2 is the opposite of this

$$\vec{\boldsymbol{F}}_{21} = -\vec{\boldsymbol{F}}_{12}.$$

The instantaneous velocities are $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$. We can find the center of mass G of the pair of particles as

$$m_{tot}\vec{\boldsymbol{r}}_G = m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2$$

with $m_{tot} = m_1 + m_2$.

Either by system linear momentum balance or by adding up $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each of the particles it is easy to see that

$$\vec{\boldsymbol{a}}_G = \vec{\boldsymbol{0}} \qquad \text{and} \qquad \vec{\boldsymbol{v}}_G = \text{constant}.$$

Thus we could put the origin of a good Newtonian reference frame at the center of mass $\vec{\boldsymbol{r}}_G$. The positions, velocities and accelerations relative to G, indicated with a prime ('), are

$$\begin{aligned}
\vec{\boldsymbol{r}}_1' &= \vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_G \\
\vec{\boldsymbol{r}}_2' &= \vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_G \\
\vec{\boldsymbol{v}}_1' &= \vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_G \\
\vec{\boldsymbol{v}}_2' &= \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_G \\
\vec{\boldsymbol{a}}_1' &= \vec{\boldsymbol{a}}_1 \\
\vec{\boldsymbol{a}}_2' &= \vec{\boldsymbol{a}}_2
\end{aligned}$$

where we can skip use of the prime for the acceleration. Now some facts.

- $m_1\vec{\boldsymbol{r}}_1' + m_2\vec{\boldsymbol{r}}_2' = \vec{\mathbf{0}}$ so $\vec{\boldsymbol{r}}_1' = -(m_2/m_1)\vec{\boldsymbol{r}}_2'$. For all time the two positions (relative to the center of mass) are in the opposite direction and proportional. Similarly $\vec{\boldsymbol{v}}_1' = -(m_2/m_1)\vec{\boldsymbol{v}}_2'$ and $\vec{\boldsymbol{a}}_1' = -(m_2/m_1)\vec{\boldsymbol{a}}_2'$.
- At a given instant there is a single plane defined by $\vec{\boldsymbol{r}}_1', \vec{\boldsymbol{r}}_2', \vec{\boldsymbol{v}}_1', \vec{\boldsymbol{v}}_2', \vec{\boldsymbol{a}}_1'$ and $\vec{\boldsymbol{a}}_2'$ because the positions and accelerations are all parallel (or antiparallel) and the two velocities are (anti) parallel.
- The plane above is constant in time. This is because neither the velocity nor the acceleration has a component orthogonal to the plane, thus there is no tendency to leave the plane.
- Each particle moves as if it was a single particle attracted to a central force at G. Why? Lets look at the force on mass 1

$$\vec{\boldsymbol{F}}_{12} = F(r_{12})\frac{\vec{\boldsymbol{r}}_{12}}{r_{12}} = -F(r_{12})\frac{\vec{\boldsymbol{r}}_1'}{r_1'}$$

  because the relative position of the masses passes through the origin G.
- In the special case of inverse-square gravitational attraction

$$F = \frac{Gm_1m_2}{r_{12}^2} = \frac{Gm_1m_2}{(r_1' + r_2')^2} = \frac{Gm_1M}{{r_1'}^2}$$

  where $M = m_2/(1 + 2(m_1/m_2) + (m_1/m_2)^2)$ is a fictitious mass at G we find using the substitution $r_2' = (m_1/m_2)r_1'$.

What we have found here is somewhat remarkable. Two particles are flying around in space attracted to each other by inverse-square gravitational attraction. Instead of doing something wild, they each move, relative to their joint center of mass, as if they were in central force motion with a fixed mass. That is, the 3D two-body problem reduces, exactly, to the 2D one body problem. You just have to use a coordinate system that is on the plane of motion and whose origin is at the center of mass.

Thus, the moon doesn't really go around the earth. Rather the moon and earth go around their common center of mass (a point about 3/4 of the way out towards the earth's surface from its center). And Jupiter doesn't go around the sun, the sun and Jupiter go around their combined center of mass just outside the sun. But both of these examples are, in detail, wrong. Because the earth-moon system is affected by the sun and jupiter. And the Jupiter-sun system is affected by the earth and moon.

The three-body problem

With inverse-square attraction, one body goes around a fixed point on one or another conic section. Two bodies go around each other in exactly the same way as one body about a fixed point. The two-body problem reduced to the one-body problem. What about lots of bodies? Let's start with three. How, in general, do three bodies move that are all mutually attracted with inverse-square gravitation? Great question. Lots of people have asked it. And no-one knows the answer. Given any three masses and their initial conditions we could use a computer program to find out their subsequent positions and velocities. But no-one knows how to categorize all the possible motions of such systems.

Some things are known about 'the three body problem.' One is that it is hard, the best minds haven't been able to solve it in general. Another is that the solutions can be pretty wild. For example, three particles might tumble around each other for a long time and, with no change in the equations, all of a sudden one of the particles will be ejected at high speed and never return (as if on a hyperbolic trajectory relative to the other two particles). A few special solutions of the three-body problem are known. For example, with the right initial conditions, three identical particles can move in either a circle or in Montgomery's figure 8.

Despite the difficulty of analytic description, there is no special impediment to finding solutions to any 3-body problem with computer simulation.

The n-body problem

With many particles all manner of complicated motion is possible. And there are few solutions which are known analytically. One solution has the n particles chasing each-other around in a circle, with the particles forming a regular polygon. Another amazing approximate solution, the Buck solution, is that a string of thousands of particles will all chase each-other around an arbitrary curve in 3-dimensional space. At least approximately, for a while.

By applying $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ to 3 or 1000 interacting particles you can see all manner of n-body solutions on your computer.

SAMPLE 11.1 Location of the center-of-mass. A structure is made up of three point masses, $m_1 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$ and $m_3 = 3\,\mathrm{kg}$. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\mathrm{m/s}\hat{\imath}$, $2\,\mathrm{m/s}(\hat{\imath} - 1.5\hat{\jmath})$ and $1\,\mathrm{m/s}\hat{\jmath}$, respectively.

1. Find the coordinates of the center-of-mass of the structure.
2. Find the velocity of the center-of-mass.

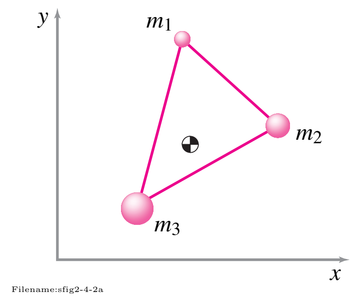

Figure 11.2:

Solution

1. Let $(\bar{x}, \bar{y})$ be the coordinates of the mass-center. Then from the definition of mass-center

$$\begin{aligned}
\bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} \\
&= \frac{1\,\mathrm{kg}\cdot 1.25\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.75\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}} \\
&= \frac{7.5\,\cancel{\mathrm{kg}}\cdot\mathrm{m}}{6\,\cancel{\mathrm{kg}}} = 1.25\,\mathrm{m}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\bar{y} &= \frac{\sum m_i y_i}{\sum m_i} \\
&= \frac{1\,\mathrm{kg}\cdot 3\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.5\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}} \\
&= \frac{8.5\,\cancel{\mathrm{kg}}\cdot\mathrm{m}}{6\,\cancel{\mathrm{kg}}} = 1.42\,\mathrm{m}.
\end{aligned}$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\mathrm{m},\ 1.42\,\mathrm{m})}$$

2. For a system of particles, the linear momentum

$$\begin{aligned}
\vec{L} &= \sum m_i \vec{v}_i = m_{\mathrm{tot}} \vec{v}_{cm} \\
\Rightarrow \quad \vec{v}_{cm} &= \frac{\sum m_i \vec{v}_i}{m_{\mathrm{tot}}} \\
&= \frac{1\,\mathrm{kg}\cdot(2\,\mathrm{m/s}\hat{\imath}) + 2\,\mathrm{kg}\cdot(2\hat{\imath} - 3\hat{\jmath})\,\mathrm{m/s} + 3\,\mathrm{kg}\cdot(1\,\mathrm{m/s}\hat{\jmath})}{6\,\mathrm{kg}} \\
&= \frac{(6\hat{\imath} - 3\hat{\jmath})\,\cancel{\mathrm{kg}}\cdot\mathrm{m/s}}{6\,\cancel{\mathrm{kg}}} \\
&= 1\,\mathrm{m/s}\hat{\imath} - 0.5\,\mathrm{m/s}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{v}_{cm} = 1\,\mathrm{m/s}\hat{\imath} - 0.5\,\mathrm{m/s}\hat{\jmath}}$$

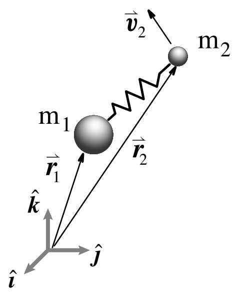

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Figure 11.3:

SAMPLE 11.2 A spring-mass system in space. A spring-mass system consists of two masses, $m_1 = 10\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a weak spring with stiffness $k = 1\,\text{N/m}$. The spring has zero relaxed length. The system is in 3-D space where there is no gravity. At the instant of observaion, *i.e.,* at $t = 0$, $\vec{r}_1 = \vec{\mathbf{0}}$, $\vec{r}_2 = 1\,\text{m}(\hat{\imath} + \hat{\jmath} + \hat{k})$, $\dot{\vec{r}}_1 = \vec{\mathbf{0}}$, and $\dot{\vec{r}}_2 = \sqrt{6}\,\text{m/s}(-\hat{\imath} + \hat{\jmath})$. Track the motion of the system for the next 20 seconds. In particular,

1. Plot the trajectory of the two masses in space.
2. Plot the trajectory of the center-of-mass of the system.
3. Plot the trajectory of the two masses as seen by an observer sitting at the center-of-mass.
4. Compute and plot the total energy of the system and show that it remains constant during the entire motion.

Solution The free-body diagrams of the two masses are shown in fig. 11.4. The only force acting on each mass is the force due to the spring which is directed along the line joining the two masses. Thus, the system represents a central force problem. From the linear momentum balance of the two masses, we can write the equations of motion as follows.

$$\begin{aligned} m_1\ddot{\vec{r}}_1 &= k(\vec{r}_2 - \vec{r}_1) \\ m_2\ddot{\vec{r}}_2 &= -k(\vec{r}_2 - \vec{r}_1) \end{aligned}$$

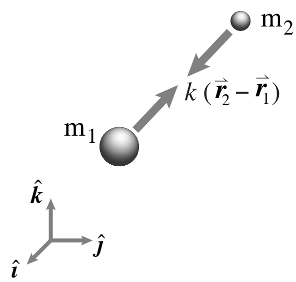

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Figure 11.4: Free-body diagram of the two masses.

Let $\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$. Substituting above and dotting the two equations with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, we get

$$\begin{aligned} \ddot{x}_1 &= \frac{k}{m_1}(x_2 - x_1); & \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1) \\ \ddot{y}_1 &= \frac{k}{m_1}(y_2 - y_1); & \ddot{y}_2 &= -\frac{k}{m_2}(y_2 - y_1) \\ \ddot{z}_1 &= \frac{k}{m_1}(z_2 - z_1); & \ddot{z}_2 &= -\frac{k}{m_2}(z_2 - z_1) \end{aligned}$$

Thus we get six second order coupled linear ODEs as equations of motion.

1. To plot the trajectory of the two masses, we need to solve for $\vec{r}_1(t)$ and $\vec{r}_2(t)$, *i.e.,* for $x_1(t), y_1(t), z_1(t)$, and $x_2(t), y_2(t), z_2(t)$. We can do this by first writing the six second order equations as a set of 12 first order equations and then solving them using a numerical ODE solver. Here is a pseudocode to accomplish this task.

```
ODEs = {x1dot = u1,
        u1dot = k/m1*(x2-x1),
        y1dot = v1,
        v1dot = k/m1*(y2-y1),
        z1dot = w1,
        w1dot = k/m1*(z2-z1),
        x2dot = u2,
        u2dot = -k/m2*(x2-x1),
        y2dot = v2,
        v2dot = -k/m2*(y2-y1),
        z2dot = w2,
        w2dot = -k/m2*(z2-z1) }
IC   = {x1(0)=0, y1(0)=0, z1(0)=0,
        u1(0)=0, v1(0)=0, w1(0)=0,
        x2(0)=1, y2(0)=1, z2(0)=1,
        u2(0)=-sqrt(6), v2(0)=sqrt(6), w2(0)=0}
Set  k=1, m1=10, m2=1
Solve ODEs with IC for t=0 to t=20
Plot {x1,y1,z1} and {x2,y2,z2}
```

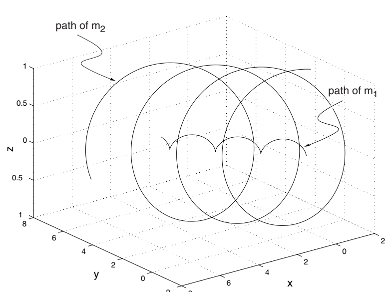

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Figure 11.5: 3-D trajectory of m_1 and m_2 plotted from numerical solution of the equations of motion.

The 3-D plot showing the trajectory of the two masses obtained from the numerical solution is shown in fig. 11.5. From the plot, it seems like the smaller mass goes around the bigger mass as the bigger mass moves on its trajectory.

2. We can find the trajectory of the center-of-mass using the following relationships.

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_{\text{cm}} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}.$$

Since there is no external force on the system if we consider the two masses and the spring together, the center-of-mass of the system has zero acceleration. Therefore, we expect the center-of-mass to move on a straight path with constant velocity. The center-of-mass coordinates x_{cm}, y_{cm}, and z_{cm} are plotted against time in fig. 11.6 which show that the center-of-mass moves on a straight line in a plane parallel to the xy-plane (z is constant). This is expected since the initial velocity of the center of has no z-component:

$$\begin{aligned} \vec{v}_{\text{cm}} &= \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \\ &= \frac{m_1 \cdot \vec{0} + 1\,\text{kg} \cdot \sqrt(6)\,\text{m/s}(-\hat{\imath} + \hat{\jmath})}{10\,\text{kg} + 1\,\text{kg}} \\ &= 0.22\,\text{m/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

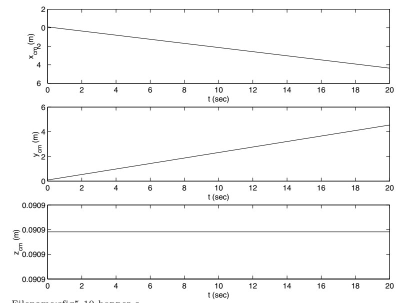

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Figure 11.6: The center-of-mass coordinates $x_{\text{cm}}(t)$, $y_{\text{cm}}(t)$, and $z_{\text{cm}}(t)$. The center-of-mass moves on a straight line in a plane parallel to the xy-plane.

3. The trajectory of the two masses with respect to the center-of-mass can be easily obtained by the following relationships.

$$\begin{aligned} x_{1/cm} &= x_1 - x_{\text{cm}}, & y_{1/cm} &= y_1 - y_{\text{cm}}, & z_{1/cm} &= z_1 - z_{\text{cm}} \\ x_{2/cm} &= x_2 - x_{\text{cm}}, & y_{2/cm} &= y_2 - y_{\text{cm}}, & z_{2/cm} &= z_2 - z_{\text{cm}} \end{aligned}$$

The trajectories thus obtained are shown in fig. 11.7. It is clear that the two masses have closed orbits with respect to the center-of-mass. These closed orbits are actually conic sections as we would expect in a central force problem.

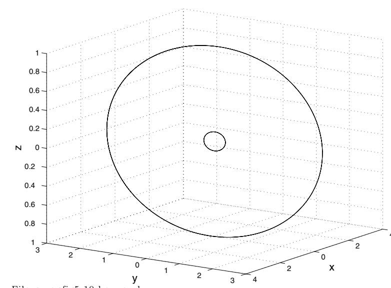

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Figure 11.7: The paths of m_1 and m_2 as seen from the center-of-mass. The two masses are on closed orbits with respect to the center-of-mass.

4. We can calculate the kinetic energy of the two masses and the potential energy of the spring at each instant during the motion and add them up to find the total energy.

$$\begin{aligned} (E_k)_{m_1} &= \frac{1}{2} m_1 (u_1^2 + v_1^2 + w_1^2) \\ (E_k)_{m_2} &= \frac{1}{2} m_2 (u_2^2 + v_2^2 + w_2^2) \\ E_p &= \frac{1}{2} k[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \\ E_{\text{total}} &= (E_k)_{m_1} + (E_k)_{m_2} + E_p \end{aligned}$$

The energies so calculated are plotted in fig. 11.8. It is clear from the plot that the total energy remains constant during the entire motion.

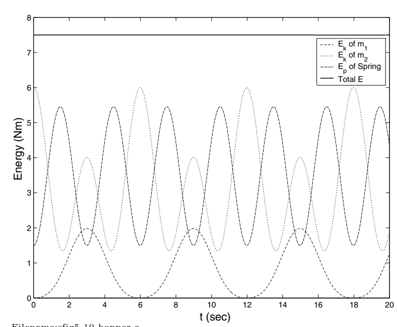

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Figure 11.8: The kinetic energy of the two masses and the potential energy of the spring sum up to the constant total energy of the system.

11.2 Collisions and explosions of particles in 2D and 3D

When two things bump into each other there is often a big interaction force. Think about a ball bouncing off the ground, two pool balls collidng, a baseball hitting a bat, two cars crashing, or the big forces when a satellite gravitationally slingshots around a planet it passes close by. Similarly there are big short-lived forces when things explode into two or more pieces. A big and short-lived force is often described by

$$\text{its net impulse } \vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}\, dt$$

rather than its detailed time-history $\vec{\boldsymbol{F}}(t)$. The collision modeling assumption is that these interaction forces are so big that all other forces on the particles can be ignored. For a two-particle collision the impulses are $\vec{\boldsymbol{P}}_1 = \vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{P}}_2 = -\vec{\boldsymbol{P}}$ acting on m_1 and m_2. Rather than looking at the acceleration of mass during the collision one just calculates

$$\text{the net change in velocity} = \Delta\vec{\boldsymbol{v}}.$$

Before the collision two particles m_1 and m_2 have velocities $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ (see fig. 11.9). The superscript "$^-$" means *just* before the collision. Then the particles collide. Even though we ignore the spatial extent of the particles for most of the mechanics analysis, we note that the two particles have a common tangent plane. The normal of that plane, pointing out of particle 1, say, is $\hat{\boldsymbol{n}}$. Just after the collision the particles have velocities $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$ with the superscript"$^+$" indicating *just* after the collision.

The general collision problem is

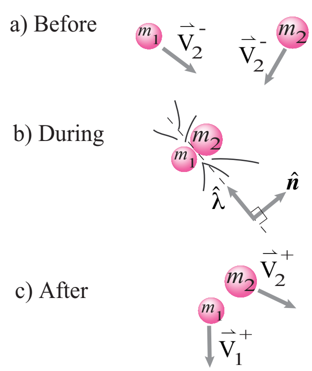

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Figure 11.9: Two particles collide. Their velocities before the collision are $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$. When they collide their common tangent plane has normal $\hat{\boldsymbol{n}}$. After the collision their velocities are $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$.

> Given some information about the motion before the collision, the motion after the collision, and the collisional impulse, find other information about these same quantities.

We find the unknowns using

- Momentum balance for each particle: $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}(t)\, dt = m\Delta\vec{\boldsymbol{v}}$; and
- Some information about the collisional impulse, usually a constitutive law for the collision.

For collisional modeling the constitutive law for interaction involves impulse and change of velocity. We only consider two such constitutive models:

- *plastic sticking* collisions where $\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_2^+$.
- frictionless restitution with $(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+) \cdot \hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-) \cdot \hat{\boldsymbol{n}}$ and $\vec{\boldsymbol{P}} \cdot \hat{\boldsymbol{\lambda}} = 0$.

The constitutive models are discussed further below in the context of the the three idealized collisions we treat here:

- sticking collisions
- frictionless collisions with restitution
- explosions.

The only expansion in this section over the 1D collissions in section 9.5 is the need for 2D and 3D geometry.

Sticking collisions

The conceptually simplest collision is a *sticking* collision also called a *perfectly plastic no-slip* collision (see fig. 11.10). Here the word 'plastic' is used in its old latin meaning malleable or 'clay like'. Imagine two lumps of wet clay colliding in space and just sticking together. The plastic collision model applies, for example,

- when a projectile gets embedded in its target,
- when two cars crash and get entangled so they move together after the collision, or
- when two machine parts engage at contact because of a mechanism like a door catch.

In short, the constitutive law for plastic collisions is

$$\text{Plastic (sticking collision)} \quad \Rightarrow \quad \vec{v}_1^{\,+} = \vec{v}_2^{\,+}$$

And the impulse is what it is, as determined by momentum balance for the two particles. Here's the simplest collision problem.

Example: A particle collides with an immovable object.
The impulse on the particle is

$$\vec{P} = m\Delta\vec{v} = m(\vec{0} - \vec{v}^{\,-}) = -m\vec{v}^{\,-}.$$

And here is the general two-particle sticking collision problem.

Example: Two particles collide and stick.
There are three velocities to consider, the before-collision velocities $\vec{v}_1^{\,-}$ and $\vec{v}_2^{\,-}$ and the common after-collision velocity $\vec{v}^{\,+}$. Also relevant is the interaction impulse $\vec{P}$. That's 4 vector quantities (8 scalars in 2D, 12 in 3D). The governing equations are momentum balance for the two particles

$$\vec{P} = m_1(\vec{v}^{\,+} - \vec{v}_1^{\,-}) \qquad \text{and} \qquad \vec{P} = m_2(\vec{v}^{\,+} - \vec{v}_2^{\,-})$$

making up 2 vector equations (4 scalar equations in 2D, 6 in 3D). Thus to solve a problem in 2D, 4 scalar quantities need to be given so that the other quantities can be found from the momentum balance equations. In 3D, 6 scalar quantities have to be given.

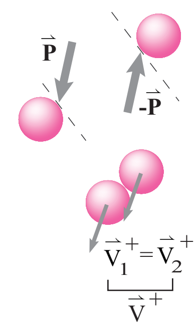

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Figure 11.10: Two particles collide and stick together so in their subsequent motion $\vec{v}_1^{\,+} = \vec{v}_1^{\,-} = \vec{v}^{\,+}$. The action-reaction impulse pair is in whatever direction it needs to be to get this sticking; the impulses need not be in just the $\hat{n}$ direction.

There are all different ways to involute such problems, say by taking one of the masses as unknown. Here is the most straightforward example.

Example: Find the post-collision velocities for a sticking collision.
Given $m_1, m_2, \vec{v}_1^-$ and $\vec{v}_2^-$ we find by solving the momentum balance equations that

$$\vec{v}^+ = \frac{m_1\vec{v}_1^- + m_2\vec{v}_2^-}{m_1+m_2} \qquad \text{and} \qquad \vec{P}_1 = -\vec{P}_2 = \frac{m_1 m_2}{m_1+m_2}\left(\vec{v}_2^- - \vec{v}_1^-\right).$$

The answer can be interpreted like this. The final velocity is the same as the pre-collision average velocity. This is also the system's initial (and final) center of mass velocity. The impulsive interaction is associated with the change of velocity of $\vec{v}_1^- - \vec{v}_2^-$ of an effective 'reduced mass' with a value of $m_{red} = m_1 m_2/(m_1 + m_2)$ (see box 11.1 on page 590).

Frictionless collision with restitution

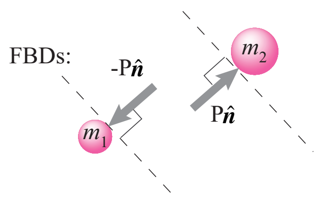

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Figure 11.11: Two particles collide and bounce off each other frictionlessly. The assumption is that their relative separation speed is the coefficient of restitution e times their approach speed. Even though for momentum balance we treat the masses as particles, for considering the collision we look at the normal $\hat{n}$ of the common contacting tangent plane. The separation speed is $(\vec{v}_1^+ - \vec{v}_2^+)\cdot\hat{n}$ and the approach speed is $-(\vec{v}_1^- - \vec{v}_2^-)\cdot\hat{n}$.

Frictionless collisions with restitution

This is the most common model used in elementary mechanics courses. It is originally due to Newton, at least in the 1D case we discussed in section 9.5. Two particles collide and then separate. There is no interaction force in their common contact tangent plane (hence 'frictionless'). See fig. 11.11. The impulse is such that the particles separate at a speed that is a fixed ratio e of the speed at which they approached. The speed of approach and separation are measured in the $\hat{n}$ direction.

The speed of approach is the rate at which the distance between the particles decreases just before the collision. Really, this only makes precise sense if

- the masses are round, or
- the masses are not rotating.

11.1 Effective mass

In two-particle collisions the forces of interest are in the action-reaction pair between the particles: $\vec{F}_1$ acts on m_1 and $\vec{F}_2 = -\vec{F}_1$ acts on particle 2. If we know one we know the other, so let's call $\vec{F}$ the force $\vec{F}_1$ on m_1. The two-particle system has no net acceleration, meaning the center of mass does not accelerate. All that the interaction force does is affect the relative motion of the particles.

So consider the relative acceleration of particle 1, say, relative to particle 2:

$$\vec{a}_{\rm rel} = \vec{a}_1 - \vec{a}_2 = \frac{\vec{F}_1}{m_1} - \frac{\vec{F}_2}{m_2} = \frac{\vec{F}_1}{m_1} - \frac{-\vec{F}_1}{m_2} = \vec{F}\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

Thus we can write

$$\vec{F} = m_{\rm eff}\vec{a}_{\rm rel}$$

$$\text{with} \quad m_{\rm eff} = \left(\tfrac{1}{m_1} + \tfrac{1}{m_2}\right)^{-1} = \frac{m_1 m_2}{m_1 + m_2}$$

The 'reduced mass' or 'effective mass' $m_{\rm eff}$ is that which connects the interaction force with the relative acceleration $\vec{a}_{\rm rel} = \vec{a}_1 = \vec{a}_2$ of the masses. To personalize it, imagine you have negligible mass and are floating in space between two big masses holding a handle on each. Then the relation between the tension in your arms and the relative acceleration of the masses is determined by the reduced effective mass $m_{\rm eff}$. The effective mass is less than either of the masses separately (because the relative acceleration comes from the addition of the two accelerations). For two equal masses $m = m_1 = m_2$ the effective mass is $m_{\rm eff} = m/2$.

Integrating in time the effective mass also relates the interaction impulse with the change in relative velocity.

$$\vec{P} = m_{\rm eff}\Delta\vec{v}_{\rm rel}.$$

where $\vec{P} = \int \vec{F}\,dt$ acts on m_1 and $\vec{v}_{\rm rel} = \vec{v}_1 - \vec{v}_2$.

The approach speed is the relative velocity dotted with the $\hat{\boldsymbol{n}}$ direction

$$v_{approach} = (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-) \cdot \hat{\boldsymbol{n}}.$$

The separation speed, measured just after the collision, has the same definition but with a sign change

$$v_{\text{sep}} = -\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}}.$$

Newton's law of collisional restitution① is

①Why the word *restitution*? The particles approach each other with some momentum relative to their common center of mass. At some point during the collision they have none. Then some of the momentum is restituted, paid back, and they bounce. No restitution, $e = 0$, and there is no bounce. Full restitution, $e = 1$, and they separate with the as much relative momentum as they had when they approached.

$$v_{\text{sep}} = e v_{appraoch} \quad \text{or} \quad \left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}} = -e\left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}. \quad (11.4)$$

We use the coefficient of restitution for approximate collisional modeling but,

> the 'coefficient of restitution' restitution equation is not an accurate law of nature.

Example: Two-particle elastic collision.
Two particles m_1 and m_2 have pre-collision velocities of $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ and collide frictionlessly with coefficient of restitution e on the tangent plane with normal $\hat{\boldsymbol{n}}$. The post collision velocities $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$, as well as the impulse $\vec{\boldsymbol{P}}$ are found by simultaneously solving these equations.

$$\begin{aligned}
\vec{\boldsymbol{P}} &= m_1\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_1^-\right)\\
-\vec{\boldsymbol{P}} &= m_2\left(\vec{\boldsymbol{v}}_2^+ - \vec{\boldsymbol{v}}_2^-\right)\\
\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}} &= -e\left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}
\end{aligned}$$

In 2D this makes up 5 scalar equations for 5 scalar unknowns. In 3D its 7 equations for 7 unknowns. The most direct solution is to set up and solve these equations using a computer.

Rather it is an approximate empirical observation. Or, to put it another way, the value of the coefficient of restitution e depends on the material, the shape, the orientation and the speed of the colliding particles. It is not a true constant. Nonetheless, eqn. (11.4) is a reasonable approximation for some engineering purposes. Just don't assume that predictions it makes will generally be highly accurate.

The 'frictionless' part of this collision law is expressed by the assumption that the net impulse of interaction is in the $\hat{\boldsymbol{n}}$ direction. So $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ with no component in the $\hat{\boldsymbol{\lambda}}$ direction.

Generally one assumes that the coefficient of restitution is between zero and one:

$$0 \leq e \leq 1.$$

For $e < 0$ the masses have to pass through each other. For $e > 1$ the

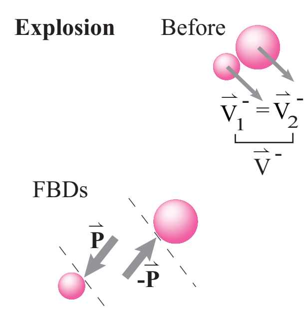

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Figure 11.12: An explosion is like a sticking collision run backwards in time. The particles initially have the same velocity $\vec{\boldsymbol{v}}_1^- = \vec{\boldsymbol{v}}_2^- = \vec{\boldsymbol{v}}^-$ and then separate due to an action-reaction impulse pair in any direction.

collision would involve a gain in energy. This might happen if there was explosive gunpowder in the contact region of the collision.

In the case $e = 0$ the collision is perfectly plastic but still frictionless. This is generally not a sticking collision because the masses can enter and hence leave the collision with some relative velocity in the common contact plane (the $\hat{\boldsymbol{\lambda}}$ direction).

Explosions

If one particle explodes into pieces it's as if the pieces had a collision. It's just that the initial velocities of the pieces were all the same and the total kinetic energy of the system increases during the 'collision'. See fig. 11.12. The overall treatment is extremely similar to that for sticking collisions, but in some sense backwards. Instead of the particles entering the collision with different velocities and leaving with the same velocity, they enter with the same velocity and leave with different velocities. But the same momentum principles apply. There is no colllision law or coefficient of restitution to apply, all of the post-collision relative velocity is restituted from nothing. Rather one just has to know (or find) the action-reaction impulse between the masses.

Example: An explosion.
Two particles m_1 and m_2 are stuck together and moving at $\vec{\boldsymbol{v}}^-$ when they explode

11.2 Energetics of collisions

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. We'd like to treat these cases in a unified way. First let's calculate the total kinetic energy.

$$\begin{aligned} 2E_\text{K} &= m_1 v_1^2 + m_2 v_2^2 \\ &= m_\text{tot} v_\text{cm}^2 + m_1 |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_\text{cm}|^2 + m_2 |\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_\text{cm}|^2 \\ &= m_\text{tot} v_\text{cm}^2 + m_\text{eff} |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|^2 \\ &= m_\text{tot} v_\text{cm}^2 + m_\text{eff} v_\text{rel}^2 \end{aligned}$$

where $m_\text{tot} = m_1 + m_2$, $m_\text{eff} = m_1 m_2/(m_1 + m_2)$ and $v_\text{rel} = |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|$. There are a few algebra steps needed to go from line to line above (see section C for related calculations). The concept of effective mass m_eff is introduced in box 11.1. The key result is that the kinetic energy of a two-particle system can be written as the sum of two terms, one involving center of mass velocity and one involving the relative velocity of the two masses.

This is a special result for two-particle systems. For any system the kinetic energy is a center of mass term $(mv_\text{cm}^2/2)$ plus a term for motion relative to the center of mass. But generally the relative motion term is written as a sum of terms, one for each particle, and the motion of each particle is measured relative to the center of mass $(mv_{i/\text{cm}})$. What is special for two-particle systems is that the relative motion part can be written in terms of the motion of the two particles relative to each other. Because that is not the velocity of any real thing, it only gives the right kinetic energy when used with the corrected effective mass (m_eff).

What about energy and collisions? The center of mass velocity and energy do not change in the collision. So the only change in kinetic energy is that associated with changes in $m_\text{eff} v_\text{rel}^2$.

$$\begin{aligned} 2\Delta E_\text{K} &= m_\text{eff}\left((v_\text{rel}^+)^2 - (v_\text{rel}^-)^2\right) \\ &= 2\vec{\boldsymbol{v}}_\text{rel}^- \cdot \vec{\boldsymbol{P}} + |\vec{\boldsymbol{P}}|^2/m_\text{eff} \end{aligned}$$

where we used that $\vec{\boldsymbol{P}} = m_\text{eff}(\vec{\boldsymbol{v}}_\text{rel}^+ - \vec{\boldsymbol{v}}_\text{rel}^-)$. This formula applies for both sticking collisions, in which case $\vec{\boldsymbol{P}} = -m_\text{eff}\vec{\boldsymbol{v}}_\text{rel}^-$ and $2\Delta E_\text{K} = -m_\text{eff}(v_\text{rel}^-)^2$, and to explosions where $\vec{\boldsymbol{P}} = m_{eff}\vec{\boldsymbol{v}}_\text{rel}^+$ and $2\Delta E_\text{K} = m_\text{eff}(v_\text{rel}^+)^2$. It also applies to interactions in-between.

All that enters the change-of-energy equations above is the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction. Thus the issue of energy loss or gain is determined by whether the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction decreases or increases in magnitude. Thus a collision with $-1 < e < 1$ loses energy and a collision with $|e| > 1$ increases energy. We included $e < 0$ for completeness even though it is sometimes considered 'non-physical' in that it involves the particles passing by or passing through each other.

and an impulse $\vec{P}$ separates them. After the collision

$$\vec{v}_1^+ = \vec{v}^- + \vec{P}/m_1 \qquad \text{and} \qquad \vec{v}_2^+ = \vec{v}^- - \vec{P}/m_2.$$

The full range of behavior for sticking collisions to explosions can be captured with a single restitution coefficient e_g (see box 11.3).

Frictional collisions

Our avoiding of frictional collisions is not because there generally is no friction during collisions. Friction is a fact of the mechanical world. We avoid friction here because a host of special assumptions are needed to make frictional problems deterministic. And no given set of assumptions is known to yield accurate predictions. Frictional collision models have too dis-satisfyingly low a ratio of accuracy to complexity for inclusion in a book at this level.

Simultaneous collisions

If one particle is involved in two collisions at one time then we have not explained how to calculate the resulting motion. In an attempt to make the situation clear one is tempted to say "Let's make it ideal and assume the collisions are exactly instantaneous and at *exactly* the same time." Then, unfortunately, one is making the situation *exactly* ambiguous.

Unfortunately for our hope of making reliable predictions, simultaneous collisions are *not* rare events. Why? Imagine B is touching C and both are stationary. Then A comes and bangs into B. Because B and C are already touching one must assume that there are impulsive forces not just between A and B, but also between B and C. And we have no reliable rules for sorting out the result. Nor will we find such rules if we make it a life's work.

Example: A triangular array of identical spheres.
Imagine 15 accurately-machined nominally-identical spheres laid out in a tight triangle (5 in one row, 4 in the next, then 3,2, and 1) on a very flat smooth surface. Then imagine a 16th ball rolls in and hits the apex of the triangle. How do the 15 balls move?

This experiment is performed in smoky rooms full of intoxicated people night after night. Its the 'break' in a pool game. And the game depends on the result being unpredictable. Each time, due to tiny differences, the results are different.

And, according to theory, the more rigid and perfect the balls are, the more sensitive are the results to the smallest of differences in the initial conditions.

What is the source of the problem?

Example: Three balls in a line.
Consider the one-dimensional collision of three identical particles. B and C are in line, stationary and touching and then A comes along with $v_{A0} = v^-$. Let's assume that the collision(s) whatever they are, are completely elastic and conserve energy ($e = 1, e_g = 0$). Here are two ways to predict the outcome:

- A hits B and C, being all the way at the other side of B, is oblivious to the interaction between A and B until it is complete. Thus A comes to rest and B is moving to the right with v^-. Then B colllides elastically with C and B comes to rest and C shoots off with the v^-.

- B and C are touching and act as a single rigid object throughout the collision with A. Thus the result is like that between a particle with mass m and another with mass $2m$. Such an elastic collision would leave B and C going forwards at $2v^-/3$ and A going the other way with $v_A^+ = -v^-/3$. A different result.
- actually there is a one-parameter family of results that are consistent with energy conservation and momentum balance. We have three outcomes (the velocities of the three particles) and only 2 scalar equations restricting them (momentum and energy balance).

But what would *really* happen? That would depend on details that are not stated. Of course if the *exact* shape and configuration of the balls was known, and the exact rules for elastic and inelastic deformation, then one could calculate the resulting motion solving partial differential equations or with atomic simulations. In principle. But we generally do not know such details nor have have such calculation abilities. And crowding that which we don't know into concepts like 'rigid-object' and 'exactly simultaneous' crowds the prediction of the outcome to dependence on infinitesimal things.

So, as an engineer, what are you supposed to do when calculating in situations involving simultaneous collisions?

- first relax and remember that no collision calculation is likely to be very accurate (unless the result only depends on balance of momentum). So simultaneous collisions, while philosophically worse in that even the equations are indeterminate, are not that much worse than the usual deterministic, but not accurate, collisional relations.
- do experiments, and
- take account the range of outcomes depending on assumptions about the collision details.

Samples 11.4 and 11.5 starting on page 598 illustrate the ambiguity of simultaneous collisions in 2D.

Final comments

This is the second of three sections about collisions. Section 9.5 was about collisions in 1D, then this section about particles in 2D and 3D, and finally the ideas in this section will be extended from particles to rigid objects in section 14.5.

11.3 Coefficient of generation

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. For passive frictionless collisions one can characterize the collision by the coefficient of restitution

$$e \equiv \frac{\text{(separation speed)}}{\text{(approach speed)}} = \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}}}{(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \tag{11.5}$$

However, for explosions the coefficient of restitution is $e = \infty$. If one is equally interested in energy absorbing or energy creating collisions one can use a more democratic coefficient of generation

$$e_g \equiv \frac{\text{(separation speed)-(approach speed)}}{\text{(separation speed)+(approach speed)}} = \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} - (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}}{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} + (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \tag{11.6}$$

We can write the collisional coefficient of generation in terms of the restitution coefficient, and *vice versa*, as

$$e_g = \frac{e-1}{e+1} \qquad \text{and} \qquad e = \frac{1+e_g}{1-e_g}.$$

The generation coefficient is -1 for sticking collisions and 1 for explosions. This coefficient is zero for energetically neutral collisions (no gain, no loss, $e = 1$). And the coefficient of generation does not allow for passing-through or passing-by collisions ($e < 0$).

As a replacement for the conventional coefficient of restitution the coefficient of generation e_g is more complex to use in simple calculations in that eqn. (11.6) is more complex than eqn. (11.5). On the other hand the coefficient of generation is convenient for describing situations which are a mix of passive ($e < 1$ and $e_g < 0$) and active ($e > 1$ and $e_g > 0$). Such is the case, for example, in simple models of legged locomotion (see box 11.4 on page 596).

Note that in all the collisional restitution formulas we could replace $\hat{\boldsymbol{n}}$ with $-\hat{\boldsymbol{n}}$ without affecting the validity of the equations. Similarly all the subscript 2's could be replaced with 1's and *vice versa* without affecting the validity of the equations. Knowing this relieves anxiety about the choice of normal $\hat{\boldsymbol{n}}$ (towards m_1 or towards m_2?) or which particle to call 1 and which to call 2.

11.4 A particle collision model of running

At every step a running person flies through the air, hits the ground with a foot and pushes on the ground. By action and reaction, the ground pushes back on the foot which pushes on the leg which pushes on the body which causes the body to slow its descent and then go from moving forward and somewhat down to moving forward and somewhat up. Then the foot leaves the ground and the person flies through the air again readying for the next foot contact.

Human bodies are somewhat bigger than human legs so one approximation is that the legs have negligible mass. Human bodies don't tumble about much during a running step, so a next approximation is to neglect all distortion and rotation of the body and think of it as a particle. Finally, one might imagine that the ground contact time is short, and that the step on the ground is like a bounce. Thus running is like a sequence of collisions between a body and the ground. Obviously a running person is not a bouncing particle in all regards. Nonetheless, this model gives a means for making various estimates about running.

In the flight phase of running, neglecting air friction, the body moves in a parabolic arc according to:

$$\vec{F} = m\vec{a} \quad \Rightarrow \quad -mg\,\hat{\jmath} = m\vec{a}$$

This has solution that the time of flight is

$$t_f = 2v_{y0}/g$$

where v_{y0} is the vertical component of the velocity at the start of flight. The distance of flight is

$$d = v_x t_f = 2v_x v_y/g$$

where v_x is the constant horizontal component of velocity.

What happens in the 'collision' with the ground?

The horrible leap-frog model of running

We could think of each step as independent. Each running step would be a jump at the end of which the body would come to rest and then jump again. That is, each step would start with an explosion and, after a period of flight, end with a plastic no-slip collision. Then immediately after there would be another jump. How much energy would it take to run like that? Each jump would involve an impulse to get the body from zero velocity to $\vec{v} = v_x\hat{\imath} + v_{y0}\hat{\jmath}$. The work of the legs would be the increase in kinetic energy.

$$W = mv^2/2 = m(v_x^2 + v_{y0}^2)/2$$

Then the legs would absorb that much energy at landing. Muscles, unlike generators, are not regenerative. If muscles were regenerative you would feel especially peppy after you walked down a long stair case. On the other hand, walking down stairs is not that tiring. So let's approximate that there is no metabolic cost for absorbing work. So the energetic cost of locomotion per unit time would be

$$P = W/t_f = \frac{m\left(v_x^2 + v_{y0}^2\right)/2}{2v_{y0}/g} = mg\,v_x\frac{\tan\theta + \cot\theta}{4}$$

where $\tan\theta = v_{y0}/v_x$ is the angle of the trajectory at liftoff. The function $\tan\theta + \cot\theta$ has its minimum value of 2 at $\theta = 45°$ so the cost of such locomotion, in terms of average power, is $mg\,v_x/2$. Muscles use about 4 time as much chemical energy as they can produce work (ie, about 25% efficient at best) so the chemical energy to run by jumping and landing, over and over again, would be about

$$\text{Metabolic Cost per unit time} = P_{\text{met}} = 2mg\,v_x,$$

that twice the weight times the speed. The chemical energy needed per unit distance would be about $2 \cdot (\text{weight})$.

Obviously this seems like a tiring way to run. You shouldn't stop and start your horizontal motion at every step. Real people don't do that. Furthermore, the energy cost we have just predicted is bigger than what people use by a factor of about 5; the rate at which people use chemical energy to run is more like $mg\,v_x/2$ or $mg\,v_x/3$. Notice that the energetic cost of this mode of 'running' does not depend on the step length or flight time but only on the initial angle of the trajectories. Smaller steps involve smaller collisions and hence smaller energy cost per collision. But with smaller jumps there are more collisions per unit distance. The two effects exactly cancel in this model. Only the angle of liftoff matters, not the length of the jumps.

Frictionless collision model of running

Although shoes generally have high friction, the legs pivot under the body during ground contact. The result is that the main force transmitted by the leg to the body is vertical. In effect the leg mediates an effectively frictionless collision. At least that's an extreme idealization of what a leg does. Perhaps a better model of running is then a sequence of vertical frictionless collisions.

At each step there is, in effect, a plastic frictionless collision which absorbs energy immediately followed by an energetically generative collision that sends the body back up again. Together they look like a single frictionless elastic collision, but in this model we want to take account of the work absorbed in landing and the work needed to take off again. To start we will neglect that humans do have springs in their legs (e.g., tendons).

Thus at each step the energy needed to take off is

$$W = mv_{y0}^2/2.$$

The time of flight is again $t_f = 2v_{y0}/g$ and so, for this model the average work per unit time is

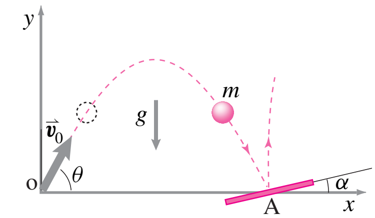
$$P = W/t_f = \frac{mv_{y0}^2/2}{2v_{y0}/g} = mg\,v_{y0}/4$$

and the work per unit distance would be

$$\text{work per unit distance} = mg\frac{v_{y0}/v_x}{4} = mg\tan\theta/4$$

and the metabolic cost per unit distance, taking muscle efficiency as 25% again, would be the weight times $\tan\theta$. So, at a given horizontal speed, the energy cost per unit distance can be made arbitrarily small by having the flight angle small and there being, consequently, more and more small collisions. But for a person to try to save energy that way she would have to swing her legs in impossibly tiring small rapid steps. To complete this model so thqat it would not predict that people should choose infinite frequency and infinitesimal steps we would have to add in a formula for the cost of swinging the legs rapidly. If we evaluate this model with the step length of real human running, and the consequent launch angle θ we over-estimate the actual energetic cost of running by about a factor of 2. Why is that? Probably because people do use their springs to bounce. They don't just throw away all their energy at each ground landing and then jump vertically. Rather their tendons store energy and release it at each step, doing something like half of the work needed to get airborne again.

SAMPLE 11.3 **Projectile hits a slanted floor:** A ball of mass $m = 0.2\,\mathrm{kg}$ is thrown in the air at an angle $\theta = 60°$ with initial speed $v_0 = 10\,\mathrm{m/s}$. The ball lands on a hard, frictionless floor that is tilted at angle $\alpha = 20°$ with the horizontal. The coefficient of restitution between the floor and the ball is $e = 0.85$. Ignore air resistance. Find the height of the ball after the rebound from the floor.



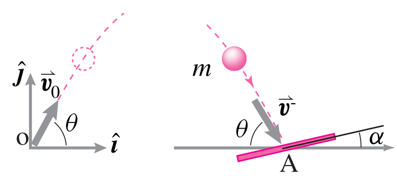
Filename:sfig11-2-tiltedfloor

Figure 11.13:

Solution This problem has two parts to it. In order to figure out the height after rebound, we need to find the rebound velocity. But to find the rebound velocity, we need to know the velocity of the ball before impact with the floor. Let the velocity just before the impact be $\vec{v}^-$ and the velocity of rebound (just after impact be $\vec{v}^+$. Let us first find $\vec{v}^-$.

The ball undergoes projectile motion before it lands at A. Its initial (launch) velocity is $\vec{v}_0 = v_0(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$. From energy conservation, we know that the kinetic energy just before impact at A, $m|\vec{v}^-|^2/2$, must be the same as kinetic energy at launch, $mv_0^2/2$. Thus $|\vec{v}^-| = v_0$. And, from the symmetry of the flight, we can conclude that $\vec{v}^-$ must make the same angle θ with the horizontal that $\vec{v}_0$ does. Thus, using fig. 11.14, we have

$$\vec{v}^- = v_0(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}).$$



Filename:sfig11-2-tiltedfloor-a

Figure 11.14: Trajectory of the ball through air: From the symmetry of the trajectory, we can conclude that the angle the velocity vector makes with the horizontal at point A is the same as the angle of launch, θ.

Now we are ready to do collision mechanics at point A. We need to determine $\vec{v}^+$ given $\vec{v}^-$ and the coefficient of restitution for the collision at A. From collision law, we know that the velocity component normal to the floor changes because of the normal impulse during collision, while the tangential velocity remains the same because there is no force or impulse parallel to the floor. Thus,

$$\begin{aligned} \vec{v}^+ \cdot \hat{\boldsymbol{\lambda}} &= \vec{v}^- \cdot \hat{\boldsymbol{\lambda}} \\ \vec{v}^+ \cdot \hat{\boldsymbol{n}} &= -e(\vec{v}^- \cdot \hat{\boldsymbol{n}}). \end{aligned}$$

Writing out $\vec{v}^+ = v_x^+\hat{\imath} + v_y^+\hat{\jmath}$, and noting that $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ and $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, we get, from the equations above,

$$\begin{aligned} \cos\alpha v_x^+ + \sin\alpha v_y^+ &= v_0\cos\theta\cos\alpha - v_0\sin\theta\sin\alpha = v_0\cos(\theta+\alpha) \\ -\sin\alpha v_x^+ + \cos\alpha v_y^+ &= -e\,v_0(-\cos\theta\sin\alpha - \sin\theta\cos\alpha) = e\,v_0\sin(\theta+\alpha). \end{aligned}$$

These are two equations in two unknowns, v_x^+ and v_y^+. Writing them in a matrix form and solving the matrix equation, we get

$$\begin{pmatrix} v_x^+ \\ v_y^+ \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} v_0\cos(\theta+\alpha) \\ e\,v_0\sin(\theta+\alpha) \end{pmatrix} = \begin{pmatrix} v_0\cos(\theta+2\alpha) \\ e\,v_0\sin(\theta+2\alpha) \end{pmatrix}.$$

Thus, we know the rebound velocity $\vec{v}+ = v_0[\cos(\theta+2\alpha)\hat{\imath} + e\sin(\theta+2\alpha)\hat{\jmath}]$.

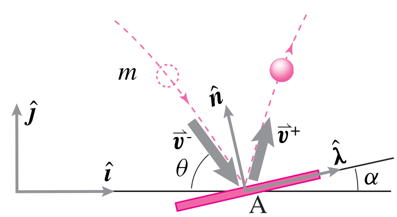
To find the maximum height reached by the ball on the rebound, we only need the vertical component of the rebound velocity. Since the ball has a constant deceleration g, we can use the formula $(v_y)^2 = (v_{y_0})^2 - 2gh$ with $v_y = 0$ at the maximum height $h_{\max}$ to get,

$$h_{\max} = \frac{(v_y^+)^2}{2g} = \frac{e^2\,v_0^2\,\sin^2(\theta+2\alpha)}{2g}$$

Substituting the given values of v_0, e, θ, and α, and using $g = 9.81\,\mathrm{m/s^2}$, we get,

$$h_{\max} = 3.57\,\mathrm{m}.$$

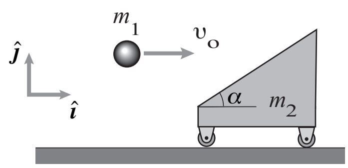
$$\boxed{h_{\max} = 3.57\,\mathrm{m}}$$



Filename:sfig11-2-tiltedfloor-b

Figure 11.15: Collision of the ball with the tilted floor: The ball strikes the floor with velocity $\vec{v}^-$. It rebounds with velocity $\vec{v}^+$. The impulse during the collision acts in the normal direction to the floor at A. The unit normal and the unit tangent vectors at A are $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, and $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ respectively.

You can see that the inclined plane helps in getting the ball reach higher on the bounce. If the floor were flat ($\alpha = 0$), we will get $h_{\max} = 2.76\,\mathrm{m}$. It should be obvious that for maximum height, we should have $\sin(\theta+2\alpha) = 1$ which gives $\alpha = \frac{1}{2}(\frac{\pi}{2} - \theta)$.



Filename:sfig11-2-two-collisions

Figure 11.16:

SAMPLE 11.4 Simultaneous collisions: This problem involves two simultaneous collisions. In general, such problems are hard to solve. We are going to show one way of solving such problems by treating the collisions successively. However, this leads to nonuniqueness of solution. Here we solve the problem in one way and in the next sample, we solve the same problem in another way.

A 12 kg cart with an inclined face rests on a frictionless floor. A ball of mass 3 kg is shot horizontally with speed 30 m/s at the inclined face of the cart. The coefficient of restitution between the cart and the ball is 0.9. The cart subsequently moves horizontally on the floor. Find the velocity of the ball and that of the cart after the collision.

Solution There are two simultaneous collisions in this problem. One collision is between the ball and the cart and the other is between the cart and the ground. Here, we will treat the two collisions one after the other, the one between the ball and the cart preceding the one between the cart and the ground. In Sample 11.5, we treat the ground collision first.

Collision between the ball and the cart: Here we assume that the ball hits the cart and both are free to move in any direction immediately after the collision. Let the mass of the ball be m_1 and that of the cart be m_2. Let their after collision velocities be $\vec{v}_1^+$ and $\vec{v}_2^+$, respectively. Let the impulse during this collision be P_1.

Let us consider the cart and the ball as a single system during the collision. Then, the impulse becomes internal to this system and there is no net impulse on this system. Therefore, the linear momentum is conserved; that is, $\vec{L}^+ = \vec{L}^-$. From this relationship, we have,

$$\mathrm{m}_1\vec{v}_1^+ + \mathrm{m}_2\vec{v}_2^+ = \mathrm{m}_1\vec{v}_1^- + \mathrm{m}_2\vec{v}_2^- = m_1 v_0 \hat{\imath}.$$

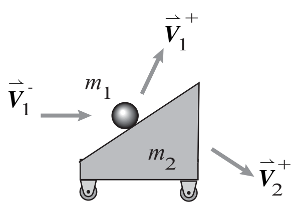


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Figure 11.17: The ball and the cart as a system during the collision. There are no external forces or impulses acting on the system. Therefore, the linear momentum must be conserved during the collision; *i.e.*, $\vec{L}^- = \vec{L}^+$.

Writing out the unknown velocities in terms of their x and y components and dotting the resulting equation with $\hat{\imath}$ and $\hat{\jmath}$ separately, we get the following two scalar equations:

$$m_1 v_{1x}^+ + m_2 v_{2x}^+ = m_1 v_0 \quad (11.7)$$

$$m_1 v_{1y}^+ + m_2 v_{2y}^+ = 0. \quad (11.8)$$

We have four unknowns here, v_{1x}^+, v_{1y}^+, v_{2x}^+, and v_{2y}^+. So, far we have just two equations. We need more equations. We can write restitution equation relating the relative velocities of the ball and the cart in the normal direction before and after the collision:

$$(\vec{v}_2^+ - \vec{v}_1^+)\cdot\hat{\boldsymbol{n}} = -e(\vec{v}_2^- - \vec{v}_1^-)\cdot\hat{\boldsymbol{n}} = e(v_0\hat{\imath})\cdot\hat{\boldsymbol{n}}.$$

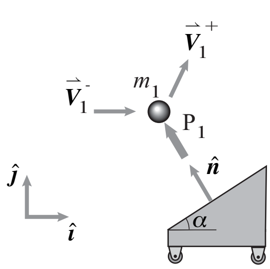
Now, writing $\hat{\boldsymbol{n}} = n_x\hat{\imath} + n_y\hat{\jmath}$ and carrying out the dot products (after writing $\vec{v}_1^+$ and $\vec{v}_2^+$ in terms of their components), we get,

$$(v_{2x}^+ - v_{1x}^+)n_x + (v_{2y}^+ - v_{1y}^+)n_y = -e\, v_0\, n_x. \quad (11.9)$$

We still need another equation. Let us now consider the impulse acting on the ball during the collision. From the free body diagram shown in fig. 11.2, we can write the change in momentum of the ball as,

$$P_1\hat{\boldsymbol{n}} = m_1\vec{v}_1^+ - m_1\vec{v}_1^-$$

$$\text{or} \quad P_1(n_x\hat{\imath} + n_y\hat{\jmath}) = m_1(v_{1x}^+\hat{\imath} + v_{1y}^+\hat{\jmath} - v_0\hat{\imath}).$$



Filename:sfig11-2-two-collisions-b

Figure 11.18: Free body diagram of the ball during the collision. The impulse P_1 acts normal to the plane of collision which is the inclined surface of the cart.

Again, separating out this equation in scalar equations (by dotting the equation with $\hat{\imath}$ and $\hat{\jmath}$ separately), we get,

$$P_1 n_x - m_1 v_{1x}^+ = -m_1 v_0 \quad (11.10)$$

$$P_1 n_y - m_1 v_{1y}^+ = 0. \quad (11.11)$$

Now, we have added another unknown P_1, but fortunately, we have got an extra equation too. We now have five unknowns and five independent equations. So we should be able to solve for all the unknowns.

For solving these equations, we first write them in matrix form and then use a computer to solve them. We write eqn. (11.7)–eqn. (11.11) as,

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & m_2 & 0 \\ -n_x & -n_y & n_x & n_y & 0 \\ -m_1 & 0 & 0 & 0 & n_x \\ 0 & -m_1 & 0 & 0 & n_y \end{bmatrix} \begin{pmatrix} v_{1x}^+ \\ v_{1y}^+ \\ v_{2x}^+ \\ v_{2y}^+ \\ P_1 \end{pmatrix} = \begin{pmatrix} m_1 v_0 \\ 0 \\ -e v_0 n_x \\ -m_1 v_0 \\ 0 \end{pmatrix}.$$

Here is the pseudo computer code to solve this matrix equation:

```
m1 = 3, m2 = 12
theta = pi/6                  % angle in radians
nx = -sin(theta), ny = cos(theta) % components of the normal
v0 = 30
e =  0.9

A = [ m1   0   m2  0    0    % x comp of lin mom bal
      0    m1  0   m2   0    % y comp of lin mom bal
     -nx  -ny  nx  ny   0    % restitution equation
     -m1   0   0   0   -nx   % impulse-momentum for m1, x comp
      0   -m1  0   0   -ny]  % impulse-momentum for m1, y comp
b = [m1*v0  0  -e*v0*nx  -m1*v0  0]' % the known right hand side
solve A*x = b for x
```

The solution thus computed gives us

$$v_{1x}^+ = 18.60\,\text{m/s}, \qquad v_{1y}^+ = 19.74\,\text{m/s},$$
$$v_{2x}^+ = 2.85\,\text{m/s}, \qquad v_{2y}^+ = -4.94\,\text{m/s},$$
$$P_1 = -68.40\,\text{N}\cdot\text{s}.$$

$$\begin{aligned} \vec{v}_1^+ &= (18.60\,\text{m/s})\hat{\imath} + (19.74\,\text{m/s})\hat{\jmath} \\ \vec{v}_2^+ &= (2.85\,\text{m/s})\hat{\imath} + (-4.94\,\text{m/s})\hat{\jmath} \end{aligned}$$

Collision between the cart and the ground: Now, we consider the collision between the cart and the ground, taking $\vec{v}_2^+$ as the velocity of the cart just before the collision. Figure 11.19 shows the impulse from the ground acting on the cart. We know the final velocity of the cart has to be in the $\hat{\imath}$ direction. Just to keep our notations straight, let us denote the velocity of the cart after collision as $\vec{v}_2^{++}$ (after the second collision) and keep the incoming velocity as $\vec{v}_2^+$. Then, from impulse momentum, we have,

$$P_2\hat{\jmath} = m_2\vec{v}_2^{++} - m_2\vec{v}_2^+.$$

This is a vector equation which we can write as two scalar equations in the $\hat{\imath}$ and $\hat{\jmath}$ directions. Note that $\vec{v}_2^{++} = v_2^{++}\hat{\imath}$ and we already know $\vec{v}_2^+ = (2.85\,\text{m/s})\hat{\imath} + (-4.94\,\text{m/s})\hat{\jmath}$ as found before. Thus,

$$\begin{aligned} v_2^{++} &= \; = \vec{v}_2^+ \cdot \hat{\imath} = 2.85\,\text{m/s} \\ P_2 &= -m_2\vec{v}_2^+ \cdot \hat{\jmath} = -(12\,\text{kg})(-4.94\,\text{m/s}) = 59.24\,\text{kg}\cdot\text{m/s} = 59.24\,\text{N}\cdot\text{s}. \end{aligned}$$

$$\vec{v}_2^{++} = 2.85\,\text{m/s}\hat{\imath}$$

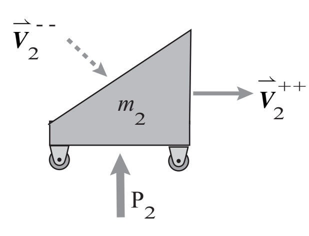


Figure 11.19:

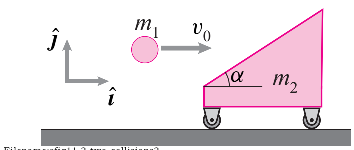


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Figure 11.20:

SAMPLE 11.5 Simultaneous collisions again: Consider the ball and the cart collision problem of Sample 11.4 again. This time, consider the ball and the cart together to have a collision with the ground first. Then consider the collision between the cart and the ball. Once again, you are to find the final horizontal velocity of the cart. The problem parameters are the same — mass of the ball $m_1 = 3\,\text{kg}$, mass of the cart $m_2 = 12\,\text{kg}$, $e = 0.9$ between the ball and the cart, and the velocity of the ball before impact, $v_0 = 30\,\text{m/s}$.

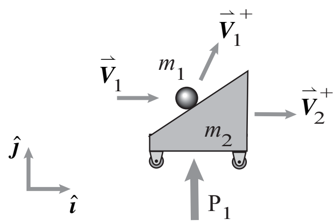

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Figure 11.21: Free body diagram of the ball-cart system during the collision with the ground. The ground impulse is P, the velocities of the ball and the cart before the collision are given — $\overline{\boldsymbol{v}_1^-} = v_0\hat{\boldsymbol{\imath}}$, $\overline{\boldsymbol{v}_2^-} = \vec{\mathbf{0}}$, and the velocities after the collision $\overline{\boldsymbol{v}_2^+} = v_2\hat{\boldsymbol{\imath}}$ and $\overline{\boldsymbol{v}_1^+} = v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}$ are unknown.

Solution Let us consider the ball and the cart as a system colliding with the ground as shown in fig. 11.21. There is an unknown external impulse P_2 from the ground acting on this system in the $\hat{\jmath}$ direction. Using this information, we now write impulse-momentum equation for this system:

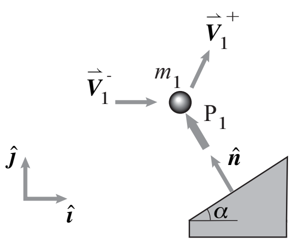
$$P_2\hat{\boldsymbol{\jmath}} = \overline{\boldsymbol{L}}_2 - \overline{\boldsymbol{L}}_1 = m_1\overline{\boldsymbol{v}_1^+} + m_2\overline{\boldsymbol{v}_2^+} - m_1\overline{\boldsymbol{v}_1^-}$$

Assuming that $\overline{\boldsymbol{v}_1^+} = v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}$ and $\overline{\boldsymbol{v}_2^+} = v_2^+\hat{\boldsymbol{\imath}}$, and using the given information $\overline{\boldsymbol{v}_1^-} = v_0\hat{\boldsymbol{\imath}}$, we obtain the following two scalar equations from the vector impulse-momentum equation:

$$\begin{aligned} m_1 v_{1x}^+ + m_2 v_2^+ &= m_1 v_0 && (11.12)\\ m_1 v_{1y}^+ - P_2 &= 0 && (11.13) \end{aligned}$$

So far, we have two equations and four unknowns — v_{1x}^+, v_{1y}^+, v_2^+ and P_2. Obviously, we need more equations. Now, let us consider the collision between the cart and the ball. Let the impulse of this collision be P_1. Then the impulse-momentum equation for the ball gives us,

$$P_1\hat{\boldsymbol{n}} = m_1(v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}) - m_1 v_0\hat{\boldsymbol{\imath}}.$$



Filename:sfig11-2-two-collisions2-b

Figure 11.22: Free body diagram of the ball during the collision with the cart. The cart impulse P_1 acts along the normal $\hat{\boldsymbol{n}}$. From geometry, we find that $\hat{\boldsymbol{n}} \equiv n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}} = -\sin\alpha\,\hat{\boldsymbol{\imath}} + \cos\alpha\,\hat{\boldsymbol{\jmath}}$.

Once again, we separate out the scalar equations from this vector equation, using the information $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}$:

$$\begin{aligned} m_1 v_{1x}^+ - P n_x &= m_1 v_0 && (11.14)\\ m_1 v_{1y}^+ - P n_y &= 0 && (11.15) \end{aligned}$$

Thus, we have now four equations; we still need one more. We now use the restitution equation to relate the normal components of the relative velocities of approach and departure of the ball and the cart:

$$\begin{aligned} (\overline{\boldsymbol{v}_1^+} - \overline{\boldsymbol{v}_2^+})\cdot\hat{\boldsymbol{n}} &= -e(v_1^- - v_2^-)\cdot\hat{\boldsymbol{n}} \\ \Rightarrow\quad v_{1x}^+ n_x + v_{1y}^+ n_y - v_2 n_x &= -e\, v_0 n_x. && (11.16) \end{aligned}$$

Now we have five equations in five unknowns. All we need to do now is to solve these linear equations for all the unknowns. We do so by first writing the five equations (eqn. (11.12) to eqn. (11.16) in matrix form and then solving the matrix equation on a computer. The matrix equation is:

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & 0 & -1 \\ m_1 & 0 & 0 & -n_x & 0 \\ 0 & m_1 & 0 & -n_y & 0 \\ n_x & n_y & -n_x & 0 & 0 \end{bmatrix} \begin{pmatrix} v_{1x}^+ \\ v_{1y}^+ \\ v_2^+ \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} m_1 v_0 \\ 0 \\ m_1 v_0 \\ 0 \\ -e\, v_0 n_x. \end{pmatrix}.$$

Solving this equation as in the previous sample, we get,

$$v_{1x}^+ = 16.59\,\text{m/s},\ v_{1y}^+ = 23.23\,\text{m/s},\ v_2^+ = 3.35\,\text{m/s},\ P_1 = 80.47\,\text{N}\cdot\text{s},\ P_2 = 69.69\,\text{N}\cdot\text{s}.$$

$$\boxed{\overline{\boldsymbol{v}_2^+} = (3.35\,\text{m/s})\hat{\boldsymbol{\imath}}}$$

Note that the answer obtained here is not the same as that found in Sample 11.4; the cart moves a bit faster to the right in this answer. Depending on the mass ratios and the angle of impact, the two methods can give very different answers or very close answers. Welcome to the world of modeling!

Problems for Chapter 11

Coupled motions for particles in space

11.1 Coupled motions of particles in space

Preparatory Problems

11.1.1 Linear momentum balance applied to the whole of a system consisting of multiple interacting particles reduces to $\vec{F} = m\vec{a}$ *if you interpret the terms correctly.* What are the correct interpretations of $\vec{F}$, m and $\vec{a}$?

11.1.2 A particle of mass $m_1 = 6\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in the xy-plane. At a particular instant of interest, particle 1 has position $\vec{r}_1 = 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}$, velocity $\vec{v}_1 = -16\,\text{m/s}\hat{\imath} + 6\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_1 = 10\,\text{m/s}^2\hat{\imath} - 24\,\text{m/s}^2\hat{\jmath}$; and particle 2 has position $\vec{r}_2 = -6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$, velocity $\vec{v}_2 = 8\,\text{m/s}\hat{\imath} + 4\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_2 = 5\,\text{m/s}^2\hat{\imath} - 16\,\text{m/s}^2\hat{\jmath}$.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of each particle at the instant of interest.

b) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of the system of the two particles at the instant of interest.

c) Find the center of mass of the system at the instant of interest.

d) Find the velocity and acceleration of the center of mass.

11.1.3 A particle of mass $m_1 = 5\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in space. At a particular instant of interest, particle 1 has position, velocity, and acceleration

$$\begin{aligned}
\vec{r}_1 &= 1\,\text{m}\hat{\imath} + 1\,\text{m}\hat{\jmath} \\
\vec{v}_1 &= 2\,\text{m/s}\hat{\jmath} \\
\vec{a}_1 &= 3\,\text{m/s}^2\hat{k}
\end{aligned}$$

respectively, and particle 2 has position, velocity, and acceleration

$$\begin{aligned}
\vec{r}_2 &= 2\,\text{m}\hat{\imath} \\
\vec{v}_2 &= 1\,\text{m/s}\hat{k} \\
\vec{a}_2 &= 1\,\text{m/s}^2\hat{\jmath}
\end{aligned}$$

respectively. For the system of particles at the instant of interest, find its

a) linear momentum $\vec{L}$,

b) rate of change of linear momentum $\dot{\vec{L}}$,

c) angular momentum about the origin $\vec{H}_{/\text{O}}$,

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/\text{O}}$,

e) kinetic energy E_K, and

f) rate of change of kinetic energy.

11.1.4 If you are given the total mass, the position, the velocity, and the acceleration of the center of mass of a system of particles can you find the angular momentum $\vec{H}_{/\text{O}}$ of the system, where O is not at the center of mass? If so, how and why? If not, then give a reason and/or a counter example. *

11.1.5 Seventeen particles are interaction with the force on particle i from particle j being $\vec{F}_{ij}$ with all $\vec{F}_{ij}$ known.

a) What is the commonly assumed assumption about the relation between, say, $\vec{F}_{36}$ and $\vec{F}_{63}$?

b) What is the total force on particle 5?

More-Involved Problems

11.1.6 Two particles each of mass m are connected by a massless elastic spring of spring constant k and unextended length $2R$. The system slides without friction on a horizontal table, so that no net external forces act.

a) Is the total linear momentum conserved? Justify your answer.

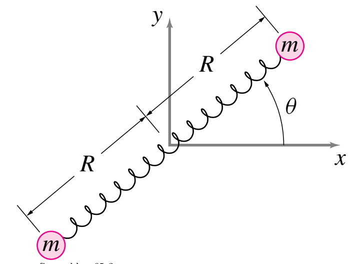
b) Can the center of mass accelerate? Justify your answer.

c) Draw free body diagrams for each mass.

d) Derive the equations of motion for each mass in terms of cartesian coordinates.

e) What are the total kinetic and potential energies of the system?

f) For constant values and initial conditions of your choosing, plot the trajectories of the two particles and of the center of mass (on the same plot).



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Problem 11.1.6

11.1.7 Two ice skaters whirl around one another. They are connected by a linear elastic cord whose center is stationary in space. We wish to consider the motion of one of the skaters by modeling her as a mass m held by a cord that exerts k Newtons for each meter it is extended from the central position.

a) Draw a free body diagram showing the forces that act on the mass is at an arbitrary position.

b) Write the differential equations that describe the motion.

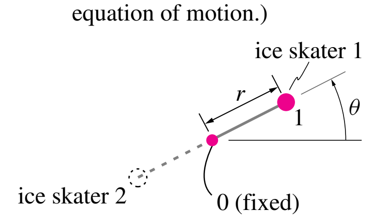
c) Describe in physical and mathematical terms the nature of the motion for the three cases

   a) $\omega < \sqrt{k/m}$;

   b) $\omega = \sqrt{k/m}$;-

   c) $\omega > \sqrt{k/m}$.

(You are not asked to solve the equation of motion.)



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Problem 11.1.7

11.1.8 n identical particles with mass m are on the vertices of an n sided regular polygon. Equivalently, n particles are equally spaced on a circle with radius R. At $t = 0$ they all have velocities tangent to the circle and equal in magnitude v_0. All the particles are attracted to each other with an inverse square gravitational attraction. For the numerical simulations below pick values of n, m, G and R any way that pleases you.

a) Find an initial value for v_0 so that all the masses spiral in and then bounce out again. Plot the trajectories of all the masses on one plot for a long-enough time so the plot is pleasing to the eye.

b) Find a value for v_0 so all the particles travel on circular trajectories.

c) Can you find a formula for v_0 above in terms of the other parameters in the problem?

11.1.9 Two masses, both with $M = 1000m$ travel in circles on the xy plane according two

$$\vec{\boldsymbol{r}}_1 = -\vec{\boldsymbol{r}}_2 = R(\cos\omega t\hat{\boldsymbol{\imath}} + \sin\omega t\hat{\boldsymbol{\imath}})$$

a) Assume inverse square attraction and find a set of values for m, G, R and ω so the assumed circular path is a solution of the equations of motion.

b) A third mass m is introduced which is gravitationally attracted to the other two. Pick initial conditions for the two big masses that are consistent with their circular motion solution. For the third mass use initial conditions $[\vec{\boldsymbol{r}}_0]_{xyz} = [00z_0]$. Run a simulation for some time.

- Does the third mass stay *exactly* on the z axis for all time in the simulation? Would it if the simulation was exact? Is so, why? If not, why not? *
- Is the motion of the third mass exactly periodic in the computer simulation? Would it be if the solution was exact? If so, why? If not, why not? *

11.1.10 Montgomery's eight. Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$\begin{aligned}(x1, y1) &= (-0.97000436, 0.24308753)\\ (x2, y2) &= (-x1, -y1)\\ (x3, y3) &= (0, 0)\end{aligned}$$

and initial velocities

$$\begin{aligned}(vx3, vy3) &= (0.93240737, 0.86473146)\\ (vx1, vy1) &= -(vx3, vy3)/2\\ (vx2, vy2) &= -(vx3, vy3)/2.\end{aligned}$$

For each of the problems below show accurate computer plots and explain any curiosities.

a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units. *

b) Same as above, but run for 10 time units. *

c) Same as above, but change the initial conditions slightly. *

d) Same as above, but change the initial conditions more and run for a much longer time. *

11.2 Collisions and explosions

Preparatory Problems

11.2.1 Assuming θ, v_0, and e to be known quantities, write the following equations in matrix form set up to solve for v'_{Ax} and v'_{Ay}:

$$\begin{aligned}\sin\theta v'_{Ax} + \cos\theta v'_{Ay} &= ev_0\cos\theta\\ \cos\theta v'_{Ax} - \sin\theta v'_{Ay} &= v_0\sin\theta.\end{aligned}$$

11.2.2 The equation $(\vec{\boldsymbol{v}}'_1 - \vec{\boldsymbol{v}}'_2)\cdot\hat{\boldsymbol{n}} = e(\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_1)\cdot\hat{\boldsymbol{n}}$ relates relative velocities of two point masses before and after frictionless impact in the normal direction $\hat{\boldsymbol{n}}$ of the impact. If $\vec{\boldsymbol{v}}'_1 = v_{1_x}\hat{\boldsymbol{\imath}} + v_{1_y}\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}}'_2 = -v_0\hat{\boldsymbol{\imath}}$, $e = 0.5$, $\vec{\boldsymbol{v}}_2 = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{v}}_1 = 2\,\mathrm{ft/s}\hat{\boldsymbol{\imath}} - 5\,\mathrm{ft/s}\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{n}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$, find the scalar equation relating the velocities in the normal direction.

11.2.3 The following three equations are obtained by applying the principle of conservation of linear momentum on some system.

$$\begin{aligned}m_0v_0 &= 24\,\mathrm{m/s}\,m_A - .67m_Bv_B - .58m_Cv_C\\ 0 &= 36\,\mathrm{m/s}\,m_A + .33m_Bv_B + 0.3m_Cv_C\\ 0 &= 23\,\mathrm{m/s}\,m_A - .67m_Bv_B - .58m_Cv_C.\end{aligned}$$

Assume v_0, v_B, and v_C are the only unknowns. Write the equations in matrix form set up to solve for the unknowns.

11.2.4 The following three equations are obtained to solve for v'_{Ax}, v'_{Ay}, and v'_{Bx}:

$$\begin{aligned}(v'_{Bx} - v'_{Ax})\cos\theta &= v'_{Ay}\sin\theta - 10\,\mathrm{m/s}\\ v'_{Ax}\sin\theta &= v'_{Ay}\cos\theta - 36\,\mathrm{m/s}\\ m_Bv'_{Bx} + m_Av'_{Ax} &= (-60\,\mathrm{m/s})\,m_A.\end{aligned}$$

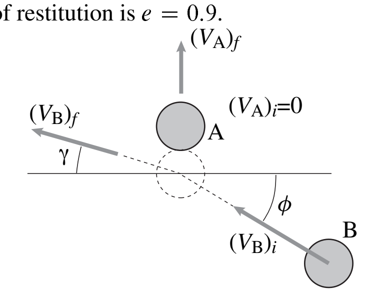
Set up these equations in matrix form.

11.2.5 Solve for the unknowns v'_{A_x}, v'_{A_y}, and v'_{B_x} in problem 11.2.4 taking $\theta = 50^\circ$, $m_A = 1.5\,\mathrm{m}_B$ and $m_B = 0.8\,\mathrm{kg}$. Use any computer program.

11.2.6 Using the matrix form of equations in Problem 11.2.1, solve for v'_{Ax} and v'_{Ay} if $\theta = 20^\circ$ and $v_0 = 5\,\mathrm{ft/s}$.

More-Involved Problems

11.2.7 Two frictionless equal-mass pucks sliding on a plane collide as shown below. Puck A is initially at rest. Given that $(V_B)_i = 1.0\,\mathrm{m/s}$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\mathrm{m/s}$, find the approach angle ϕ and rebound angle γ. The coefficient of restitution is $e = 0.9$.



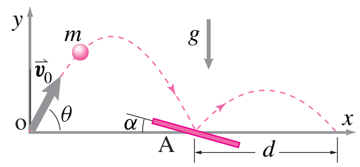
Problem 11.2.7

11.2.8 Reconsider problem 11.2.7. Given instead that $\gamma = 30°$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\mathrm{m/s}$, find the initial velocity of puck B.

11.2.9 A ball of mass $m = 0.5\,\mathrm{kg}$ is thrown up in the air with initial speed $v_0 = 50\,\mathrm{m/s}$ at an angle $\theta = 60°$. The ball lands on and bounces off a slanted floor that makes an angle $\alpha = 15°$ with the horizontal. Assume the collision with the floor to be elastic and ignore air drag on the ball.

a) Find the impulse of the collision of the ball,

b) After bouncing off the slanted floor, how much horizontal distance does the ball travel before landing on the ground again? Is this distance more, less, or the same as it would have travelled had the floor been not slanted?



Filename:pfig11-2-tiltedfloor1
Problem 11.2.9

11.2.10 Solve the general two-particle frictionless collision problem. For example, write computer code that has lines like this near the start :

`m1=3; m2=19`	Set values of masses
`v1zero=[10 20]`	Initial velocity of mass 1
`v2zero=[-5 3]`	Initial velocity of mass 2
`e=.5`	Set coefficient of restitution
`theta=pi/4`	Angle that the normal to contact plane makes, measured CCW from +x axis, in radians

Your program (function, code, script) should calculate the impulse of mass 1 on mass 2, and the velocities of the two masses after the collision. Your program should assume consistent units for all quantities.

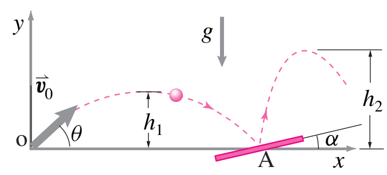
a) You should demonstrate that your program works by solving at least 4 different problems for which you can check your answer by simple pencil-and-paper calculations. These problems should have as much variety as possible. Sketch these problems clearly, show their analytic solution, and show that the computer agrees. *

b) Solve the problem given in the sample text given in the initial problem statement.

11.2.11 A projectile is launched at $\theta = 40°$ with speed $v_0 = 25\,\text{m/s}$. The projectile lands on a steel plate that can be adjusted to make any angle α with the horizontal. The projectile bounces off the steel plate without loosing any energy. The projectile is required to reach a height after rebound twice as much it did during its flight before hitting the plate. Ignore air resistance.

a) Find the required angle α of the plate.

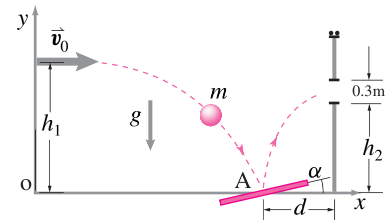
b) Can you always find some α for any launch angle $\theta < \pi/2$ such that $h_2 = 2h_1$?



Filename:pfig11-2-tiltedfloor2
Problem 11.2.11

11.2.12 Two equal mass cars approach an intersection at right angles. They crash and stick together. One of the cars was going at 30 mph before the crash. The other car's path gets deflected by 15°. How fast was it going?

A ball m is thrown horizontally at height h and speed v_0. It then has a sequence of bounces on the horizontal ground. Treating each collision as frictionless with restitution coefficient e how far has the ball travelled horizontally when it just finishes bouncing? Answer in terms of some or all of m, g, h, v_0 and e. A ball m is thrown horizontally at height h and speed v_0. It then has a sequence of bounces on the horizontal ground. Treating each collision as frictionless with restitution coefficient e how far has the ball travelled horizontally when it just finishes bouncing? Answer in terms of some or all of m, g, h, v_0 and e. **11.2.13** A game involves using a pedal to direct a falling ball into a fixed vertical slot by simply rotating the pedal when the ball hits the pedal. A model of this game is shown in the figure. The ball is thrown horizontally with an initial speed $v = 10\,\text{m/s}$ from a height $h_{\text{ball}} = 3\,\text{m}$. The pedal is located at $d = 2\,\text{m}$ from the wall that houses the slot at height $h = 2\,\text{m}$. The slot itself is 0.3 m in extent. The coefficient of restitution between the pedal and the ball is $e = 0.9$. The air resistance is negligible. Find the angle α or the range of this angle, so that the ball makes it through the slot. You can ignore the dimensions of the ball.



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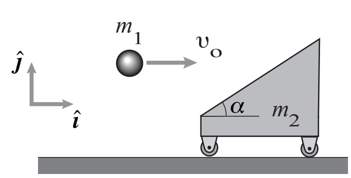
Problem 11.2.13

11.2.14 An airplane is flying steadily at an altitutude of 30,000 ft at a speed of 500 mph. It explodes into two equal pieces. One piece is found to the right of the airplane's initial trajectory and 8 miles forward of the explosion point. Where should you look for the other piece? Assume the interaction impulse is in the horizontal plane and make the approximation that the two pieces fly in frictionless parabolic trajectories.

11.2.15 Consider the simultaneous collisions problem discussed in Sample 11.4. Consider the ball to be much more massive than the cart; $m_1 = 20\,\text{kg}$ and $m_2 = 1\,\text{kg}$. The angle of the inclined face is very shallow, $\alpha = 2°$. The ball hits the cart with the velocity $\vec{v}_1 = 50\,\text{m/s}\hat{\imath}$. The impact is elastic and frictionless. Find the subsequent velocities of the ball and the cart using the two methods discussed in Sample 11.4 and Sample 11.5. Comment on the answers you get. How will your answers change if you reversed the mass ratio?



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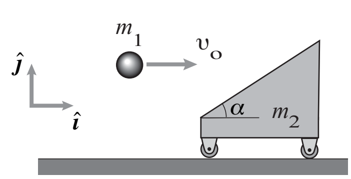
Problem 11.2.15

11.2.16 Consider the simultaneous collisions problem discussed in Sample 11.4 again. Assume that $m_1 = 5\,\text{kg}$, $m_2 = 10\,\text{kg}$, $\vec{v}_1 = 50\,\text{m/s}$, $e = 0.75$, and the angle $\alpha = 88°$. Find the final velocities of the cart and the ball assuming that the cart must move in the $\hat{\imath}$ direction only. What is the the net loss of energy in the impacts?



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Problem 11.2.16

CHAPTER 12

Constrained straight-line motion

Here is an introduction to kinematic constraint in its simplest context, systems that are constrained to move without rotation in a straight line. In one dimension pulley problems provide the main example. Two and three dimensional problems are covered, such as finding structural support forces in accelerating vehicles and the slowing or incipient capsize of a braking car or bicycle. Angular momentum balance is introduced as a needed tool but without the complexities of rotatioinal kinematics.

Contents

In the previous chapters you learned to write the equations of motion for a particle or collection of a few particles, assuming you have a model for the forces on the particles in terms of their positions, velocities, and time. Some caveats to using that approach for engineering systems that don't seem to behave like isolated particles, but rather are composed of many particles were listed at the start of chapter 11 on page 579. One way to finesse these problems is to make kinematic assumptions about how particles and collections of particles move. Why? Sometimes, often actually, the simplest model of mechanical interaction is not a law for force as a function of position, velocity and time, but just a geometric description of the relative positions or velocities of points. The reasons for this geometric, instead of force-based, approach are two-fold:

- **The minute details of the motion are often not of interest and therefore not worth tracking.** For example, the vibrations of a solid, or relative motions of atoms in a solid might be of a smaller scale than the overall motion of interest, and
- **Often one does not know an accurate force law.** For example, at the microscopic level one does not know the details of atomic interactions; or, at the machine level, one may not know exactly the relations between the small motions of one part relative to another with which it makes contact. For example, even though one knows that the axle being in a hole restricts the relative motion of the axle with the train one may not know in detail how the contact forces depend on the exact position of the axle in its hole.).

Filename:TruckStraightLine5030

Figure 12.1: A truck or car running on straight level road is in straight-line motion, neglecting, of course, the wheel rotation, the bouncing, the moving engine parts, and the wandering eyes of the passengers.

Kinematic constraints

Much mechanical modeling involves the replacement of force-interaction rules with assumptions about the geometry of the motions. Idealizing an interaction force as causing a definite geometric restriction on motion is called imposing a *kinematic constraint*.

A *kinematic constraint* is an equation that describes a restriction on allowed positions, velocities or accelerations of parts in a system. Kinematic constraints are always accompanied by one or more *a priori* unknown 'constraint' forces that maintain the geometric constraint relations.

The basic laws of forces and mechanics apply to all systems, no matter how they are or are not constrained. But, if objects are treated as kinematically constrained the methods in mechanics have a slightly different flavor. To get the idea we start with simple systems that have simple constraints and that move in simple ways. In this short chapter, we will discuss the mechanics of things where every point in each object has the same velocity and acceleration as every other point (so called *parallel motion*) and with the further restriction that every point moves in a straight line.

Example: Train on Straight Level Tracks
Consider a train on straight level tracks. If we focus on the body of the train, we can approximate the motion as parallel straight-line motion. All parts move the same amount, with the same velocities and accelerations in the same fixed direction.

We start with 1-D mechanics and constraint with string and pulleys, and then move on to 2-D and 3-D mechanics (of systems in 1D motion).

12.1 1-D constrained motion and pulleys

In this section masses are connected together with bars or ropes. These connections are idealized as being inextensible. Consider a car towing another with a strong light chain. We may not want to consider the elasticity of the chain but instead idealize the chain as having a fixed length. Although the idealization of zero deformation is a simplification, it is a simplification that requires special treatment. It is the simplest example of a *kinematic constraint*.

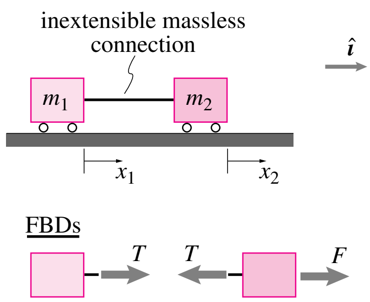

Figure 12.2: A schematic of one car pulling another, or of a boat pulling a barge. Also shown are FBDs of the bodies separately. Because our analysis is only in one spatial dimension, forces with no component in $\hat{\imath}$ direction are not shown.

Figure 12.2 shows a schematic of one car pulling another. One-dimensional free body diagrams are also shown. The force F is the force transmitted from the road to the front car through the tires. The tension T is the tension in the connecting chain. From linear momentum balance for each of the objects (modeled as particles):

$$T = m_1\ddot{x}_1 \quad \text{and} \quad F - T = m_2\ddot{x}_2. \tag{12.1}$$

These equations are exactly the same as for cars connected by a spring, a dashpot, or any idealized-as-massless connector. And all these systems have the same free body diagrams but different motions. If the connection were with a spring or dashpot the equations above would be supplemented with

$$T = k(x_2 - x_1 - \ell_0) \quad \text{or} \quad T = c(\dot{x}_2 - \dot{x}_1)$$

In this case we need our equations to somehow indicate that the two particles are not allowed to move independently. We need a constraint equation to replace these constitutive laws.

Kinematic constraint: two approaches

There are two basic ways of dealing with kinematic constraints:

1. Use separate free body diagrams and equations of motion for each particle and then add extra kinematic constraint equations, or
2. do something clever to avoid having to find the constraint forces.

Methad 1: Finding the accelerations and the constraint forces together

The geometric (or kinematic) constraint that two masses move together is

$$x_1 = x_2 + \text{Constant}.$$

We can differentiate the kinematic constraint twice to get

$$\ddot{x}_1 = \ddot{x}_2. \tag{12.2}$$

If we take F and the two masses as given, equations 12.1 and 12.2 are three equations for the unknowns $\ddot{x}_1$, $\ddot{x}_2$, and T. In matrix form,we have:

$$\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

We can solve these equations to find $\ddot{x}_1$, $\ddot{x}_2$, and T in terms of F.

Method 2: Finesse having to find the constraint force

On the other hand, if all we are interested in are the accelerations of the cars it would be nice to avoid even having to think about the constraint force. One way to avoid dealing with the constraint force is to draw a free body diagram of the entire system as in fig. 12.3. If we just call the acceleration of the system $\ddot{x}$ we get, from linear momentum balance, one equation in one unknown:

$$F = (m_1 + m_2)\ddot{x}.$$

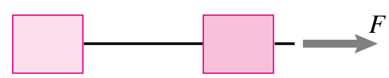

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Figure 12.3: A free body diagram of the whole system. Note that the unknown tension (constraint) force does not show. As usual for 1D mechanics, vertical forces are left off for simplicity (although it would be more correct to include them).

Kinematic constraints

A generalization of the 1D inextensible-cable constraint example above is the rigid-object constraint where not just two, but many particles are assumed to keep constant distance from one another, and in one, two or three dimensions. Another important constraint is an ideal hinge connection between two objects. Much of the theory of mechanics after Newton has been motivated by a desire to deal easily with these and other kinematic constraints. In fact, one way of characterizing the primary difficulty of dynamics as a subject is the difficulty of dealing with kinematic constraints.

Pulleys

Pulleys are used to redirect force to amplify or attenuate force and to amplify or attenuate motion. Like a lever, a pulley system is an example of a mechanical transmission. Objects connected by inextensible ropes around ideal pulleys are also examples of kinematic constraint.

Constant length and constant tension

Problems with pulleys are solved by using two facts about idealized strings. First, an ideal string is inextensible so the sum of the string lengths, over the different inter-pulley sections, adds to a constant (not varying in time):

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 + ... = \text{constant}. \tag{12.3}$$

Second, for round pulleys of negligible mass and no bearing friction, tension is constant along the length of the string①.The tension on one side of a pulley is the same as the tension on the other side:

①See fig. 4.25 on page 205 and the related text which shows why $T_1 = T_2$ for one round pulley idealized as frictionless and massless.

$$T_1 = T_2 = T_3 \ldots. \tag{12.4}$$

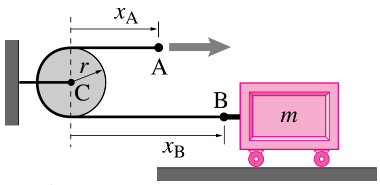

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Figure 12.4: One mass, one pulley, and one string

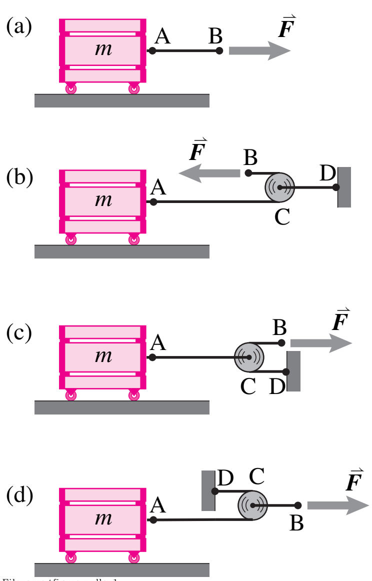

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Figure 12.5: The four classic cases: (a) no pulley, (b) a pulley system with no mechanical advantage, (c) a pulley system that multiplies force and attenuates motion, and (d) a pulley system that attenuates force and amplifies motion.

Example: Length of string calculation

We use the trivial pulley example in fig. 12.4. Starting from point A, we add up the lengths of string

$$\ell_{tot} = x_A + \pi r + x_B \equiv constant. \tag{12.5}$$

One portion of the string touches half of the pulley circumference, πr, even if x_A and x_B change in time and different portions of string wrap around the pulley at different times.

We now formally deduce the relations between the velocities and accelerations of points A and B. Differentiating equation 12.5 with respect to time once and then again, we get

$$\begin{aligned} \dot{\ell}_{tot} = 0 &= \dot{x}_A + 0 + \dot{x}_B \\ \Rightarrow \dot{x}_A &= -\dot{x}_B \\ \Rightarrow \ddot{x}_A &= -\ddot{x}_B \end{aligned} \tag{12.6}$$

When point A is displaced to the right by an amount Δx_A, point B is displaced exactly the same amount but to the left; that is, $\Delta x_A = -\Delta x_B$. Note, to find the kinematic relations 12.6 for the pulley system, we never need to know the total length of the string, only that it is constant in time. The constant-in-time quantities (the pulley half-circumference and the string length) get 'killed' in the process of differentiation.

Commonly we think of pulleys as small and thus never account for the pulley-contacting string length. Luckily this approximation generally leads to no error because we most often are interested in displacements, velocities, and accelerations. And in these cases, as in the example above, the pulley contact length drops out of the equations anyway.

The classic simple uses of pulleys

First imagine trying to move a load with no pulley as in fig. 12.5a. The force you apply goes right to the mass. This is like direct drive with no transmission.

Now you would like to use pulleys to help you move the mass. In the cases we consider here the mass is on a frictionless support and we are trying to accelerate it. But the concepts are the same if there are also resisting forces on the mass. What can we do with one pulley? Three possibilities are shown in fig. 12.5b-d which might, at a blinking glance, look roughly the same. But they are quite different. Here we discuss each design qualitatively.

In fig. 12.5b we pull one direction and the mass accelerates the other way. This illustrates

> the simplest use of a pulley, to redirect an applied force.

The force on the mass has magnitude $|\vec{F}|$ and there is no mechanical advantage.

Figure 12.5c shows

> the classic use of a pulley, to multiply a force.

Here's a detailed solution to this problem.

Example: Pulley in figure 12.5c

The methods here will solve every pulley problem②. First draw the FBDs and rope geometry sketch (fig. 12.6). First linear momentum balance (and force balance for the negligible-mass pulley):

$$\text{LMB} \quad \Rightarrow \quad m\ddot{x}_A = T_A \quad \text{and} \quad T_A = 2T_B. \tag{12.7}$$

Next the rope length kinematics:

$$\ell_1 = x_C - x_A \quad \text{and} \quad \ell_2 = (x_B - x_C) + (x_D - x_C) + \pi r$$

Differentiate the rope-length equations twice, remembering that ℓ_1, ℓ_2, r and x_D are all constants, to get

$$\ddot{x}_C = \ddot{x}_A \quad \text{and} \quad \ddot{x}_B = 2\ddot{x}_C. \tag{12.8}$$

The second equation above ($\ddot{x}_B = 2\ddot{x}_C$) is the only one that you couldn't write down at a glance. Solve these equations in terms of $F = T_A$ to get:

$$T_A = 2F \quad \text{and} \quad \ddot{x}_A = 2F/m \quad \text{and} \quad \ddot{x}_B = 4F/m.$$

Figure 12.5d shows

> a less common use of a pulley, to multiply motion.

In this case an analysis similar to that above (that you are asked to do in problem 12.1.6 on page 632) shows that:

$$T_A = 2F/2 \quad \text{and} \quad \ddot{x}_A = F/(2m) \quad \text{and} \quad \ddot{x}_B = F/(4m)$$

Despite the superficial similarity, this setup is the opposite to the use in fig. 12.5c.

② **How to solve pulley problems** Follow these rules, and you can solve all pulley problems (see the example from fig. 12.5c in the text).

1. Draw a free body diagram of each pulley and each mass, taking account that the tension in any given rope is constant along its length.
2. Write linear momentum balance for each pulley and mass (this might just be force balance if you neglect the mass of, say, a pulley).
3. Do length accounting for each rope, taking account that its length is constant. To avoid sign errors, *use a reference point that is off to the side of the whole mechanism,* even if there is no such point in the original problem (the wall at 0 in the solution to fig. 12.5c above).
4. Differentiate the string length equations (from the step above), twice.
5. Solve the equations from parts 2 and 4 (above) for desired unknowns.

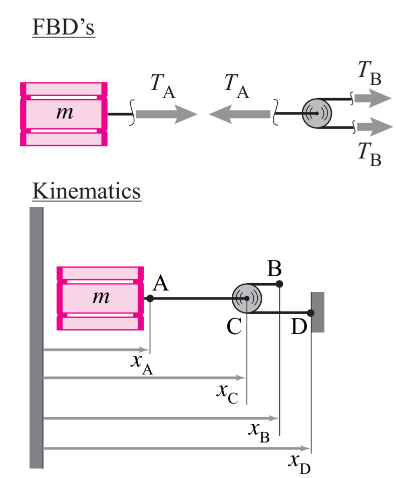

Figure 12.6: FBDs and kinematics for the mechanism in fig. 12.5c

Force amplification is motion attenuation

In fig. 12.5c a force F is applied at B. The resulting force on the mass at A is $2F$ and point A moves with half the acceleration of point B. However, in fig. 12.5d the resulting force on the mass at A is $F/2$, and point A moves with twice the acceleration of point B.

These illustrate a general duality rule

> If force is amplified then motion is equally attenuated. If motion is amplified the force is equally attenuated ③.

③Caveats: Negligible dissipation and negligible mass of the transmission parts.

High gear and low gear. This rule applies to all frictionless passive transmissions (*e.g.,* levers, gear trains, hydraulic systems) with negligible inertia: In a low gear in a car (or bicycle) the force at the wheel is large for a given force of the engine (leg muscle) but the wheel doesn't turn much for a given displacement of the engine (foot). In a high gear the force at the wheel is small but the wheel turns a lot for a given amount of engine rotation (or foot displacement). This are, of course a special cases of the much more general rule: you can't win!

Power balance

In every special case we can derive the duality rule (above) using momentum balance and kinematics. But the general result is best understood with energy balance:

> the power of the applied force is the power applied to the mass.

Example: Power balance and the pulley in figure 12.5c
There are various ways of setting up the energy (or power) balance equations. Here is one:

$$\text{Power into transmission} = \text{Power supplied by transmission}$$

The power in is $F \cdot \dot{x}_B$. The power supplied is $T_A \cdot \dot{x}_A$. So,

$$F \cdot \dot{x}_B = T_A \cdot \dot{x}_A \quad \Rightarrow \quad \frac{T_A}{T_B} = \frac{\dot{x}_B}{\dot{x}_A}.$$

But because we can differentiate the kinematics relationships, $\ddot{x}_A/\ddot{x}_B = \dot{x}_a/\dot{x}_B$. So we also have

$$\frac{T_A}{T_B} = \frac{\ddot{x}_B}{\ddot{x}_A}.$$

The 'effective mass' of a point of force application The feel of the machine is of concern for machines that people handle. One aspect of feel is the *effective mass* (sometimes called 'reflected inertia') is defined by the response of a point to an applied force.

$$m_{\text{eff}} = \frac{|\vec{\boldsymbol{F}}_{\text{B}}|}{|\vec{\boldsymbol{a}}_{\text{B}}|}.$$

For the case of fig. 12.5a and fig. 12.5b the effective mass of point B is just m. For the case of fig. 12.5c the block at A has $2|\vec{\boldsymbol{F}}|$ acting on it and point B has twice the acceleration of point A. So the acceeration of poing B is $4F/m = F/(m/4)$ and the effective mass of point B is $m/4$. For the case of fig. 12.5d, the mass only has $|\vec{\boldsymbol{F}}|/2$ acting on it and point B only has half the acceleration of point A, so the effective mass is $4m$.

These special cases exemplify the general rule:

The effective mass of one end of a transmission is the mass of the other end multiplied by the square of the motion amplification ratio.

In terms of the effective mass, the systems shown infig. 12.5c and fig. 12.5d which look so similar to a novice, actually differ by a factor of $2^2 \cdot 2^2 = 16$. With a given F and m point B in fig. 12.5c has 16 times the acceleration of point B in fig. 12.5d.

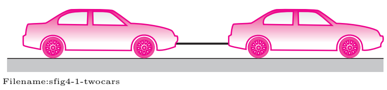

Figure 12.7:

SAMPLE 12.1 Find the motion of two cars. One car is towing another of equal mass on level ground. The thrust of the wheels of the first car is F. The second car rolls frictionlessly. Find the acceleration of the system two ways:

1. using separate free body diagrams,
2. using a system free body diagram.

Solution

1. The free body diagram of each car is shown below, in fig. 12.8.

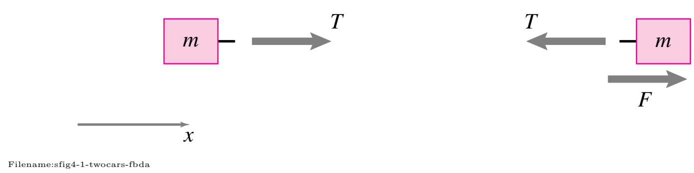

Figure 12.8: Partial free body diagrams of the two cars (the vertical ground reactions are not shown as they are of no interest to us for the horizontal motion.

From the linear momentum balance of each car, we get

$$m\ddot{x}_1 = T \tag{12.9}$$
$$F - T = m\ddot{x}_2 \tag{12.10}$$

The kinematic constraint of towing (the cars move together, *i.e.,* no relative displacement between the cars) gives

$$\ddot{x}_1 - \ddot{x}_2 = 0 \tag{12.11}$$

Solving eqns. (12.9), (12.10), and (12.11) simultaneously, we get

$$\ddot{x}_1 = \ddot{x}_2 = \frac{F}{2m} \quad (T = \frac{F}{2})$$

2. The free body diagram of the two cars together is shown below, in fig. 12.9.

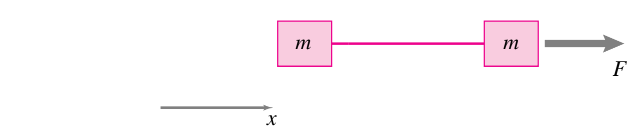

Figure 12.9:

From the linear momentum balance of the two cars as one system, we get

$$\begin{aligned} m\ddot{x} + m\ddot{x} &= F \\ \ddot{x} &= F/2m \end{aligned}$$

$$\boxed{\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = F/2m}$$

SAMPLE 12.2 Driving a pile into the ground. A cylindrical wooden pile of mass 10 kg and cross-sectional diameter 20 cm is driven into the ground with the blows of a hammer. The hammer is a block of steel with mass 50 kg which is dropped from a height of 2 m to deliver the blow. At the nth blow the pile is driven into the ground by an additional 5 cm. Assuming the impact between the hammer and the pile to be totally inelastic (*i.e.,* the two stick together), find the average resistance of the soil to penetration of the pile.

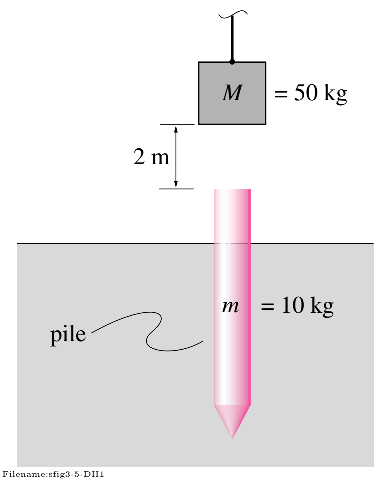

Figure 12.10:

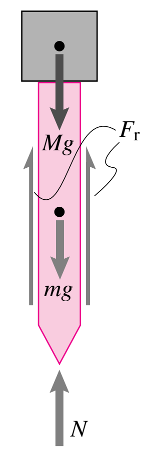

Figure 12.11: Free body diagram of the hammer and pile system. F_r is the total resistance of the ground.

Solution Let F_r be the average (constant over the period of driving the pile by 5 cm) resistance of the soil. From the free body diagram of the pile and hammer system, we have

$$\sum \vec{F} = -mg\hat{\boldsymbol{j}} - Mg\hat{\boldsymbol{j}} + N\hat{\boldsymbol{j}} + F_r\hat{\boldsymbol{j}}.$$

But N is the normal reaction of the ground, which from static equilibrium, must be equal to $mg + Mg$. Thus,

$$\sum \vec{F} = Fr\hat{\boldsymbol{j}}.$$

Therefore, from linear momentum balance ($\sum \vec{F} = m\vec{a}$),

$$\vec{a} = \frac{F_r}{M+m}\hat{\boldsymbol{j}}.$$

Now we need to find the acceleration from given conditions. Let v be the speed of the hammer just before impact and V be the combined speed of the hammer and the pile immediately after impact. Then, treating the hammer and the pile as one system, we can ignore all other forces *during* the impact (none of the external forces: gravity, soil resistance, ground reaction, is comparable to the impulsive impact force, see page 816). The impact force is internal to the system. Therefore, during impact, $\sum \vec{F} = \vec{0}$ which implies that linear momentum is conserved. Thus

$$\begin{aligned} -Mv\hat{\boldsymbol{j}} &= -(m+M)V\hat{\boldsymbol{j}} \\ \Rightarrow \qquad V &= \left(\frac{M}{m+M}\right)v = \frac{50\,\text{kg}}{60\,\text{kg}}v = \frac{5}{6}v. \end{aligned}$$

The hammer speed v can be easily calculated, since it is the free fall speed from a height of 2 m:

$$v = \sqrt{2gh} = \sqrt{2\cdot(9.81\,\text{m/s}^2)\cdot(2\,\text{m})} = 6.26\,\text{m/s} \quad \Rightarrow \quad V = \frac{5}{6}v = 5.22\,\text{m/s}.$$

The pile and the hammer travel a distance of $s = 5\,\text{cm}$ under the deceleration a. The initial speed $V = 5.22\,\text{m/s}$ and the final speed = 0. Plugging these quantities into the one-dimensional kinematic formula

$$v^2 = v_0^2 + 2as,$$

we get,

$$\begin{aligned} 0 &= V^2 - 2as \quad \text{(Note that } a \text{ is negative)} \\ \Rightarrow \quad a &= \frac{V^2}{2s} = \frac{(5.22\,\text{m/s})^2}{2\times 0.05\,\text{m}} = 272.48\,\text{m/s}^2. \end{aligned}$$

Thus $\vec{a} = 272.48\,\text{m/s}^2\,\hat{\boldsymbol{j}}$. Therefore,

$$F_r = (m+M)a = (60\,\text{kg})\cdot(272.48\,\text{m/s}^2) = 1.635\times 10^4\,\text{N}$$

$$\boxed{F_r \approx 16.35\,\text{kN}}$$

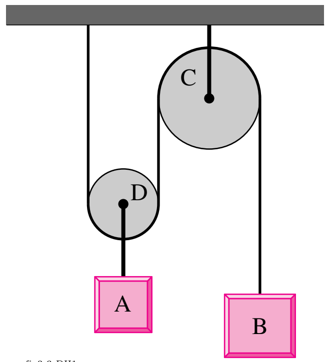

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Figure 12.12:

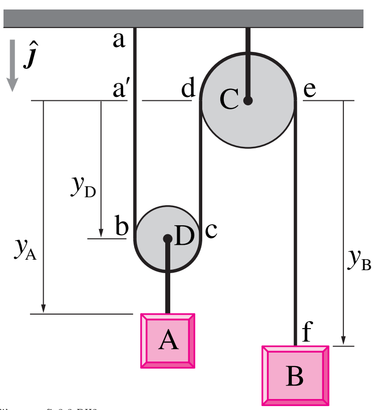

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Figure 12.13:

④ We have done an elaborate calculation of ℓ_{tot} here. Usually, the constant lengths over the pulleys and some constant segments such as aa′ are ignored in calculating ℓ_{tot}. These constant length segments can be ignored because they drop out of the equation when we take time derivatives to relate velocities and accelerations of different points, such as B and D here.

SAMPLE 12.3 **Pulley kinematics.** For the masses and ideal-massless pulleys shown in figure 12.12, find the acceleration of mass A in terms of the acceleration of mass B. Pulley C is fixed to the ceiling and pulley D is free to move vertically. All strings are inextensible.

Solution Let us measure the position of the two masses from a fixed point, say the center of pulley C. (Since C is fixed, its center is fixed too.) Let y_A and y_B be the vertical distances of masses A and B, respectively, from the chosen reference (C). Then the position vectors of A and B are:

$$\vec{r}_A = y_A\hat{\jmath} \quad \text{and} \quad \vec{r}_B = y_B\hat{\jmath}.$$

Therefore, the velocities and accelerations of the two masses are

$$\begin{aligned} \vec{v}_A &= \dot{y}_A\hat{\jmath}, & \vec{v}_B &= \dot{y}_B\hat{\jmath}, \\ \vec{a}_A &= \ddot{y}_A\hat{\jmath}, & \vec{a}_B &= \ddot{y}_B\hat{\jmath}. \end{aligned}$$

Since all quantities are in the same direction ($\hat{\jmath}$), we can drop $\hat{\jmath}$ from our calculations and just do scalar calculations. We are asked to relate $\ddot{y}_A$ to $\ddot{y}_B$.

In all pulley problems, the trick in doing kinematic calculations is to relate the variable positions to the fixed length of the string. Here, the length of the string ℓ_{tot} is: ④

$$\begin{aligned} \ell_{tot} &= \text{ab + bc + cd + de + ef = constant} \\ \text{where}\quad \text{ab} &= \underbrace{\text{aa}'}_{\text{constant}} + \underbrace{\text{a}'\text{b}}_{(=\text{cd}=y_D)} \\ \text{bc} &= \text{string over the pulley D = constant} \\ \text{de} &= \text{string over the pulley C = constant} \\ \text{ef} &= y_B \\ \text{thus}\quad \ell_{tot} &= 2y_D + y_B + \overbrace{\text{constant}}^{(\text{aa}'+\text{bc}+\text{de})}. \end{aligned}$$

Taking the time derivative on both sides, we get

$$\overbrace{\frac{d}{dt}(\ell_{tot})}^{\boxed{=\ 0,\ \text{because } \ell_{tot} \text{ does not change with time}}} = 2\dot{y}_D + \dot{y}_B \quad \Rightarrow \quad \dot{y}_D = -\frac{1}{2}\dot{y}_B \tag{12.12}$$

$$\Rightarrow \quad \ddot{y}_D = -\frac{1}{2}\ddot{y}_B. \tag{12.13}$$

$$\begin{aligned} \text{But}\quad & y_D = y_A - \text{AD and AD = constant} \\ \Rightarrow\quad & \dot{y}_D = \dot{y}_A \quad \text{and} \quad \ddot{y}_D = \ddot{y}_A. \end{aligned}$$

Thus, substituting $\dot{y}_A$ and $\ddot{y}_A$ for $\dot{y}_D$ and $\ddot{y}_D$ in (12.12) and (12.13) we get

$$\dot{y}_A = -\frac{1}{2}\dot{y}_B \quad \text{and} \quad \ddot{y}_A = -\frac{1}{2}\ddot{y}_B$$

$$\boxed{\ddot{y}_A = -\tfrac{1}{2}\ddot{y}_B}$$

SAMPLE 12.4 **A two-mass pulley system.** The two masses shown in Fig. 12.14 have frictionless bases and round frictionless pulleys. The inextensible cord connecting them is always taut. Given that $F = 130\,\mathrm{N}$, $m_A = m_B = m = 40\,\mathrm{kg}$, find the acceleration of the two blocks using:

1. linear momentum balance and
2. energy balance.

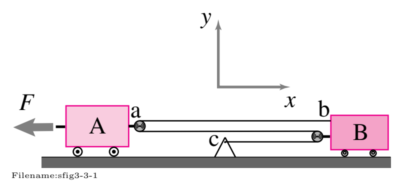

Figure 12.14: A two-mass pulley system.

Solution

1. **Using Linear Momentum Balance:** The free-body diagrams of the two masses A and B are shown in Fig. 12.15.

Linear momentum balance for mass A gives (assuming $\vec{a}_A = a_A\hat{\imath}$ and $\vec{a}_B = a_B\hat{\imath}$):

$$\begin{aligned} (2T - F)\hat{\imath} + (2N_A - mg)\hat{\jmath} &= m\vec{a}_A = -ma_A\hat{\imath} \\ \text{(dotting with } \hat{\jmath}) \quad \Rightarrow \quad 2N_A &= mg \\ \text{(dotting with } \hat{\imath}) \quad \Rightarrow \quad 2T - F &= ma_A \end{aligned} \tag{12.14}$$

Similarly, linear momentum balance for mass B gives:

$$\begin{aligned} -3T\hat{\imath} + (2N_B - mg)\hat{\jmath} &= m\vec{a}_B = ma_B\hat{\imath} \\ \Rightarrow \quad 2N_B &= mg \\ \text{and} \quad -3T &= ma_B. \end{aligned} \tag{12.15}$$

From (12.14) and (12.15) we have three unknowns: T, a_A, a_B, but only 2 equations!. We need an extra equation to solve for the three unknowns.⑤

We can get the extra equation from kinematics. Since A and B are connected by a string of fixed length, their accelerations must be related. For simplicity, and since these terms drop out anyway, we neglect the radius of the pulleys and the lengths of the little connecting cords. We use the left wall as the reference position to get

$$\begin{aligned} \ell_{tot} &\equiv \text{length of the string connecting A and B} \\ &= (3x_B - X_C) + 2(X_C - x_A) \\ \Rightarrow \quad \overbrace{\dot{\ell}_{tot}}^{0} &= 3\dot{x}_B + 2(-\dot{x}_A) \\ \Rightarrow \quad \dot{x}_B &= -\frac{2}{3}(-\dot{x}_A) \\ \Rightarrow \quad \ddot{x}_B &= \frac{2}{3}\ddot{x}_A \end{aligned} \tag{12.16}$$

(12.17)

Since

$$\begin{aligned} \vec{v}_A &= v_A\hat{\imath} = -\dot{x}_A\hat{\imath}, \\ \vec{a}_A &= a_A\hat{\imath} = \ddot{x}_A\hat{\imath}, \\ \vec{v}_B &= v_B\hat{\imath} = \dot{x}_B\hat{\imath}, \text{ and} \\ \vec{a}_B &= a_B\hat{\imath} = \ddot{x}_B\hat{\imath}, \end{aligned}$$

we get

$$a_B = \frac{2}{3}a_A. \tag{12.18}$$

Substituting (12.18) into (12.15), we get

$$9T = -2m_B a_A. \tag{12.19}$$

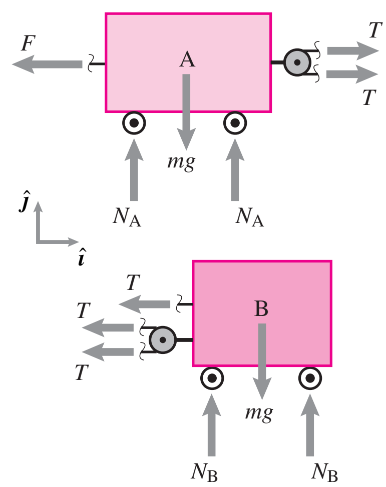

Figure 12.15: Free body diagrams of the two masses.

⑤ You may be tempted to use angular momentum balance (AMB) to get an extra equation. In this case AMB could help determine the vertical reactions, but offers no help in finding the rope tension or the accelerations.

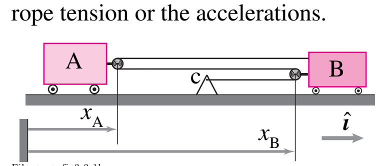

Figure 12.16: Pulley kinematics. The left wall is added as a reference point, off to the side of all pulleys and masses, to help prevent sign errors.

Now solving (12.14) and (12.19) for T, we get

$$T = \frac{2F}{13} = \frac{2 \cdot 130\,\mathrm{N}}{13} = 20\,\mathrm{N}.$$

Therefore,

$$\begin{aligned} a_A &= -\frac{9T}{2m} = -\frac{9 \cdot 20\,\mathrm{N}}{2 \cdot 40\,\mathrm{kg}} = -2.25\,\mathrm{m/s^2} \\ a_B &= \frac{2}{3} a_A = -1.5\,\mathrm{m/s^2} \end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_A = -2.25\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{a}}_B = -1.5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}.}$$

2. **Using Power Balance (III):** We have,

$$P = \dot{E}_\mathrm{K}.$$

The power balance equation becomes

$$\sum \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = m\, a_A\, v_A + m_B\, a_B\, v_B.$$

Because the force at A is the only force that does work on the system, ⑥ when we apply power balance to the whole system (see the FBD in fig. 12.17), we get,

$$\begin{aligned} -F v_A - T \overbrace{v_q}^{=v_c=0} &= m_A v_A a_A + m_B v_B a_B \\ \text{or} \qquad F &= -m a_A - m \frac{v_B}{v_A} a_B \\ &= -a_A \left(m + m \frac{v_B}{v_A} \frac{a_B}{a_A}\right). \end{aligned}$$

Substituting $a_B = 2/3 a_A$ and $v_B = 2/3 v_A$ from eqn. (12.18),

$$\begin{aligned} a_A &= \frac{-F}{m + \frac{4}{9}m} \\ &= \frac{-130\,\mathrm{N}}{40\,\mathrm{kg}(1 + \frac{4}{9})} \\ &= -2.25\,\mathrm{m/s^2}, \end{aligned}$$

and since $a_B = 2/3 a_A$,

$$a_B = -1.5\,\mathrm{m/s^2},$$

which are the same accelerations as found before.

$$\boxed{a_A = -2.25\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}},\ a_B = -1.5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}}$$

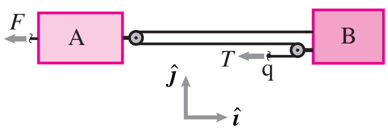

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Figure 12.17: 1D free body diagram of the whole system. Note that except F, no other forces do any work.

⑥It may not be obvious why T shown in the FBD in fig. 12.17 does no work. T acts on a material point q' somewhere on the inextensible string from the fixed point 'c'. Even though mass B moves, point 'q' on the string has no displacement. Because the material point q on which it acts does not move, the work done by T is zero.

SAMPLE 12.5 In static equilibrium the spring in fig. 12.18 is compressed by y_s from its unstretched length ℓ_0. Now, the spring is compressed by an additional amount y_0 and released with no initial velocity.

1. Find the force on the top mass m exerted by the lower mass M.
2. When does this force become minimum? Can this force become zero?
3. Can the force on m due to M ever be negative?

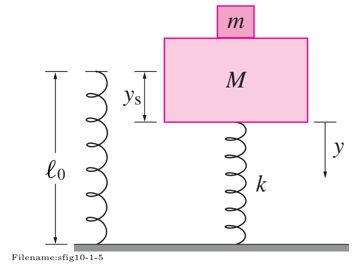

Figure 12.18:

Solution

1. The free body diagram of the two masses is shown in Figure 12.19 when the system is in static equilibrium. From linear momentum balance we have

$$\sum \vec{F} = \vec{0} \quad \Rightarrow \quad k y_s = (m + M) g. \tag{12.20}$$

The free body diagrams of the two masses at an arbitrary position y during motion are shown in Figure 12.20. Since the two masses oscillate together, they have the same acceleration. From linear momentum balance for mass m we get (note that we have chosen y to be positive downwards),

$$mg - N = m\ddot{y}. \tag{12.21}$$

We are interested in finding the normal force N. Clearly, we need to find $\ddot{y}$ to calculate N. Now, from linear momentum balance for mass M we get

$$Mg + N - k(y + y_s) = M\ddot{y}. \tag{12.22}$$

Adding eqn. (12.21) with eqn. (12.22) we get

$$(m + M)g - ky - ky_s = (m + M)\ddot{y}.$$

But $ky_s = (m + M)g$ from eqn. (12.20). Therefore, the equation of motion of the system is

$$\begin{aligned} -ky &= (m + M)\ddot{y} \\ \text{or} \quad \ddot{y} + \frac{k}{(m + M)} y &= 0. \end{aligned} \tag{12.23}$$

As you recall from your study of the harmonic oscillator, the general solution of this differential equation is

$$y(t) = A \sin \lambda t + B \cos \lambda t \tag{12.24}$$

$$\text{where} \quad \lambda = \sqrt{\frac{k}{m + M}}. \tag{12.25}$$

The constants A and B are to be determined from the initial conditions. From eqn. (12.24) we obtain

$$\dot{y}(t) = A\lambda \cos \lambda t - B\lambda \sin \lambda t. \tag{12.26}$$

Substituting the given initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$ in eqns. (12.24) and (12.26), respectively, we get

$$\begin{aligned} \overbrace{y(0)}^{y_0} &= A \sin(\lambda \cdot 0) + B \cos(\lambda \cdot 0) \quad \Rightarrow \quad B = y_0 \\ \underbrace{\dot{y}(0)}_{0} &= A\lambda \cos(\lambda \cdot 0) - B\lambda \sin(\lambda \cdot 0) \quad \Rightarrow \quad A = 0. \end{aligned}$$

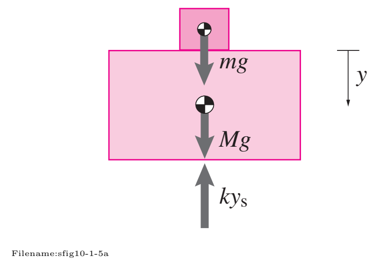

Figure 12.19: Free body diagram of the two masses as one system in static equilibrium (this special case could be skipped as it follows from the free body diagram below).

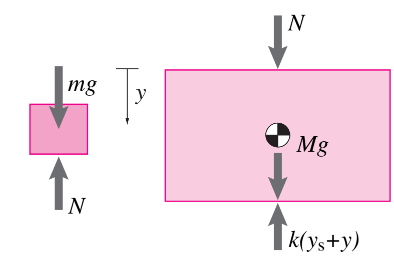

Figure 12.20: Free body diagrams of the individual masses.

Thus,

$$y(t) = y_0 \cos \lambda t. \tag{12.27}$$

Now we can find the acceleration by differentiating eqn. (12.27) twice :

$$\ddot{y} = -y_0 \lambda^2 \cos \lambda t.$$

Substituting this expression in eqn. (12.21) we get the force applied by mass M on the smaller mass m:

$$\begin{aligned} mg - N &= m \overbrace{(-y_0 \lambda^2 \cos \lambda t)}^{\ddot{y}} \\ \Rightarrow \quad N &= mg + m y_0 \lambda^2 \cos \lambda t \\ &= mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t \right) \end{aligned} \tag{12.28}$$

$$\boxed{N = mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t \right)}$$

2. Since $\cos \lambda t$ varies between ± 1, the value of the force N varies between $mg \pm y_0 \lambda^2$. Clearly, N attains its minimum value when $\cos \lambda t = -1$, *i.e.,* when $\lambda t = \pi$. This condition is met when the spring is fully stretched and the mass is at its highest vertical position. At this point,

$$N \equiv N_{\text{min}} = mg \left(1 - \frac{y_0 \lambda^2}{g} \right).$$

If y_0, the initial displacement from the static equilibrium position, is chosen such that $y_0 \lambda^2 = g$ (that is, the amplitude of the harmonically varying acceleration equals g), then $N = 0$ when $\cos \lambda t = -1$, *i.e.,* at the topmost point in the vertical motion. This condition, $N = 0$, means that the two masses momentarily lose contact with each other; and it happens precisely when they are about to begin their downward motion.

3. From eqn. (12.28) we can get a negative value of N when $\cos \lambda t = -1$ and $y_0 \lambda^2 > g$. However, a negative value for N is nonsense unless the blocks are glued. Without glue the bigger mass M cannot apply a negative force (or a compression) on m, *i.e.,* it cannot "suck" m. When $y_0 \lambda^2 > g$ then N becomes zero before $\cos \lambda t$ decreases to -1. That is, assuming no bonding, the two masses lose contact on their way to the highest vertical position but before reaching the highest point. Beyond that point, the equations of motion derived above are no longer valid for unglued blocks because the equations assume contact between m and M. Equation (12.28) is inapplicable when $N \leq 0$.

12.2 1D motion with 2D and 3D forces

Even if all the motion is in a single direction, an engineer may still have to consider two- or three-dimensional forces.

Example: Piston in a cylinder.
A piston slides vertically in a cylinder with coefficient of friction μ between the piston and the cylinder wall. Assune the connecting rod has negligible mass so it can be treated as a two-force member as discussed in section 4.2b. The free body diagram of the piston (with a bit of the connecting rod) is shown in fig. 12.21. If the piston is moving up so the friction force resists the motion and points down. Linear momentum balance for this system is:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ -N\hat{\imath} - \mu N\hat{\jmath} + T\hat{\boldsymbol{\lambda}}_{rod} &= m_{piston}\, a\hat{\jmath}. \end{aligned}$$

If we assume that the acceleration $a\hat{\jmath}$ of the piston is known, as is its mass m_{piston}, the coefficient of friction μ, and the orientation of the connecting rod $\hat{\boldsymbol{\lambda}}_{rod}$, then we can solve for the rod tension T and the normal reaction N.

Note: *even though the piston moves in one direction, the momentum balance equation is a two-dimensional vector equation.*

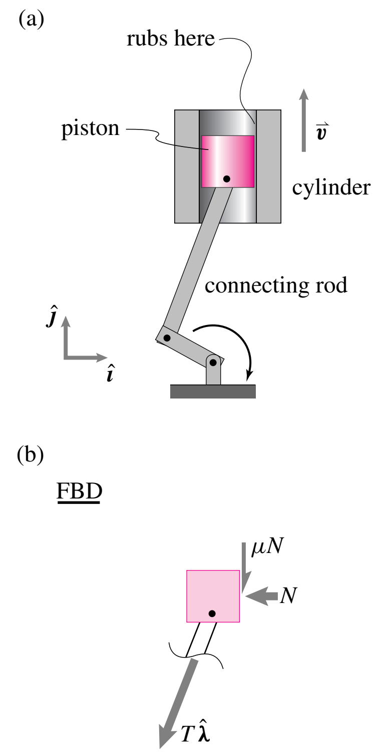

Figure 12.21: (a) shows a piston in a cylinder. (b) shows a free body diagram of the piston. To draw this FBD, we have assumed: (1) a coefficient of friction μ between the piston and cylinder wall, and (2) negligible mass for the connecting rod, and (3) ignored the spatial extent of the cylinder.

Unlike the 1D mechanics of the previous chapter, in this section on 1D motion, the momentum balance equations are 2D and 3D vector equations. Compared to more general 2D and 3D motion, the 1-D motions we assume in this chapter allow an easy introduction to 2D and 3D dynamics calculations.

Highly constrained bodies

This chapter is about rigid objects that move in straight lines. Most objects will not agree to be the topic of such discussion without being forced into doing so. Without being held in place they would rotate and move in a curvy way. To keep an object that is subject to various forces from rotating or curving takes some constraint by wires, rods, rails, hinges, welds, *etc.*. Of course the presence of constraint is not always associated with the disallowance of rotation — constraints could even cause rotation. But in this chapter, constraints keep a rigid object in straight-line motion.

Constraint forces are of interest

Of common interest is making sure that static and dynamic loads do not cause failure of parts that enforce constraints. For example, suppose a truck hauls a very heavy load that is held down by chains or straps. When the truck accelerates, what is the tension in the chains, and will it exceed their strength limits?

1D mechanics with 1D forces, moving on

This all is in contrasts with the situation in 1D ”unconstrained” dynamics of the previous chapter. For one-dimensional mechanics we assumed that everything of interest mechanically happened in, say, the $\hat{\imath}$ (x) direction. That is, we ignored *all* torques and angular momenta, and only consider the $\hat{\imath}$

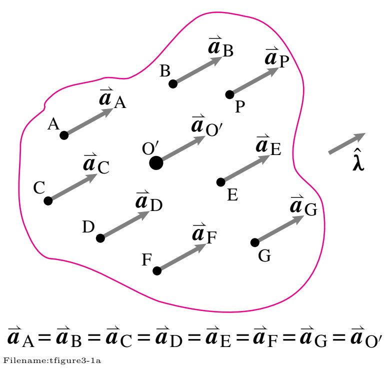

Figure 12.22: Parallel motion: all points on the body have the same acceleration $\vec{a} = a\hat{\boldsymbol{\lambda}}$. For straight-line motion: $\hat{\boldsymbol{\lambda}}(t)$=constant in time and $\vec{v} = v\hat{\boldsymbol{\lambda}}$.

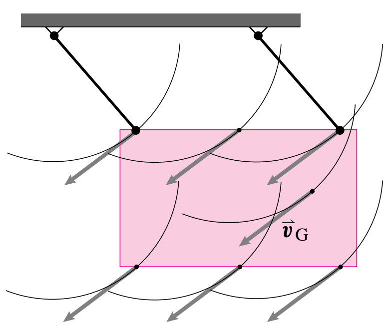

Figure 12.23: A swing showing instantaneous parallel motion which is *curvilinear*. At every instant, each point has the same velocity as the others, but the motion is not in a straight line.

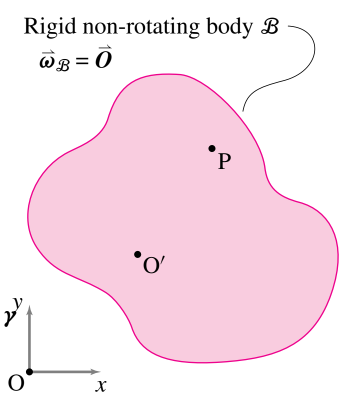

Figure 12.24: A non-rotating body $\mathcal{B}$ with points O' and P.

components of the forces (*i.e.*, $\vec{\boldsymbol{F}} \cdot \hat{\imath}$) and linear momentum ($\vec{\boldsymbol{L}} \cdot \hat{\imath}$), namely F_x and L_x.

Kinematics of parallel motion and straight line motion

Let's consider a set of points in the system of interest. Let's call them A to G, or generically, P. For convenience we distinguish a reference point O'. O' may be the center-of-mass, the origin of a local coordinate system, or a fleck of dirt that serves as a marker. By *parallel motion*, we mean that the system happens to move in such a way that $\vec{a}_P = \vec{a}_{O'}$, and $\vec{v}_P = \vec{v}_{O'}$ (fig. 12.22). That is,

$$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_P = \vec{a}_{O'}$$

at every instant in time. We also assume that $\vec{v}_A = \ldots = \vec{v}_P = \vec{v}_{O'}$.

A special case of parallel motion is straight-line motion.

a system moves with straight-line motion if it moves like a non-rotating rigid body, in a straight line.

For straight-line motion, the velocity of the body is in a fixed unchanging direction. If we call a unit vector in that direction $\hat{\boldsymbol{\lambda}}$, then we have

$$\vec{v}(t) = v(t)\hat{\boldsymbol{\lambda}}, \qquad \vec{a}(t) = a(t)\hat{\boldsymbol{\lambda}} \qquad \text{and} \qquad \vec{r}(t) = \vec{r}_0 + s(t)\hat{\boldsymbol{\lambda}}$$

for every point in the system. $\vec{r}_0$ is the position of a point at time 0 and s is the distance the point moves in the $\hat{\boldsymbol{\lambda}}$ direction. Every point in the system has the same s, v, a, and $\hat{\boldsymbol{\lambda}}$ as the other points. There are a variety of problems of practical interest that can be idealized as fitting into this class, notably, the motions of things constrained to move on belts, roads, and rails, like the train on a straight track.

Example: Parallel swing is not straight-line motion

The swing shown does not rotate — all points on the swing have the same velocity. The velocity of all particles are parallel but, since paths are curved, this motion is not straight-line motion. Such curvilinear parallel motion will be discussed later in the book.

Velocity of a point

The velocity of any point P on a non-rotating rigid body (such as for straight-line motion) is the same as that of any reference point on the body (see Fig. 12.24).

$$\vec{v}_P = \vec{v}_{O'}$$

A more general case, which you will learn in later chapters, is shown as 5b in Table II at the back of the book. This formula concerns rotational rate which we will measure with the vector $\vec{\omega}$. For now all you need to know is that $\vec{\omega} = \vec{0}$ when something is not rotating. In 5b in Table II, if you set $\vec{\omega}_{\mathcal{B}} = 0$ and $\vec{v}_{P/\mathcal{B}} = \vec{0}$ it says that $\vec{v}_P = \dot{\vec{r}}_{O'/O}$ or in shorthand, $\vec{v}_P = \vec{v}_{O'}$, as we have written above.

Acceleration of a point

Similarly, the acceleration of every point on a non-rotating rigid body is the same as every other point. The more general case, not needed in this chapter, is shown as entry 5c in Table II at the back of the book.

General results

Before we proceed with discussion of the details of the mechanics of straight-line motion we present some ideas that are also more generally applicable. That is, the concept of the center-of-mass allows some useful simplifications of the general expressions for $\vec{\boldsymbol{L}}$, $\dot{\vec{\boldsymbol{L}}}$, $\vec{\boldsymbol{H}}_{/\mathrm{C}}$, $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ and E_{K}.

Linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$

Although we are dealing with zillions of atoms in a given object, the linear momentum and angular momentum are simple to evaluate:

$$\vec{\boldsymbol{L}} = m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}}.$$

Actually, as the front inside cover states, these formulas are good for any motion of any system. The nice simplification for the straight-line motion of this chapter is that all points on a given object have the same velocity and acceleration. So we don't need to find or track the center of mass, but can track the motion of any point on the object.

Angular momentum $\vec{H}_{/\mathrm{C}}$ and its rate of change, $\dot{\vec{H}}_{C}$ for straight-line motion

For the motions in this chapter, where $\vec{\boldsymbol{a}}_i = \vec{\boldsymbol{a}}_{cm}$ and thus $\vec{\boldsymbol{a}}_{i/cm} = \vec{\boldsymbol{0}}$, angular momentum considerations are simplified, as explained in Box 12.1 on page 621①.

$$\vec{\boldsymbol{H}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times \vec{\boldsymbol{v}}_{\mathrm{cm}} m_{\mathrm{tot}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times \vec{\boldsymbol{a}}_{\mathrm{cm}} m_{\mathrm{tot}}$$

① Calculating rate of change of angular momentum will get more difficult as the book progresses. For a rigid body $\mathcal{B}$ in more general motion, the calculation of rate of change of angular momentum involves the angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$, its rate of change $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}}$, and the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{cm}}]$. If you look in the back of the book at Table I, entries 6c and 6d, you will see formulas that reduce to the formulas below if you assume no rotation and thus use $\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{0}}$ and $\dot{\vec{\boldsymbol{\omega}}} = \vec{\boldsymbol{0}}$. On the other hand, rate of change of linear momentum is simple, at least in concept, in this chapter, as well as in the rest of this book. For all motions of all systems we have

$$\dot{\vec{\boldsymbol{L}}} = m_{tot}\vec{\boldsymbol{a}}_{cm}.$$

12.1 Angular momentum for straight-line motion

For straight-line motion, and parallel motion in general, we can derive the simplification in the calculation of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{C}} &\equiv \sum \vec{\boldsymbol{r}}_{i\,/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{v}}_i \text{(definition)} \\
&= \sum \vec{\boldsymbol{r}}_{i\,/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{v}}_{\mathrm{cm}} \quad \text{(since, } \vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{\mathrm{cm}}\text{)} \\
&= \left(\sum \vec{\boldsymbol{r}}_{i\,/\mathrm{C}}\, \mathrm{m}_i\right) \times \vec{\boldsymbol{v}}_{\mathrm{cm}}, \\
&= \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times (m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{\mathrm{cm}}), \\
&\quad \text{(since, } \textstyle\sum \vec{\boldsymbol{r}}_{i\,/\mathrm{C}}\, \mathrm{m}_i \equiv m_{\mathrm{tot}} \vec{\boldsymbol{r}}_{\mathrm{cm/C}}\text{)}.
\end{aligned}$$

The derivation that $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times (m\vec{\boldsymbol{a}}_{\mathrm{cm}})$ follows from $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \equiv \sum \vec{\boldsymbol{r}}_{i\,/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{a}}_i$ by the same reasoning.

②**Caution:** The special motions in this chapter are almost the only cases where the angular momentum and its rate of change are so easy to calculate.

But for straight-line motion (and, slightly more generally, for any parallel motion), the calculations turn out to be the same as we would get if we put a single point mass at the center-of-mass②:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{C}} &\equiv \sum(\vec{\boldsymbol{r}}_{i/\mathrm{C}} \times m_i \vec{\boldsymbol{v}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\mathrm{total}} \vec{\boldsymbol{v}}_{\mathrm{cm}}),\\
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} &\equiv \sum(\vec{\boldsymbol{r}}_{i/\mathrm{C}} \times m_i \vec{\boldsymbol{a}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\mathrm{total}} \vec{\boldsymbol{a}}_{cm}).
\end{aligned}$$

Note, there is some subtlety in the definition of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$, as explained in section C.

Kinetic energy

Generally things will not be so simple, but for straight-line motion, or any parallel motion where all points on an object have the same velocity and acceleration, kinetic energy and its rate of change are also easy to calculate:

$$E_{\mathrm{K}} = m_{\mathrm{tot}} v_{\mathrm{cm}}^2/2 \qquad \text{and} \qquad \dot{E}_{\mathrm{K}} = m_{\mathrm{tot}} v_{\mathrm{cm}} a_{\mathrm{cm}}.$$

The kinetic energy works the same as if all the mass was concentrated at the center of mass. This result does not generalize to more complex motions.

Approach

To study systems in straight-line motion (as always) we:

- draw a free body diagram, showing the appropriate forces and couples at places where connections are ‘cut’,
- state reasonable kinematic assumptions based on the motions that the constraints allow,
- write linear and/or angular momentum balance equations and/or energy balance, and
- solve for quantities of interest.

Angular momentum balance about a judiciously chosen axis is a particularly useful tool for reducing the number of equations that need to be solved.

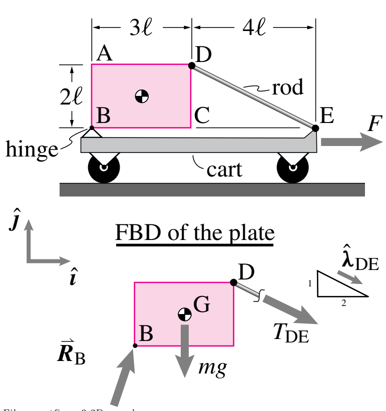

Figure 12.25: Uniform plate supported by a hinge and a rod on an accelerating cart.

Example: Plate on a cart

A uniform rectangular plate $ABCD$ of mass m is supported by a light rigid rod DE and a hinge joint at point B. The dimensions are as shown. The cart has acceleration $a_x\hat{\imath}$ due to a force $F\hat{\imath}$ and the constraints of the wheels. Referring to the free body diagram in fig. 12.25 and writing angular momentum balance for the plate about point B, we can get an equation for the tension in the rod T_{DE} in terms of m and a_x:

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/B} &= \dot{\vec{\boldsymbol{H}}}_{/B}\\
\{\vec{\boldsymbol{r}}_{D/B} \times (T_{DE}\hat{\boldsymbol{\lambda}}_{DE}) + \vec{\boldsymbol{r}}_{G/B} \times (-mg\hat{\boldsymbol{\jmath}}) &= \vec{\boldsymbol{r}}_{G/B} \times (ma_x\hat{\boldsymbol{\imath}})\}\\
\{\} \cdot \hat{\boldsymbol{k}} \Rightarrow \qquad T_{DE} &= \frac{\sqrt{5}}{7} m(a_x - \frac{3}{2}g).
\end{aligned}$$

Summarizing note:

angular momentum balance is important even when there is no rotation.

Sliding and pseudo-sliding objects

A car coming to a stop can be roughly modeled as a rigid body that translates and does not rotate. That is, at least for a first approximation, the rotation of the car due to the suspension and tire deformation, can be neglected. The free body diagram will show various forces with lines of action that do not all act through a single point so that angular momentum balance must be used to analyze the system. Similarly, a bicycle which is braking or a box that is skidding (if not tipping) may be analyzed by assuming straight-line motion.

Example: Car skidding
Consider the accelerating four-wheel drive car in fig. 12.26. The motion quantities for the car are $\dot{\vec{L}} = m_{car}\vec{a}_{car}$ and $\dot{\vec{H}}_{/C} = \vec{r}_{cm/C} \times \vec{a}_{car}m_{car}$. We could calculate angular momentum balance relative to the car's center of mass in which case $\sum \vec{M}_{cm} = \dot{\vec{H}}_{cm} = \vec{0}$ (because the position of the center-of-mass relative to the center-of-mass is $\vec{0}$).

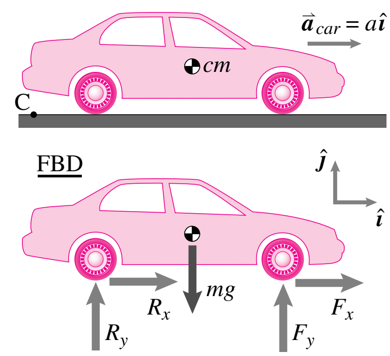

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Figure 12.26: A four-wheel drive car accelerating but not tipping. See fig. 3.35 on page 176 for more about FBDs involving wheel contact.

As mentioned, it is often useful to calculate angular momentum balance of sliding objects about points of contact (such as where tires contact the road) or about points that lie on lines of action of applied forces when writing angular momentum balance to solve for forces or accelerations. To do so usually eliminates some unknown reactions from the equations to be solved. For example, the angular momentum balance equation about the rear-wheel contact of a car does not contain the rear-wheel contact forces.

Wheels

The function of wheels is to allow easy sliding-like (pseudo-sliding) motion between objects, at least in the direction they are pointed. On the other hand, wheels do sometimes slip due to:

- being overpowered (as in a screeching accelerating car),
- being braked hard, or
- having very bad bearings (like a rusty toy car).

How wheels are treated when analyzing cars, bikes, and the like depends on both the application and on the level of detail one requires. In *this chapter*, we will always assume that wheels have negligible mass. Thus, when we treat the special case of un-driven and un-braked wheels our free body diagrams will be as in fig. 3.36a on page 176 and *not* like the one in fig. 3.36b. With the ideal wheel approximation, all of the various cases for a car traveling to the right are shown with partial free body diagrams of a wheel in fig. 3.35. For the purposes of actually solving problems, we have accepted Coulomb's law of friction as a model for contacting interaction (see page 172 in sec. 3.2).

3-D forces in straight-line motion

The ideas we have discussed apply as well in three dimensions as in two. As you learned from doing statics problems, working out the details in 3D, where vector methods must be used carefully, is more involved than in 2D. As for statics, three dimensional problems often yield simple results and simple intuitions by considering angular momentum balance about an axis.

Angular momentum balance about an axis

The simplest way to think of angular momentum balance about an axis is to look at angular momentum balance about a point and then take a dot product with a unit vector along an axis:

$$\hat{\boldsymbol{\lambda}} \cdot \left\{ \sum \vec{\boldsymbol{M}}_{/\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \right\}.$$

Note that the axis need not correspond to any mechanical device in any way resembling an axle. The equation above applies for any point C and any vector $\hat{\boldsymbol{\lambda}}$. If you choose C and $\hat{\boldsymbol{\lambda}}$ judiciously many terms in your equations may drop out.

SAMPLE 12.6 Force in braking. A front-wheel-drive car of mass $m = 1200\,\text{kg}$ is cruising at $v = 60\,\text{mph}$ on a straight road when the driver slams on the brake. The car slows down to 20 mph in 4 s while maintaining its straight path.

1. What is the average force (average in time) applied on the car during braking?
2. What is the average power of braking?

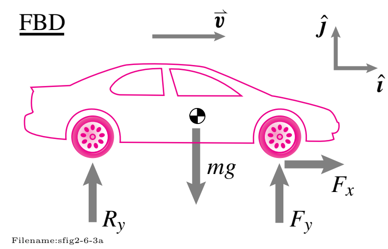

Figure 12.27: Free body diagram of a front-wheel-drive car during braking. Note that we have (arbitrarily) pointed F_x to the right. The algebra in this problem will tell us that $F_x < 0$.

Solution

1. Let us assume that we have an xy coordinate system in which the car is traveling along the x-axis during the entire time under consideration. Then, the velocity of the car before braking, $\vec{v}_1$, and after braking, $\vec{v}_2$, are

$$\vec{v}_1 = v_1\hat{\imath} = 60\,\text{mph}\,\hat{\imath} \quad \text{and} \quad \vec{v}_2 = v_2\hat{\imath} = 20\,\text{mph}\,\hat{\imath}.$$

The linear impulse during braking is $\vec{F}_{\text{ave}}\Delta t$ where $\vec{F} \equiv F_x\hat{\imath}$ (see free body diagram of the car). Now, from the impulse-momentum relationship,

$$\vec{F}\Delta t = \vec{L}_2 - \vec{L}_1,$$

where $\vec{L}_1$ and $\vec{L}_2$ are linear momenta of the car before and after braking, respectively, and $\vec{F}$ is the average applied force. Therefore,

$$\begin{aligned}
\vec{F} &= \frac{1}{\Delta t}(\vec{L}_2 - \vec{L}_1) = \frac{m}{\Delta t}(\vec{v}_2 - \vec{v}_1)\\
&= \frac{1200\,\text{kg}}{4\,\text{s}}(20-60)\,\text{mph}\hat{\imath}\\
&= -12000\frac{\text{kg}}{\text{s}}\cdot\frac{\cancel{\text{mi}}}{\cancel{\text{hr}}}\cdot\frac{1600\ \text{m}}{1\ \cancel{\text{mi}}}\cdot\frac{1\ \cancel{\text{hr}}}{3600\,\text{s}}\hat{\imath}\\
&= -\frac{16,000}{3}\,\text{kg}\cdot\text{m/s}^2\hat{\imath} = -5.33\,\text{kN}\hat{\imath}.
\end{aligned}$$

Thus

$$F_x\hat{\imath} = -5.33\,\text{kN}\hat{\imath} \quad \Rightarrow \quad F_x = -5.33\,\text{kN}.$$

$$\boxed{F_x = -5.33\,\text{kN}}$$

2. Let the average power during breaking be P_{ave}. Then the work done during breaking is $W = \int P_{\text{ave}}dt$. From work-energy principle, we have

$$\begin{aligned}
W &= \Delta E_K\\
\int_{t_1}^{t_2} P_{\text{ave}}dt &= \frac{1}{2}m(v_2^2 - v_1^2)\\
P_{\text{ave}}(t_2 - t_1) &= \frac{1}{2}m(v_2^2 - v_1^2)\\
P_{\text{ave}} &= \frac{m}{2\Delta t}(v_2^2 - v_1^2)
\end{aligned}$$

Substituting $m = 1200\,\text{kg}$, $\Delta t = 4\,\text{s}$, $v_1 = 60\,\text{mph} = 26.67\,\text{m/s}$ and $v_2 = 20\,\text{mph} = 8.89\,\text{m/s}$, we get

$$P_{\text{ave}} = -94815\,\text{N·m/ s} = -94.815\,\text{kW}.$$

It is easy to check that if we take the average force F_{ave} calculated above and the average speed $v_{\text{ave}} = (v_1 + v_2)/2 = 40\,\text{mph} = 17.77\,\text{m/s}$, then

$$P_{\text{ave}} = F_{\text{ave}}v_{\text{ave}} = -5.33\,\text{kN}\cdot 17.77\,\text{m/s} = -94.815\,\text{kW},$$

as obtained above.

$$\boxed{P_{\text{ave}} = -94.815\,\text{kW}}$$

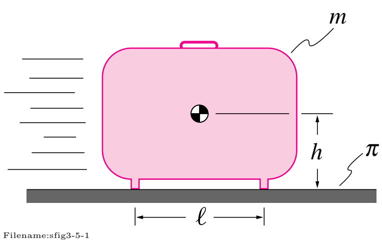

Figure 12.28: A suitcase in motion.

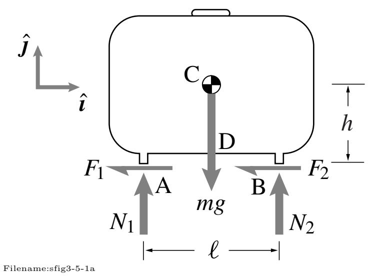

Figure 12.29: FBD of the suitcase.

SAMPLE 12.7 A suitcase skidding on frictional ground. A suitcase of mass m is pushed and sent sliding on a horizontal surface. The suitcase slides without any rotation. A and B are the only contact points of the suitcase with the ground. If the coefficient of friction between the suitcase and the ground is μ, find all the forces applied by the ground on the suitcase. Discuss the results obtained for normal forces.

Solution As usual, we first draw a free body diagram of the suitcase. The FBD is shown in Fig. 12.29. Assuming Coulomb's law of friction holds, we can write

$$\vec{F}_1 = -\mu N_1 \hat{\imath} \quad \text{and} \quad \vec{F}_2 = -\mu N_2 \hat{\imath}. \tag{12.29}$$

Now we write the balance of linear momentum for the suitcase:

$$\sum \vec{F} = m\vec{a}_{\text{cm}}$$

$$\Rightarrow \quad -(F_1 + F_2)\hat{\imath} + (N_1 + N_2 - mg)\hat{\jmath} = ma_C\hat{\imath} \tag{12.30}$$

where $\vec{a}_C = a_C\hat{\imath}$ is the unknown acceleration. Dotting eqn. (12.30) with $\hat{\imath}$ and $\hat{\jmath}$ and substituting for F_1 and F_2 from eqn. (12.29) we get

$$-\mu(N_1 + N_2) = ma_C \tag{12.31}$$

$$N_1 + N_2 = mg. \tag{12.32}$$

Equations (12.31) and (12.32) represent 2 scalar equations in three unknowns N_1, N_2 and a. Obviously, we need another equation to solve for these unknowns.

We can write the balance of angular momentum about any point. Points A or B are good choices because they each eliminate some reaction components. Let us write the balance of angular momentum about point A:

$$\sum \vec{M}_A = \dot{\vec{H}}_A$$

$$\begin{aligned} \sum \vec{M}_A &= \vec{r}_{B/A} \times N_2\hat{\jmath} + \vec{r}_{D/A} \times (-mg)\hat{\jmath} \\ &= \ell\hat{\imath} \times N_2\hat{\jmath} + \frac{\ell}{2}\hat{\imath} \times (-mg)\hat{\jmath} \\ &= (\ell N_2 - mg\frac{\ell}{2})\hat{k} \end{aligned} \tag{12.33}$$

and

$$\begin{aligned} \dot{\vec{H}}_A &= \vec{r}_{C/A} \times m\vec{a}_C \\ &= (\frac{\ell}{2}\hat{\imath} + h\hat{\jmath}) \times ma_C\hat{\imath} \\ &= -ma_C h\hat{k}. \end{aligned} \tag{12.34}$$

Equating (12.33) and (12.34) and dotting both sides with $\hat{k}$ we get the following third scalar equation:

$$\ell N_2 - mg\frac{\ell}{2} = -ma_C h. \tag{12.35}$$

Solving eqns. (12.31) and (12.32) for a we get

$$a_C = -\mu g$$

and substituting this value of a_C in eqn. (12.35) we get

$$\begin{aligned} N_2 &= \frac{m\mu gh + mg\ell/2}{\ell} \\ &= mg\left(\frac{1}{2} + \frac{h}{\ell}\mu\right). \end{aligned}$$

Substituting the value of N_2 in either of the equations (12.31) or (12.32) we get

$$N_1 = mg\left(\frac{1}{2} - \frac{h}{\ell}\mu\right).$$

$$N_1 = mg(\tfrac{1}{2} - \tfrac{h}{\ell}\mu),\ N_2 = mg(\tfrac{1}{2} + \tfrac{h}{\ell}\mu),\ f_1 = \mu N_1,\ f_2 = \mu N_2.$$

Discussion: From the expressions for N_1 and N_2 we see that

1. $N_1 = N_2 = \frac{1}{2}mg$ if $\mu = 0$ because without friction there is no deceleration. The problem becomes equivalent to a statics problem.
2. $N_1 = N_2 \approx \frac{1}{2}mg$ if $\ell >> h$. In this case, the moment produced by the friction forces is too small to cause a significant difference in the magnitudes of the normal forces. For example, take $\ell = 20h$ and calculate moment about the center-of-mass to convince yourself.

Graphically, N_1, N_2 and their difference $N_1 - N_2$ are shown in the plot below as a function of h/ℓ for a particular value of μ and mg. As the equations indicate, $N_1 - N_2$ increases steadily as h/ℓ increases, showing how the moment produced by the friction forces makes a bigger and bigger difference between N_1 and N_2 as this moment gets bigger.

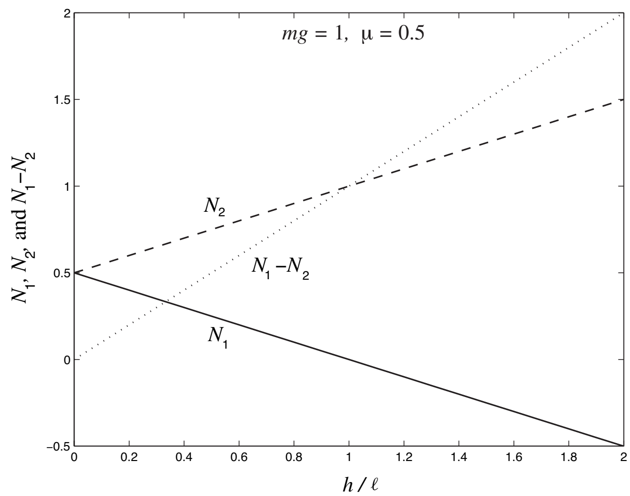

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Figure 12.30: The normal forces N_1 and N_2 differ from each other more and more as h/ℓ increases.

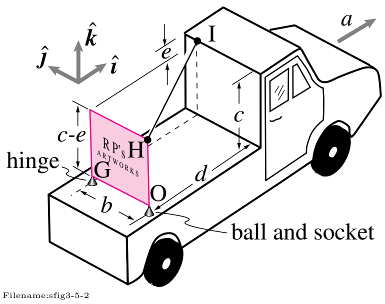


Figure 12.31: An accelerating board in 3-D

SAMPLE 12.8 Uniform acceleration of a board in 3-D. A uniform signboard of mass $m = 20\,\text{kg}$ sits in the back of an accelerating flatbed truck. The board is supported with a ball-and-socket joint at O and a hinge at G. A light rod from H to I keeps the board from falling over. The truck is on level ground and has forward acceleration $\vec{a} = 0.6\,\text{m/s}^2\hat{\boldsymbol{\imath}}$. The relevant dimensions are $b = 1.5\,\text{m}$, $c = 1.5\,\text{m}$, $d = 3\,\text{m}$, $e = 0.5\,\text{m}$. There is gravity ($g = 10\,\text{m/s}^2$).

1. Draw a free body diagram of the board.
2. Set up equations to solve for all the unknown forces shown on the FBD.
3. Use the balance of angular momentum about an axis to find the tension in the rod.

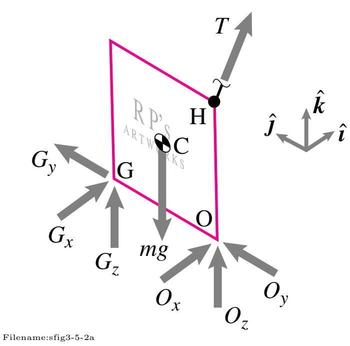


Figure 12.32: FBD of the board

Solution

1. The free body diagram of the board is shown in Fig. 12.32.
2. Linear momentum balance for the board:

$$\sum \vec{F} = m\vec{a}, \quad \text{or}$$

$$(G_x + O_x)\hat{\boldsymbol{\imath}} + (G_y + O_y)\hat{\boldsymbol{\jmath}} + (G_z + O_z - mg)\hat{\boldsymbol{k}} + T\hat{\boldsymbol{\lambda}}_{HI} = ma\hat{\boldsymbol{\imath}} \tag{12.36}$$

where

$$\hat{\boldsymbol{\lambda}}_{HI} = \frac{d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}}{\sqrt{d^2 + b^2 + e^2}} = \frac{d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}}{\ell},$$

and ℓ is the length of the rod HI.

Dotting eqn. (12.36) with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ we get the following three scalar equations:

$$G_x + O_x + T\frac{d}{\ell} = ma \tag{12.37}$$

$$G_y + O_y + T\frac{b}{\ell} = 0 \tag{12.38}$$

$$G_z + O_z + T\frac{e}{\ell} = mg \tag{12.39}$$

Angular momentum balance about point G:

$$\sum \vec{M}_G = \dot{\vec{H}}_G$$

$$\begin{aligned}
\sum \vec{M}_G &= \vec{r}_{C/G} \times (-mg\hat{\boldsymbol{k}}) + \vec{r}_{O/G} \times (O_x\hat{\boldsymbol{\imath}} + O_z\hat{\boldsymbol{k}}) + \vec{r}_{H/G} \times T\hat{\boldsymbol{\lambda}}_{HI} \\
&= (-\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}) \times (-mg\hat{\boldsymbol{k}}) - b\hat{\boldsymbol{\jmath}} \times (O_x\hat{\boldsymbol{\imath}} + O_z\hat{\boldsymbol{k}}) \\
&\quad + [-b\hat{\boldsymbol{\jmath}} + (c-e)\hat{\boldsymbol{k}}] \times \frac{T}{\ell}(d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}) \\
&= \left(\frac{b}{2}mg - bO_z - be\frac{T}{\ell} - (c-e)b\frac{T}{\ell}\right)\hat{\boldsymbol{\imath}} \\
&\quad + (c-e)d\frac{T}{\ell}\hat{\boldsymbol{\jmath}} + \left(bO_x + bd\frac{T}{\ell}\right)\hat{\boldsymbol{k}}
\end{aligned} \tag{12.40}$$

and

$$\begin{aligned}
\dot{\vec{H}}_G &= \vec{r}_{C/G} \times ma\hat{\boldsymbol{\imath}} \\
&= (-\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}) \times ma\hat{\boldsymbol{\imath}} \\
&= \frac{b}{2}ma\hat{\boldsymbol{k}} + \frac{c-e}{2}ma\hat{\boldsymbol{\jmath}}.
\end{aligned} \tag{12.41}$$

Equating (12.40) and (12.41) and dotting both sides with $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ we get the following three additional scalar equations:

$$O_z + \frac{c}{\ell}T = \frac{1}{2}mg \tag{12.42}$$

$$\frac{d}{\ell}T = \frac{1}{2}ma \tag{12.43}$$

$$O_x + \frac{d}{\ell}T = \frac{1}{2}ma \tag{12.44}$$

Now we have six scalar equations in seven unknowns — O_x, O_y, O_z, G_x, G_y, G_z, and T. From basic linear algebra, we know that we cannot find unique solutions for all these unknowns from the given equations. A closer inspection of eqns. (12.37–12.39) and (12.42–12.44) shows that we can easily solve for O_x, O_z, G_x, G_z, and T, but O_y and G_y cannot be determined uniquely because they appear together as the sum $G_y + O_y$. ③ Fortunately, we can find the tension in the wire HI without worrying about the values of O_y and G_y as we show below.

③ Note that G_y and O_y will always appear together as the sum $G_y + O_y$ even if you took the angular momentum balance about some other point. This is because they have the same line of action. Thus, they cannot be found independently. This mathematical problem corresponds to the physical reality that the supports at points O and G could be squeezing the plate along the line OG with, say, $O_y = 1000\,\text{N}$ and $G_y = -1000\,\text{N}$ even if there were no gravity, and the truck was not accelerating. To make prestress problems like this tractable, people often make assumptions like, 'Assume $G_y = 0$', that is, they try to get rid of the redundancy in supports to make the problem statically determinate.

3. Balance of angular momentum about axis OG gives:

$$\begin{aligned} \hat{\boldsymbol{\lambda}}_{OG} \cdot \sum \vec{\boldsymbol{M}}_G &= \hat{\boldsymbol{\lambda}}_{OG} \cdot \dot{\vec{\boldsymbol{H}}}_G \\ &= \hat{\boldsymbol{\lambda}}_{OG} \cdot (\vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}}). \end{aligned} \tag{12.45}$$

Since all reaction forces and the weight go through axis OG, they do not produce any moment about this axis (convince yourself that the forces from the reactions have no torque about the axis by calculation or geometry). Therefore,

$$\begin{aligned} \hat{\boldsymbol{\lambda}}_{OG} \cdot \sum \vec{\boldsymbol{M}}_G &= \hat{\boldsymbol{\jmath}} \cdot (\vec{\boldsymbol{r}}_{H/G} \times T\hat{\boldsymbol{\lambda}}_{HI}) \\ &= T\frac{d(c-e)}{\ell}. \end{aligned} \tag{12.46}$$

$$\begin{aligned} \hat{\boldsymbol{\lambda}}_{OG} \cdot (\vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}}) &= \hat{\boldsymbol{\jmath}} \cdot \left[\left(\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}\right) \times ma\hat{\boldsymbol{\imath}}\right] \\ &= ma\frac{(c-e)}{2}. \end{aligned} \tag{12.47}$$

Equating (12.46) and (12.47), as required by eqn. (12.45), we get

$$\begin{aligned} T &= \frac{ma\ell}{2d} \\ &= \frac{20\,\text{kg} \cdot 0.6\,\text{m/s}^2 \cdot 3.39\,\text{m}}{2 \cdot 3\,\text{m}} \\ &= 6.78\,\text{N}. \end{aligned}$$

$$\boxed{T_{HI} = 6.78\,\text{N}}$$

SAMPLE 12.9 Computer solution of algebraic equations. In the previous sample problem (Sample 12.8), six equations were obtained to solve for the six unknown forces (assuming $G_y = 0$). (i) Set up the six equations in matrix form and (ii) solve the matrix equation on a computer. Check the solution by substituting the values obtained in one or two equations.

Solution

1. The six scalar equations — (12.37), (12.38), (12.39), (12.42), (12.43), and (12.44) are amenable to hand calculations. We, however, set up these equations in matrix form and solve the matrix equation on the computer. The matrix form of the equations is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{d}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{b}{\ell} \\ 0 & 0 & 1 & 0 & 1 & \frac{e}{\ell} \\ 0 & 0 & 1 & 0 & 0 & \frac{c}{\ell} \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \\ 1 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \end{bmatrix} \begin{Bmatrix} O_x \\ O_y \\ O_z \\ G_x \\ G_z \\ T \end{Bmatrix} = \begin{Bmatrix} ma \\ 0 \\ mg \\ mg/2 \\ ma/2 \\ ma/2 \end{Bmatrix}. \tag{12.48}$$

The above equation can be written, in matrix notation, as

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

where **A** is the coefficient matrix, **x** is the vector of the unknown forces, and **b** is the vector on the right hand side of the equation. Now we are ready to solve the system of equations on the computer.

2. We use the following pseudo-code to solve the above matrix equation. ④

```
m = 20, a = 0.6,
b = 1.5, c = 1.5, d = 3, e = 0.5, g = 10,
l = sqrt(b^2 + d^2 + e^2),

A = [1 0 0 1 0 d/l
     0 1 0 0 0 b/l
     0 0 1 0 1 e/l
     0 0 1 0 0 c/l
     0 0 0 0 0 d/l
     1 0 0 0 0 d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =                  % this is the computer output
          0
   -3.0000
   97.0000
    6.0000
  102.0000
    6.7823
```

The solution obtained from the computer means:

$$O_x = 0,\ O_y = -3\,\mathrm{N},\ O_z = 97\,\mathrm{N},\ G_x = 6\,\mathrm{N},\ G_z = 102\,\mathrm{N},\ T = 6.78\,\mathrm{N}.$$

④ Be careful with units. Most computer programs will not take care of your units. They only deal with numerical input and output. You should, therefore, make sure that your variables have proper units for the required calculations. Either do dimensionless calculations or use consistent units for all quantities.

We now hand-check the solution by substituting the values obtained in, say, Eqns. (12.38) and (12.43). Before we substitute the values of forces, we need to calculate the length ℓ.

$$\begin{aligned} \ell &= \sqrt{d^2 + b^2 + e^2} \\ &= 3.3912\,\mathrm{m}. \end{aligned}$$

Therefore,

$$\text{Eqn. (12.38):} \quad O_y + T\frac{b}{\ell} = -3\,\text{N} + 6.78\,\text{N} \cdot \frac{1.5\,\text{m}}{3.3912\,\text{m}} \stackrel{\checkmark}{=} 0,$$

$$\text{Eqn. (12.43):} \quad \frac{d}{\ell}T - \frac{1}{2}ma = \frac{3\,\text{m}}{3.3912\,\text{m}}6.78\,\text{N} - \frac{1}{2}20\,\text{kg}\,0.6\,\text{m/s}^2 \stackrel{\checkmark}{=} 0.$$

Thus, the computer solution agrees with our equations.
Comments: We could have solved the six equations for seven unknowns without assuming $G_y = 0$ if our computer program or package allows us to do so. We will, of course, not get a unique solution. For example, by taking the following **A**, a 6×7 matrix, and solving **A x = b** for $\mathbf{x} = [O_x \; O_y \; O_z; G_x \; G_y \; G_z \; T]^T$ with the same **b** as input above, we get the solution as shown below.

```
A = [1  0  0  1  0  0  d/l
     0  1  0  0  1  0  b/l
     0  0  1  0  0  1  e/l
     0  0  1  0  0  0  c/l
     0  0  0  0  0  0  d/l
     1  0  0  0  0  0  d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =               % this is the computer output
        0
   -3.0000
   97.0000
    6.0000
        0
  102.0000
    6.7823
```

This is the same solution as we got before except that it includes $G_y = 0$ in the solution. Now, if we add a vector $\Delta\, \mathbf{x} = [0 \; \alpha \; 0 \; 0 \; -\alpha \; 0 \; 0]^T$ to **x** where α is any number, and compute **A (x+Δx)** , we get back **b**. That is, the six equilibrium conditions are satisfied irrespective of the actual values of O_y and G_y as long as the value of $O_y + G_y$ remains the same.

Problems for Chapter 12

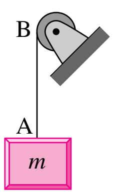
1D constrained motion

12.1 1D constrained motion and pulleys

For all problems, unless stated otherwise, treat all strings as inextensible, flexible and massless. Treat all pulleys and wheels as round, frictionless and massless. Assume all massive objects are prevented from rotating (e.g., wheels stay on the ground, *etc.*). When numbers are called for use $g = 10\,\text{m/s}^2$ or $g = 32\,\text{ft/s}^2$.

Preparatory Problems

12.1.1 A motor at B allows the block of mass $m = 3\,\text{kg}$ shown in the figure to accelerate downwards at $2\,\text{m/s}^2$. There is gravity. What is the tension in the string AB?



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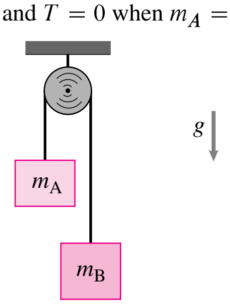
Problem 12.1.1

12.1.2 Two masses connected by an inextensible string hang from an ideal pulley.

a) Find the downward acceleration of mass B. Answer in terms of any or all of m_A, m_B, g, and the present velocities of the blocks. As a check, your answer should give $a_B = g$ when $m_A = 0$ and $a_B = 0$ when $m_A = m_B$. *.

b) Find the tension in the string. As a check, your answer should give $T = m_B g = m_A g$ when $m_A = m_B$ and $T = 0$ when $m_A = 0$. *



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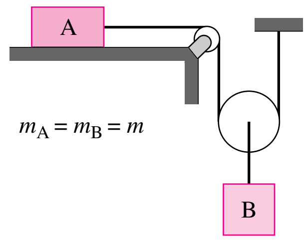
Problem 12.1.2

12.1.3 The blocks shown are released from rest.

a) What is the acceleration of block A at $t = 0^+$ (just after release)?

b) What is the speed of block B after it has fallen 2 meters?



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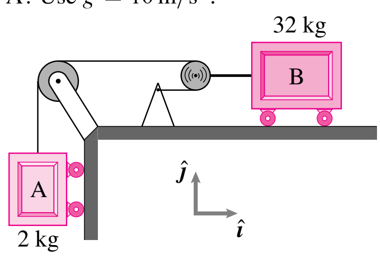
Problem 12.1.3

12.1.4 What is the acceleration of block A? Use $g = 10\,\text{m/s}^2$. *



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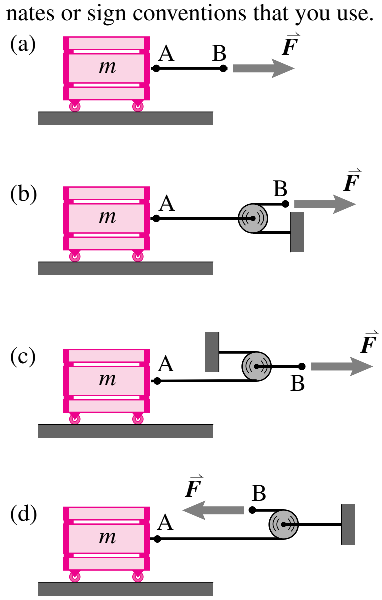
Problem 12.1.4

12.1.5 For the system shown in problem 12.1.2, find the acceleration of mass B using energy balance ($P = \dot{E}_K$).

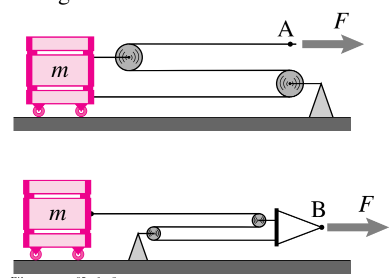
12.1.6 For the various situations pictured, find the acceleration of mass A and point B. Clearly define any variables, coordinates or sign conventions that you use. *



Filename:pulley1
Problem 12.1.6: Four different ways to pull a mass.

12.1.7 For each of the situations in problem 12.1.6 find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.1.8 What is the ratio of the acceleration of point A to that of point B in each configuration? $m = m$ and $F = F$. *



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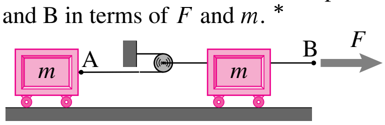
Problem 12.1.8

12.1.9 Find the acceleration of points A and B in terms of F and m. *



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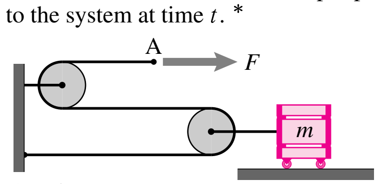
Problem 12.1.9

12.1.10 For the situation pictured in problem 12.1.9 find the accelerations of the two masses using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.1.11 The point of application A of the force moves twice as fast as the mass. At some instant in time t, the speed of the mass is $\dot{x}$ to the left. Find the input power to the system at time t. *



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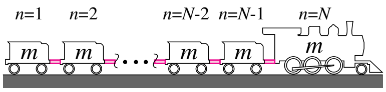
Problem 12.1.11

More-Involved Problems

12.1.12 A train engine of mass m pulls and accelerates N cars, each of mass m. The power of the engine is P_t and its speed is v_t. Find the tension T_n between car n and car n+1. Assume there is no resistance and the ground is level. Assume the cars are connected with rigid links. *



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Problem 12.1.12

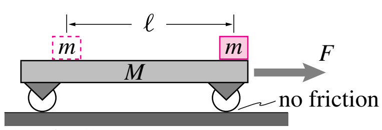
12.1.13 A cart of mass M, initially at rest, can move horizontally along a frictionless track. When $t = 0$, a force F is applied as shown to the cart. During the acceleration of M by the force F, a small box of mass m slides along the cart from the front to the rear. The coefficient of friction between the cart and the box is μ, and it is assumed that the acceleration of the cart is sufficient to cause sliding.

a) Draw free body diagrams of the cart, the box, and the cart and box together.

b) Write the equation of linear momentum balance for the cart, the box, and the system of cart and box.

c) Show that the equations of motion for the cart and box can be combined to give the equation of motion of the mass center of the system of two bodies.

d) Find the displacement of the cart at the time when the box has moved a distance ℓ along the cart. *



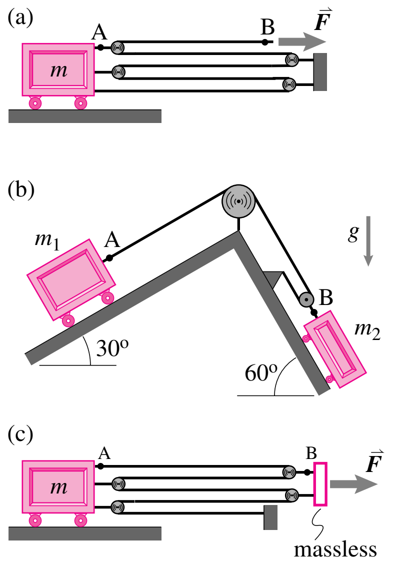
Filename:pfigure-blue-28-1
Problem 12.1.13

12.1.14 For the situations pictured, find the accelerations of mass A and of point B. Clearly define any variables, coordinates or sign conventions that you use.

a) A single mass and four pulleys. *

b) Two masses and two pulleys. *

c) A single mass and four pulleys. *



Filename:pulley4
Problem 12.1.14: Various pulley arrangements.

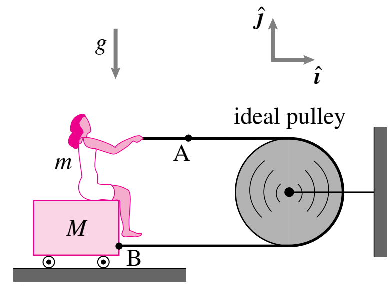
12.1.15 For the situations pictured in problem 12.1.14, find the acceleration of the mass using energy balance ($P = \dot{E}_K$).

12.1.16 A person of mass m, modeled as a rigid object, is sitting on a cart of mass $M > m$ and pulling the string towards herself. The coefficient of friction between her seat and the cart is μ. Point B is attached to the cart and point A is attached to the rope.

a) If you are given that she is pulling rope in with acceleration a_0 relative to herself (that is, $\vec{a}_{A/B} \equiv \vec{a}_A - \vec{a}_B = -a_0\hat{\imath}$) and that she is not slipping relative to the cart, find $\vec{a}_A$. (Answer in terms of some or all of $m, M, g, \mu, \hat{\imath}$ and a_0.) *

b) Find the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)

c) If instead, $M < m$, what is the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)

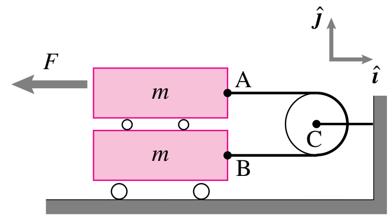


Filename:s97p2-2
Problem 12.1.16: **Pulley.**

12.1.17 Two blocks and a pulley. Two identical blocks are stacked and tied together by the pulley as shown. Find

a) the acceleration of point A, and

b) the tension in the line.



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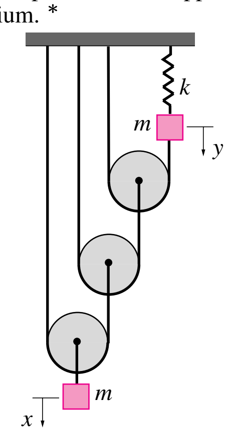
Problem 12.1.17

12.1.18 What is the natural frequency of vibration of this system? Include gravity. x measures the vertical position of the lower mass from equilibrium. y measures the vertical position of the upper mass from equilibrium. *



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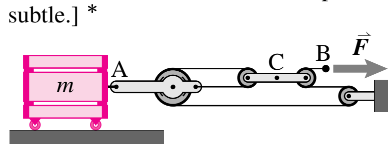
Problem 12.1.18

12.1.19 For the situation pictured, find the acceleration of mass A and points B and C shown. [Hint: the situation with point C is subtle.] *



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Problem 12.1.19

12.1.20 For the situation pictured in problem 12.1.19, find the acceleration of point A using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.1.21 Design a pulley system. You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\vec{F} = F\hat{\imath}$ pulling it to the right. Mass B has mass m_B. You can connect point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). You must design the system so that mass B accelerates to the left with $\frac{F}{2m_B}$ (i.e., $\vec{a}_B = -\frac{F}{2m_B}\hat{\imath}$).

a) Draw the system clearly. Justify your answer with enough words or equations so that a reasonable person, say a grader, can tell that you understand your solution.

b) Find the acceleration of point A.

12.1.22 Design a pulley system. You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\vec{F} = F\hat{\imath}$ pulling it to the right. Mass B has mass m_B. You can connect the point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). Draw the system clearly. Justify your answer with enough words or equations to convince a skeptical person that your solution is correct. You must design the system so that the mass B accelerates .

a) to the left with $\frac{F}{m_B}$ (i.e., $\vec{a}_B = -\frac{F}{m_B}\hat{\imath}$)

b) to the left with $\frac{2F}{m_B}$

c) to the left with $\frac{F}{2m_B}$

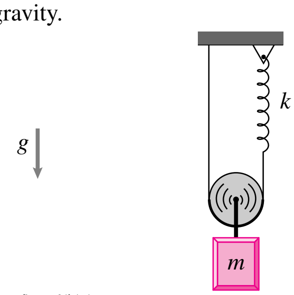
d) to the right with $\frac{2F}{m_B}$

e) to the right with $\frac{F}{2m_B}$

f) to the left with $\frac{8F}{m_B}$

g) to the right with $\frac{F}{5m_B}$

12.1.23 Pulley and spring. For the hanging mass find the period of oscillation. Only vertical motion is of interest. There is gravity.



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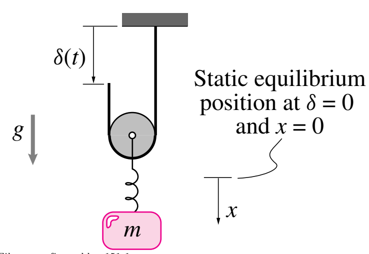
Problem 12.1.23

12.1.24 The spring-mass system shown (m = 10 slugs ($\equiv$ lb $\cdot$ sec^2/ft), k = 10 lb/ft) is excited by moving the free end of the cable vertically according to $\delta(t) = 4\sin(\omega t)$ in, as shown in the figure.

a) Derive the equation of motion for the block in terms of the displacement x from the static equilibrium position, as shown in the figure.

b) If $\omega = 0.9$ rad/s, check to see if the pulley is always in contact with the cable (ignore the transient solution).



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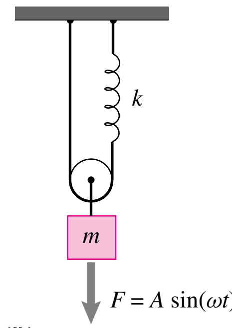
Problem 12.1.24

12.1.25 The block of mass m hanging on the spring with constant k and a string shown in the figure is forced by $F = A\sin(\omega t)$. Do not neglect gravity. The pulley has negligible mass.

a) What is the differential equation governing the motion of the block? You may assume that the only motion is vertical motion. *

b) Given A, m and k, for what values of ω would the string go slack at some point in the cyclical motion? (The common assumption in such problems, which you can use, is to neglect the homogeneous solution to the differential equation. It is assumed that the damping, small enough to be neglected in the governing equations is large enough so that the particular solution will have damped out at the time of observation.) *



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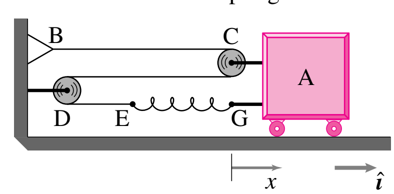
Problem 12.1.25

12.1.26 Block A, with mass m_A, is pulled to the right a distance d from the position it would have if the spring were relaxed. It is then released from rest. Assume ideal string, pulleys and wheels. The spring has constant k.

a) What is the acceleration of block A just after it is released (in terms of k, m_A, and d)? *

b) What is the speed of the mass when the mass passes through the position where the spring is relaxed? *.



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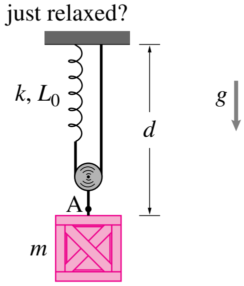
Problem 12.1.26

12.1.27 What is the static displacement of the mass from the position where the spring is just relaxed?

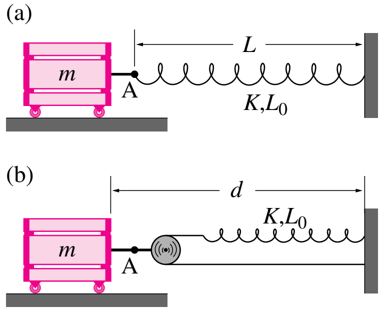


Filename:pulley3-a

Problem 12.1.27

12.1.28 For the two situations pictured, find the acceleration of point A shown using balance of linear momentum ($\sum \vec{F} =$

$m\vec{a}$). Assuming both masses are deflected an equal distance from the position where the spring is just relaxed, how much smaller or bigger is the acceleration of block (b) than that of block (a). Define any variables, coordinate system origins, coordinates or sign conventions that you need to do your calculations and to define your solution.



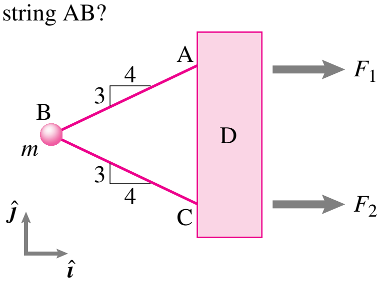
Filename:pulley3
Problem 12.1.28

12.1.29 For each of the situations pictured in problem 12.1.28, find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

12.2 1D motion with 2D and 3D forces

Preparatory Problems

12.2.1 Mass pulled by two strings. F_1 and F_2 are applied so that the system shown accelerates to the right at $5\,\mathrm{m/s^2}$ (*i.e.,* $\boldsymbol{a} = 5\,\mathrm{m/s^2}\hat{\imath} + 0\hat{\jmath}$) and has no rotation. The mass of D and forces F_1 and F_2 are unknown. What is the tension in string AB?



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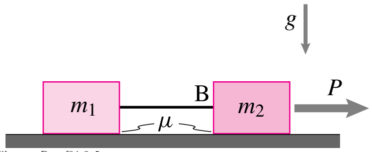
Problem 12.2.1

12.2.2 The two blocks, $m_1 = m_2 = m$, are connected by an inextensible string AB. The string can only withstand a tension T_{cr}. Find the maximum value of the applied force P so that the string does not break. The sliding coefficient of friction between the blocks and the ground is μ.



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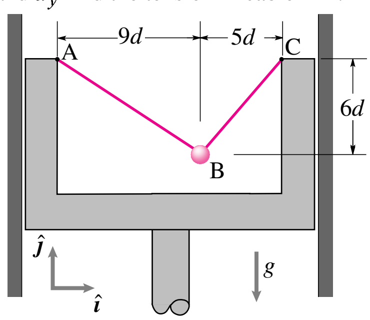
Problem 12.2.2

12.2.3 A point mass m is attached to a piston by two inextensible cables. The piston has upwards acceleration of $a_y\hat{\jmath}$. There is gravity. In terms of some or all of m, g, d, and a_y find the tension in cable AB. *



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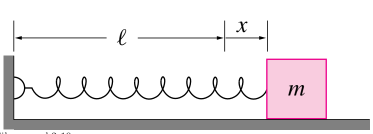
Problem 12.2.3

12.2.4 A point mass of mass m moves on a frictional surface with coefficient of friction μ and is connected to a spring with constant k and unstretched length ℓ. There is gravity. At the instant of interest, the mass is at a distance x to the right from its position where the spring is unstretched and is moving with $\dot{x} > 0$ to the right.

a) Draw a free body diagram of the mass at the instant of interest.

b) At the instant of interest, write the equation of linear momentum balance for the block evaluating the left hand side as explicitly as possible. Let the acceleration of the block be $\vec{a} = \ddot{x}\hat{\imath}$.



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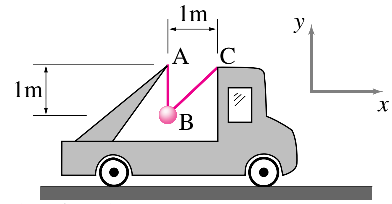
Problem 12.2.4

12.2.5 Consider the mass at B (2 kg) supported by two strings in the back of a truck which has acceleration of $3\,\mathrm{m/s^2}$. Use $g = 10\,\mathrm{m/s^2}$. What is the tension T_{AB} in the string AB in Newtons?



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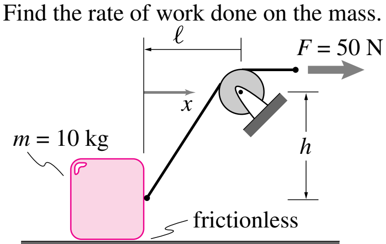
Problem 12.2.5:

12.2.6 At the instant shown, the mass is moving to the right at speed $v = 3\,\mathrm{m/s}$. Find the rate of work done on the mass.



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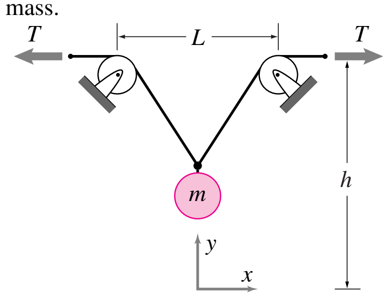
Problem 12.2.6

12.2.7 A point mass 'm' is pulled straight up by two strings. The two strings pull the mass symmetrically about the vertical axis with constant and equal force T. At an instant in time t, the position and the velocity of the mass are $y(t)\hat{\jmath}$ and $\dot{y}(t)\hat{\jmath}$, respectively. Find the power input to the moving mass.



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Problem 12.2.7

More-Involved Problems

12.2.8 Two blocks, each of mass m, are connectedby a rod of length S. They slide down a slope of angle θ. Do not neglect gravity but do neglect friction.

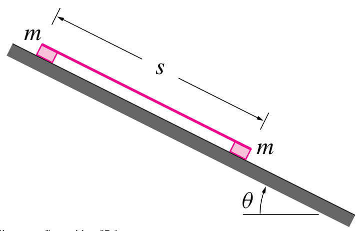
a) Draw separate free body diagrams of each block, the rod, and the system of the two blocks and rod.

b) Write separate equations for linear momentum balance for each block, the rod, and the system of blocks and rod.

c) What is the acceleration of the center of mass of the two blocks? *

d) What is the force in the rod? *

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest. [Hint: Dot your momentum balance equations with a unit vector along the ramp in order to reduce this problem to a problem in one dimensional mechanics.] *



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Problem 12.2.8

12.2.9 Two blocks, each of mass m, are connected by a massless rod of length S; the blocks' dimensions are small compared to S. They slide down a slope of angle θ. The coefficient of friction of the top block is μ and of the bottom block is $\mu/2$.

a) Draw separate free body diagrams of each block, the string, and the system of the two blocks and rod.

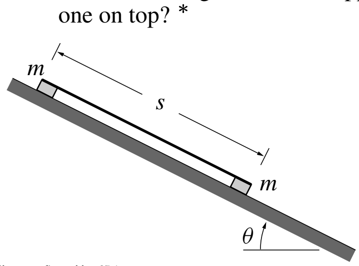
b) Write separate equations for linear momentum balance for each block, the string, and the system of blocks and rod.

c) What is the acceleration of the center of mass of the two blocks? *

d) What is the force in the rod? *

e) What is the speed of the center of mass for the two blocks after they have traveled a distance d down the slope, having started from rest. *

f) How would your solutions to parts (a) and (c) differ if the two blocks were interchanged with the slippery one on top? *



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Problem 12.2.9

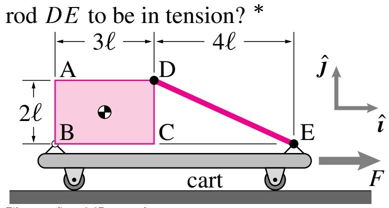
12.2.10 Coin on a car on a ramp. A student engineering design course asked students to build a cart (mass $= m_c$) that rolls down a ramp with angle θ. A small weight (mass $m_w \ll m_c$) is placed on top of the cart on a surface tipped with respect to the cart (angle ϕ). Assume the small mass does not slide. Assume massless wheels with frictionless bearings. $\hat{\imath}$ is horizontal and $\hat{\jmath}$ is vertical up.

a) Find the acceleration of the cart. Answer in terms of some or all of $m_c, g, \hat{\imath}, \theta$ and $\hat{\jmath}$.

b) What coefficient of friction μ is required (the smallest that will work) to keep the small mass from sliding as the cart rolls down the slope? Answer in terms of some or all of m_c, m_w, g, θ, and ϕ.

c) What angle ϕ will allow a small mass to ride on the cart with the smallest coefficient of friction? Answer in terms of some or all of m_c, m_w, g, and θ.

12.2.11 Guyed plate on a cart A uniform rectangular plate $ABCD$ of mass m is supported by a rod DE and a hinge joint at point B. The dimensions are as shown. There is gravity. What must the acceleration of the cart be in order for massless rod DE to be in tension? *



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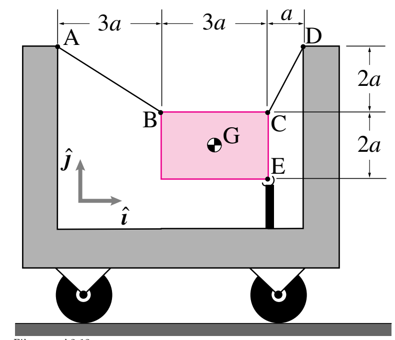
Problem 12.2.11: Uniform plate supported by a hinge and a cable on an accelerating cart.

12.2.12 A uniform rectangular plate of mass m is supported by two inextensible cables AB and CD and by a hinge at point E on the cart as shown. The cart has acceleration $a_x\hat{\imath}$ due to a force not shown. There is gravity.

a) Draw a free body diagram of the plate.

b) Write the equation of linear momentum balance for the plate and evaluate the left hand side as explicitly as possible.

c) Write the equation for angular momentum balance about point E and evaluate the left hand side as explicitly as possible.



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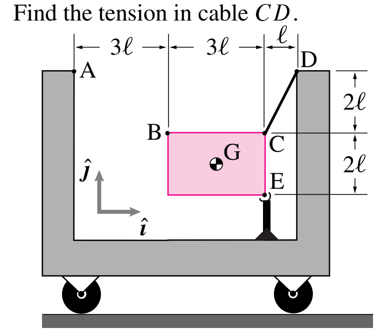
Problem 12.2.12

12.2.13 A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart is at rest. There is gravity. Find the tension in cable CD.



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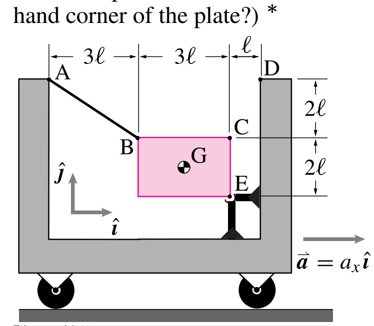
Problem 12.2.13

12.2.14 A uniform rectangular plate of mass m is supported by an inextensible cable AB and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\imath}$. There is gravity. Find the tension in cable AB. (What's 'wrong' with this problem? What if instead point B were at the bottom left hand corner of the plate?) *



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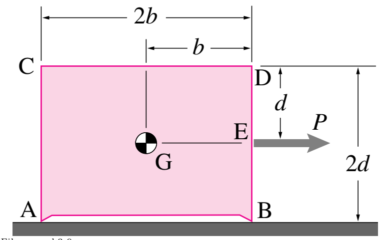
Problem 12.2.14

12.2.15 A block of mass m is sitting on a frictionless surface and acted upon at point E by the horizontal force P through the center of mass. Draw a free body diagram of the block. There is gravity. Find a) the acceleration of the block and b) reactions on the block at points A and B. *



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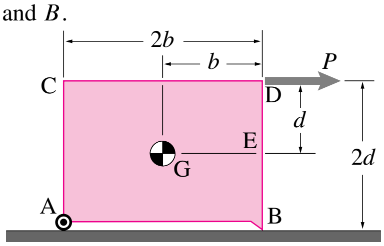
Problem 12.2.15

12.2.16 Reconsider the block in problem 12.2.15. This time, find the acceleration of the block and the reactions at A and B if the force P is applied instead at point D. Are the acceleration and the reactions on the block different from those found when P is applied at point E? *

12.2.17 A block of mass m is sitting on a frictional surface and acted upon at point D by the horizontal force P. The block is resting on a sharp edge at point B and is supported by an ideal wheel at point A. There is gravity. Assuming the block is sliding with coefficient of friction μ at point B, find the acceleration of the block and the reactions on the block at points A and B.



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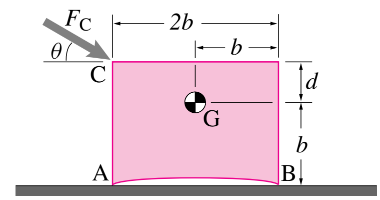
Problem 12.2.17

12.2.18 A force F_C is applied to the corner C of a box of weight W with dimensions and center of gravity at G as shown in the figure. The coefficient of sliding friction between the floor and the points of contact A and B is μ. Assuming that the box slides when F_C is applied, find the acceleration of the box and the reactions at A and B in terms of W, F_C, θ, b, and d.



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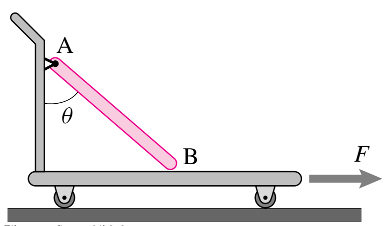
Problem 12.2.18

12.2.19 A uniform rod with mass m_r rests on a cart (mass m_c) which is being pulled to the right. The rod is hinged at one end (with a frictionless hinge) and has no friction at the contact with the cart. The cart is rolling on wheels that are modeled as having no mass and no bearing friction (ideal massless wheels). Answer in terms of g, m_r, m_c, θ and F. Find:

a) The force on the rod from the cart at point B.

b) The force on the rod from the cart at point A.



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Problem 12.2.19

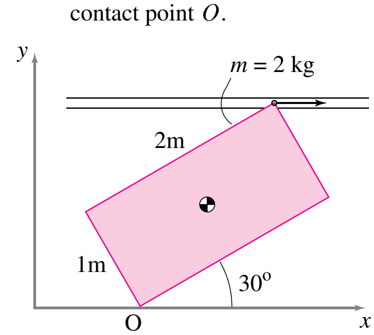
12.2.20 The box shown in the figure is dragged in the x-direction with a constant acceleration $\vec{\boldsymbol{a}} = 0.5\,\mathrm{m/s^2}\hat{\boldsymbol{\imath}}$. At the instant shown, the velocity of (every point on) the box is $\vec{\boldsymbol{v}} = 0.8\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$.

a) Find the linear momentum of the box.

b) Find the rate of change of linear momentum of the box.

c) Find the angular momentum of the box about the contact point O.

d) Find the rate of change of angular momentum of the box about the contact point O.



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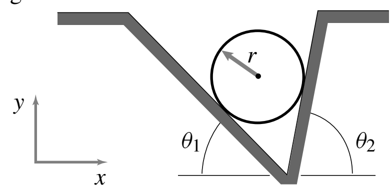
Problem 12.2.20

12.2.21 The groove and disk accelerate upwards, $\vec{\boldsymbol{a}} = a\hat{\boldsymbol{\jmath}}$. Neglecting gravity, what are the forces on the disk due to the groove?



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Problem 12.2.21

12.2.22 The following problems concern a box that is in the back of a pickup truck. The pickup truck is moving forward with acceleration of a_t. The truck's speed is v_t. The box has sharp feet at the front and back ends so the only place it contacts the truck is at the feet. The center of mass of the box is at the geometric center of the box. The box has height h, length ℓ and depth w (into the paper.) Its mass is m. There is gravity. The friction coefficient between the truck and the box edges is μ.

In the problems below you should express your solutions in terms of the variables given in the figure, ℓ, h, μ, m, g, a_t, and v_t. If any variables do not enter the expressions comment on why they do not. In all cases you may assume that the box does not rotate (though it might be on the verge of doing so).

a) Assuming the box does not slide, what is the total force that the truck exerts on the box (i.e. the sum of the reactions at A and B)?

b) Assuming the box does not slide what are the reactions at A and B? [Note: You cannot find both of them without additional assumptions.]

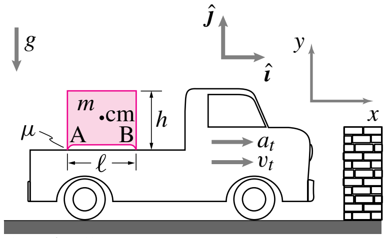
c) Assuming the box does slide, what is the total force that the truck exerts on the box?

d) Assuming the box does slide, what are the reactions at A and B?

e) Assuming the box does not slide, what is the maximum acceleration of the truck for which the box will not tip over (hint: just at that critical acceleration what is the vertical reaction at B?)?

f) What is the maximum acceleration of the truck for which the block will not slide?

g) The truck hits a brick wall and stops instantly. Does the block tip over? Assuming the block does not tip over, how far does it slide on the truck before stopping (assume the bed of the truck is sufficiently long)?



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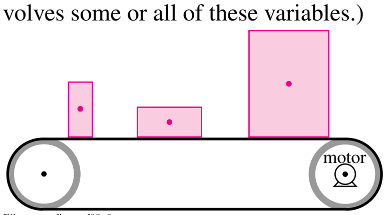
Problem 12.2.22

12.2.23 A collection of uniform boxes with various heights h and widths w and masses m sit on a horizontal conveyer belt. The acceleration $a(t)$ of the conveyer belt gets extremely large sometimes due to an erratic over-powered motor. Assume the boxes touch the belt at their left and right edges only and that the coefficient of friction there is μ. It is observed that some boxes never tip over. What is true about μ, g, w, h, and m for the boxes that always maintain contact at both the right and left bottom edges? (Write an inequality that involves some or all of these variables.)



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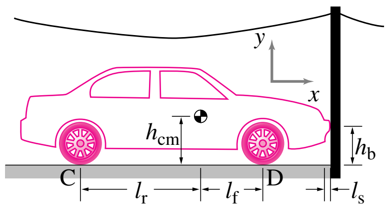
Problem 12.2.23

12.2.24 After failure of her normal brakes, a driver pulls the emergency brake of her old car. This action locks the rear wheels (friction coefficient = μ) but leaves the well lubricated and light front wheels spinning freely. The car, braking inadequately as is the case for rear wheel braking, hits a stiff and slippery phone pole which compresses the car bumper. The car bumper is modeled here as a linear spring (constant = k, rest length = l_0, present length = l_s). The car is still traveling forward at the moment of interest. The bumper is at a height h_b above the ground. Assume that the car, excepting the bumper, is a non-rotating rigid body and that the wheels remain on the ground (that is, the bumper is compliant but the suspension is stiff).

- What is the acceleration of the car in terms of g, m, μ, l_f, l_r, k, h_b, h_{cm}, l_0, and l_s (and any other parameters if needed)?



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Problem 12.2.24

12.2.25 Car braking: front brakes versus rear brakes versus all four brakes. What is the peak deceleration of a car when you apply: the front brakes till they skid, the rear brakes till they skid, and all four brakes till they skid? Assume that the coefficient of friction between rubber and road is $\mu = 1$ (about right, the coefficient of friction between rubber and road varies between about .7 and 1.3) and that $g = 10\,\mathrm{m/s^2}$ (2% error). Pick the dimensions and mass of the car, but assume the center of mass height h is greater than zero but is less than half the wheel base w, the distance between the front and rear wheel. Also assume that the CM is halfway between the front and back wheels (*i.e.,* $l_f = l_r = w/2$). The car has a stiff suspension so the car does not move up or down or tip appreciably during braking. Neglect the mass of the rotating wheels in the linear and angular momentum balance equations. Treat this problem as two-dimensional problem; i.e., the car is symmetric left to right, does not turn left or right, and that the left and right wheels carry the same loads. To organize your work, here are some steps to follow.

a) Draw a FBD of the car assuming rear wheel is skidding. The FBD should show the dimensions, the gravity force, what you know *a priori* about the forces on the wheels from the ground (i.e., that the friction force $F_r = \mu N_r$, and that there is no friction at the front wheels), and the coordinate directions. Label points of interest that you will use in your momentum balance equations. (Hint: also draw a free body diagram of the rear wheel.)

b) Write the equation of linear momentum balance.

c) Write the equation of angular momentum balance relative to a point of your choosing. Some particularly useful points to use are:
   - the point above the front wheel and at the height of the center of mass;
   - the point at the height of the center of mass, behind the rear wheel that makes a 45 degree angle line down to the rear wheel ground contact point; and
   - the point on the ground straight under the front wheel that is as far below ground as the wheel base is long.

d) Solve the momentum balance equations for the wheel contact forces and the deceleration of the car. If you have used any or all of the recommendations from part (c) you will have the pleasure of only solving one equation in one unknown at a time. *

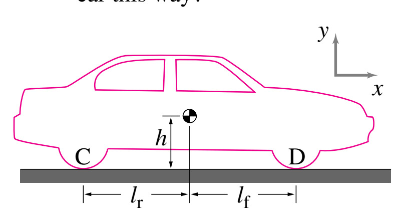
e) Repeat steps (a) to (d) for front-wheel skidding. Note that the advantageous points to use for angular momentum balance are now different. Does a car stop faster or slower or the same by skidding the front instead of the rear wheels? Would your solution to (e) be different if the center of mass of the car were at ground level(h=0)? *

f) Repeat steps (a) to (d) for all-wheel skidding. There are some shortcuts here. You determine the car deceleration without ever knowing the wheel reactions (or using angular momentum balance) if you look at the linear momentum balance equations carefully. *

g) Does the deceleration in (f) equal the sum of the decelerations in (d) and (e)? Why or why not? *

h) What peculiarity occurs in the solution for front-wheel skidding if the wheel base is twice the height of the CM above ground and $\mu = 1$? *

i) What impossibility does the solution predict if the wheel base is shorter than twice the CM height? What wrong assumption gives rise to this impossibility? What would really happen if one tried to skid a car this way? *



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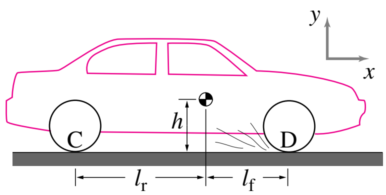
Problem 12.2.25

12.2.26 Assuming massless wheels, an infinitely powerful engine, a stiff suspension (*i.e.,* no rotation of the car) and a coefficient of friction μ between tires and road,

a) what is the maximum forward acceleration of this front wheel drive car? *

b) what is the force of the ground on the rear wheels during this acceleration?

c) what is the force of the ground on the front wheels?



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Problem 12.2.26

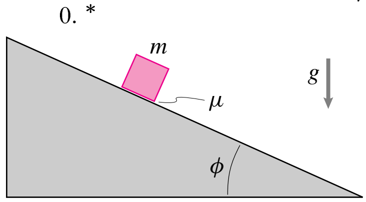
12.2.27 At time $t = 0$, the block of mass m is released from rest on the slope of angle ϕ. The coefficient of friction between the block and slope is μ.

a) What is the acceleration of the block for $\mu > 0$? *

b) What is the acceleration of the block for $\mu = 0$? *

c) Find the position and velocity of the block as a function of time for $\mu > 0$. *

d) Find the position and velocity of the block as a function of time for $\mu = 0$. *



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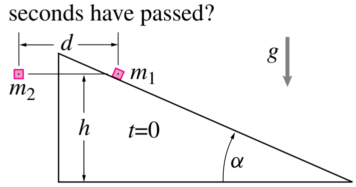
Problem 12.2.27

12.2.28 A small block of mass m_1 is released from rest at altitude h on a frictionless slope of angle α. At the instant of release, another small block of mass m_2 is dropped vertically from rest at the same altitude. The second block does not interact with the ramp. What is the velocity of the first block relative to the second block after t seconds have passed?



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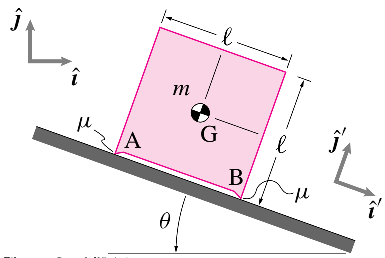
Problem 12.2.28

12.2.29 Block sliding on a ramp with friction. A square box is sliding down a ramp of angle θ with instantaneous velocity $v\hat{\imath}'$. Assume it does not tip over.

a) What is the force on the block from the ramp at point A? Answer in terms of any or all of θ, ℓ, m, g, μ, v, $\hat{\imath}'$, and $\hat{\jmath}'$. As a check, your answer should reduce to $\frac{mg}{2}\hat{\jmath}'$ when $\theta = \mu = 0$. *

b) In addition to solving the problem by hand, see if you can write a set of computer commands that, if θ, μ, ℓ, m, v and g were specified, would give the correct answer.

c) Assuming $\theta = 80^\circ$ and $\mu = 0.9$, can the box slide this way or would it tip over? Why? *



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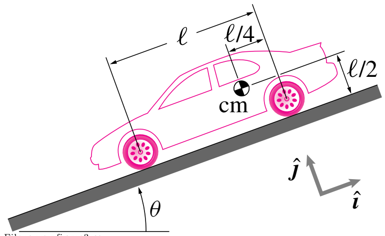
Problem 12.2.29

12.2.30 A coin is given a sliding shove up a ramp with angle ϕ with the horizontal. It takes twice as long to slide down as it does to slide up. What is the coefficient of friction μ between the coin and the ramp. Answer in terms of some or all of m, g, ϕ and the initial sliding velocity v.

12.2.31 A skidding car. What is the braking acceleration of the front-wheel braked car as it slides down hill. Express your answer as a function of any or all of the following variables: the slope θ of the hill, the mass of the car m, the wheel base ℓ, and the gravitational constant g. Use $\mu = 1$. *



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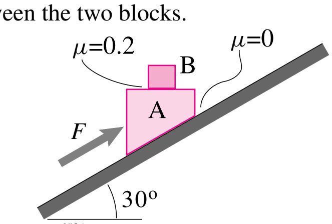
Problem 12.2.31: A car skidding downhill on a slope of angle θ

12.2.32 Two blocks A and B are pushed up a frictionless inclined plane by an external force F as shown in the figure. The coefficient of friction between the two blocks is $\mu = 0.2$. The masses of the two blocks are $m_A = 5\,\text{kg}$ and $m_B = 2\,\text{kg}$. Find the magnitude of the maximum allowable force such that no relative slip occurs between the two blocks.



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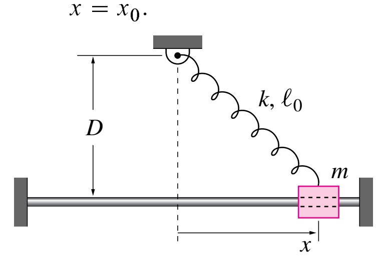
Problem 12.2.32

12.2.33 A bead slides on a frictionless rod. The spring has constant k and rest length ℓ_0. The bead has mass m.

a) Given x and $\dot{x}$ find the acceleration of the bead (in terms of some or all of $D, \ell_0, x, \dot{x}, m, k$ and any base vectors that you define).

b) If the bead is allowed to move, as constrained by the slippery rod and the spring, find a differential equation that must be satisfied by the variable x. (Do not try to solve this somewhat ugly non-linear equation.)

c) In the special case that $\ell_0 = 0$* find how long it takes for the block to return to its starting position after release with no initial velocity at $x = x_0$.



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Problem 12.2.33

12.2.34 A bead oscillates on a straight frictionless wire. The spring obeys the equation $F = k\,(\ell - \ell_0)$, where ℓ = length of the spring and ℓ_0 is the 'rest' length. Assume

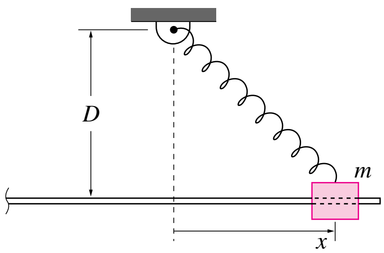
$$x(t = 0) = x_0,\ \dot{x}(t = 0) = 0.$$

a) Write a differential equation satisfied by $x(t)$.

b) What is $\dot{x}$ when $x = 0$? [hint: Don't try to solve the equation in (a)!]

c) What is the simplification in (a) if $\ell_o = 0$ (spring is then a so-called "zero–length" spring).

d) For this special case ($\ell_o = 0$) solve the equation in (a) and show the result agrees with (b) in this special case.



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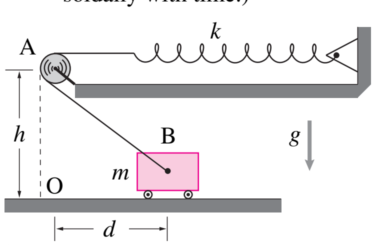
Problem 12.2.34

12.2.35 A cart on an elastic leash. A cart B (mass m) rolls on a frictionless level floor. One end of an inextensible string is attached to the the cart. The string wraps around a pulley at point A and the other end is attached to a spring with constant k. When the cart is at point O, it is in static equilibrium. The spring relaxed length, rope length, and room height h are such that the spring would be relaxed if the end of rope at B were disconnected from the cart and brought up to point A. The gravitational constant is g. The cart is pulled a horizontal distance d from the center of the room (at O) and released.

a) Assuming that the cart never leaves the floor, what is the speed of the cart when it passes through the center of the room, in terms of m, h, g and d. *

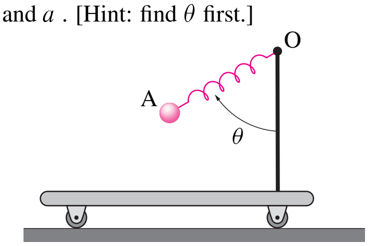
b) Does the cart undergo simple harmonic motion for small or large oscillations (specify which if either)? (Simple harmonic motion occurs when position varies inusoidally with time.) *



Filename:cartosc

Problem 12.2.35

12.2.36 The cart moves to the right with constant acceleration a. The ball has mass m. The spring has unstretched length ℓ_0 and spring constant k. Assuming the ball is stationary with respect to the cart find the distance from O to A in terms of k, ℓ_0, and a . [Hint: find θ first.]



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Problem 12.2.36

12.2.37 Consider a person, modeled as a rigid body, riding an accelerating motorcycle (2-D). The person is sitting on the seat and cannot slide fore or aft, but is free to rock in the plane of the motorcycle (as if there were a hinge connecting the motorcycle to the rider at the seat). The person's feet are off the pegs and the legs are sticking down and not touching anything. The person's arms are like cables (they are massless and only carry tension). Assume all dimensions and masses are known (you have to define them carefully with a sketch and words). Assume the forward acceleration of the motorcycle is known. You may use numbers and/or variables to describe the quantities of interest.

a) Draw a clear sketch of the problem showing needed dimensional information and the coordinate system you will use.

b) Draw a Free Body Diagram of the rider.

c) Write the equations of linear and angular momentum balance for the rider.

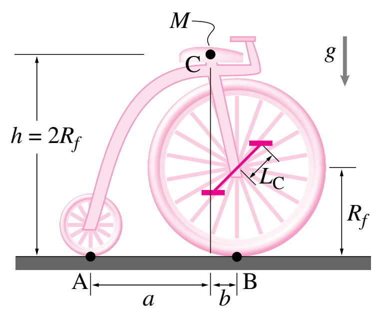
d) Find all forces on the rider from the motorcycle (i.e., at the hands and the seat).

e) What are the forces on the motorcycle from the rider?

12.2.38 Acceleration of a bicycle on level ground. 2-D . A very compact bicycler (modeled as a point mass M at the bicycle seat C with height h, and distance b behind the front wheel contact), rides a very light old-fashioned bicycle (all components have negligible mass) that is well maintained (all bearings have no frictional torque) and streamlined (neglect air resistance). The rider applies a force F_p to the pedal perpendicular to the pedal crank (with length L_c). No force is applied to the other pedal. The radius of the front wheel is R_f.

a) Assuming no slip, what is the forward acceleration of the bicycle? [Hint: draw a FBD of the front wheel and crank, and another FBD of the whole bicycle-rider system.] *

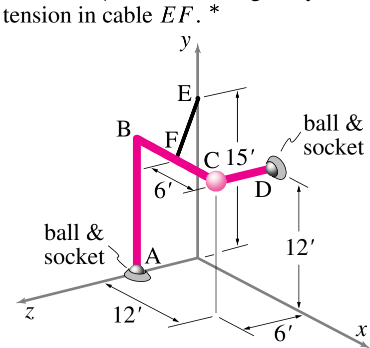
b) (Harder) Assuming the rider can push arbitrarily hard but that $\mu = 1$, what is the maximum possible forward acceleration of the bicycle. *



Filename:f-3

Problem 12.2.38

12.2.39 A 320 lbm mass is attached at the corner C of a light rigid piece of pipe bent as shown. The pipe is supported by ball-and-socket joints at A and D and by cable EF. The points A, D, and E are fastened to the floor and vertical sidewall of a pick-up truck which is accelerating in the z-direction. The acceleration of the truck is $\vec{a} = 5\,\text{ft/s}^2\hat{\boldsymbol{k}}$. There is gravity. Find the tension in cable EF. *



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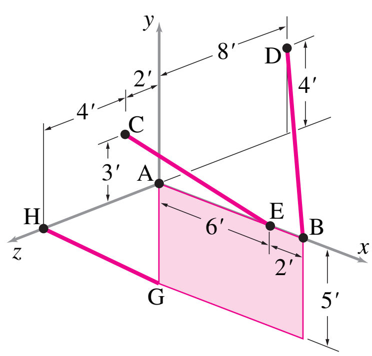
Problem 12.2.39

12.2.40 A 5 ft by 8 ft rectangular plate of uniform density has mass $m = 10$ lbm and is supported by a ball-and-socket joint at point A and the light rods CE, BD, and GH. The entire system is attached to a truck which is moving with acceleration $\vec{a}_T$. The plate is moving without rotation or angular acceleration relative to the truck. Thus, the center of mass acceleration of the plate is the same as the truck's. Dimensions are as shown. Points A, C, and D are fixed to the truck but the truck is not touching the plate at any other points. Find the tension in rod BD.

a) If the truck's acceleration is $\boldsymbol{a}_{cm} = (5\,\mathrm{ft/s^2})\boldsymbol{k}$, what is the tension or compression in rod BD? *

b) If the truck's acceleration is $\vec{a}_{\mathrm{cm}} = (5\,\mathrm{ft/s^2})\hat{\boldsymbol{j}} + (6\,\mathrm{ft/s^2})\hat{\boldsymbol{k}}$, what is the tension or compression in rod GH? *



Filename:Mikesp92p1
Problem 12.2.40

12.2.41 Hanging a shelf. A shelf with negligible mass supports a 0.5 kg mass at its center. The shelf is supported at one corner with a ball and socket joint and at the other three corners with strings. At the moment of interest the shelf is in a rocket in outer space and accelerating at $10\,\mathrm{m/s^2}$ in the $\boldsymbol{k}$ direction. The shelf is in the xy plane.

a) Draw a FBD of the shelf.

b) Challenge: without doing any calculations on paper can you find one of the reaction force components or the tension in any of the cables? Give yourself a few minutes of staring to try this approach. If you can't, then come back to this question after you have done all the calculations. *

c) Write down the linear momentum balance equation (a vector equation). *

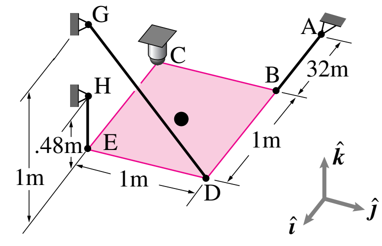
d) Write down the angular momentum balance equation using the center of mass as a reference point. *

e) By taking components, turn (b) and (c) into six scalar equations in six unknowns. *

f) Solve these equations by hand or on the computer. *

g) Instead of using a system of equations try to find a single equation which can be solved for T_{EH}. Solve it and compare to your result from before. *

h) Challenge: For how many of the reactions can you find one equation which will tell you that particular reaction without knowing any of the other reactions? [Hint, try angular momentum balance about various axes as well as linear momentum balance in an appropriate direction. It is possible to find five of the six unknown reaction components this way.] Must these solutions agree with (d)? Do they?



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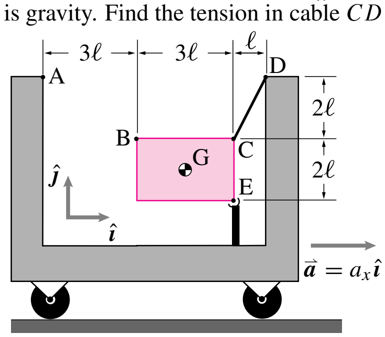
Problem 12.2.41

12.2.42 A uniform rectangular plate of mass m is supported by an inextensible cable CD and a hinge joint at point E on the cart as shown. The hinge joint is attached to a rigid column welded to the floor of the cart. The cart has acceleration $a_x\hat{\boldsymbol{\imath}}$. There is gravity. Find the tension in cable CD.



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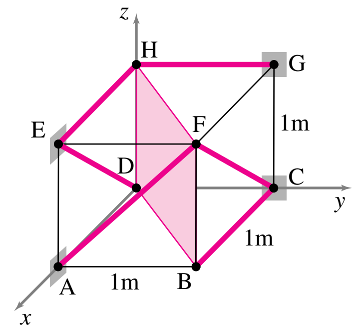
Problem 12.2.42

12.2.43 The uniform 2 kg plate DBFH is held by six massless rods (AF, CB, CF, GH, ED, and EH) which are hinged at their ends. The support points A, C, G, and E are all accelerating in the x-direction with acceleration $\boldsymbol{a} = 3\,\mathrm{m/s^2}\boldsymbol{\imath}$. There is no gravity.

a) What is $\{\sum \vec{\boldsymbol{F}}\}\cdot\hat{\boldsymbol{\imath}}$ for the forces acting on the plate?

b) What is the tension in bar CB?



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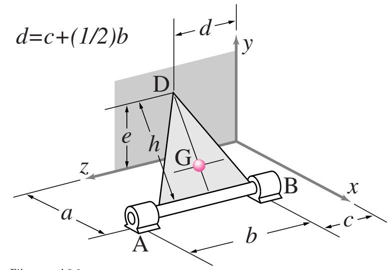
Problem 12.2.43

12.2.44 A massless triangular plate rests against a frictionless wall of a pick-up truck at point D and is rigidly attached to a massless rod supported by two ideal bearings fixed to the floor of the pick-up truck. A ball of mass m is fixed to the centroid of the plate. There is gravity. The pick-up truck skids across a road with acceleration $\vec{a} = a_x\hat{\boldsymbol{\imath}} + a_z\hat{\boldsymbol{k}}$. What is the reaction at point D on the plate?



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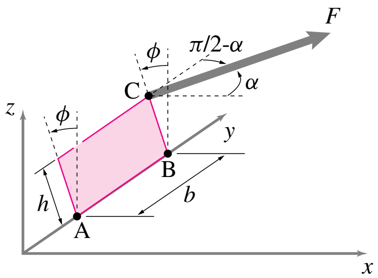
Problem 12.2.44

12.2.45 Towing a bicycle. A bicycle on the level xy plane is steered straight ahead and is being towed by a rope. The bicycle and rider are modeled as a uniform plate with mass m (for the convenience of the artist). The tow force F applied at C has no z component and makes an angle α with the x axis. The rolling wheel contacts are at A and B. The bike is tipped an angle ϕ from the vertical. The towing force F is the magnitude needed to keep the bike accelerating in a straight line (along the y axis) without tipping any more or less than the angle ϕ. What is the acceleration of the bicycle? Answer in terms of some or all of b, h, α, ϕ, m, g and $\hat{\boldsymbol{j}}$ (Note: F should *not* appear in your final answer.)



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Problem 12.2.45

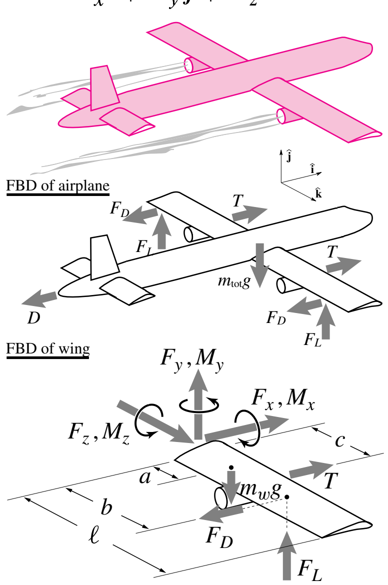
12.2.46 An airplane is in straight level flight but is accelerating in the forward direction. In terms of some or all of the following parameters,

- m_{tot} $\equiv$ the total mass of the plane (including the wings),
- D= the drag force on the fuselage,
- F_D= the drag force on each wing,
- g= gravitational constant, and,
- T = the thrust of one engine.

a) What is the lift on each wing F_L? *

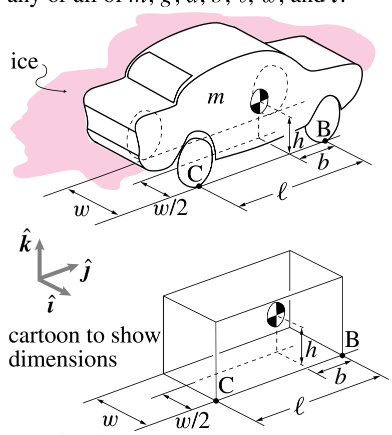
b) What is the acceleration of the plane $\vec{a}_P$? *

c) A free body diagram of one wing is shown. The mass of one wing is m_w. What, in terms of m_{tot}, m_w, F_L, F_D, g, a,b, c, and ℓ are the reactions at the base of the wing (where it is attached to the plane), $\vec{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$ and $\vec{M} = M_x\hat{\imath} + M_y\hat{\jmath} + M_z\hat{k}$? *



Filename:pfigure3-airplane
Problem 12.2.46

12.2.47 A rear-wheel drive car on level ground. The two left wheels are on perfectly slippery ice. The right wheels are on dry pavement. The negligible-mass front right wheel at B is steered straight ahead and rolls without slip. The right rear wheel at C also rolls without slip and drives the car forward with velocity $\vec{v} = v\hat{\jmath}$ and acceleration $\vec{a} = a\hat{\jmath}$. Dimensions are as shown and the car has mass m . What is the sideways force from the ground on the right front wheel at B? Answer in terms of any or all of m, g, a, b, ℓ, w, and $\hat{\imath}$. *



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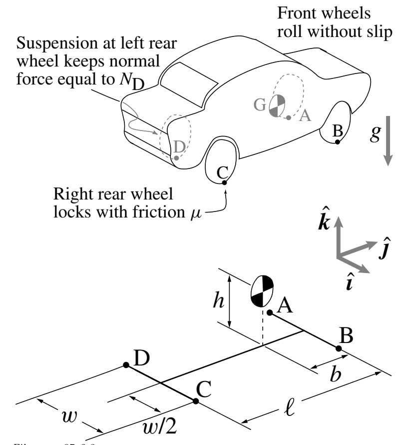
Problem 12.2.47: The left wheels of this car are on ice.

12.2.48 A somewhat crippled car slams on the brakes. The suspension springs at A, B, and C are frozen and keep the car level and at constant height. The normal force at D is kept equal to N_D by the only working suspension spring which is on the left rear wheel at D. The only brake which is working is that of the right rear wheel at C which slides on the ground with friction coefficient μ. Wheels A, B, and D roll freely without slip. Dimensions are as shown.

a) Find the acceleration of the car in terms of some or all of $m, w, \ell, b, h, g, \mu, \hat{\jmath}$, and N_D.

b) From the information given could you also find all of the reaction forces at all of the wheels? If so, why? If not, what can't you find and why? (No credit for correct answer. Credit depends on clear explanation.)



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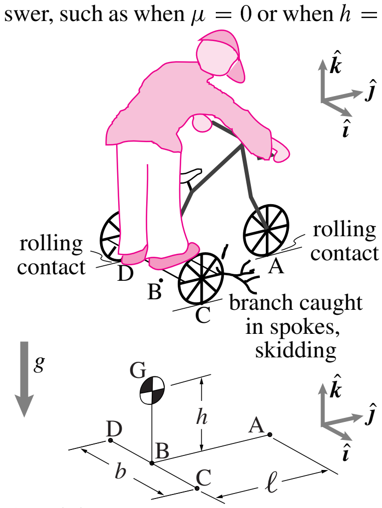
Problem 12.2.48: Given normal force at D, skidding at C.

12.2.49 Speeding tricycle gets a branch caught in the right rear wheel. A scared-stiff tricyclist riding on level ground gets a branch stuck in the right rear wheel so the wheel skids with friction coefficient μ. Assume that the center of mass of the tricycle-person system is directly above the rear axle. Assume that the left rear wheel and the front wheel have negligible mass, good bearings, and have sufficient friction that they roll in the $\hat{\jmath}$ direction without slip, thus constraining the overall motion of the tricycle. Dimensions are shown in the lower sketch. *Find the acceleration of the tricycle* (in terms of some or all of ℓ, h, b, m, $[I^{cm}]$, μ, g, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$). [Hint: check your answer against special cases for which you might guess the answer, such as when $\mu = 0$ or when $h = 0$.]



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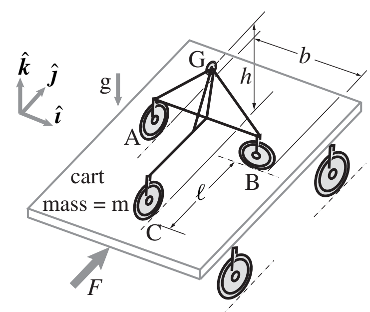
Problem 12.2.49: Right rear wheel skids.

12.2.50 A 3-wheeled robot. A 3-wheeled robot with mass m is being transported on a level flatbed trailer also with mass m. The trailer is being pushed with a force $F\hat{\boldsymbol{\jmath}}$. The ideal massless trailer wheels roll without slip. The ideal massless robot wheels also roll without slip. The robot steering mechanism has turned the wheels so that wheels at A and C are free to roll in the $\hat{\boldsymbol{\jmath}}$ direction and the wheel at B is free to roll in the $\hat{\boldsymbol{\imath}}$ direction. The center of mass of the robot at G is h above the trailer bed and symmetrically above the axle connecting wheels A and B. The wheels A and B are a distance b apart. The length of the robot is ℓ.

Find the force vector $\vec{\boldsymbol{F}}_A$ of the trailer on the robot at A in terms of some or all of $m, g, \ell, F, b, h, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$. [Hints: Use a free body diagram of the cart with robot to find their acceleration. With reference to a free body diagram of the robot, use angular momentum balance about axis BC to find F_{Az}.]



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Problem 12.2.50

CHAPTER 13

Circular motion

After movement on straight-lines the second important special case of motion is rotation on a circular path. Polar coordinates and base vectors are introduced in this simplest possible context. The key new idea is that not just coordinates, but base vectors, can change with time. The primary applications are pendulums, gear trains, and rotationally accelerating motors or brakes.

Contents

We covered the special case of straight-line motion in the previous chapter. But an unconstrained particle, such as a thrown ball, generally moves on a curved path as pushed by gravity and aerodynamic forces. Also, when a rigid object moves, it translates and rotates while the points on the object move on complicated curved paths. Now we consider the archetypal curved motion, motion on along a circular path. Circular motion deserves special attention because

- the most common connection between moving parts on a machine is with a bearing (or hinge or axle) (fig. 13.1), if the axle on one part is fixed then all points on the part move in circles;
- circular motion is the simplest case of curved-path motion;
- circular motion provides a simple way to introduce time-varying base vectors;
- circular motion includes most of the conceptual ingredients of more general curved motions;
- at least in 2 dimensions, the only way two particles on one rigid object can move relative to each other is by circular motion (no matter how the object is moving); and
- circular motion is the simplest case with which to introduce two important rigid-object concepts:
  - angular velocity, and
  - moment of inertia.

Many useful calculations can be made by approximating the motion of particles as circular. For example, the motions of points on a jet engine's turbine blade, a car engine's crank shaft, a car's wheel, a windmill's propeller, the earth spinning about its axis, a clock pendulum or watch balance wheel, all the points on a bicycle when it is going around a corner, a satellite orbiting the earth or a spinning satellite going around its spin axis, might all be approximately described as having circular motion about some appropriate point or axis.

This chapter concerns only motion in two dimensions. The first two sections consider the kinematics and mechanics of a single particle going in circles. The later sections concern the kinematics and mechanics of rigid objects. A later chapter discusses circular motion, which is always planar, in a three-dimensional context.

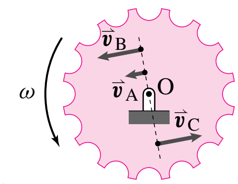

Figure 13.1: All the points on a gear move in circles, assuming the axle is not translating.

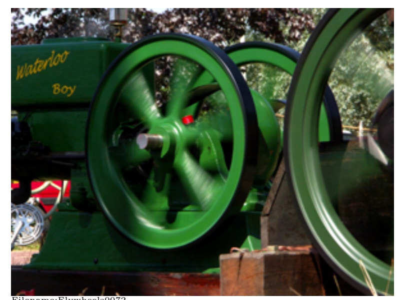

Figure 13.2: All points on a flywheel move in circles. The points further from the center move faster, so they mover further while the camera shutter is open, so they show a bigger motion blur (the constant-thickness spokes look fatter near the rim).

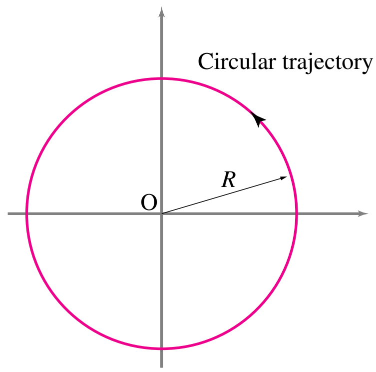

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Figure 13.3: Trajectory of particle for circular motion.

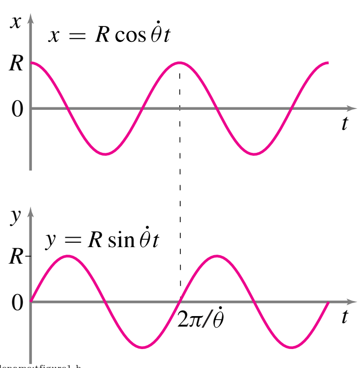

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Figure 13.4: Plots of x versus t and y versus t for a particle going in a circle of radius R at constant rate. For simplicity we assumed constant $\dot{\theta}$ with $\theta = \dot{\theta}t$. So both x and y vary as sinusoidal functions of time: $x = R\cos(\dot{\theta}t)$ and $y = R\sin(\dot{\theta}t)$.

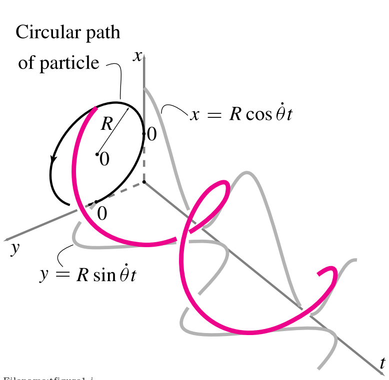

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Figure 13.5: Plot of x and y versus time for a particle going in circles at constant rate. x versus t is a cosine curve, y versus t is a sine curve. Together they make up the 3D helix.

Mechanics of circular motion

For the systems in this chapter, for every system we show in a free body diagram we have, as always,

$$\text{linear momentum balance,} \qquad \sum \vec{F}_i = \dot{\vec{L}}$$

$$\text{angular momentum balance,} \qquad \sum \vec{M}_{i/C} = \dot{\vec{H}}_{/C},$$

$$\text{and power balance:} \qquad P = \dot{E}_K + \dot{E}_P + \dot{E}_{int}.$$

Because you already know how to work with forces and moments (the left sides of the top two equations), the primary new skill in this chapter is the evaluation of $\dot{\vec{L}}$, $\dot{\vec{H}}_{/C}$, and $\dot{E}_K$ for a rotating particle or rigid object. That is, you need to understand the position, velocity and acceleration of points moving in circles. The rest of the skills used are universal, for example solving algebraic or differential equations, plotting, etc.

13.1 Kinematics of a particle in planar circular motion

This section concerns the position, velocity and accelerations of one point going in circles. The essence of the content here is this:

> If $\hat{\boldsymbol{e}}_R$ is a unit vector in the plane that is rotating counter-clockwise (CCW) at a rate of $\dot{\theta}$ its rate of change is
> $$\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$$
> where $\hat{\boldsymbol{e}}_\theta$ is a unit vector given by rotating $\hat{\boldsymbol{e}}_R$ 90° CCW.

If you learn this idea inside and out then either you will have picked up all the other key facts on the way, or you will be able to learn them in a flash. Note, we use $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{r}}$ interchangeably. Likewise for $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_R$.

Circular motion

The position of a particle going in circles around the origin on the xy plane is

$$\vec{r} = R\cos\theta\,\hat{\imath} + R\sin\theta\,\hat{\jmath},$$

with the radius R a constant. Or, in terms of components,

$$x = R\cos\theta \qquad \text{and} \qquad y = R\sin\theta.$$

A natural graphical representation of this motion is a circle (fig. 13.3). Unfortunately, a picture of the circular trajectory doesn't give any information about the speed of the particle on the circle. A plot of a particle moving in circles slowly looks just like a plot of a particle moving quickly.

To get a sense of how position changes in time one can plot the functions $x(t)$ and $y(t)$ (fig. 13.4). Unfortunately this figure only indirectly conveys that the particle is going in circles.

If you want to see both the trajectory and the time history of both variables one can make a 3-D plot of xy position versus time (fig. 13.4). The shadows of this helix on the three coordinate planes are the three graphs just discussed.

Finally, rather than representing time as a spatial coordinate, one can represent time with time itself. How? Make an animated movie showing a particle on the xy plane as it moves. Move your finger around in circles on the table. That's it. Similarly, you can make a dot move in circles on your screen. How do you make all these plots? Using a calculator or computer you can evaluate x and y for a range of values of t. Then, using pencil and paper, a plotting calculator, or a computer, plot x *vs* t, y *vs* t, and y *vs* x. For animations plot x and y over and over again for a sequence of values of t, and show these on your screen at a sequence of times.

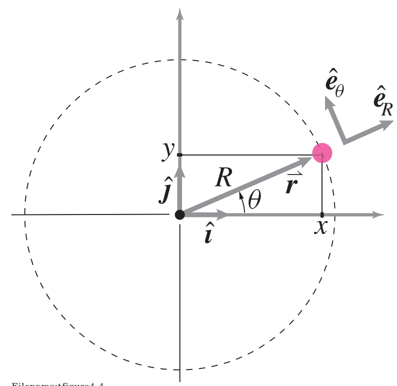

Figure 13.6: The position vector $\vec{r}$ of the particle relative to the center of the circle is $\vec{r}$ (or $\vec{R}$) which is both
$\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and
$\vec{r} = R\hat{e}_R$ (or $r\hat{e}_r$).
$\vec{r}$ makes an angle θ measured counterclockwise from the positive x-axis. The unit vectors $\hat{e}_R$ and $\hat{e}_\theta$ are in the radial and tangential directions, the directions of increasing R and increasing θ.

Polar coordinates R and θ and unit vectors $\hat{e}_R$ and $\hat{e}_\theta$

Especially for circular motion, it is convenient to represent position, velocity and acceleration with polar, rather than rectangular, coordinates. With polar coordinates we use polar base vectors which, unlike the fixed $\hat{\imath}$ and $\hat{\jmath}$, rotate as the particle goes around. Let's redraw fig. 13.3 and show the unit base vectors

$\hat{e}_R$ ('e R') and $\hat{e}_\theta$ ('e theta').

The radial unit vector $\hat{e}_R$ is directed from the center of the circle towards the point of interest and the transverse vector $\hat{e}_\theta$, perpendicular to $\hat{e}_R$, is tangent to the circle at that point in the direction of increasing θ. As the particle goes around, its $\hat{e}_R$ and $\hat{e}_\theta$ unit vectors change accordingly. Two different particles both going in circles with the same center at the same rate each have their own $\hat{e}_R$ and $\hat{e}_\theta$ vectors. To be precise we can define $\hat{e}_R$ and $\hat{e}_\theta$ as

$$\hat{e}_R \equiv \vec{R}/|\vec{R}| = \vec{R}/R \qquad \text{and} \qquad \hat{e}_\theta \equiv \hat{k} \times \hat{e}_R. \tag{13.1}$$

Note that also $\hat{e}_R \times \hat{e}_\theta = \hat{k}$.

The velocity and acceleration of a point going in circles, using polar coordinates

In dynamics we are interested in velocity and acceleration so we need to know how to represent these in polar coordinates. First, observe that the position of the particle is (see fig. 13.6)

$$\vec{R} = R\hat{e}_R. \tag{13.2}$$

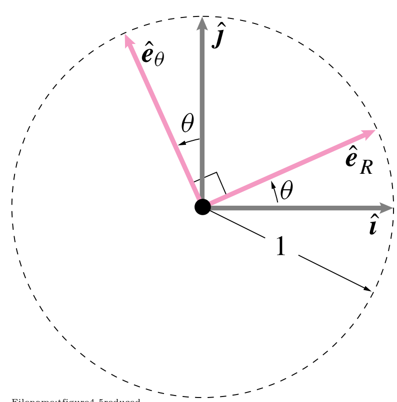

Figure 13.7: You can think of the unit vectors $\hat{e}_R$ and $\hat{e}_\theta$ as the set $\hat{\imath}$ and $\hat{\jmath}$ rotated counter-clockwise by the angle θ.

That is, the position vector is the distance from the origin times a unit vector in the direction of the particle's position. Given the position, it is just a

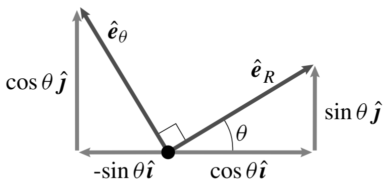

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Figure 13.8: Projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y directions. From this picture you can immediately extract that
$\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$ and that
$\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$.
A similar picture showing the projections of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in the $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ d directions would show that
$\hat{\boldsymbol{\imath}} = \cos\theta\hat{\boldsymbol{e}}_R - \sin\theta\hat{\boldsymbol{e}}_\theta$ and that
$\hat{\boldsymbol{\jmath}} = \sin\theta\hat{\boldsymbol{e}}_R + \cos\theta\hat{\boldsymbol{e}}_\theta$.

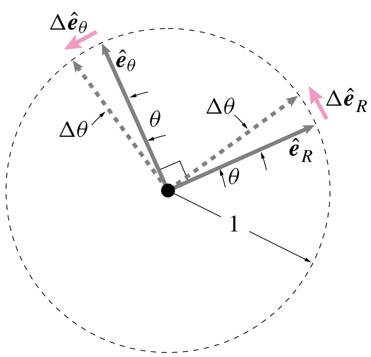

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Figure 13.9: A close up view of the unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$. They make an angle θ with the positive x and y-axis, respectively. As the particle advances an amount $\Delta\theta$ both $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ change. In particular, for small $\Delta\theta$, $\Delta\hat{\boldsymbol{e}}_R$ is approximately in the $\hat{\boldsymbol{e}}_\theta$ direction and $\Delta\hat{\boldsymbol{e}}_\theta$ is approximately in the $-\hat{\boldsymbol{e}}_R$ direction.

matter of *careful* differentiation to find velocity and acceleration. Here is one of many possible ways to derive the polar-coordinate expressions for velocity and acceleration. First, velocity is the time derivative of position, so

$$\vec{\boldsymbol{v}} = \frac{d}{dt}\vec{\boldsymbol{R}} = \frac{d}{dt}(R\hat{\boldsymbol{e}}_R) = \underbrace{\dot{R}}_{0}\,\hat{\boldsymbol{e}}_R + R\dot{\hat{\boldsymbol{e}}}_R. \tag{13.3}$$

Because a circle has constant radius R, $\dot{R}$ is zero. But how do we calculate the rate of change of $\hat{\boldsymbol{e}}_R$ with respect to time, $\dot{\hat{\boldsymbol{e}}}_R$,?

Derivatives of $\hat{e}_R$ and of $\hat{e}_\theta$

To find the velocity in polar coordinates we were just confronted with the problem of finding $\dot{\hat{\boldsymbol{e}}}_R$.

Method 1: One way to find $d\hat{\boldsymbol{e}}_R/dt = \dot{\hat{\boldsymbol{e}}}_R$ uses the geometry of fig. 13.9 and the informal calculus of finite differences (represented by Δ). $\Delta\hat{\boldsymbol{e}}_R$ is evidently (about) in the direction $\hat{\boldsymbol{e}}_\theta$ and has magnitude $\Delta\theta$ so $\Delta\hat{\boldsymbol{e}}_R \approx (\Delta\theta)\hat{\boldsymbol{e}}_\theta$. Dividing by Δt, we have $\Delta\hat{\boldsymbol{e}}_R/\Delta t \approx (\Delta\theta/\Delta t)\hat{\boldsymbol{e}}_\theta$. So, using this sloppy calculus, we get $\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$. Similarly, and we will need this shortly, we could get $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$.

Method 2 This method is a little less geometric and a little more algebraic. We start with the decomposition of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ into cartesian coordinates. These decompositions are found by looking at the projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y-directions (see fig. 13.8).

$$\begin{aligned} \hat{\boldsymbol{e}}_R &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} \end{aligned} \tag{13.4}$$

We can find $\dot{\hat{\boldsymbol{e}}}_R$ by differentiating, taking into account that θ is changing with time but that the unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are fixed (so they don't change with time).

$$\begin{aligned} \dot{\hat{\boldsymbol{e}}}_R &= \frac{d}{dt}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\sin\theta\hat{\boldsymbol{\imath}} + \dot{\theta}\cos\theta\,\hat{\boldsymbol{\jmath}} = \dot{\theta}\hat{\boldsymbol{e}}_\theta \\ \dot{\hat{\boldsymbol{e}}}_\theta &= \frac{d}{dt}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\hat{\boldsymbol{e}}_R \end{aligned}$$

We had to use the chain rule, that is

$$\frac{d\,\sin\theta(t)}{dt} = \frac{d\,\sin\theta}{d\theta}\frac{d\theta(t)}{dt} = \dot{\theta}\cos\theta.$$

Now, two different ways, we know

$$\dot{\hat{e}}_R = \dot{\theta}\hat{e}_\theta \quad \text{and} \quad \dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_R. \tag{13.5}$$

Continuing the quest for velocity and acceleration

Now that we know how $\hat{e}_R$ changes in time we can continue our quest for $\vec{v}$. Continuing from eqn. (13.3) we now have

$$\vec{v} = \dot{\vec{R}} = R\dot{\hat{e}}_R = R\dot{\theta}\hat{e}_\theta. \tag{13.6}$$

Similarly we can find the acceleration $\ddot{\vec{R}}$ by differentiating once again,

$$\vec{a} = \ddot{\vec{R}} = \dot{\vec{v}} = \frac{d}{dt}(R\dot{\theta}\hat{e}_\theta) = \underbrace{\dot{R}\dot{\theta}\hat{e}_\theta}_{\vec{0}} + R\ddot{\theta}\hat{e}_\theta + R\dot{\theta}\dot{\hat{e}}_\theta \tag{13.7}$$

The first term on the right hand side is zero because $\dot{R}$ is 0 for circular motion. The third term is evaluated using the formula we just found for the rate of change of $\hat{e}_\theta$: $\dot{\hat{e}}_\theta = -\dot{\theta}\hat{e}_R$. So, using that $\vec{R} = R\hat{e}_R$,

$$\vec{a} = -\dot{\theta}^2\vec{R} + R\ddot{\theta}\hat{e}_\theta = -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta \tag{13.8}$$

The velocity $\vec{v}$ and acceleration $\vec{a}$ for a particle going in circles at constant rate are shown in fig. 13.10.

Example: A person standing on the earth's equator
A person standing on the equator has velocity

$$\begin{aligned} \vec{v} = \dot{\theta}R\hat{e}_\theta &\approx \left(\frac{2\pi \text{ rad}}{24 \text{ hr}}\right) 4000 \text{ mi}\hat{e}_\theta \\ &\approx 1050 \text{ mph}\hat{e}_\theta \approx 1535 \text{ ft/s}\hat{e}_\theta \end{aligned}$$

and acceleration

$$\begin{aligned} \vec{a} = -\dot{\theta}^2 R\hat{e}_R &\approx -\left(\frac{2\pi \text{ rad}}{24 \text{ hr}}\right)^2 4000 \text{ mi}\hat{e}_R \\ &\approx -274 \text{ mi/hr}^2\hat{e}_R \approx -0.11 \text{ ft/s}^2\hat{e}_R. \end{aligned}$$

The velocity of a person standing on the equator, due to the earth's rotation, is about 1000 mph tangent to the earth. Her acceleration is about $0.11 \text{ ft/s}^2 \approx 0.03 \text{ m/s}^2$ towards the center of the earth, about $1/300$ of g, about $1/300$ the acceleration of a an object in near-earth-surface frictionless free-fall.

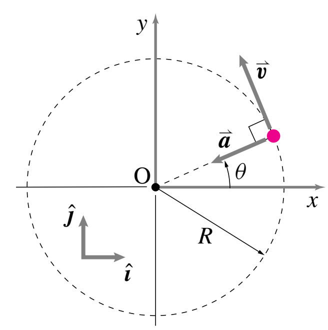

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Figure 13.10: The directions of velocity $\vec{v}$ and acceleration $\vec{a}$ are shown for a particle going in circles at *constant* rate. The velocity is tangent to the circle and the acceleration is directed towards the center of the circle.

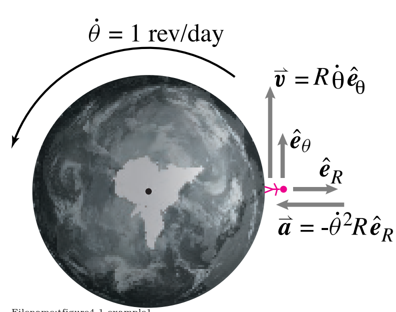

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Figure 13.11:

Alternate expressions for the velocity and acceleration formulas

Note that we can define a scalar velocity $v = R\dot{\theta}$. We informally call this scalar the speed even though it can be positive or negative. So

$$\vec{v} = R\dot{\theta}\hat{e}_\theta = v\hat{e}_\theta.$$

Similarly the acceleration is

$$\vec{\boldsymbol{a}} = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta = -\frac{v^2}{R}\hat{\boldsymbol{e}}_R + \dot{v}\hat{\boldsymbol{e}}_\theta.$$

where $\dot{v}$ is the rate of change of tangential speed①. Thus the acceleration is made of two terms. One proportional to the speed squared and directed towards the center of the circle, and one proportional to the rate of change of speed and directed tangent to the circle.

①**Caution:** Note that the rate of change of speed is not the magnitude of the acceleration: $\dot{v} \neq |\vec{\boldsymbol{a}}|$ or in other words: $\frac{d}{dt}|\vec{\boldsymbol{v}}| \neq |\frac{d}{dt}\vec{\boldsymbol{v}}|$. Consider the case of a car driving in circles at constant rate. Its rate of change of speed is zero, yet it has an acceleration.

Centripetal acceleration

The term

$$-R\dot{\theta}^2\hat{\boldsymbol{e}}_R = -\dot{\theta}^2\vec{\boldsymbol{R}}$$

is called the *centripetal* acceleration. Why, intuitively, is the centripetal acceleration proportional to the speed *squared*? Well, the acceleration is the change in the velocity vector per unit time. There are two effects

1. If the speed is twice as big then the velocity is twice as big.
2. And, for a given radius, the angle it rotates per unit time is twice as big.

These two proportionalities with speed, the size of the velocity vector which rotates and the rate at which it rotates both apply. So if the speed is twice as big the acceleration is 4 times as big. Hence the v^2.

13.1 Summary: the motion quantities

We can use our results for velocity and acceleration to better evaluate the momenta and energy quantities. These results will allow us to do mechanics problems associated with circular motion. For one particle in circular motion we have:

Linear momentum:

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}m = R\dot{\theta}\hat{\boldsymbol{e}}_\theta m,$$

Rate of change of linear momentum:

$$\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}m = (-\dot{\theta}^2\vec{\boldsymbol{R}} + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta)m,$$

Angular momentum:

$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{v}}m = R^2\dot{\theta}m\hat{\boldsymbol{k}},$$

Rate of change of angular momentum:

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{a}}m = R^2\ddot{\theta}m\hat{\boldsymbol{k}},$$

Kinetic energy:

$$E_\mathrm{K} = \frac{1}{2}v^2m = \frac{1}{2}R^2\dot{\theta}^2m, \text{and}$$

Rate of change of kinetic energy:

$$\dot{E}_\mathrm{K} = \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}}\,m = mR^2\dot{\theta}\ddot{\theta}$$

SAMPLE 13.1 The velocity vector in circular motion. A particle executes circular motion in the xy plane with constant speed $v = 5\,\mathrm{m/s}$. At $t = 0$ the particle is at $\theta = 0$. Given that the radius of the circular orbit is $2.5\,\mathrm{m}$, find the velocity of the particle at $t = 2\,\mathrm{sec}$.

Solution It is given that

$$\begin{aligned} R &= 2.5\,\mathrm{m} \\ v &= \text{constant} = 5\,\mathrm{m/s} \\ \theta(t=0) &= 0. \end{aligned}$$

The velocity of a particle in constant-rate circular motion is:

$$\begin{aligned} \vec{v} &= R\dot{\theta}\hat{e}_\theta \\ \text{where} \quad \hat{e}_\theta &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \end{aligned}$$

Since R is constant and $v = |\vec{v}| = R\dot{\theta}$ is constant,

$$\dot{\theta} = \frac{v}{R} = \frac{5\,\mathrm{m/s}}{2.5\,\mathrm{m}} = 2\,\mathrm{rad/s}$$

is also constant. Thus,

$$\vec{v}(t=2\,\mathrm{s}) = \underbrace{R\dot{\theta}}_{v}\,\hat{e}_\theta\Big|_{t=2\,\mathrm{s}} = 5\,\mathrm{m/s}\,\hat{e}_\theta(t=2\,\mathrm{s}).$$

Clearly, we need to find $\hat{e}_\theta$ at $t = 2\,\mathrm{sec}$.

$$\begin{aligned} \text{Now} \quad \dot{\theta} &\equiv \frac{d\theta}{dt} = 2\,\mathrm{rad/s} \\ \Rightarrow \quad \int_0^\theta d\theta &= \int_0^{2\,\mathrm{s}} 2\,\mathrm{rad/s}\,dt \\ \Rightarrow \quad \theta &= (2\,\mathrm{rad/s})\,t\Big|_0^{2\,\mathrm{s}} \\ &= 2\,\mathrm{rad/s}\cdot 2\,\mathrm{s} \\ &= 4\,\mathrm{rad}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{e}_\theta &= -\sin 4\hat{\imath} + \cos 4\hat{\jmath} \\ &= 0.76\hat{\imath} - 0.65\hat{\jmath}, \end{aligned}$$

and

$$\begin{aligned} \vec{v}(2\,\mathrm{s}) &= 5\,\mathrm{m/s}(0.76\hat{\imath} - 0.65\hat{\jmath}) \\ &= (3.78\hat{\imath} - 3.27\hat{\jmath})\,\mathrm{m/s}. \end{aligned}$$

$$\boxed{\vec{v} = (3.78\hat{\imath} - 3.27\hat{\jmath})\,\mathrm{m/s}}$$

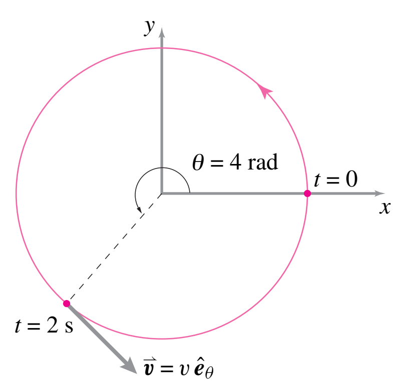

Figure 13.12: The velocity vector $\vec{v}$ at $t = 2\,\mathrm{s}$.

SAMPLE 13.2 Basic kinematics: A point mass executes circular motion with angular acceleration $\ddot{\theta} = 5\,\text{rad/s}^2$. The radius of the circular path is 0.25 m. If the mass starts from rest at $\theta = 0^\circ$, find and draw

1. the velocity of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°,
2. the acceleration of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°.

Solution We are given, $\ddot{\theta} = 5\,\text{rad/s}^2$, and $R = 0.25\,\text{m}$.

1. The velocity $\vec{v}$ in circular (constant or non-constant rate) motion is given by:

$$\vec{v} = R\dot{\theta}\hat{e}_\theta.$$

So, to find the velocity at different positions we need $\dot{\theta}$ at those positions. Here the angular acceleration is constant, i.e., $\ddot{\theta} = 5\,\text{rad/s}^2$. Therefore, we can use the formula ②

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\ddot{\theta}\theta$$

to find the angular speed $\dot{\theta}$ at various θ's. But $\dot{\theta}_0 = 0$ (mass starts from rest), therefore $\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$. Now we make a table for computing the velocities at different positions:

Position (θ)	θ in radians	$\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$	$\vec{v} = R\dot{\theta}\hat{e}_\theta$
0°	0	0 rad/s	$\vec{0}$
30°	$\pi/6$	$\sqrt{10\pi/6} = 2.29\,\text{rad/s}$	$0.57\,\text{m/s}\,\hat{e}_\theta$
90°	$\pi/2$	$\sqrt{10\pi/2} = 3.96\,\text{rad/s}$	$0.99\,\text{m/s}\,\hat{e}_\theta$
210°	$7\pi/6$	$\sqrt{70\pi/6} = 6.05\,\text{rad/s}$	$1.51\,\text{m/s}\,\hat{e}_\theta$

The computed velocities are shown in Fig. 13.13.

② We use this formula because we need $\dot{\theta}$ at different values of θ. In elementary physics books, the same formula is usually written as

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\alpha\theta$$

where α is the constant angular acceleration and $\dot{\theta}(= \dot{\theta})$ is the angular speed.

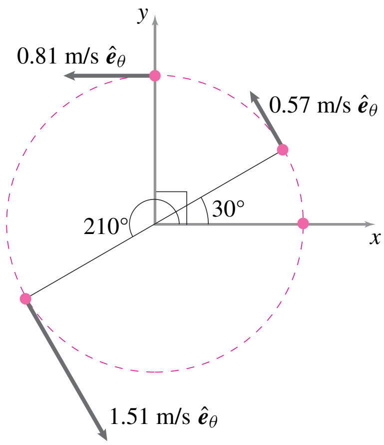

Figure 13.13: Velocity of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°.

2. The acceleration of the mass is given by

$$\begin{aligned}\vec{a} &= \overbrace{a_R\hat{e}_R}^{\text{radial}} + \overbrace{a_\theta\hat{e}_\theta}^{\text{tangential}}\\ &= -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta.\end{aligned}$$

Since $\ddot{\theta}$ is constant, the tangential component of the acceleration is constant at all positions. We have already calculated $\dot{\theta}$ at various positions, so we can easily calculate the radial (also called the normal) component of the acceleration. Thus we can find the acceleration. For example, at $\theta = 30^\circ$,

$$\begin{aligned}\vec{a} &= -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta\\ &= -0.25\,\text{m}\cdot\frac{10\pi}{6}\frac{1}{\text{s}^2}\hat{e}_R + 0.25\,\text{m}\cdot 5\frac{1}{\text{s}^2}\hat{e}_\theta\\ &= -1.31\,\text{m/s}^2\hat{e}_R + 1.25\,\text{m/s}^2\hat{e}_\theta.\end{aligned}$$

Similarly, we find the acceleration of the mass at other positions by substituting the values of R, $\ddot{\theta}$ and $\dot{\theta}$ in the formula and tabulate the results in the table below.

Position (θ)	$a_r = -R\dot{\theta}^2$	$a_\theta = R\ddot{\theta}$	$\vec{a} = a_r \hat{e}_R + a_\theta \hat{e}_\theta$
0°	0	$1.25\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}\,\hat{e}_\theta$
30°	$-1.31\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-1.31\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$
90°	$-3.93\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-3.93\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$
210°	$-9.16\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-9.16\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$

The accelerations computed are shown in Fig. 13.14. The acceleration vector as well as its tangential and radial components are shown in the figure at each position.

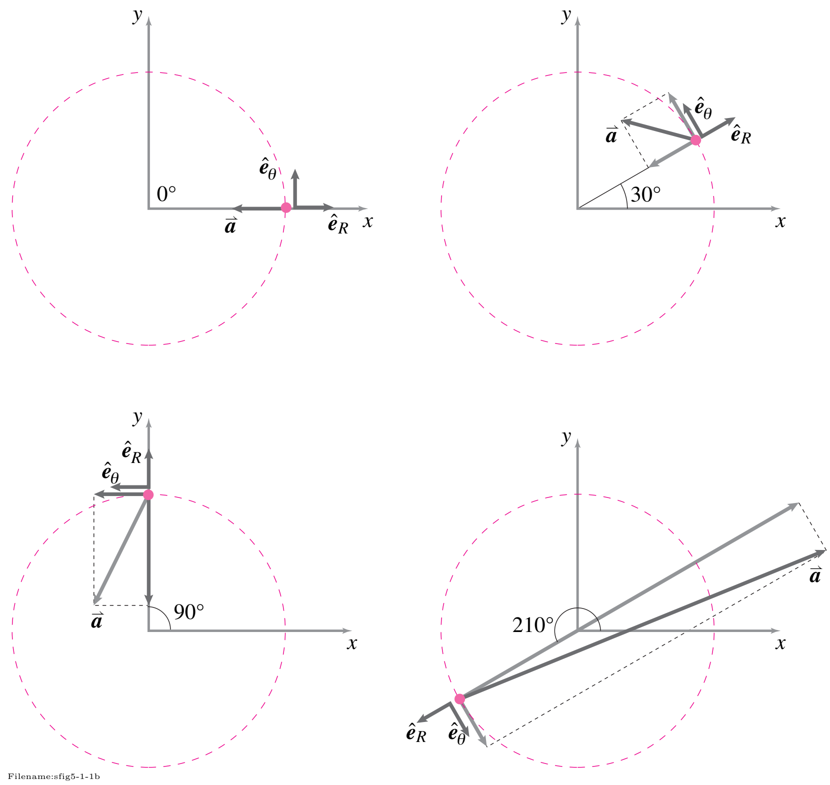

Figure 13.14: Acceleration of the mass at $\theta = 0^\circ$, 30°, 90°, and 210°. The radial and tangential components are shown with grey arrows. As the angular velocity increases, the radial component of the acceleration increases; therefore, the total acceleration vector leans more and more towards the radial direction.

SAMPLE 13.3 In an experiment, the magnitude of angular deceleration of a rotating ball is found to be proportional to its angular speed $\dot{\theta}$ (*i.e.*, $\ddot{\theta} \propto -\dot{\theta}$). Assume that the proportionality constant is k.

1. Find $\dot{\theta}$ as a function of t, given that $\dot{\theta}(t=0) = \dot{\theta}_0$.
2. Given that $k = 0.1/\,\text{s}$, how much time does it take for $\dot{\theta}$ to reduce to half the initial value?

Solution The equation given is:

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = -k\dot{\theta}. \tag{13.9}$$

1. We can solve this equation in a couple of ways.

Method-1: Let us guess a solution of the exponential form with arbitrary constants and plug it into eqn. (13.9) to check if our solution works. Let $\dot{\theta}(t) = C_1 e^{C_2 t}$. Substituting in eqn. (13.9), we get

$$\begin{aligned} C_1 C_2 e^{C_2 t} &= -k C_1 e^{C_2 t} \\ \Rightarrow \quad C_2 &= -k, \\ \text{also,} \quad \dot{\theta}(0) &= \dot{\theta}_0 = C_1 e^{C_2 \cdot 0} \\ \Rightarrow \quad C_1 &= \dot{\theta}_0. \end{aligned}$$

Therefore,

$$\dot{\theta}(t) = \dot{\theta}_0 e^{-kt}. \tag{13.10}$$

$$\boxed{\dot{\theta}(t) = \dot{\theta}_0\, e^{-kt}}$$

Method-2: Equation (13.9) can also be solved by direct integration as follows.

$$\begin{aligned} \frac{d\dot{\theta}}{\dot{\theta}} &= -k\,dt \\ \Rightarrow \quad \int_{\dot{\theta}_0}^{\dot{\theta}(t)} \frac{d\dot{\theta}}{\dot{\theta}} &= -\int_0^t k\,dt \\ \Rightarrow \quad \ln\dot{\theta}\Big|_{\dot{\theta}_0}^{\dot{\theta}(t)} &= -kt \\ \Rightarrow \quad \ln\left(\frac{\dot{\theta}(t)}{\dot{\theta}_0}\right) &= -kt \end{aligned}$$

Therefore,

$$\dot{\theta}(t) = \dot{\theta}_0\, e^{-kt},$$

which is the same solution as equation (13.10).

2. We need to find t for $\dot{\theta} = \dot{\theta}_0/2$, given that $k = 0.1$. From eqn. (13.10), we get

$$\begin{aligned} \frac{\dot{\theta}}{\dot{\theta}_0} &= e^{-kt} \\ \Rightarrow \quad t &= \frac{1}{-k}\ln\left(\frac{\dot{\theta}}{\dot{\theta}_0}\right) \\ &= \frac{1}{-0.1}\ln\left(\frac{1}{2}\right) = \frac{-0.693}{-0.1/\,\text{s}} = 6.93\,\text{s}. \end{aligned}$$

$$\boxed{t = 6.93\,\text{s for } \dot{\theta}(t) = \dot{\theta}_0/2}$$

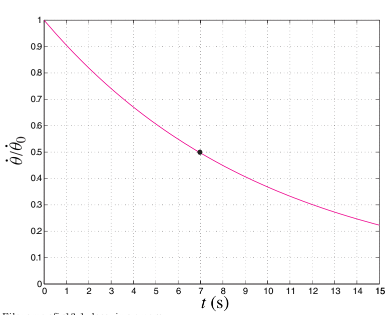

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Figure 13.15: Plot of $\dot{\theta}(t)/\dot{\theta}_0 = e^{-kt}$ for $k = 0.1/\,\text{s}$. The angular speed $\dot{\theta}$ reduces to half of its initial value in 6.93 s. Note that this time is the same for $\dot{\theta}$ to reduce to half its value at any given time t (not just at $t = 0$).

SAMPLE 13.4 **Using kinematic formulae:** The spinning wheel of a stationary exercise bike is brought to rest from 100 rpm by applying brakes over a period of 5 seconds.

1. Find the average angular deceleration of the wheel.
2. Find the number of revolutions it makes during the braking.

Solution We are given,

$$\dot{\theta}_0 = 100\,\text{rpm}, \quad \dot{\theta}_{\text{final}} = 0, \quad \text{and} \quad t = 5\,\text{s}.$$

1. Let α be the average (constant) deceleration. Then

$$\dot{\theta}_{\text{final}} = \dot{\theta}_0 - \alpha t.$$

Therefore,

$$\begin{aligned}
\alpha &= \frac{\dot{\theta}_0 - \dot{\theta}_{\text{final}}}{t} \\
&= \frac{100\,\text{rpm} - 0\,\text{rpm}}{5\,\text{s}} \\
&= \frac{100\,\text{rev}}{60\,\text{s}} \cdot \frac{1}{5\,\text{s}} \\
&= 0.33\frac{\text{rev}}{\text{s}^2}.
\end{aligned}$$

$$\boxed{\alpha = 0.33\frac{\text{rev}}{\text{s}^2}}$$

2. To find the number of revolutions made during the braking period, we use the formula

$$\theta(t) = \underbrace{\theta_0}_{0} + \dot{\theta}_0 t + \frac{1}{2}(-\alpha)t^2 = \dot{\theta}_0 t - \frac{1}{2}\alpha t^2.$$

Substituting the known values, we get

$$\begin{aligned}
\theta &= \frac{100\,\text{rev}}{60\,\text{s}} \cdot 5\,\text{s} - \frac{1}{2} 0.33\frac{\text{rev}}{\text{s}^2} \cdot 25\,\text{s}^2 \\
&= 8.33\,\text{rev} - 4.12\,\text{rev} \\
&= 4.21\,\text{rev}.
\end{aligned}$$

$$\boxed{\theta = 4.21\,\text{rev}}$$

Comments:

- Note the negative sign used in both the formulae above. Since α is deceleration, that is, a negative acceleration, we have used negative sign with α in the formulae.
- Note that it is not always necessary to convert rpm in rad/s. Here we changed rpm to rev/ s because time was given in seconds.

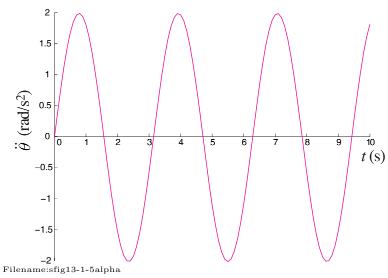

Figure 13.16: Time varying angular acceleration, $\ddot{\theta}(t) = c \sin \beta t$.

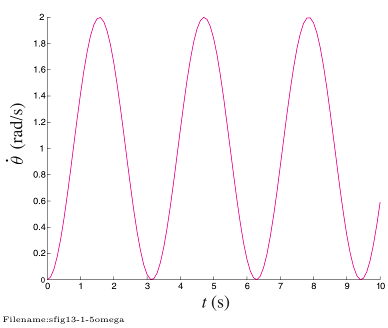

Figure 13.17: Angular speed, $\dot{\theta}(t) = \frac{c}{\beta}(1 - \cos \beta t)$, plotted against time for $c = 2\,\mathrm{rad/s^2}$ and $\beta = 2\,\mathrm{rad/s}$.

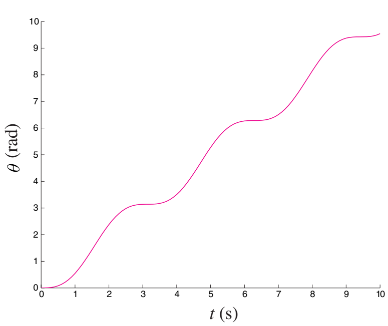

Figure 13.18: Angular position, $\theta(t) = \frac{c}{\beta^2}(\beta t - \sin \beta t)$, plotted against time for $c = 2\,\mathrm{rad/s^2}$ and $\beta = 2\,\mathrm{rad/s}$.

SAMPLE 13.5 Non-constant acceleration: A particle of mass 500 grams executes circular motion with radius $R = 100\,\mathrm{cm}$ and angular acceleration $\ddot{\theta}(t) = c \sin \beta t$, where $c = 2\,\mathrm{rad/s^2}$ and $\beta = 2\,\mathrm{rad/s}$.

1. Find the position of the particle after 10 seconds if the particle starts from rest, that is, $\theta(0) = 0$.
2. How much kinetic energy does the particle have at the position found above?

Solution

1. We are given $\ddot{\theta}(t) = c \sin \beta t$, $\dot{\theta}(0) = 0$ and $\theta(0) = 0$. We have to find $\theta(10\,\mathrm{s})$. Basically, we have to solve a second order differential equation with given initial conditions.

$$\begin{aligned} \ddot{\theta} \equiv \frac{d}{dt}(\dot{\theta}) &= c \sin \beta t \\ \Rightarrow \int_{\dot{\theta}_0 = 0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t c \sin \beta \tau \, d\tau \\ \dot{\theta}(t) &= -\frac{c}{\beta} \cos \beta \tau \Big|_0^t = \frac{c}{\beta}(1 - \cos \beta t). \end{aligned}$$

Thus, we get the expression for the angular speed $\dot{\theta}(t)$. We can solve for the position $\theta(t)$ by integrating once more:

$$\begin{aligned} \dot{\theta} \equiv \frac{d}{dt}(\theta) &= \frac{c}{\beta}(1 - \cos \beta t) \\ \Rightarrow \int_{\theta_0 = 0}^{\theta(t)} d\theta &= \int_0^t \frac{c}{\beta}(1 - \cos \beta \tau) \\ \theta(t) &= \frac{c}{\beta}\left[\tau - \frac{\sin \beta \tau}{\beta}\right]_0^t \\ &= \frac{c}{\beta^2}(\beta t - \sin \beta t). \end{aligned}$$

Now substituting $t = 10\,\mathrm{s}$ in the last expression along with the values of other constants, we get

$$\begin{aligned} \theta(10\,\mathrm{s}) &= \frac{2\,\mathrm{rad/s^2}}{(2\,\mathrm{rad/s})^2}[2\,\mathrm{rad/s} \cdot 10\,\mathrm{s} - \sin(2\,\mathrm{rad/s} \cdot 10\,\mathrm{s})] \\ &= 9.54\,\mathrm{rad}. \end{aligned}$$

$\theta = 9.54\,\mathrm{rad}$

2. The kinetic energy of the particle is given by

$$\begin{aligned} E_{\mathrm{K}} &= \frac{1}{2}mv^2 = \frac{1}{2}m(R\dot{\theta})^2 \\ &= \frac{1}{2}mR^2[\underbrace{\frac{c}{\beta}(1 - \cos \beta t)}_{\dot{\theta}(t)}]^2 \\ &= \frac{1}{2}0.5\,\mathrm{kg} \cdot 1\,\mathrm{m}^2 \cdot \left[\frac{2\,\mathrm{rad/s^2}}{2\,\mathrm{rad/s}} \cdot (1 - \cos(20))\right]^2 \\ &= 0.086\,\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^2 = 0.086\mathrm{Joule}. \end{aligned}$$

$E_{\mathrm{K}} = 0.086\mathrm{J}$

13.2 Dynamics of a particle in circular motion

The simplest examples of circular motion concern the motion of a particle constrained by a massless connection to be a fixed distance from a support point.

Example: Rock spinning on a string
Neglecting gravity, we can now deal with the familiar problem of a point mass being held in constant circular-rate motion by a massless string or rod. Linear momentum balance for the mass gives:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ \Rightarrow -T\hat{e}_R &= m\vec{a} \\ \{-T\hat{e}_R &= m(-\dot{\theta}^2\ell\hat{e}_R)\} \\ \{\}\cdot\hat{e}_R \Rightarrow & \quad T = \dot{\theta}^2\ell m = (v^2/\ell)m \end{aligned}$$

The force required to keep a mass in constant rate circular motion is mv^2/ℓ (sometimes remembered as mv^2/R).

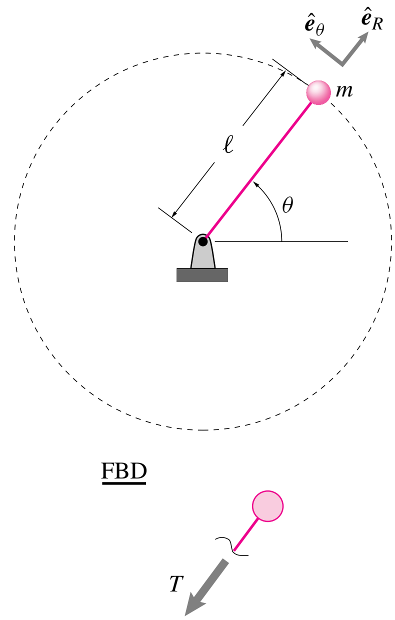

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Figure 13.19: Point mass spinning in circles. Sketch of system and a free body diagram.

The simplest example of 'celestial mechanics' is also circular motion.

Example: Geosynchronous orbit
Assuming a spherical earth, the centrally acing force of earth's gravity on a satellite is mg at the earth's surface and decays with radius squared so is

$$F = mg\frac{R_e^2}{r^2}$$

where R_e is the radius of the earth and r is the distance of the satellite from the center of the earth. Linear momentum balance for the mass gives:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ \Rightarrow -mg\frac{R_e^2}{r^2}\hat{e}_R &= m\vec{a} \\ \left\{-mg\frac{R_e^2}{r^2}\hat{e}_R \right. &= \left. m(-\dot{\theta}^2 r\hat{e}_R)\right\} \\ \{\}\cdot\hat{e}_R \Rightarrow & \quad r = \left(\frac{gR_e^2}{\dot{\theta}^2}\right)^{1/3}. \end{aligned}$$

Communication satellites in 'geosynchronous' orbits go around once a day (staying in the sites of millions of satellite dishes). So, using $g \approx 10\,\mathrm{m/s^2}$, $R_e \approx 6400\,\mathrm{km}$ and $\dot{\theta} \approx 1$ rev/day, we get $r = 42600\,\mathrm{km}$①.

Similar calculations can find the motion of low altitude sattelites, the motion of the moon around the earth and of the earth around the sun.

Centripital and centrifugal forces

Because the centrally directed part of a particle's acceleration is called the 'centripital' acceleration, the centrally directed force needed to keep a particle in circular motion is sometimes called the 'centripital' force. Thus, in the first example above the tension in the string is a centripital force, and in the

①There are various errors (approximations) in this satellite calculation, of course. The earth doesn't rotate once per day, but a little more because it goes around once per day relative to a line connecting the earth and sun and that line is itself rotating relative to the 'fixed stars' (the period of a geosynchronous sattelite is one part in 365.25 shorter than a day). And the force of gravity on a near-earth mass is a bit more than mg because 'g' actually measures the force it takes to hold up a mass on the earth's surface, which is the gravity force less the accelation from going in circles on the surface of the earth (the actual acceleration of gravity at the earth's surface is about 0.5% more than g). And the earth isn't exactly spherical, and so on. The actual geosynchronous radius is close to 42164 km. So the example calculation is off by 436 kmor about 1%.

satellite problem the gravity force is a centripital force. On the other hand, the 'centrifugal' force outwards is not really a force at all and is best dropped as a concept, at least for beginners.

Non-constant rate circular motion

Situations in which the circular rate is not constant are just slightly more complex. In these cases the part of the acceleration tangent to the circular motion is nonzero,

$$a_\theta = \ddot{\theta} r = \dot{v}_\theta,$$

so the net force on the particle has a component tangent to the circle.

Linear momentum balance in polar coordinates

The equation of linear momentum balance for a particle $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ in polar coordinates can be written as follows:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ \sum F_r \hat{\boldsymbol{e}}_R + \sum F_\theta \hat{\boldsymbol{e}}_\theta &= m\left(a_r \hat{\boldsymbol{e}}_R + a_\theta \hat{\boldsymbol{e}}_\theta\right). \end{aligned} \quad \begin{aligned} &(13.11) \\ &(13.12) \end{aligned}$$

For circular motion we have, from Section 13.1, that

$$a_r = -\dot{\theta}^2 r = -\frac{v^2}{r} \qquad \text{and} \qquad a_\theta = \ddot{\theta} r = \dot{v}.$$

Angular momentum

In general one can use angular momentum balance with respect to any point you like. But for circular motion with the circle center at 0 one is almost always concerned with angular momentum balance about 0. In this case the various torque and angular momentum expressions are particularly simple, for example

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{/0} &= \dot{\vec{\boldsymbol{H}}}_{/0} \\ \Rightarrow \quad \vec{\boldsymbol{r}} \times \left(\sum \vec{\boldsymbol{F}}\right) &= \vec{\boldsymbol{r}} \times (m\vec{\boldsymbol{a}}) \\ \Rightarrow \quad r\hat{\boldsymbol{e}}_R \times \left(\sum F_r \hat{\boldsymbol{e}}_R + \sum F_\theta \hat{\boldsymbol{e}}_\theta\right) &= r\hat{\boldsymbol{e}}_R \times m(a_r \hat{\boldsymbol{e}}_R + a_\theta \hat{\boldsymbol{e}}_\theta) \\ \hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_R = \vec{\boldsymbol{0}}, \hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}} \quad \Rightarrow \quad \left\{ r \sum F_\theta \hat{\boldsymbol{k}} \right. &= \left. r m a_\theta \hat{\boldsymbol{k}} \right\} \\ \{\} \cdot \hat{\boldsymbol{k}} \quad \Rightarrow \quad r \sum F_\theta &= r m a_\theta. \end{aligned} \qquad (13.13)$$

Energy

Kinetic energy is also particularly simple in polar coordinates for circular motion because there is only one degree of freedom:

$$\begin{aligned} E_{\mathrm{K}} &= \frac{mv^2}{2} = \frac{mv_\theta^2}{2} \\ &= m\dot{\theta}^2 r^2/2. \end{aligned} \tag{13.14}$$

The simple pendulum

Perhaps the most famous mechanics example of circular motion at non-constant rate is a simple pendulum. As a child's swing, the inside of a grandfather clock, a hypnotist's device, or a gallows, the motion of a simple pendulum is a clear image to all of us. Galileo studied the simple pendulum before Newton created Newton's laws, and the pendulum is a core topic in high-school and freshman physics.

For starters, we consider a 2-D pendulum of fixed length with no forcing other than gravity. All mass is concentrated at a point. Of primary interest is the motion of the pendulum and the tension in the string. First we find governing differential equations (the equations of motion).

First, the tension in the pendulum rod (or string) acts along the length because the rod is a massless two-force body. At least that is the idealization. For any real pendulum, where the rod is not precisely massless and where the mass is not precisely concentrated at a point, there is a small force transmitted that is not along the rod. We neglect this 'shear' force in the treatment of the ideal pendulum. One way to get the equation of motion is to use linear momentum balance in polar coordinates, eqn. (13.12), and dot both sides with $\hat{\boldsymbol{e}}_\theta$ to get

$$\begin{aligned} -T\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0} + mg\underbrace{\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{e}}_\theta}_{-\sin\theta} &= m\ell\ddot{\theta}\underbrace{\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{e}}_\theta}_{1} - \ell\dot{\theta}^2\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0} \\ \Rightarrow -mg\sin\theta &= m\ell\ddot{\theta} \\ \text{so} \quad \ddot{\theta} &= -\frac{g}{\ell}\sin\theta. \end{aligned}$$

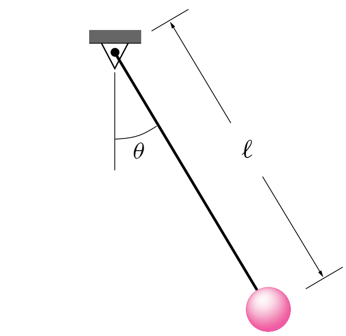

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Figure 13.20: The ideal simple pendulum.

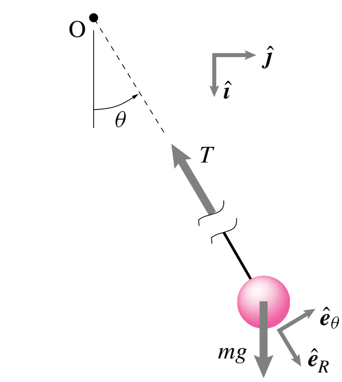

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Figure 13.21: Free body diagram of the mass in a simple pendulum.

Small angle approximation (linearization)

For small angles, $\sin\theta \approx \theta$, so we have

$$\ddot{\theta} = -\frac{g}{\ell}\theta$$

for small oscillations. This equation describes a harmonic oscillator with $\sqrt{\frac{g}{\ell}}$ replacing the $\sqrt{\frac{k}{m}}$ coefficient in a spring-mass system. Thus the general solution is

$$\theta = A\cos\sqrt{g/l}\,t + B\sin\sqrt{g/l}\,t \tag{13.15}$$

where $A = \theta_0$ and $B\sqrt{g/l} = \dot{\theta}_0$. This solution has the famous property, Galileo loved this, that the frequency is the same for big as for small oscillations. Thus, a pendulum of a given length that swings back and forth 1 degree makes about the same number of swings per minute as one that swings with an amplitude of 10 degrees. How big is the error in this constant frequency result? Well, something less than the error in the approximation that $\sin\theta = \theta$.

$$\%\ \text{error} = 100 \cdot \frac{\theta - \sin\theta}{\sin\theta} \approx 100 \cdot \frac{\theta^3/3}{\theta} \approx \frac{\theta^2}{3} \approx 1\%$$

for $\theta = 10° \approx 1/6\,\text{rad}$. The actual error in the period is less than this, as you can find by numerically solving the non-linear pendulum equation.

The inverted pendulum

A pendulum with the mass-end up is called an inverted pendulum. By methods just like we used for the regular pendulum, we find the equation of motion to be

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta$$

which, for small θ, is well approximated by

$$\ddot{\theta} = \frac{g}{\ell}\theta.$$

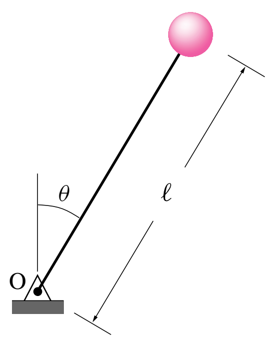

Figure 13.22: The inverted pendulum

As opposed to the simple pendulum, which has oscillatory solutions, this differential equation has exponential solutions

$$\theta = C_1 e^{\sqrt{g/\ell}\,t} + C_2 e^{-\sqrt{g/\ell}\,t},$$

one term of which has exponential growth (the implicit "+" in front of the argument of the $\sqrt{g/\ell}\,t$), indicating the inherent *instability* of the inverted pendulum. That is, as is intuitively obvious, an inverted pendulum has tendency to fall over when slightly disturbed from the vertical position②.

② After the pendulum falls a ways, say past 30 degrees from vertical, the exponential solution is not an accurate description, but the actual motion (as viewed by an experiment, a computer simulation, or the exact elliptic integral solution of the equations) shows that the pendulum keeps falling.

More about pendula

Pendula are useful as models of many phenomena from the swing of a leg in walking to the tipping of a chimney in an earthquake. Pendula also serve as a simple example for many more general concepts in mechanics. For example, the pendulum is popular as an example of "chaos"; if you push a pendulum periodically its motions can be wild.

13.2 Other derivations of the pendulum equation

The simplest derivation of the pendulum differential equation is to use linear momentum balance in polar coordinates. Here are two other derivations.

Method one: linear momentum balance in cartesian coordinates

The equation of linear momentum balance is

$$\sum \vec{F} = \overbrace{\dot{\vec{L}}}^{m\vec{a}}$$

Evaluating the left side (using the free body diagram) and right side (using the kinematics of circular motion), we get

$$-T\hat{e}_R + mg\hat{\imath} = m[\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R] \tag{13.16}$$

From the picture (or recalling) we see that $\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_\theta = \cos\theta\hat{\jmath} - \sin\theta\hat{\imath}$. So, upon substitution into the equation above, we get

$$\begin{aligned} &-T(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + mg\hat{\imath} \\ &\quad = m\left[\ell\ddot{\theta}(\cos\theta\hat{\jmath} - \sin\theta\hat{\imath}) - \ell\dot{\theta}^2(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})\right] \end{aligned}$$

Breaking this equation into its x and y components (by dotting both sides with $\hat{\imath}$ and $\hat{\jmath}$, respectively) gives

$$\begin{aligned} -T\cos\theta + mg &= -m\ell\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \text{ and} \\ -T\sin\theta &= m\ell\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right). \end{aligned} \tag{13.17}$$

Note, when deriving equations of motion, we think of both positions and the rates and velocities as knowns. For example, we take θ and $\dot{\theta}$ as known. But how do we know them? We don't. But think of them as known helps us write a set of differential equations from which we can eventually find them. Thus the equations above are two simultaneous equations that we can solve for the two unknowns T and $\ddot{\theta}$ to get

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{13.18}$$

$$T = m[\ell\dot{\theta}^2 + g\cos\theta]. \tag{13.19}$$

The first equation is the familiar pendulum differential equation, the second allows as to find the tension in the pendulum string.

Method two: angular momentum balance

Using angular momentum balance, we can 'kill' (eliminate) the tension term at the start. Taking angular momentum balance about the point O, we get

$$\begin{aligned} \sum \vec{M}_O &= \dot{\vec{H}}_{/O} \\ -mg\ell\sin\theta\hat{k} &= \underbrace{\vec{r}_{/O}}_{\ell\hat{e}_R} \times \underbrace{\vec{a}}_{\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R} m \\ -mg\ell\sin\theta\hat{k} &= m\ell^2\ddot{\theta}\hat{k} \\ \Rightarrow \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

since $\hat{e}_R \times \hat{e}_R = 0$ and $\hat{e}_R \times \hat{e}_\theta = \hat{k}$. So, the governing equation for a simple pendulum is

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

Method three: Conservation of energy

The string tension is always orthogonal to the velocity so does no work. The gravity force is conservative. So energy is conserved.

$$\begin{aligned} constant &= E_T \\ \Rightarrow \quad constant &= E_K + E_P \\ \Rightarrow \quad 0 &= \dot{E}_K + \dot{E}_P \\ \Rightarrow \quad 0 &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}(mgh) \\ \Rightarrow \quad 0 &= \frac{d}{dt}\left(\frac{1}{2}m(\ell\dot{\theta})^2\right) + \frac{d}{dt}(-mg\ell\cos\theta) \\ \Rightarrow \quad 0 &= m\ell^2\ddot{\theta}\dot{\theta} + mg\ell(\sin\theta)\dot{\theta} \end{aligned}$$

Now m cancels from both sides and we can divide through by ℓ^2. We can also divide through by $\dot{\theta}$, but for exceptional instants in time when $\dot{\theta} = 0$. Thus

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

which is the familiar differential equation for a pendulum. This method lacks some rigor in that the cancelation of $\dot{\theta}$ is not valid at exactly every instant in time. However, it is valid for all but those instants, and happens to give the right answer at the exceptional instants as well.

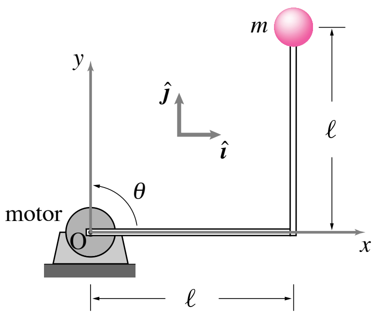

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Figure 13.23: The motor rotates the structure at a constant angular speed in the counterclockwise direction.

SAMPLE 13.6 Circular motion in 2-D. Two bars, each of negligible mass and length $\ell = 3$ ft, are welded together at right angles to form an 'L' shaped structure. The structure supports a 3.2 lbf ($= mg$) ball at one end and is connected to a motor on the other end (see Fig. 13.23). The motor rotates the structure in the vertical plane at a constant rate $\dot{\theta} = 10\,\text{rad/s}$ in the counter-clockwise direction. Take $g = 32\,\text{ft/s}^2$. At the instant shown in Fig. 13.23, find

1. the velocity of the ball,
2. the acceleration of the ball, and
3. the net force and moment applied by the motor and the support at O on the structure.

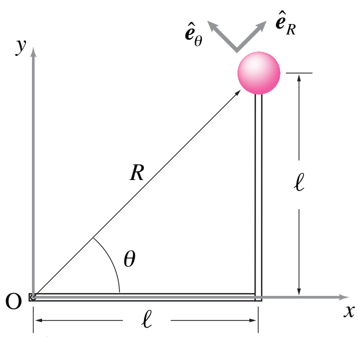

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Figure 13.24: The ball follows a circular path of radius R. The position, velocity, and acceleration of the ball can be expressed in terms of the polar basis vectors $\hat{e}_R$ and $\hat{e}_\theta$.

Solution The motor rotates the structure at a constant rate. Therefore, the ball is going in circles with angular velocity $\vec{\omega} = \dot{\theta}\hat{k} = 10\,\text{rad/s}\hat{k}$. The radius of the circle is $R = \sqrt{\ell^2 + \ell^2} = \ell\sqrt{2}$. Since the motion is in the xy plane, we use the following formulae to find the velocity $\vec{v}$ and acceleration $\vec{a}$.

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$$

$$\vec{a} = -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta,$$

where $\hat{e}_R$ and $\hat{e}_\theta$ are the polar basis vectors shown in Fig. 13.24. In Fig. 13.24, we note that $\theta = 45^\circ$. Therefore,

$$\begin{aligned}
\hat{e}_R &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} \\
&= \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}), \\
\hat{e}_\theta &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} \\
&= \frac{1}{\sqrt{2}}(-\hat{\imath} + \hat{\jmath}).
\end{aligned}$$

Here, $R = L\sqrt{2} = 3\sqrt{2}\,\text{ft}$ is constant, and $\ddot{\theta} = 0$ because $\dot{\theta} = 10\,\text{rad/s} = \text{constant}$. Thus,

1. the velocity of the ball is

$$\begin{aligned}
\vec{v} &= R\dot{\theta}\hat{e}_\theta \\
&= 3\sqrt{2}\,\text{ft} \cdot 10\,\text{rad/s}\hat{e}_\theta \\
&= 30\sqrt{2}\,\text{ft/s} \cdot \frac{1}{\sqrt{2}}(-\hat{\imath} + \hat{\jmath}) \\
&= 30\,\text{ft/s}(-\hat{\imath} + \hat{\jmath}).
\end{aligned}$$

$$\boxed{\vec{v} = 30\,\text{ft/s}(-\hat{\imath} + \hat{\jmath})}$$

2. The acceleration of the ball is

$$\begin{aligned}
\vec{a} &= -R\dot{\theta}^2\hat{e}_R \\
&= -3\sqrt{2}\,\text{ft} \cdot (10\,\text{rad/s})^2\hat{e}_R \\
&= -300\sqrt{2}\,\text{ft/s}^2 \cdot \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) \\
&= -300\,\text{ft/s}^2(\hat{\imath} + \hat{\jmath}).
\end{aligned}$$

$$\boxed{\vec{a} = -300\,\text{ft/s}^2(\hat{\imath} + \hat{\jmath})}$$

3. Let the net force and the moment applied by the motor-support system be $\vec{F}$ and $\vec{M}$ as shown in Fig. 13.25. From the linear momentum balance for the structure,

$$
\begin{aligned}
\sum \vec{F} &= m\vec{a} \\
\vec{F} - mg\hat{\jmath} &= m\vec{a} \\
\Rightarrow \quad \vec{F} &= m\vec{a} + mg\hat{\jmath} \\
&= \overbrace{\frac{3.2\,\text{lbf}}{32\,\text{ft/s}^2}}^{m}(-300\sqrt{2}\,\text{ft/s}^2)\hat{e}_R + \overbrace{3.2\,\text{lbf}}^{mg}\,\hat{\jmath} \\
&= -30\sqrt{2}\,\text{lbf}\hat{e}_R + 3.2\,\text{lbf}\,\hat{\jmath}. \\
&= -30\sqrt{2}\,\text{lbf}\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) + 3.2\,\text{lbf}\hat{\jmath} \\
&= -30\,\text{lbf}\hat{\imath} - 26.8\,\text{lbf}\hat{\jmath}.
\end{aligned}
$$

Similarly, from the angular momentum balance for the structure,

$$
\begin{aligned}
\sum \vec{M}_{\text{O}} &= \dot{\vec{H}}_{/\text{O}}, \\
\text{where} \quad \sum \vec{M}_{\text{O}} &= \vec{M} + \vec{r}_{/\text{O}} \times mg(-\hat{\jmath}) \\
&= \vec{M} + \underbrace{R\hat{e}_R}_{\ell(\hat{\imath}+\hat{\jmath})} \times mg(-\hat{\jmath}) \\
&= \vec{M} - mg\ell\hat{k}, \\
\text{and} \quad \dot{\vec{H}}_{/\text{O}} &= \vec{r}_{/\text{O}} \times m\vec{a} \\
&= R\hat{e}_R \times m(-R\dot{\theta}^2\hat{e}_R) \\
&= -mR^2\dot{\theta}^2 \underbrace{(\hat{e}_R \times \hat{e}_R)}_{\vec{0}} \\
&= \vec{0}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\vec{M} &= mg\ell\hat{k} \\
&= \underbrace{3.2\,\text{lbf}}_{mg} \cdot \underbrace{3\,\text{ft}}_{\ell}\,\hat{k} \\
&= 9.6\,\text{lbf}\cdot\text{ft}\hat{k}.
\end{aligned}
$$

$$
\boxed{\vec{F} = -30\,\text{lbf}\hat{\imath} - 26.8\,\text{lbf}\hat{\jmath}, \quad \vec{M} = 9.6\,\text{lbf}\cdot\text{ft}\hat{k}}
$$

Note: If there was no gravity, the moment applied by the motor would be zero.

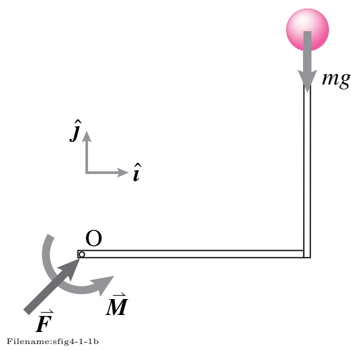

Figure 13.25: Free body diagram of the structure.

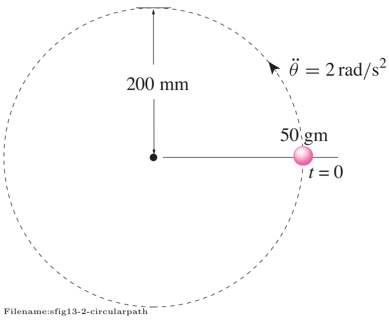

Figure 13.26:

SAMPLE 13.7 A 50 gm point mass executes circular motion with angular acceleration $\ddot{\theta} = 2\,\mathrm{rad/s^2}$. The radius of the circular path is 200 mm. If the mass starts from rest at $t = 0$, find

1. Its angular momentum $\vec{H}$ about the center at $t = 5\,\mathrm{s}$.
2. Its rate of change of angular momentum $\dot{\vec{H}}$ about the center.

Solution

1. From the definition of angular momentum,

$$\begin{aligned}
\vec{H}_{/O} &= \vec{r}_{/0} \times m\vec{v} \\
&= R\hat{e}_R \times m\dot{\theta}R\hat{e}_\theta \\
&= mR^2\dot{\theta}(\hat{e}_R \times \hat{e}_\theta) \\
&= mR^2\dot{\theta}\hat{k}
\end{aligned}$$

On the right hand side of this equation, the only unknown is $\dot{\theta}$. Thus to find $\vec{H}_{/O}$ at $t = 5\,\mathrm{s}$, we need to find $\dot{\theta}$ at $t = 5\,\mathrm{s}$. Now,

$$\begin{aligned}
\ddot{\theta} &= \frac{d\dot{\theta}}{dt} \\
d\dot{\theta} &= \ddot{\theta}\,dt \\
\int_{\dot{\theta}_0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t \ddot{\theta}\,dt \\
\dot{\theta}(t) - \dot{\theta}_0 &= \ddot{\theta}(t_t - t_0) \\
\dot{\theta} &= \dot{\theta}_0 + \ddot{\theta}(t - t_0)
\end{aligned}$$

Writing α for $\ddot{\theta}$ and substituting $t_0 = 0$ in the above expression, we get $\dot{\theta}(t) = \dot{\theta}_0 + \alpha t$, which is the angular speed version of the linear speed formula $v(t) = v_0 + at$. ③ Substituting $t = 5\,\mathrm{s}$, $\dot{\theta}_0 = 0$, and $\alpha = 2\,\mathrm{rad/s^2}$ we get $\dot{\theta} = 2\,\mathrm{rad/s^2} \cdot 5\,\mathrm{s} = 10\,\mathrm{rad/s}$. Therefore,

③ Be warned that these formulae are valid only for constant rate of change of speed.

$$\begin{aligned}
\vec{H}_{/O} &= 0.05\,\mathrm{kg} \cdot (0.2\,\mathrm{m})^2 \cdot 10\,\mathrm{rad/s}\hat{k} \\
&= 0.02\,\mathrm{kg.\,m^2/\,s} = 0.02\,\mathrm{N{\cdot}m \cdot s}.
\end{aligned}$$

$$\boxed{\vec{H}_{/O} = 0.02\,\mathrm{N{\cdot}m \cdot s}.}$$

2. Similarly, we can calculate the rate of change of angular momentum:

$$\begin{aligned}
\dot{\vec{H}}_{/O} &= \vec{r}_{/0} \times m\vec{a} \\
&= R\hat{e}_R \times m(R\ddot{\theta}\hat{e}_\theta - \dot{\theta}^2 R\hat{e}_R) \\
&= mR^2\ddot{\theta}(\hat{e}_R \times \hat{e}_\theta) \\
&= mR^2\ddot{\theta}\hat{k} \\
&= 0.05\,\mathrm{kg} \cdot (0.2m)^2 \cdot 2\,\mathrm{rad/s^2}\hat{k} \\
&= 0.004\,\mathrm{kg \cdot m^2/\,s^2} = 0.004\,\mathrm{N{\cdot}m}
\end{aligned}$$

$$\boxed{\dot{\vec{H}}_{/O} = 0.004\,\mathrm{N{\cdot}m}}$$

SAMPLE 13.8 The simple pendulum. A simple pendulum swings about its vertical equilibrium position (2-D motion) with amplitude $\theta_{\text{max}} = 10^{\circ}$. Find

1. the magnitude of the maximum angular acceleration,
2. the maximum tension in the string.

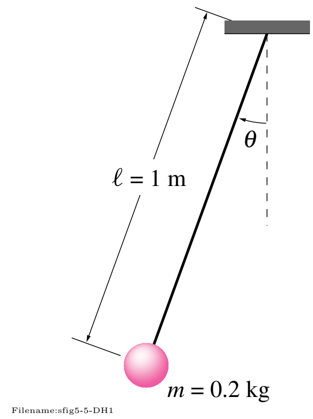

Figure 13.27:

Solution

1. The equation of motion of the pendulum is given by (see eqn. (13.18) in the text):

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

We are given that $|\theta| \leq \theta_{\text{max}}$. For $\theta_{\text{max}} = 10^{\circ} = 0.1745\,\text{rad}$, $\sin\theta_{\text{max}} = 0.1736$. Thus we see that $\sin\theta \approx \theta$ even when θ is maximum. Therefore, we can safely use linear approximation (although we could solve this problem without it); i.e.,

$$\ddot{\theta} = -\frac{g}{\ell}\theta.$$

Clearly, $|\ddot{\theta}|$ is maximum when θ is maximum. Thus,

$$|\ddot{\theta}|_{\text{max}} = \frac{g}{\ell}\theta_{\text{max}} = \frac{9.81\,\text{m/s}^2}{1\,\text{m}}\cdot(0.1745\,\text{rad}) = 1.71\,\text{rad/s}^2.$$

$$\boxed{|\ddot{\theta}|_{\text{max}} = 1.71\,\text{rad/s}^2}$$

2. The tension in the string is given by (see equation 13.19 of text):

$$T = m(\ell\dot{\theta}^2 + g\cos\theta).$$

This time, we will not make the small angle assumption. We can find T_{max} and the corresponding θ using conservation of energy. Let the position of maximum amplitude be position 1 and the position at any θ be position 2. When $\theta = \theta_{\text{max}}$, the mass comes to rest and switches its direction of motion. Thus, its angular velocity and, hence, its kinetic energy is zero at θ_{max}. Using conservation of energy, we have

$$\begin{aligned} E_{\text{K1}} + E_{\text{P1}} &= E_{\text{K2}} + E_{\text{P2}} \\ 0 + mg\ell(1-\cos\theta_{\text{max}}) &= \frac{1}{2}m(\ell\dot{\theta})^2 + mg\ell(1-\cos\theta). \end{aligned} \tag{13.20}$$

and solving for $\dot{\theta}$, we get,

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}(\cos\theta - \cos\theta_{\text{max}})}.$$

Therefore, the tension at any θ is

$$T(\theta) = m(\ell\dot{\theta}^2 + g\cos\theta) = mg(3\cos\theta - 2\cos\theta_{\text{max}}).$$

To find the maximum tension, we set $\frac{dT}{d\theta} = 0$, and find that, for $0 \leq \theta \leq \theta_{\text{max}}$, T is maximum when $\theta = 0$. Now, substituting $\theta = 0$ in $T(\theta)$, we get,

$$T_{\text{max}} = mg\big(3\cos(0) - 2\cos(\theta_{\text{max}})\big) = 0.2\,\text{kg}\cdot 9.81\,\text{m/s}^2\big(3 - 1.97\big) = 2.02\,\text{N}.$$

The maximum tension corresponds to maximum speed which occurs at the bottom of the swing where all of the potential energy is converted to kinetic energy.

$$\boxed{T_{\text{max}} = 2.02\,\text{N}}$$

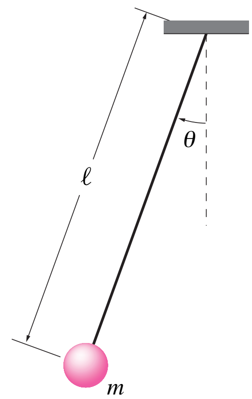

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Figure 13.28:

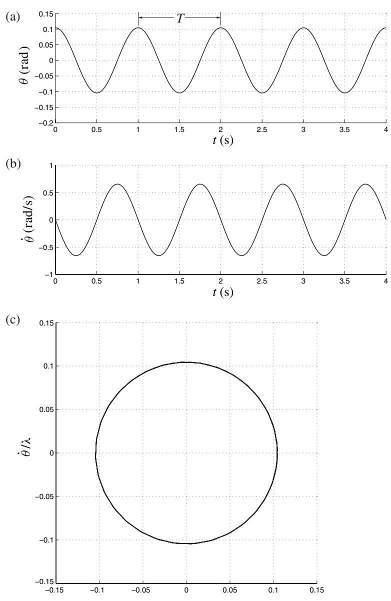

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Figure 13.29: Numerical solution of the nonlinear pendulum equation, $\ddot{\theta} = -g/ell \sin\theta$, with the initial conditions, $\theta(0) = \pi/30$ and $\dot{\theta}(0) = 0$; (a) $\theta(t)$, (b) $\dot{\theta}(t)$, and (c) $\dot{\theta}$ vs θ (phase plane).

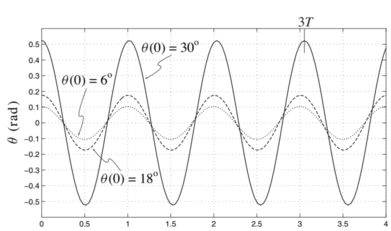

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Figure 13.30: Comparison of $\theta(t)$ obtained from three different initial launch angles, $\theta(0) = 6°,\ 18°$ and $30°$.

SAMPLE 13.9 The nonlinear pendulum: Consider the simple pendulum of Sample 13.8 again. Let the mass be m and the length of the pendulum ℓ. The equation of motion of the pendulum is $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$ as derived in the text (see eqn. (13.18)). This is a nonlinear ordinary differential equation but it can be solved easily numerically. Write a computer code using some ODE solver to solve the equation. Take g and ℓ such that $\lambda = \sqrt{g/\ell} = 2\pi$ (this makes the time period of the pendulum $T = 2\pi/\lambda = 1\,\text{s}$). Using the code, do the following calculations.

1. Solve the equation over a time interval of $t = 0$ to 4 seconds using the initial conditions $\theta(0) = 6°$ and $\dot{\theta}(0) = 0$, and plot θ vs t, $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ. How do these plots compare with the solution of the linear equation $\ddot{\theta} = -\frac{g}{\ell}\theta$?
2. Solve the equation again over the same time interval using the initial conditions $[\theta(0), \dot{\theta}(0)] = [18°, 0]$, and $[30°, 0]$. Plot $\theta(t)$ starting with all the three initial conditions used so far on the same graph and comment on the time period of oscillations.
3. Solve the equation again over t in$[0, 4\,\text{s}]$ using $\theta(0) = \pi/2$ and $\pi/1.02$ while keeping $\dot{\theta}(0) = 0$. Again plot θ vs t, $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ, for the three solutions obtained with $\theta(0) = \pi/6, \pi/2$, and $\pi/1.02$. Comment on the plots.
4. For the last three initial conditions, compute E_P, E_K, and $E_\text{T} = E_\text{P} + E_\text{K}$ from the solutions obtained. For each initial condition, plot E_P, E_K, and E_T on the same graph and show that the total energy in each case remains constant irrespective of the nature of oscillations.

Solution The equation of motion of the pendulum is (as given)

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

To solve this second order differential equation numerically, we need to first convert it into a set of two first order equations. Let $\omega = \dot{\theta}$. Then, we can write

$$\begin{aligned}\dot{\theta} &= \omega\\ \dot{\omega} &= -\frac{g}{\ell}\sin\theta.\end{aligned}$$

We are now ready to write a computer program to solve these equations numerically. We use the following pseudocode to accomplish the task.

```
ODEs = {thetadot = omega,
        omegadot = -g/l*sin(theta) }
ICs   = {theta(0) = pi/30, omega(0) = 0 }
Set  g = 1, l = g/(4*pi^2)
Solve ODEs with ICs for t=0 to t=4
plot theta vs t, and omega vs t; plot omega vs theta
```

1. **Small amplitude oscillations:** The solution obtained with $\theta(0) = 6° = \pi/30$, and $\dot{\theta}(0) = 0$ is shown in fig. 13.29. The plots of $\theta(t)$ and $\dot{\theta}(t)$ clearly show the initial conditions at $t = 0$. From the figure, we see that the motion is sinusoidal and the time period of oscillation is 1 second, as expected.
2. **Deviation from linear equation solution:** The new initial conditions involve larger initial angles ($\theta(0) = 18°$ and $30°$). That is the only difference. We use the same program as used before and get the solutions with the new initial conditions. We plot

$\theta(t)$ against t for all the three solutions on the same graph. The resulting plot is shown in fig. 13.30.

Now what we observe from this plot is that the three solutions, starting with the three different initial conditions, do not have the same time period of oscillations. The difference is not clearly visible between $\theta(0) = 6^\circ$ and $\theta(0) = 18^\circ$ solutions but it is much clearer for $\theta = 30^\circ$ (see the third peak, marked with $3T$). As the initial angle, $\theta(0)$, increases, the period of oscillation seems to increase.

The dependence of time period (or frequency) of oscillations on the amplitude is the hallmark of nonlinear oscillators. In contrast, linear oscillators have a constant period of oscillation, irrespective of the amplitude of motion. For our pendulum, as long as the initial θ is so small that $\sin\theta \approx \theta$, the equation of motion can be replaced by the linear equation, $\ddot{\theta} = -g/\ell\theta$, and all solutions will have the same time period of oscillation. As $\theta(0)$ becomes larger, the approximation $\sin\theta \approx \theta$ breaks down, and the linear equation of motion is no longer valid.

3. **Large amplitude oscillations:** We now run the program with large initial angles, $\theta(0) = \pi/2\,(90^\circ)$ and $\theta(0) = \pi/1.02$ ($\approx 176^\circ$, *i.e.,* close to the vertically upright position), and obtain the corresponding solutions. Plots of $\theta(t)$ and $\dot{\theta}(t)$ for three initial conditions, small θ ($\pi/30$), moderately large θ ($\pi/2$), and very large θ ($\pi/1.02$) are shown in fig. 13.31. From the plots it is clear that not only the period of oscillation increases drastically with larger amplitudes, but also the qualitative nature of oscillations changes. For small amplitude (small initial θ), oscillations are simple harmonic but for larger amplitudes (large initial θ) oscillations are no more simple harmonic. This fact is more evident from the velocity plot, fig. 13.31(b). The phase plot, fig. 13.31(c), shows how the three solution trajectories (also called *orbits*) look in the phase space. All simple harmonic motions lead to circular orbits (you can show that by writing the solution for $\theta(t)$ and $\dot{\theta}(t)$ and then showing that $\theta^2 + \dot{\theta}^2 =$ constant) in this phase space. However, for large amplitude motion, the orbits become oblong and approach a rather strange looking trajectory, called the *separatrix*, as the amplitude of motion grows. This separatrix marks the boundary of all possible periodic motions of the pendulum. Outside this separatrix, solutions do exist but they correspond to whirling motion of the pendulum which is not periodic (because $\theta(t)$ keeps growing without bounds).

4. **Energy conservation:** Let θ^* and $\dot{\theta}^*$ be the values of angular displacement and angular speed of the pendulum at some instant t^*. Then, assuming $\theta = 0$ to be the datum for potential energy, we can write the expressions for potential energy and kinetic energy as

$$\begin{aligned} E_{\text{P}} &= mg\ell(1-\cos\theta^*) \\ E_{\text{K}} &= \frac{1}{2}m\ell^2\dot{\theta}^{*2}. \end{aligned}$$

Therefore, the total energy at $t = t^*$ is,

$$E_{\text{T}} = E_{\text{P}} + E_{\text{K}} = mg\ell(1-\cos\theta^*) + \frac{1}{2}m\ell^2\dot{\theta}^{*2}.$$

From the numerical solutions obtained for the three initial conditions, we have values of θ and $\dot{\theta}$ at different time instants. Now, using the formulas for E_{P}, E_{K} and E_{T}, we compute the values of these quantities and plot them as shown in fig. 13.32. We see that for each initial condition, the potential and kinetic energies vary differently with time. However, the total energy remains constant at all times. This is expected as there is no dissipation in the system (not present in our mathematical model). A given initial condition determines the initial energy of the pendulum which must be preserved throughout the motion.

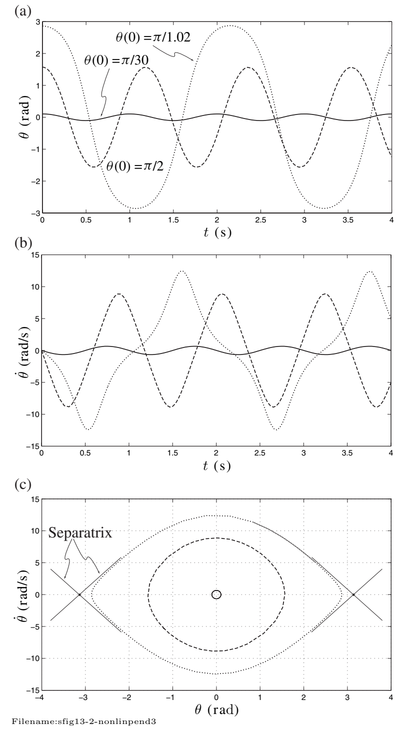

Figure 13.31: Comparison of pendulum motions when it is released from rest at small angle ($\theta(0) = \pi/30$), at horizontal position (($\theta(0) = \pi/2$), and at almost vertically upright position ($\theta(0) = \pi/1.02$); (a) θ vs t, (b) $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ (phase plane plot).

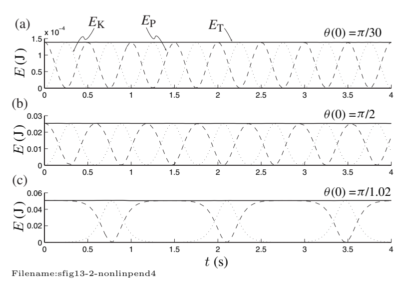

Figure 13.32: Plots of potential energy E_{P}, kinetic energy E_{K}, and total energy E_{T} during motion under three different initial conditions: (a) $\theta(0) = \pi/30$, $\dot{\theta}(0) = 0$; (b) $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$; and (c) $\theta(0) = \pi/1.02$, $\dot{\theta}(0) = 0$.

13.3 Rotation of a rigid object in planar circular motion

When two parts are glued together or attached by welding, gluing, several tight screws, bolts, rivets bolts or the like we call the connection a 'rigid attachment'. And, for the purposes of mechanics analysis, the two connected parts make up one bigger object. But most machines have various parts that are connected to each other, but not welded to each other. The most common such non-rigid attachment in engineering is a hinge. In 2D,

> a *hinge* attachment between two objects keeps two points, one from each object, on top of each other while freely allowing relative rotation of the two objects about the hinge point.

In 3D a hinge keeps two lines, one on each body, coincident and allows relative rotation about that line. The common line, or in 2D the line orthogonal to the plane through the points, is called the hinge, the hinge axis, or the axis of relative rotation.

One example of a hinge is a car axle which allows rotation of a wheel relative to the car suspension. The hinge axis is the wheel axle①. Physically, hinges are made various ways, sometimes by poking a cylindrical pin through the two objects and sometimes with ball bearings (see box 4.6 on page 215). So hinges are also called *pin connections* or *bearings* (fig. 13.33). In this chapter we limit our attention to a simple use of a hinge: one rigid part is hinged to a second fixed part that doesn't move at all. Such a non-moving part can be thought of as connected to, or an extension of, the ground or 'fixed frame'. In this simple case of interest in this chapter, one point on the moving part does not move and the rest of the part rotates about that point.

For definiteness and simplicity let's call the hinge location 0 and the hinge axis through 0 the z axis. One function of the hinge is to make the part's only possible motion to be rotation about O. Thus to understand the dynamics of a hinged part we need to understand the position, velocity and acceleration of points on a rigid object which rotates. This whole section is about the kinematics (the geometry of motion) for this rotation. We will measure the amount of rotation by the angle θ, and the rate of rotation θ by the *angular velocity* $\vec{\boldsymbol{\omega}}$ ('omega'), and of rate of change of this angular velocity $\vec{\boldsymbol{\omega}}$ by the *angular acceleration* $\vec{\boldsymbol{\alpha}}$ ('alpha').

As simple as this topic seems at first glance, you should pay close attention to the meanings and uses of these quantities. The rest of the book completely depends on the material in this section.

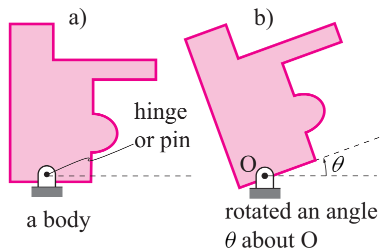

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Figure 13.33: a) A object, b) rotated counterclockwise an angle θ about 0.

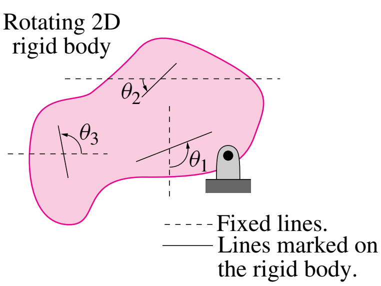

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Figure 13.34: **Rotation of lines on a rotating rigid object.** Some real or imagined lines marked on the rigid object are shown. They make the angles θ_1, θ_2, θ_3, ... with respect to various fixed lines which do not rotate. As the object rotates, each of these angles increases by the same amount.

①The word axis is obviously related to the word axle. More generally the word axis means 'line'. For example the x and y axes are generally not axles about which anything rotates, they are just lines.

Rotation of an object counterclockwise by θ

We start by imagining the object in a distinguished configuration which we call the *reference configuration*, *reference state* or *reference position*. For example we could take the left figure in fig. 13.33 as the reference configuration. If possible its usually best to pick the reference state to be one in which a prominent feature of the object is aligned with the x or y axes. The reference state may or may not be the start of the motion of interest. Even if not, we measure an object's rotation by the change, relative to the reference state, in the counterclockwise angle θ of a reference line marked in the object relative to a fixed line outside. Which reference line? Fortunately,

> *All real or imagined lines marked on a rotating rigid object rotate by the same angle, the* rotation angle, θ*. (See box 13.3).*

In three dimensions things are more complicated. General rotation of a rigid object is then represented not with a single angle θ, but rather with 3 angles, or with a unit vector and an angle, or a 3×3 matrix. So we wait to discuss which of the 2D ideas here generalize to 3D and which do not.

Rotated coordinates and base vectors $\hat{\imath}'$ and $\hat{\jmath}'$

We pick two orthogonal lines on the rotating object and give them distinguished status as *object-fixed* (or body-fixed) rotating coordinate axes x' and y'. Think of these axes as $x'y'$ coordinate axes on a piece of graph paper that

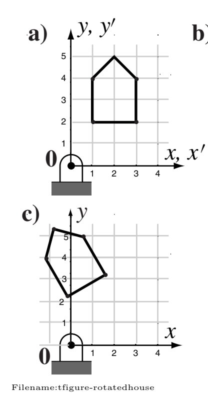


Figure 13.35: a) A house is drawn by connecting lines between 6 points, b) the house and coordinate system are rotated, thus its coordinates in the rotating system do not change c) But the coordinates in the original system do change

13.3 Rotation is uniquely defined for a rigid object (2D)

Most people will find it self-evident that, starting with a rigid object at a reference orientation, all lines marked on the object rotate by the same angle θ. Here, for the doubting, we demonstrate this fact.

A rigid object is defined this way:

> For every pair of material points A and B on a rigid object the distance |AB| between them does not change as the object moves.

In particular, when a rigid object rotates all distances between all pairs of points are preserved. Thus, by the "side-side-side" similar triangle theorem of elementary geometry, all relative angles between marked line segments are preserved by the rotation. For example, for a triangle ABC the angle at B is constant as the object rotates. Now consider any pair of line segments on the object.

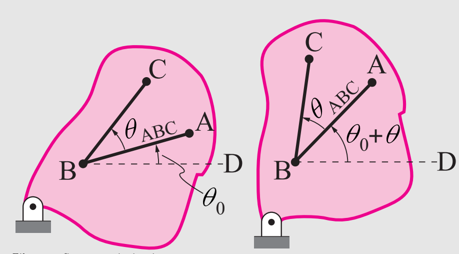
[By 'on' the object we don't mean a projected image drawn with a light pen that can move around on the object relative to the atoms. Rather, by 'on the object' we mean something defined by a particular set of atoms that make up the object.]

If the segments do not cross we can extend them to a point of intersection B. Such a pair of intersecting lines is shown here before and after rotation.Initially BA makes and angle θ_0 with a horizontal reference line. BC then makes an angle of $\theta_{ABC}+\theta_0$. After rotation we measure the angle to the line BD. BA now makes an angle of $\theta_0+\theta$. By the addition of angles in the rotated configuration line BC now makes an angle of $\theta_{ABC}+\theta_0+\theta$ which makes an increase by θ of the angle made by BC with the horizontal reference line. So both BA and AC rotate by the same angle θ.

We could use one of these two lines and compare it with an arbitrary third line through B and show that the third line also has equal rotation, and then a fourth, and so on. So all lines on the object through a point B rotate by the same angle θ. The demonstration for a pair of parallel lines, one of them through B, is easy, they stay parallel so always make a common angle with any reference line.

Because any line on the object either goes through B or is parallel to a line through B, all lines marked on a rigid object rotate by the same angle θ.

> The rotation of a rigid object in 2D is thus unambiguously defined as *the* angle through which all lines on the object rotate.



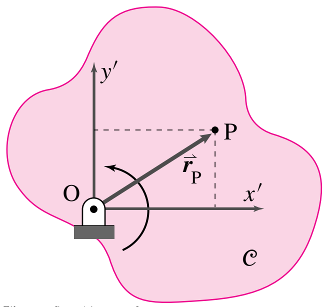


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Figure 13.36: A rotating rigid object $\mathcal{C}$ with rotating coordinates $x'y'$ rigidly attached.

is glued to the object (see fig. 13.35a&b). Its easiest if we start by assuming that the $x'y'$ axes have the same origin 0 as the xy axes and are parallel with the fixed xy axes when the object is in the reference configuration (when $\theta = 0$).

These rotating coordinate axes, x' and y', have associated rotating base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ (fig. 13.36 and 13.37). So $\hat{\imath}'$ is always in the x' direction and $\hat{\jmath}'$ always in the y' direction. We will use these rotating coordinates and base vectors to keep track of a some particle of interest P that is 'glued' to the object. To start, note that particle P which is glued to the object has x' and y' coordinates that don't change as the rotation progresses.

Example: A particle on the x' axis

If a particle P is fixed on the x'-axis at position $x' = 3\,\text{cm}$, then we have.

$$\vec{r}_{\text{P}} = 3\,\text{cm}\hat{\imath}'$$

for all time, even as the object rotates.

The position vector of a point P fixed to a rigid object hinged at O remains, as the rotation progresses,

$$\vec{r}_{\text{P}} = x'\hat{\imath}' + y'\hat{\jmath}', \tag{13.21}$$

$$\tag{13.22}$$

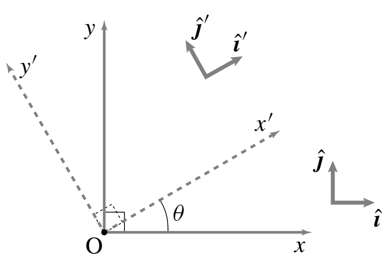

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Figure 13.37: Fixed coordinate axes and rotating coordinate axes.

with x' and y' both constant. These rotating coordinate system components, $[\vec{r}]_{x'y'} = [x', y']$, are sometimes writen as $[\vec{r}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

You will see that much of the math for rotating $x'y'$ coordinates is reminiscent of that for polar coordinates. However, the spirit is a bit different. In polar coordinates the $\hat{e}_R$ axes was picked to track a particular particle of interest. Here we pick axes that rotate with an extended object and use that one set of axes to track any and all particles of interest.

Note, even though neither x' nor y' change as θ changes, the point P they describe moves, in circles actually. How can the particle's position change if its coordinates don't change? Well, in eqn. (13.23) the change in position is represented by the base vectors changing as the object rotates. Thus we could write more explicitly that

$$\vec{r}_{\text{P}} = x'\hat{\imath}'(\theta) + y'\hat{\jmath}'(\theta). \tag{13.23}$$

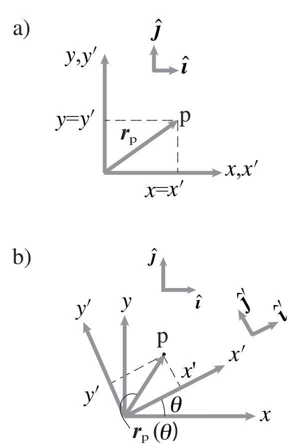

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Figure 13.38: The x' and y' coordinates of a point fixed on a rotating object stay constant while the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ rotate with the object.

Here we show more explicitly that the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ depend on θ. Just like for polar base vectors (see eqn. (13.5) on page 648) we can express the rotating base vectors in terms of the fixed base vectors and θ.

$$\begin{aligned} \hat{\imath}' &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \\ \hat{\jmath}' &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \end{aligned} \tag{13.24}$$

Also we can express the fixed basis vectors in terms of the rotating vectors like this:

$$\begin{aligned} \hat{\boldsymbol{\imath}} &= \cos\theta\,\hat{\boldsymbol{\imath}}' - \sin\theta\,\hat{\boldsymbol{\jmath}}' \\ \hat{\boldsymbol{\jmath}} &= \sin\theta\,\hat{\boldsymbol{\imath}}' + \cos\theta\,\hat{\boldsymbol{\jmath}}'. \end{aligned} \tag{13.25}$$

Please review the section on dot products, 2.2, to see one derivation of these formulae.

We will use the phrase *reference frame* or just *frame* to mean "a coordinate system attached to a rigid object". One can think of the coordinate grid as like an invisible metal framework (hence the word 'frame') that rotates with the object. We refer to a calculation based on the rotating coordinates in fig. 13.36 variously as "in the frame $\mathcal{C}$" or "using the $x'y'$ frame" or "in the $\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'$ frame②.

② Advanced aside. Sometimes a *reference frame* is defined as the set of all coordinate systems that could be attached to a rigid object. Two coordinate systems, even if rotated with respect to each other, then represent the same frame so long as they rotate together. Some of the results we will develop only depend on this more slack definition of frame, that the coordinates are glued to the object with no mind of their orientation in the reference configuration.

In computer calculations we usually manipulate lists and arrays of numbers and not geometric vectors. So on a computer we keep track of vectors by keeping track of their lists of components. Lets look at a point fixed to the object and whose coordinates we know in the reference configuration:

$$\left[\vec{\boldsymbol{r}}_{\mathrm{P}}^{\,\mathrm{ref}}\right]_{xy} = \begin{bmatrix} x^{\mathrm{ref}} \\ y^{\mathrm{ref}} \end{bmatrix}.$$

Assuming the object axes and fixed axes coincide in the reference configuration, the object coordinates of a point $[\vec{\boldsymbol{r}}_{\mathrm{P}}]_{x'y'}$ are equal to the space fixed coordinates of the point in the reference configuration $\left[\vec{\boldsymbol{r}}_{\mathrm{P}}^{\,\mathrm{ref}}\right]_{xy}$. We can think of the point as defined either way, so

$$[\vec{\boldsymbol{r}}_{\mathrm{P}}]_{x'y'} = \left[\vec{\boldsymbol{r}}_{\mathrm{P}}^{\,\mathrm{ref}}\right]_{xy}.$$

The rotation matrix $[R]$

Here is a question we often need to answer: What are the fixed basis coordinates of a point that has the rotating-frame coordinates $[\vec{\boldsymbol{r}}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$? Here is one way to find the answer:

$$\begin{aligned} \vec{\boldsymbol{r}}_{\mathrm{P}} &= x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' \\ &= x'(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) + y'(-\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}) \\ &= \underbrace{\big((\cos\theta)x' - (\sin\theta)y'\big)}_{x}\,\hat{\boldsymbol{\imath}} + \underbrace{\big((\sin\theta)x' + (\cos\theta)y'\big)}_{y}\,\hat{\boldsymbol{\jmath}} \end{aligned} \tag{13.26}$$

so we can pull out the x and y coordinates compactly as,

$$[\vec{\boldsymbol{r}}_{\mathrm{P}}]_{xy} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta\; x' + \sin\theta(-y') \\ \sin\theta\; x' + \cos\theta(y') \end{bmatrix}. \tag{13.27}$$

But this can, in turn be written in matrix notation as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{[R]} \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \text{or}$$

$$[\vec{r}_P]_{xy} = [R]\,[\vec{r}_P]_{x'y'}, \quad \text{or} \qquad (13.28)$$

$$[\vec{r}_P]_{xy} = [R]\left[\vec{r}_P^{\,\text{ref}}\right]_{xy}.$$

The matrix $[R]$ or $[R(\theta)]$ is the *rotation matrix* for counterclockwise rotations by θ. As shown above, if you know the coordinates of a point fixed on an object before rotation, you can find its coordinates after rotation by multiplying the coordinate column vector by the matrix $[R]$. You can remember what $[R]$ is by remembering its components or by remembering that

the first and second column of $[R]$ are the components of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, respectively, in the fixed coordinate system.

For example, the first column of $[R]$ consist of the x and y components of $\hat{\boldsymbol{\imath}}'$. A feature of eqn. (13.28) is that the same matrix $[R]$ prescribes the coordinate change for every different point on the object. Thus for points called 1, 2 and 3 we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [R]\begin{bmatrix} x_1' \\ y_1' \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = [R]\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} \quad \& \quad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = [R]\begin{bmatrix} x_3' \\ y_3' \end{bmatrix}.$$

A more compact way to write a matrix times a list of column vectors is to arrange the column vectors one next to the other in a matrix. By multiplying this matrix by $[R]$ we get a new matrix whose columns are the new coordinates of various points. For example,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = [R]\begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix}. \qquad (13.29)$$

Eqn. 13.29 is useful for computer animation of rotating things in video games (and in dynamics simulations too) where points 1,2, and 3 are points on an object.

Example: Rotate a picture

If a simple picture of a house is drawn by connecting the six points (fig. 13.35a) with the first point at $(x, y) = (1, 2)$, the second at $(x, y) = (3, 2)$, etc., and the sixth point on top of the first, we have,

$$[xy \text{ points BEFORE}] \equiv \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix}.$$

After a 30° counter-clockwise rotation about O, the coordinates of the house, in a coordinate system that rotates with the house, are unchanged (fig. 13.35b). But in the fixed (non-rotating, Newtonian) coordinate system the new coordinates of the rotated house points are,

$$
\begin{aligned}
[x\,y \text{ points AFTER}] &= \big[R\big]\,[x\,y \text{ points BEFORE}] = \big[R\big]\,\big[x'\,y' \text{ points}\big] \\
&= \begin{bmatrix} \sqrt{3}/2 & -.5 \\ .5 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix} \\
&\approx \begin{bmatrix} -0.1 & 1.6 & 0.6 & -0.8 & -1.1 & -0.1 \\ 2.2 & 3.2 & 5.0 & 5.3 & 4.0 & 2.2 \end{bmatrix}
\end{aligned}
$$

as shown in fig. 13.35c.

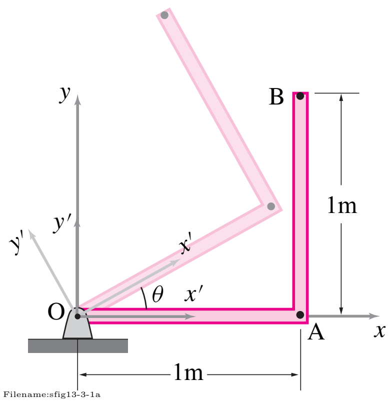

Figure 13.39:

SAMPLE 13.10 Computing rotated position of an object: A rigid object AOB is pinned at point O and is free to rotate about this point. Using the rotation matrix, find the coordinates of points A and B when $\theta = 30°$ and $110°$ respectively.

Solution Let $x'y'$ be a set of axes glued to the object with origin at O which is also the origin of the space-fixed coordinate axes xy. We first write the coordinates of points A and B in the body-fixed axes.

$$\vec{r}_A|_{x'y'} = \begin{bmatrix} 1\,\mathrm{m} \\ 0 \end{bmatrix}, \qquad \vec{r}_B|_{x'y'} = \begin{bmatrix} 1\,\mathrm{m} \\ 1\,\mathrm{m} \end{bmatrix}.$$

The rotation matrix for the $x'y'$ axes rotating counterclockwise along with the object is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

We can find the coordinates of A and B in the rotated position by multiplying their $x'y'$ coordinates with the rotation matrix. We first write the coordinates of both points A and B in a single matrix, one column for each, and multiply with R to get the new coordinates:

$$[\ \vec{r}_A \quad \vec{r}_B\ ]_{xy} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'_A & x'_B \\ y'_A & y'_B \end{bmatrix}.$$

Now, we calculate the new coordinates for the given values of θ.

- For $\theta = 30°$, we have,

$$\begin{aligned} [\ \vec{r}_A \quad \vec{r}_B\ ]_{xy} &= \begin{bmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{bmatrix} \begin{bmatrix} 1\,\mathrm{m} & 1\,\mathrm{m} \\ 0 & 1\,\mathrm{m} \end{bmatrix} \\ &= \begin{bmatrix} 0.866\,\mathrm{m} & 0.366\,\mathrm{m} \\ 0.5\,\mathrm{m} & 1.366\,\mathrm{m} \end{bmatrix}. \end{aligned}$$

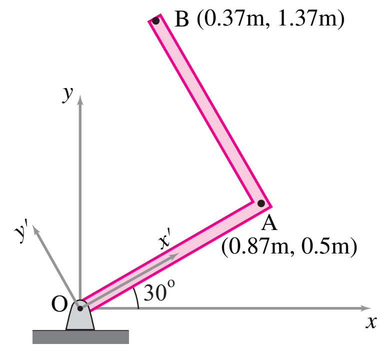

Figure 13.40: Coordinates of points A and B after a rotation of 30°.

Thus,

$$[\vec{r}_A]_{xy} = \begin{bmatrix} 0.866\,\mathrm{m} \\ 0.5\,\mathrm{m} \end{bmatrix}, \qquad [\vec{r}_B]_{xy} = \begin{bmatrix} 0.366\,\mathrm{m} \\ 1.366\,\mathrm{m} \end{bmatrix}.$$

- Similarly, for $\theta = 110°$, we get,

$$\begin{aligned} [\ \vec{r}_A \quad \vec{r}_B\ ]_{xy} &= \begin{bmatrix} -0.342 & -0.94 \\ 0.94 & -0.342 \end{bmatrix} \begin{bmatrix} 1\,\mathrm{m} & 1\,\mathrm{m} \\ 0 & 1\,\mathrm{m} \end{bmatrix} \\ &= \begin{bmatrix} -0.342\,\mathrm{m} & -1.282\,\mathrm{m} \\ 0.94\,\mathrm{m} & 0.598\,\mathrm{m} \end{bmatrix}. \end{aligned}$$

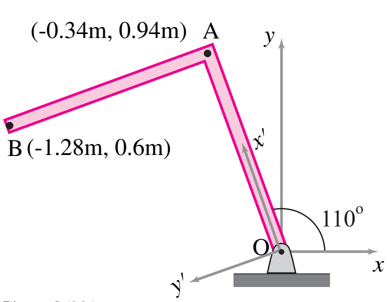

Figure 13.41: Coordinates of points A and B after a rotation of 110°.

$$[\vec{r}_A]_{xy} = \begin{bmatrix} -0.342\,\mathrm{m} \\ 0.94\,\mathrm{m} \end{bmatrix}, \qquad [\vec{r}_B]_{xy} = \begin{bmatrix} -1.282\,\mathrm{m} \\ 0.598\,\mathrm{m} \end{bmatrix}.$$

SAMPLE 13.11 Computer program for object rotation: The object shown in the figure rotates counterclockwise with its angular position θ given by $\theta(t) = \dot{\theta}_0 t$ where $\dot{\theta}_0 = 10°/s$. Write a computer program to animate the motion of the object by plotting its position at specified time instants. Use your program to plot the position of the object every two seconds for a total of 18 seconds.

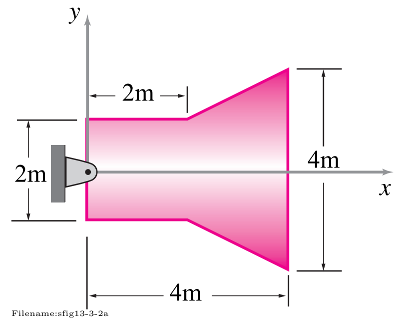

Figure 13.42:

Solution In order to draw the object on the computer screen, we need to define it by taking sufficient number of points on its boundary so that plotting all those points with connected line segments represents the object as closely as possible. In this case, we can select all the corner points on its boundary and define the object with the coordinates of these points. With the given geometry, it is fairly easy to find these coordinates. Denoting the coordinates of the kth point by (x_k, y_k), we can define this object by the following set of coordinates:

$$\text{object} = \begin{bmatrix} x_1 & x_2 & \dots & x_6 & x_1 \\ y_1 & y_2 & \dots & y_6 & y_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 4 & 4 & 2 & 0 & 0 \\ -1 & -1 & -2 & 2 & 1 & 1 & -1 \end{bmatrix}.$$

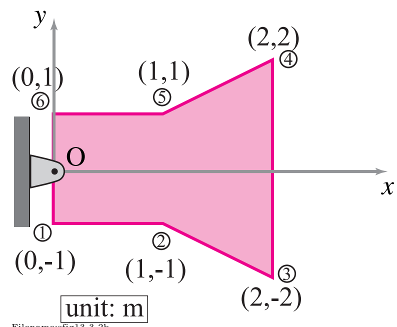

Figure 13.43: Coordinates of the six corner points are sufficient to define this object.

Note that last column is a repeat of the first column. This is essential so that when we plot the object using the x and y coordinates listed here, we get a closed boundary.

Animation of the angular motion of this object basically requires determining the rotated position of the object and plotting it at sufficiently small time intervals. To do this, we need to define the initial orientation (position), define the time increment, find the angle of rotation θ corresponding to the new time, compute the corresponding rotation matrix, multiply the object coordinates with the rotation matrix, separate out the x and y coordinates, and plot x vs y. We then repeat the whole sequence until we reach the final time.

The following pseudocode can be easily adapted to a computer program to do the required animation.

```
object = [ 0   2   4   4   2   0   0
          -1  -1  -2   2   1   1  -1]     % coordinates of points
t = 0                                     % specify initial time
t_final = 36                              % specify final time
delta_t = 3                               % specify time increment
thetadot0 = 10*pi/180                     % specify thetadot0 in rad/s

while t < t_final
      t = t + delta_t                     % increment the time
      theta = thetadot0*t                 % compute current theta
      R = [ cos(theta) -sin(theta)        % find the rotation matrix
            sin(theta)  cos(theta)]
      new_position = R*object             % find new coordinates
      x = new_position(1st row)           % extract all x-coordinates
      y = new_position(2nd row)           % extract all y-coordinates
      plot y vs x                         % plot the new position
end
```

The plot obtained by implementing this code is shown in fig. 13.44.

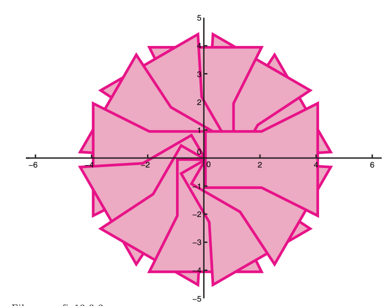

Figure 13.44: Rotated position of the object plotted at 2 second intervals.

13.4 Angular velocity and acceleration of a rigid object in planar circular motion

Angular velocity of a rigid object: $\vec{\omega}$

Thus far we have talked about rotation, but not how it varies in time. Dynamics is about motion, velocities and accelerations, so we need to think about rotation rates and rotational accelerations.

A 2D rigid object's net rotation is measured by the rotation angle θ. Thus, the simplest measure of rotation rate is $\dot{\theta} \equiv \frac{d\theta}{dt}$. Because all marked lines rotate the same amount θ they all have the same rates of change. So $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = etc.$ and, as for rotation, the concept of rotation rate of a rigid object transcends the concept of rotation rate of this or that paticular line. We give this rotation rate of a rigid object a special name, *angular velocity*, and symbol, ω (omega).

Repeating, for all lines marked on a rigid object,

$$\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \cdots = \dot{\theta}. \tag{13.30}$$

Often we think of angular velocity as a vector $\vec{\boldsymbol{\omega}}$. Its direction is the axis of the rotation which, for objects in the xy plane is $\hat{\boldsymbol{k}}$, pointed in the $+z$ direction normal to the xy plane. The scalar part of $\vec{\boldsymbol{\omega}}$ is ω. So, the *angular velocity vector* is

$$\vec{\boldsymbol{\omega}} \equiv \omega\hat{\boldsymbol{k}} \tag{13.31}$$

with ω as defined in eqn. (13.30). Note $\vec{\boldsymbol{\omega}}$ is the angular velocity of the object (and of *every* line on it)①. The usual sign convention for ω is the same as that for θ, positive is counter-clockwise (CCW) around the origin. That's the direction your fingers wrap if you point your thumb in the $+z$ direction and grab the z axis.

①We could also think of rotation rate as a derivative of the rotation matrix $[R]$. This point of view and its relation to the vector point of view here, is described in Box 13.7 on page 682.

Rate of change of $\hat{\imath}'$, $\hat{\jmath}'$

Our first use of the angular velocity vector $\vec{\boldsymbol{\omega}}$ is to calculate the rate of change of the rotating unit base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$. We can find the rate of change of, say, $\hat{\boldsymbol{\imath}}'$, by taking the time derivative of the first eqn. (13.24), and using the chain rule while recognizing that $\theta = \theta(t)$. We can also make an analogy with polar coordinates (page 647), where we think of $\hat{\boldsymbol{e}}_R$ as like $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{e}}_\theta$ as like $\hat{\boldsymbol{\jmath}}'$. We found there that $\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$ and $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$. Either from differentiating $\hat{\boldsymbol{\imath}}'$ or from the correlation with polar coordinates, we get

$$\begin{array}{lclcl} \dot{\hat{\imath}}' = \dot{\theta}\,\hat{\jmath}' & \text{or} & \dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}' & \text{and} \\ \dot{\hat{\jmath}}' = -\dot{\theta}\,\hat{\imath}' & \text{or} & \dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}' \end{array} \tag{13.32}$$

because $\hat{\jmath}' = \hat{k}' \times \hat{\imath}'$ and $\hat{\imath}' = -\hat{k}' \times \hat{\jmath}'$. Depending on the tastes of your lecturer, you may find Equations 13.32 are the most used equations from this point onward②.

② Eqn. 13.32 is sometimes considered the definition of $\vec{\omega}$. In this view, popularized by Tom Kane, $\vec{\omega}$ is that vector which determines $\dot{\hat{\imath}}'$ and $\dot{\hat{\jmath}}'$ by the formulas $\dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}'$ and $\dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}'$. Then one needs to show that such a vector exists and that it is $\vec{\omega} = \hat{\imath}' \times \dot{\hat{\imath}}'$. Luckily this reasoning leads to the same $\vec{\omega}$ as our $\vec{\omega} = \dot{\theta}\hat{k}$.

Velocity of a point fixed on a rigid object

Lets call some rotating object $\mathcal{B}$ (script capital B) to which is glued a coordinate system $x'y'$ with base vectors $\hat{\imath}'$ and $\hat{\jmath}'$. We now introduce the concept of *derivative in a frame* which we write, for the frame $\mathcal{B}$, as

$$\frac{{}^{\mathcal{B}}d}{dt}$$

which means, in words, the rate of change of something as viewed in the rotating frame $\mathcal{B}$. Now consider a point P at $\vec{r}_P$ that is glued to the object. That is, the x' and y' coordinates of $\vec{r}_P$ do not change in time.

$$\frac{{}^{\mathcal{B}}d\vec{r}_P}{dt} \equiv {}^{\mathcal{B}}\dot{\vec{r}}_P = \quad \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' = \vec{0}.$$

That is, relative to a moving frame, the velocity of a point glued to the frame is zero (no surprise).

We would like to know the velocity of such a point in the fixed frame. We just take the derivative, using the product rule and the differentiation rules we have developed for the rotating base vectors:

13.4 The fixed Newtonian reference frame $\mathcal{F}$

Now we can reconsider the concept of a Newtonian frame, a concept which we had to assume to write the equations of dynamics in the first place. All of mechanics depends, of course, on the laws of mechanics. The laws of mechanics are equations which involve, in part, the positions of things as a function of time. But how position is perceived to change in time depends on your reference frame. And some reference frames are better than others. The best, from our point of view, are reference frames in which Newton's laws are accurate. Such a reference frame is called a *Newtonian frame*. In engineering practice the frames we use as approximations of a Newtonian frame often seem, loosely speaking, somehow still. So we sometimes call such a frame *the fixed frame* and label it with a script capital $\mathcal{F}$. When we talk about velocity and acceleration of mass points, for use in the equations of mechanics, we are always talking about the velocity and acceleration relative to a $\mathcal{F}$ixed, or equivalently, Newtonian frame.

Assume x and y are the coordinates of a vector $\vec{r}_P$ and $\mathcal{F}$ is a fixed frame with fixed axis (with associated constant base vectors $\hat{\imath}$ and $\hat{\jmath}$). When we write $\dot{\vec{r}}_P$ we mean $\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. But we could be more explicit (and notationally ornate) and write the velocity of P in the Newtonian frame as

$$\frac{{}^{\mathcal{F}}d\vec{r}_P}{dt} \equiv {}^{\mathcal{F}}\dot{\vec{r}}_P \quad \text{by which we mean} \quad \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}.$$

The $\mathcal{F}$ in front of the time derivative (or in front of the dot) means that when we calculate a derivative we hold the base vectors of $\mathcal{F}$ constant. This is no surprise, because for $\mathcal{F}$ the base vectors *are* constant. In general, however, when taking a derivative in a given frame you

- write vectors in terms of base vectors stuck to the frame, and
- only differentiate the components.

But only for the $\mathcal{F}$ixed or $\mathcal{N}$ewtonian frame will accelerations calculated this way be directly applicable to Newton's laws.

We will avoid the ornate notation of labeling frames when it is not needed. For example, if you don't see any script capital letters floating around in front of derivatives, you can assume that we are taking derivatives relative to a fixed Newtonian frame.

$$
\begin{aligned}
\vec{r}_P &= x'\hat{\imath}' + y'\hat{\jmath}' \\
\Rightarrow \quad \vec{v}_P = \dot{\vec{r}}_P &= \frac{d}{dt}\left(x'\hat{\imath}' + y'\hat{\jmath}'\right) = x'\dot{\hat{\imath}} + y'\dot{\hat{\jmath}} = x'(\vec{\omega}\times\hat{\imath}') + y'(\vec{\omega}\times\hat{\jmath}') \\
&= \vec{\omega}\times(x'\hat{\imath}' + y'\hat{\jmath}')
\end{aligned}
$$

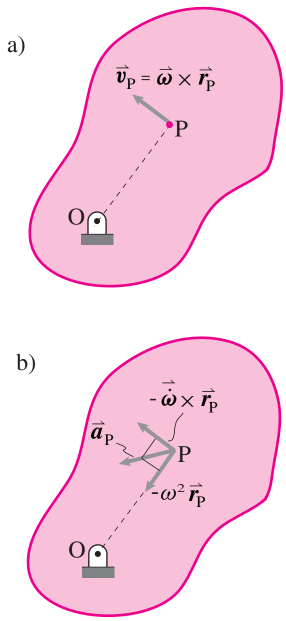

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Figure 13.45: Velocity and acceleration of two points on a rigid object rotating about 0.

where $\dot{\vec{r}}_P$ is the simple way to write $\frac{{}^{\mathcal{F}}d\vec{r}_P}{dt}$. Thus,

$$\vec{v}_P = \vec{\omega}\times\vec{r}_P \tag{13.38}$$

We can rewrite eqn. (13.38) in a minimalist or elaborate notation as

$$
\begin{aligned}
\vec{v} &= \vec{\omega}\times\vec{r} \qquad \text{or} \\
\frac{{}^{\mathcal{F}}d\vec{r}_P}{dt} &= \vec{\omega}_{\mathcal{B}/\mathcal{F}}\times\vec{r}_{P/O}.
\end{aligned}
$$

Both are correct. In the first case you have to use common sense to know what point you are talking about and that it is on a object rotating with absolute angular velocity $\vec{\omega}$. In the second case everything is laid out perfectly clearly (which is why it looks perfectly confusing). On the left side of the equation it says that we are interested in how point P moves relative to, not just any frame, but the fixed frame $\mathcal{F}$. On the right side we make clear that the rotation rate we are looking at is that of object $\mathcal{B}$ relative to $\mathcal{F}$ and not some other relative rotation. We further make clear that the formula only makes sense if the position of the point P is measured relative to a point which doesn't move, namely 0.

What we have just found largely duplicates what we already learned in section 7.1 for points moving in circles. The slight generalization is that the same angular velocity $\vec{\omega}$ can be used to calculate the velocities of multiple points on one rigid object. But the key idea remains: the velocity of a point

13.5 Plato's discussion of spinning in circles as motion (or not)

Plato imagines a discussion between Socrates and Glaucon about how an object can maintain contradictory attributes simultaneously:

" Socrates: Now let's have a more precise agreement so that we won't have any grounds for dispute as we proceed. If someone were to say of a human being standing still, but moving his hands and head, that the same man at the same time stands still and moves, I don't suppose we'd claim that it should be said like that, but rather that one part of him stands still and another moves. Isn't that so?
Glaucon: Yes it is.
Socrates: Then if the man who says this should become still more charming and make the subtle point that tops as wholes stand still and move at the same time when the peg is fixed in the same place and they spin, or that anything else going around in a circle on the same spot does this too, we wouldn't accept it because it's not with respect to the same part of themselves that such things are at the same time both at rest and in motion. But we'd say that they have in them both a straight and a circumference; and with respect to the straight they stand still since they don't lean in any direction –while with respect to the circumference they move in a circle; and when the straight inclines to the right the left, forward, or backward at the same time that it's spinning, then in no way does it stand still.
Glaucon: And we'd be right."

This chapter is about things that are still with respect to their own parts (they do not distort) but in which the points do move in circles.

going in circles is tangent to the circle it is going around and with magnitude proportional both to distance from the center and to the angular rate of rotation (fig. 13.45a).

Acceleration of a point on a rotating rigid object

Let's again consider a point stuck on a rotating object and with position

$$\vec{r}_P = x'\hat{\imath}' + y'\hat{\jmath}'.$$

Relative to the frame $\mathcal{B}$ to which a point is attached, its acceleration is zero (again no surprise). But what is its acceleration in the fixed frame? We find this acceleration by writing the position vector and then differentiating twice, repeatedly using the product rule and eqn. (13.32) we get (see Box 13.6 for the details).

$$\vec{a}_P = \dot{\vec{\omega}} \times \vec{r}_P + \vec{\omega} \times (\vec{\omega} \times \vec{r}_P) \tag{13.39}$$

which is hardly intuitive at a glance③. Recalling that in 2D $\vec{\omega} = \omega\hat{k}$ we can use either the right hand rule or manipulation of unit vectors to rewrite eqn. (13.39) as

$$\vec{a}_P = \dot{\omega}\hat{k} \times \vec{r}_P - \omega^2\vec{r}_P \tag{13.40}$$

where $\omega = \dot{\theta}$ and $\dot{\omega} = \ddot{\theta}$④.

Thus, as we found in section 13.1 for a particle going in circles, the acceleration can be written as the sum of two terms, a tangential acceleration $\dot{\omega}\hat{k} \times \vec{r}_P$ due to increasing tangential speed, and a centrally directed (centripetal) acceleration $-\omega^2\vec{r}_P$ due to the direction of the velocity continuously changing towards the center (see fig. 13.45b). The generalization we have

③ Although the form eqn. (13.39) is not of much immediate use, if you are going to continue on to the mechanics of mechanisms or three dimensional mechanics, you should follow the derivation of eqn. (13.39) carefully.

④ The derivation of eqn. (13.40) can be written more informally and briefly like this (using minimalist notation):

$$\begin{aligned}
\vec{a} &= \dot{\vec{v}} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) \\
&= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \dot{\vec{r}} \\
&= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \\
&= \dot{\vec{\omega}} \times \vec{r} - \omega^2\vec{r}.
\end{aligned}$$

13.6 Acceleration of a point on a rotating body, using $\vec{\omega}$

Leaving off the ornate pre-super-script $\mathcal{F}$ for simplicity, we have

$$\begin{aligned}
\vec{a}_P = \dot{\vec{v}}_P &= \frac{d}{dt}\left(\frac{d}{dt}(x'\hat{\imath}' + y'\hat{\jmath}')\right) \\
&= \frac{d}{dt}\left(x'(\vec{\omega} \times \hat{\imath}') + y'(\vec{\omega} \times \hat{\jmath}')\right).
\end{aligned} \tag{13.33}$$

To continue we need to use the product rule of differentiation for the cross product of two time dependent vectors like this:

$$\frac{d}{dt}(\vec{\omega} \times \hat{\imath}') = \dot{\vec{\omega}} \times \hat{\imath}' + \vec{\omega} \times \dot{\hat{\imath}}' \tag{13.34}$$

$$= \dot{\vec{\omega}} \times \hat{\imath}' + \vec{\omega} \times (\vec{\omega} \times \hat{\imath}'), \tag{13.35}$$

$$\frac{d}{dt}(\vec{\omega} \times \hat{\jmath}') = \dot{\vec{\omega}} \times \hat{\jmath}' + \vec{\omega} \times \dot{\hat{\jmath}}' \tag{13.36}$$

$$= \dot{\vec{\omega}} \times \hat{\jmath}' + \vec{\omega} \times (\vec{\omega} \times \hat{\jmath}'). \tag{13.37}$$

Substituting back into eqn. (13.33) we get

$$\begin{aligned}
\vec{a}_P &= x'\left(\dot{\vec{\omega}} \times \hat{\imath}' + \vec{\omega} \times (\vec{\omega} \times \hat{\imath}')\right) \\
&\quad + y'\left(\dot{\vec{\omega}} \times \hat{\jmath}' + \vec{\omega} \times (\vec{\omega} \times \hat{\jmath}')\right) \\
&= \dot{\vec{\omega}} \times (x'\hat{\imath}' + y'\hat{\imath}') + \vec{\omega} \times (\vec{\omega} \times (x'\hat{\imath}' + y'\hat{\imath}')) \\
&= \dot{\vec{\omega}} \times \vec{r}_P + \vec{\omega} \times (\vec{\omega} \times \vec{r}_P)
\end{aligned}$$

which derives eqn. (13.39).

made in this section is that the same $\vec{\omega}$ can be used to calculate the acceleration for all the different points on one rotating object.

Relative motion of points on a rigid object

As you well know by now, the position of point B relative to point A is $\vec{r}_{B/A} \equiv \vec{r}_B - \vec{r}_A$. Similarly the relative velocity and acceleration of two points A and B is defined to be

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A \quad \text{and} \quad \vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A \qquad (13.41)$$

So, the relative velocity (as calculated relative to a fixed frame) of two points glued to one spinning rigid object $\mathcal{B}$ is given by

$$\begin{aligned} \vec{v}_{B/A} &\equiv \vec{v}_B - \vec{v}_A \\ &= \vec{\omega} \times \vec{r}_{B/O} - \vec{\omega} \times \vec{r}_{A/O} \\ &= \vec{\omega} \times (\vec{r}_{B/O} - \vec{r}_{A/O}) \\ &= \vec{\omega} \times \vec{r}_{B/A}, \end{aligned}$$

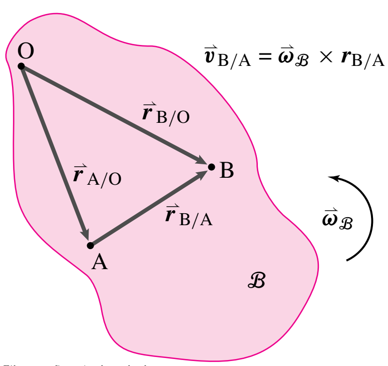

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Figure 13.46: The acceleration of B relative to A if they are both on the same rotating rigid object.

where point O is the point in the Newtonian frame on the fixed axis of rotation and $\vec{\omega} = \vec{\omega}_{\mathcal{C}}$ is the angular velocity of $\mathcal{C}$. Repeating,

$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}. \qquad (13.42)$$

Because points A and B are fixed on $\mathcal{B}$ their velocities and hence their relative velocity as observed in a reference frame fixed to $\mathcal{C}$ are all $\vec{\mathbf{0}}$. But, point A has an absolute velocity that is different from that of point B. So they have a relative velocity as seen in the fixed frame. And it is what you would get if B was just going in circles around A. Similarly, the relative acceleration of two points glued to one rigid object spinning at constant rate is

$$\vec{a}_{B/A} \quad \equiv \quad \vec{a}_B - \vec{a}_A \quad = \quad \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}). \quad (13.43)$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point O fixed on the axis. These kinematics results, 13.42 and 13.43, are useful for calculating angular momentum relative to the center-of-mass. They are also sometimes useful for the understanding of the motions of machines with moving connected parts.

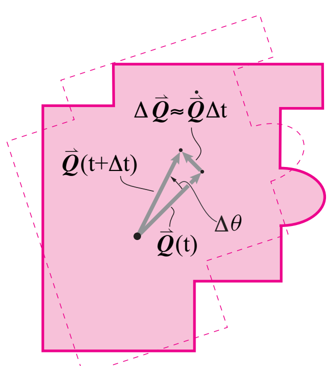

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Figure 13.47: A vector $\vec{Q}$ is fixed to a rotating object. So its rate of change is the velocity of its tip. The magnitude is $\dot{\theta}|\vec{Q}|$ and the direction is orthogonal to $\vec{Q}$.

The fundamental $\vec{\omega}$ equation

Equation 13.42 actually applies to any vector $\vec{Q}$ that has the property that it is fixed in the frame which is rotating at $\vec{\omega}$ (fig. 13.47). That is, all we needed in the derivation was that $\vec{Q}$'s components $(Q_{x'}, Q_{y'})$ in the $\hat{\imath}'$-$\hat{\jmath}'$ reference frame be constant. Examples of such $\vec{Q}$ would include more than just the relative position of two points fixed on the object, $\vec{r}_{B/A}$. Both $\hat{\imath}'$ and $\hat{\jmath}'$ are also are fixed in the rotating frame. Also imagine a bug moving at a constant speed on a straight line marked on a body. That bug's velocity relative to the rotating frame would also be constant as seen in the frame. For any $\vec{Q}$ whose representation as an arrow moves like a line segment drawn on a rigid object rotating at $\vec{\omega}$:

$$\dot{\vec{Q}} = \vec{\omega} \times \vec{Q}. \tag{13.44}$$

This is the most fundamental equation of rigid-object kinematics. Know it well. For future reference, equation (13.44) is also valid in three dimensions. Equation 13.44 is the generalization of the three equations

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}, \qquad \dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}' \qquad \text{and} \qquad \dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}'.$$

Calculating relative velocity directly, using rotating frames

A coordinate system $x'y'$ attached to a rotating rigid object $\mathcal{C}$, defines a reference frame $\mathcal{C}$ (fig. 13.36 on page 670). Recall, the base vectors in this frame change in time just like any other vector fixed in the rotating frame (eqn. (13.44))

$$\frac{d}{dt}\hat{\imath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\imath}' \quad \text{and} \quad \frac{d}{dt}\hat{\jmath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}'.$$

Once we know how the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ change with time, as per the above equations, we can find velocity and acceleration by differentiation of position. For example, consider two points A and B fixed on a rotating object. The position of B relative to A is

$$\vec{r}_{B/A} = x'\hat{\imath}' + y'\hat{\jmath}'.$$

where we are using x' and y' as a shorthand notation for $x'_{B/A}$ and $y'_{B/A}$ (see fig. 13.48). The coordinates x' and y' are constant so

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

Now we just differentiate the relative position $\vec{r}_{B/A}$ with respect to time,

$$\begin{aligned}
\frac{d}{dt}(\vec{r}_{B/A}) &= \frac{d}{dt}\left(x'\hat{\imath}' + y'\hat{\jmath}'\right) \\
&= \underbrace{\dot{x}'}_{0}\hat{\imath}' + x'\frac{d}{dt}\hat{\imath}' + \underbrace{\dot{y}'}_{0}\hat{\jmath}' + y'\frac{d}{dt}\hat{\jmath}' \\
&= x'(\vec{\omega}_{\mathcal{C}} \times \hat{\imath}') + y'(\vec{\omega}_{\mathcal{C}} \times \hat{\jmath}') \\
&= \vec{\omega}_{\mathcal{C}} \times \underbrace{(x'\hat{\imath}' + y'\hat{\jmath}')}_{\vec{r}_{B/A}} \\
&= \vec{\omega}_{\mathcal{C}} \times \vec{r}_{B/A}.
\end{aligned}$$

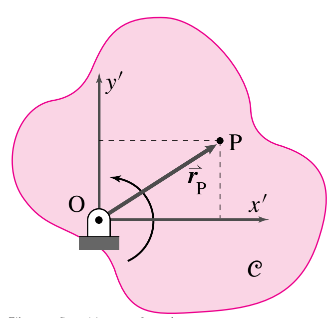


Figure 13.48: Coordinates in a rotating frame.

We could similarly calculate $\vec{a}_{B/A}$ by taking another derivative to get, after a calculation much like that above,

$$\vec{a}_{B/A} = \vec{\omega}_{\mathcal{C}} \times \left(\vec{\omega}_{\mathcal{C}} \times \vec{r}_{B/A}\right) + \dot{\vec{\omega}}_{\mathcal{C}} \times \vec{r}_{B/A}.$$

For the above calculation the points A and B were fixed on a rotating part. Most machines have many parts, at least some of which move in more complex ways than just circular motion. We will be able to understand such machines by considering points and parts that are moving relative to a part that is itself rotating. Such will be considered in Chapter 16.

13.7 Angular velocity $\vec{\omega}$ and the rotation matrix $[R]$

The rotation matrix $[R]$ and the angular velocity vector $\vec{\omega} = \omega\hat{k}$ are related. Because angular velocity is a rotation rate, in some sense it must be the derivative of the rotation. But the situation is a bit subtle because $\vec{\omega}$ is a vector and $[R]$ is a matrix.

On the one hand we have the rotation equation:

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{[\vec{r}]_{xy}} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{[R]} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{[\vec{r}^{\text{ref}}]_{xy}}.$$

This gives the coordinates of $\vec{r}$ after rotation in terms of its coordinates before the rotation. Because the coordinates $\left[\vec{r}^{\text{ref}}\right]_{xy}$ before rotation are fixed we can take the time derivative of both sides to get

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}_{[\dot{\vec{r}}]_{xy}} = \dot{\theta}\underbrace{\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}}_{[\dot{R}]} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{[\vec{r}^{\text{ref}}]_{xy}}.$$

On the other hand we have the vector equation

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}.$$

where $\vec{\omega} = \omega\hat{k}$. We can relate this vector equation to the matrix component equation above it by expressing the cross product with a matrix multiplication as described in box 2.6 on 73. Applying that result to this 2D case where

$$[\vec{\omega}]_{xyz} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \quad \text{and} \quad [\vec{r}]_{xyz} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

we get

$$\underbrace{\begin{bmatrix} -\omega y \\ \omega x \\ 0 \end{bmatrix}}_{[\dot{\vec{r}}]_{xy}} = \underbrace{\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{S}(\vec{\omega})]} \underbrace{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}}_{[\vec{r}^{\text{ref}}]_{xy}}.$$

Just keeping the 2D terms we have

$$\begin{aligned} [\vec{\omega} \times \vec{r}]_{xy} &= [\mathcal{S}(\vec{\omega})]\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

But $[R]$ tells us the coordinates in terms of the reference coordinates. So

$$[\vec{\omega} \times \vec{r}]_{xy} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \underbrace{[R]\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\begin{bmatrix} x \\ y \end{bmatrix}}.$$

Comparing the equations above we see that

$$[\dot{R}] = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}[R]. \qquad (13.45)$$

Using the notation from box 2.6 we can rewrite this as

$$[\dot{R}] = [\mathcal{S}(\vec{\omega})]\,[R]. \qquad (13.46)$$

Equation 13.46 is the way that angular velocity $\vec{\omega}$ and the rate of change $[\dot{R}]$ of the rotation matrix are related.

We can also solve for $[\mathcal{S}(\vec{\omega})]$ as

$$[\mathcal{S}(\vec{\omega})] = [\dot{R}][R]^{-1}$$

from which we can find $\pm\omega$ in the off diagonal elements if we are given $[R]$ and $[\dot{R}]$.

We can understand eqn. (13.46) better, perhaps, by thinking of the rotation matrix $[R]$ as having two columns which are, respectively, the fixed-coordinate-system components of $\hat{\imath}'$ and $\hat{\jmath}'$:

$$[R] = [[\hat{\imath}']_{xy} \mid [\hat{\jmath}']_{xy}]$$

Thus the columns of $[\dot{R}]$ are the rates of change of these two unit vectors:

$$[\dot{R}] = \left[[\dot{\hat{\imath}}']_{xy} \mid [\dot{\hat{\jmath}}']_{xy}\right] = [[\vec{\omega} \times \hat{\imath}']_{xy} \mid [\vec{\omega} \times \hat{\jmath}']_{xy}].$$

With a total abuse of notation (in general one does not allow cross products of a vectors with matrices) we can write the above equation more memorably as

$$\text{“}\,[\dot{R}] = \vec{\omega} \times [R].\,\text{”}$$

Thus eqn. (13.46) is just a version of the fundamental kinematic equation 13.44 on page 681.

SAMPLE 13.12 A uniform bar AB of length $\ell = 50\,\mathrm{cm}$ rotates counter-clockwise about point A with constant angular speed ω. At the instant shown in fig. 13.49 the linear speed v_C of the center-of-mass C is $7.5\,\mathrm{cm/\,s}$.

1. What are the angular speed and angular velocity of the bar?
2. What is the linear velocity of point B?
3. By what angles do the angular positions of points C and B change in 2 seconds?

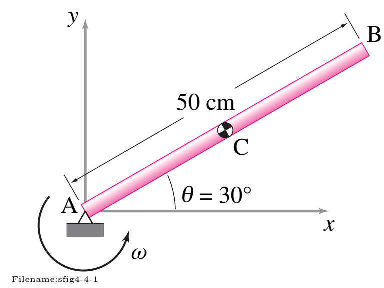


Figure 13.49:

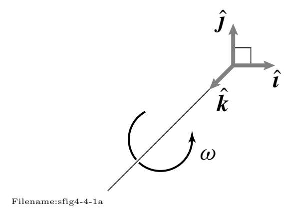


Figure 13.50:

Solution Let the angular velocity of the bar be $\vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\boldsymbol{k}}$ where $\dot{\theta}$ is the angular speed. We first need to find $\dot{\theta}$.

1. The linear speed of point C is given, $v_C = 7.5\,\mathrm{cm/\,s}$. Now,

$$\begin{aligned} v_C &= \dot{\theta}\, r_C \\ \Rightarrow \quad \dot{\theta} &= \frac{v_C}{r_C} = \frac{7.5\,\mathrm{cm/\,s}}{25\,\mathrm{cm}} = 0.3\,\mathrm{rad/s}. \end{aligned}$$

Therefore, the angular velocity of the bar is $\vec{\boldsymbol{\omega}} = \dot{\theta}\hat{\boldsymbol{k}} = 0.3\,\mathrm{rad/s}\hat{\boldsymbol{k}}$.

$$\boxed{\dot{\theta} = 0.3\,\mathrm{rad/s}, \text{ and } \vec{\boldsymbol{\omega}} = 0.3\,\mathrm{rad/s}\hat{\boldsymbol{k}}}$$

2. Point B is at distance ℓ from the pivot point A. Thus it goes around a circle of radius ℓ (see fig. 13.51). Therefore,

$$\begin{aligned} \vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_B = \dot{\theta}\hat{\boldsymbol{k}} \times \ell(\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) \\ &= \dot{\theta}\ell(\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{\imath}}) \\ &= 0.3\,\mathrm{rad/s}\cdot 50\,\mathrm{cm}\left(\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}\right) \\ &= 15\,\mathrm{cm/\,s}\left(\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}\right). \end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_B = 15\,\mathrm{cm/\,s}\left(\tfrac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \tfrac{1}{2}\hat{\boldsymbol{\imath}}\right)}$$

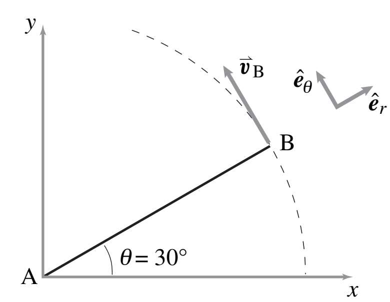


Figure 13.51: From the given geometry, $\hat{\boldsymbol{e}}_r = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{e}}_\theta = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{v}}_B = |\vec{\boldsymbol{v}}_B|\hat{\boldsymbol{e}}_\theta$.

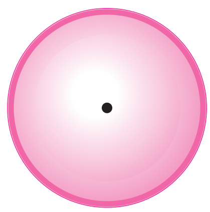
We can also write $\vec{\boldsymbol{v}}_B = 15\,\mathrm{cm/\,s/}, \hat{\boldsymbol{e}}_\theta$ where $\hat{\boldsymbol{e}}_\theta = \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}$.

3. Let θ_1 be the position of point C at some time t_1 and θ_2 be the position at time t_2. We want to find $\Delta\theta = \theta_2 - \theta_1$ for $t_2 - t_1 = 2\,\mathrm{s}$.

$$\begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} = \text{constant} = 0.3\,\mathrm{rad/s}. \\ \Rightarrow \quad d\theta &= (0.3\,\mathrm{rad/s})dt. \\ \Rightarrow \quad \int_{\theta_1}^{\theta_2} d\theta &= \int_{t_1}^{t_2} (0.3\,\mathrm{rad/s})dt. \\ \Rightarrow \quad \theta_2 - \theta_1 &= 0.3\,\mathrm{rad/s}(t_2 - t_1) \\ \text{or} \quad \Delta\theta &= 0.3\frac{\mathrm{rad}}{\cancel{\mathrm{s}}}\cdot 2\cancel{\mathrm{s}} = 0.6\,\mathrm{rad}. \end{aligned}$$

The change in angular position of point B is the same as that of point C. In fact, all points on AB undergo the same change in angular position because AB is a rigid body.

$$\boxed{\Delta\theta_C = \Delta\theta_B = 0.6\,\mathrm{rad}}$$



Filename:sfig13-3-wheel

Figure 13.52:

SAMPLE 13.13 A flywheel of diameter 2 ft is made of cast iron. To avoid extremely high stresses and cracks it is recommended that the peripheral speed not exceed 6000 to 7000 ft/min. What is the corresponding rpm rating for the wheel?

Solution

$$\begin{aligned} \text{Diameter of the wheel} &= 2\,\text{ft.} \\ \Rightarrow \quad \text{radius of wheel} &= 1\,\text{ft.} \end{aligned}$$

Now,

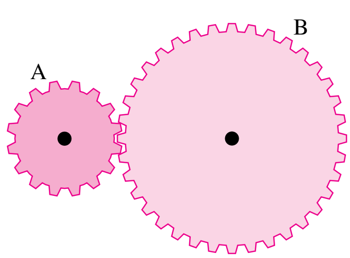
$$\begin{aligned} v &= \omega r \\ \Rightarrow \quad \omega &= \frac{v}{r} = \frac{6000\,\cancel{\text{ft}}/\text{min}}{1\,\cancel{\text{ft}}} \\ &= 6000\frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\text{ß}\,\text{rad}} \\ &= 955\,\text{rpm.} \end{aligned}$$

Similarly, corresponding to $v = 7000\,\text{ft}/\text{min}$

$$\begin{aligned} \omega &= \frac{7000\,\cancel{\text{ft}}/\text{min}}{1\,\cancel{\text{ft}}} \\ &= 7000\frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\text{ß}\,\text{rad}} \\ &= 1114\,\text{rpm.} \end{aligned}$$

Thus the rpm rating of the wheel should read 955 – 1114 rpm.

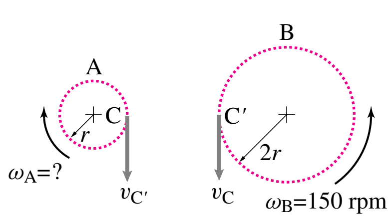
$\omega = 955$ to $1114\,\text{rpm}$.



Filename:sfig4-4-3

Figure 13.53:

SAMPLE 13.14 Two gears A and B have the diameter ratio of 1:2. Gear A drives gear B. If the output at gear B is required to be 150 rpm, what should be the angular speed of the driving gear? Assume no slip at the contact point.



Filename:sfig4-4-3a

Figure 13.54:

Solution Let C and C′ be the points of contact on gear A and B respectively at some instant t. Since there is no relative slip between C and C′, both points must have the same linear velocity at instant t. If the velocities are the same, then the linear speeds must also be the same. Thus

$$\begin{aligned} v_C &= v_{C'} \\ \Rightarrow \quad \omega_A r_A &= \omega_B r_B \\ \Rightarrow \quad \omega_A &= \omega_B \cdot \frac{r_B}{r_A} \\ &= \omega_B \cdot \frac{2\cancel{r}}{\cancel{r}} = 2\omega_B \\ &= (2)\cdot(150\,\text{rpm}) \\ &= 300\,\text{rpm.} \end{aligned}$$

$\omega_A = 300\,\text{rpm}$

SAMPLE 13.15 A uniform rigid rod AB of length $\ell = 0.6\,\text{m}$ is connected to two rigid links OA and OB. The assembly rotates at a constant rate about point O in the xy plane. At the instant shown, when rod AB is vertical, the velocities of points A and B are $\vec{v}_A = -4.64\,\text{m/s}\,\hat{\jmath} - 1.87\,\text{m/s}\hat{\imath}$, and $\vec{v}_B = 1.87\,\text{m/s}\hat{\imath} - 4.64\,\text{m/s}\,\hat{\jmath}$. Find the angular velocity of bar AB. What is the length R of the links?

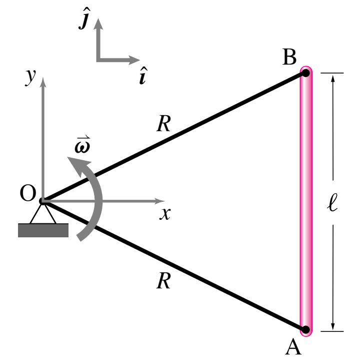


Figure 13.55:

Solution Let the angular velocity of the rod AB be $\vec{\omega} = \omega\hat{k}$. ⑤ Since we are given the velocities of two points on the rod we can use the relative velocity formula to find $\vec{\omega}$:

$$
\begin{aligned}
\vec{v}_{B/A} = \vec{\omega}\times\vec{r}_{B/A} &= \vec{v}_B - \vec{v}_A\\
\text{or}\quad \underbrace{\omega\hat{k}}_{\vec{\omega}} \times \underbrace{\ell\hat{\jmath}}_{\vec{r}_{B/A}} &= (1.87\hat{\imath} - 4.64\hat{\jmath})\,\text{m/s} - (-4.64\hat{\jmath} - 1.87\hat{\imath})\,\text{m/s}\\
\text{or}\quad \omega\ell(-\hat{\imath}) &= (1.87\hat{\imath} + 1.87\hat{\imath})\,\text{m/s} - (4.\cancel{6}4\hat{\jmath} - 4.\cancel{6}4\hat{\jmath})\,\text{m/s}\\
&= 3.74\hat{\imath}\,\text{m/s}\\
\Rightarrow\qquad \omega &= -\frac{3.74\,\text{m/s}}{\ell}\\
&= -\frac{3.74}{0.6}\,\text{rad/s}\\
&= -6.233\,\text{rad/s} \qquad (13.47)
\end{aligned}
$$

Thus,

$$\vec{\omega} = -6.233\,\text{rad/s}\hat{k}. \qquad (13.48)$$

$$\boxed{\vec{\omega} = -6.23\,\text{rad/s}\hat{k}}$$

⑤ We know that the rod rotates about the z-axis but we do not know the sense of the rotation *i.e.,* $+\hat{k}$ or $-\hat{k}$. Here we have assumed that $\vec{\omega}$ is in the positive $\hat{k}$ direction, although just by sketching $\vec{v}_A$ we can easily see that $\vec{\omega}$ must be in the $-\hat{k}$ direction.

Let θ be the angle between link OA and the horizontal axis. Now,

$$
\begin{aligned}
\vec{v}_A = \vec{\omega}\times\vec{r}_A &= \omega\hat{k} \times \underbrace{R(\cos\theta\hat{\imath} - \sin\theta\,\hat{\jmath})}_{\vec{r}_A}\\
\text{or}\quad (-4.64\hat{\jmath} - 1.87\hat{\imath})\,\text{m/s} &= \omega R(\cos\theta\,\hat{\jmath} + \sin\theta\hat{\imath})
\end{aligned}
$$

Dotting both sides of the equation with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$
\begin{aligned}
-1.87\,\text{m/s} &= \omega R\sin\theta \qquad (13.49)\\
-4.64\,\text{m/s} &= \omega R\cos\theta \qquad (13.50)
\end{aligned}
$$

Squaring and adding Eqns (13.49) and (13.50) together we get

$$
\begin{aligned}
\omega^2R^2 &= (-4.64\,\text{m/s})^2 + (-1.187\,\text{m/s})^2\\
&= 25.026\,\text{m}^2/\,\text{s}^2\\
\Rightarrow\qquad R^2 &= \frac{25.026\,\text{m}^2/\,\text{s}^2}{(-6.23\,\text{rad/s})^2}\\
&= 0.645\,\text{m}^2\\
\Rightarrow\qquad R &= 0.8\,\text{m}
\end{aligned}
$$

$$\boxed{R = 0.8\,\text{m}}$$

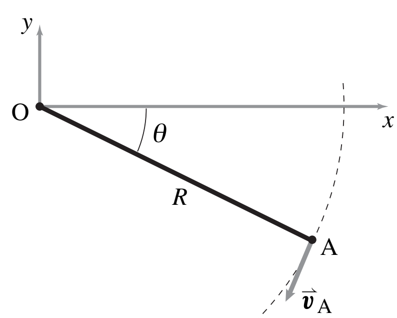


Figure 13.56: The given velocity vector $\vec{v}_A$. Since $\vec{v}_A = \vec{\omega}\times\vec{r}_A$ where $\vec{r}_A = R\cos\theta\hat{\imath} + R\sin\theta\,\hat{\jmath}$, we can find R from the given $\vec{v}_A$.

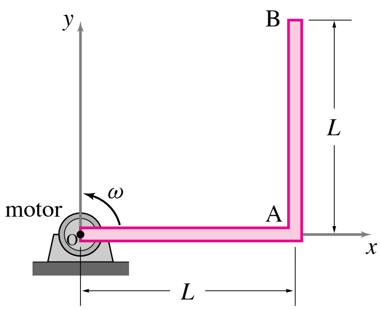


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Figure 13.57: An ‘L’ shaped bar rotates at speed ω about point O.

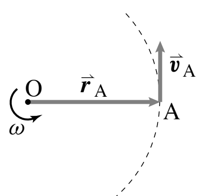

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Figure 13.58: $\vec{v}_A = \vec{\omega} \times \vec{r}_A$ is tangential to the circular path of point A.

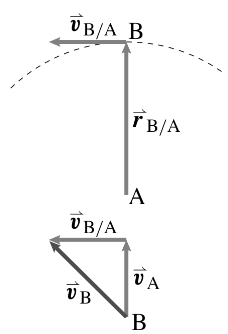

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Figure 13.59: $\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$ and $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$.

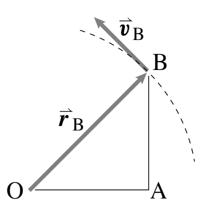

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Figure 13.60: $\vec{v}_B = \vec{\omega} \times \vec{r}_B$.

SAMPLE 13.16 **Verify the relative velocity formula:** The motor at O in Fig. 13.57 rotates the ‘L’ shaped bar OAB in counterclockwise direction at an angular speed which increases at $\dot{\omega} = 2.5\,\mathrm{rad/s^2}$. At the instant shown, the angular speed $\omega = 4.5\,\mathrm{rad/s}$. Each arm of the bar is of length L = 2 ft.

1. Find the velocity of point A.
2. Find the relative velocity $\vec{v}_{B/A}$ $(= \vec{\omega} \times \vec{r}_{B/A})$ and use the result to find the absolute velocity of point B $(\vec{v}_B = \vec{v}_A + \vec{v}_{B/A})$.
3. Find the velocity of point B directly. Check the answer obtained in part (b) against the new answer.

Solution

1. As the bar rotates, every point on the bar goes in circles centered at point O. Therefore, we can easily find the velocity of any point on the bar using circular motion formula $\vec{v} = \vec{\omega} \times \vec{r}$. Thus,

$$\begin{aligned} \vec{v}_A &= \vec{\omega} \times \vec{r}_A = \omega\hat{k} \times L\hat{\imath} = \omega L\hat{\jmath} \\ &= 4.5\,\mathrm{rad/s} \cdot 2\,\mathrm{ft}\hat{\jmath} = 9\,\mathrm{ft/s}\hat{\jmath}. \end{aligned}$$

The velocity vector $\vec{v}_A$ is shown in Fig. 13.58.

$$\boxed{\vec{v}_A = 9\,\mathrm{ft/s}\hat{\jmath}}$$

2. Point B and A are on the same rigid body. Therefore, with respect to point A, point B goes in circles about A. Hence the relative velocity of B with respect to A is

$$\begin{aligned} \vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} \\ &= \omega\hat{k} \times L\hat{\jmath} = -\omega L\hat{\imath} \\ &= -4.5\,\mathrm{rad/s} \cdot 2\,\mathrm{ft}\hat{\imath} = -9\,\mathrm{ft/s}\hat{\imath}. \\ \text{and} \qquad \vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\ &= 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

These velocities are shown in Fig. 13.59.

$$\boxed{\vec{v}_{B/A} = -9\,\mathrm{ft/s}\hat{\imath},\ \vec{v}_B = 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath})}$$

3. Since point B goes in circles of radius OB about point O, we can find its velocity directly using circular motion formula:

$$\begin{aligned} \vec{v}_B &= \vec{\omega} \times \vec{r}_B \\ &= \omega\hat{k} \times (L\hat{\imath} + L\hat{\jmath}) = \omega L(\hat{\jmath} - \hat{\imath}) \\ &= 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

The velocity vector is shown in Fig. 13.60. Of course this velocity is the same velocity as obtained in part (b) above.

$$\boxed{\vec{v}_B = 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath})}$$

Note: Nothing in this sample uses $\dot{\omega}$!

SAMPLE 13.17 **Verify the relative acceleration formula:** Consider the 'L' shaped bar of Sample 13.16 again. At the instant shown, the bar is rotating at 4 rad/s and is slowing down at the rate of 2 rad/s^2.

(i) Find the acceleration of point A.

(ii) Find the relative acceleration $\vec{a}_{B/A}$ of point B with respect to point A and use the result to find the absolute acceleration of point B ($\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$).

(iii) Find the acceleration of point B directly and verify the result obtained in (ii).

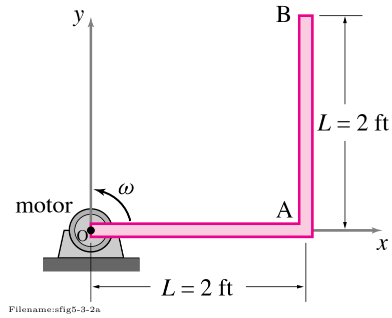


Figure 13.61: The 'L' shaped bar is rotates counterclockwise while slowing down.

Solution We are given:

$$\vec{\omega} = \omega\hat{k} = 4\,\text{rad/s}\,\hat{k}, \quad \text{and} \quad \dot{\vec{\omega}} = -\dot{\omega}\hat{k} = -2\,\text{rad/s}^2\,\hat{k}.$$

(i) Point A is going in circles of radius L. Hence,

$$\begin{aligned}
\vec{a}_A &= \dot{\vec{\omega}} \times \vec{r}_A + \vec{\omega} \times (\vec{\omega} \times \vec{r}_A) = \dot{\vec{\omega}} \times \vec{r}_A - \omega^2 \vec{r}_A \\
&= -\dot{\omega}\hat{k} \times L\hat{\imath} - \omega^2 L\hat{\imath} = -\dot{\omega}L\hat{\jmath} - \omega^2 L\hat{\imath} \\
&= -2\,\text{rad/s} \cdot 2\,\text{ft}\hat{\jmath} - (4\,\text{rad/s})^2 \cdot 2\,\text{ft}\hat{\imath} \\
&= -(4\hat{\jmath} + 32\hat{\imath})\,\text{ft/s}^2.
\end{aligned}$$

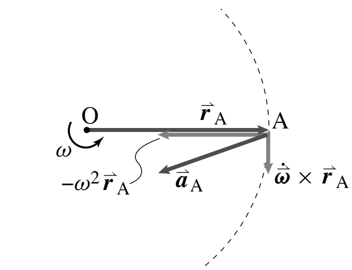


Figure 13.62:

$$\boxed{\vec{a}_A = -(4\hat{\jmath} + 32\hat{\imath})\,\text{ft/s}^2}$$

(ii) The relative acceleration of point B with respect to point A is found by considering the motion of B with respect to A. Since both the points are on the same rigid body, point B executes circular motion with respect to point A. Therefore,

$$\begin{aligned}
\vec{a}_{B/A} &= \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) = \dot{\vec{\omega}} \times \vec{r}_{B/A} - \omega^2 L\hat{\jmath} \\
&= -\dot{\omega}\hat{k} \times L\hat{\jmath} - \omega^2 L\hat{\jmath} \\
&= \dot{\omega}L\hat{\imath} - \omega^2 L\hat{\jmath} = 2\,\text{rad/s}^2 \cdot 2\,\text{ft}\hat{\imath} - (4\,\text{rad/s})^2 \cdot 2\,\text{ft}\hat{\jmath} \\
&= (4\hat{\imath} - 32\hat{\jmath})\,\text{ft/s}^2,
\end{aligned}$$

and

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A} = (-28\hat{\imath} - 36\hat{\jmath})\,\text{ft/s}^2.$$

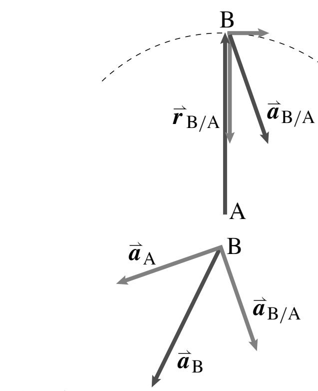


Figure 13.63:

$$\boxed{\vec{a}_B = -(28\hat{\imath} + 36\hat{\jmath})\,\text{ft/s}^2}$$

(iii) Since point B is going in circles of radius OB about point O, we can find the acceleration of B as follows.

$$\begin{aligned}
\vec{a}_B &= \dot{\vec{\omega}} \times \vec{r}_B + \vec{\omega} \times (\vec{\omega} \times \vec{r}_B) \\
&= \dot{\vec{\omega}} \times \vec{r}_B - \omega^2 \vec{r}_B \\
&= -\dot{\omega}\hat{k} \times (L\hat{\imath} + L\hat{\jmath}) - \omega^2 (L\hat{\imath} + L\hat{\jmath}) \\
&= (-\dot{\omega}L - \omega^2 L)\hat{\jmath} + (\dot{\omega}L - \omega^2 L)\hat{\imath} \\
&= (-4 - 32)\,\text{ft/s}^2\,\hat{\jmath} + (4 - 32)\,\text{ft/s}^2\,\hat{\imath} \\
&= (-36\hat{\jmath} - 28\hat{\imath})\,\text{ft/s}^2.
\end{aligned}$$

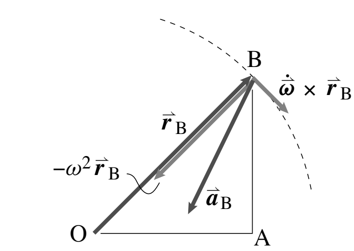


Figure 13.64:

This acceleration is, naturally again, the same acceleration as found in (ii) above.

$$\boxed{\vec{a}_B = -(28\hat{\imath} + 36\hat{\jmath})\,\text{ft/s}^2}$$

13.5 Polar moment of inertia

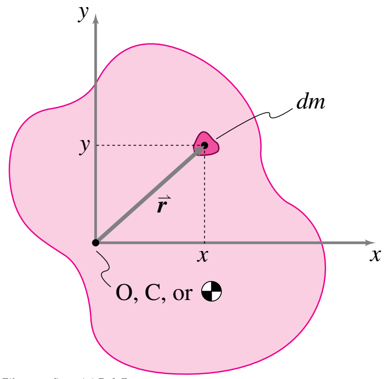


Figure 13.65: A bit of mass dm on a general planar body.

As you can see on the inside cover, all of the basic equations of mechanics concern the motion of mass. In particular, all of the terms in all of momentum, angular momentum, and energy equations concern sums over all the bits of mass in a system, with each bit of mass multiplied by some terms concerning position, velocity and acceleration. From the earlier sections in this chapter we know how to find the velocity and acceleration of every bit of mass on a 2-D rigid body as it spins about a fixed axis. So it is just a matter of doing integrals or sums to calculate the various momentum and energy quantities of interest. As a body moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid body one has to evaluate a different integral or sum at every different configuration. But there is a shortcut: for a rotating rigid object a sum (over all atoms, say), or a difficult integral (for example, over the complex region representing a machine part) is reduced to simple multiplication.

The *moment of inertia* I^{cm}①simplifies the expressions for the angular momentum, the rate of change of angular momentum, and the energy of a rigid body. For more general motions the shortcuts need a 3×3 matrix $[\boldsymbol{I}^{\text{cm}}]$ But for 2D mechanics only one component of the matrix $[\boldsymbol{I}^{\text{cm}}]$ is relevant, it is I_{zz}^{cm}, called just I or J for short.

Here are the main results. A flat object spinning with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ in the xy plane has a mass distribution which gives, by means of a calculation which we will discuss shortly, a moment of inertia I^{cm} so that:

$$\vec{\boldsymbol{H}}_{\text{cm}} = I^{\text{cm}}\omega\hat{\boldsymbol{k}} \tag{13.51}$$

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = I^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \tag{13.52}$$

$$E_{\text{K}/cm} = \frac{1}{2}\omega^2 I^{\text{cm}}. \tag{13.53}$$

There are two main skills you need to develop associated with I.

- Using the formulas above properly in angular momentum and energy balance. This is covered in sec. 13.6.
- Finding I of an object, as is covered in in most freshman calculus texts and is reviewed in this section.

Some facts about moment of inertia are summarized on the inside back cover.

The moments of inertia in 2-D : $[\boldsymbol{I}^{\text{cm}}]$ and $[\boldsymbol{I}^{\text{O}}]$.

The definition of moment of inertia② I^{cm} is

①**The COM is the most common reference point.** The moment of inertia of a given object depends on what reference point is used. Most often when people say 'the' moment of inertia they mean the moment of inertia with respect to the center-of-mass. For clarity, this moment of inertia is often notated as I^{cm} in this book. If a different reference point, say point O is used, the moment of inertia is notated as I^0.

②**What do the words 'moment of inertia' mean?** The word inertia, in this case, can be translated to mean mass. So it's 'moment of mass'. The word 'moment' is similar to the word 'moment' as in torque, or (force)·(distance), but slightly more general. The more general word moment is *'the product of a quantity and its distance, raised to a power, from a reference point,'*

$$\text{the n'th moment of a quantity rel to 0} = \int x^n (\text{quantity at } x)\, dx.$$

So, what we used to call 'moment' (meaning torque) we can think of as short for 'the first moment of perpendicular force'. And the moment of inertia is 'the second moment of mass'. The words 'first' and 'second' refer to the exponent n.

$$I^{cm} \equiv \int \underbrace{(x^2 + y^2)}_{r^2}\, dm$$

$$= \int\int r^2 \overbrace{\underbrace{\left(\frac{m_{\text{tot}}}{A}\right)}_{\text{The mass per unit area.}} dA}^{dm} \qquad \text{for a uniform planar object}$$

where $x = x_{/\text{cm}}$, $y = y_{/\text{cm}}$ and $r = r_{/\text{cm}}$ are the distances in the x and y directions of the bits of mass (dm) from the center of mass and $r = r_{/cm}$ is the total distance. Similarly if all distances are measured relative to a point C or O (instead of relative to the center of mass) then similar formulas as above calculate I^C or I^O. See box 13.9 on page 692 for the calculation of the moments of inertia of some simple objects.

The term $I^{\text{cm}} = I^{\text{cm}}_{zz}$ is sometimes called the *polar moment of inertia*, or *polar mass moment of inertia* to distinguish it from the I_{xx} and I_{yy} terms which have little utility in planar dynamics (I_{xx} and I_{yy} are all important when calculating the bending stiffness of or stress in elastic beams).

What, physically, is the polar moment of inertia? It is a measure of the extent to which mass is far from the given reference point. Every bit of mass contributes to I in proportion to the square of its distance from the reference point. As we will show in sec. 13.6 on page 699 (see eqn. (13.56) on page 701) I^{cm} is just the quantity we need to do mechanics problems.

Radius of gyration

Another measure of the extent to which mass is spread from the reference point, besides the moment of inertia I^{cm}, is the *radius of gyration*, r_{gyr}③. The radius of gyration is defined as:

$$r_{\text{gyr}} \equiv \sqrt{I/m} \quad \Rightarrow \quad r_{\text{gyr}}^2 m = I.$$

That is, the radius of gyration of an object is the radius of an equivalent ring of mass that has the same I and the same mass as the given object. It is an equivalent single radius for the whole mass distribution④.

Example: Radius of gyration of a hoop
The radius of gyration about its center of a circular hoop of mass m and radius R about its center is $r_{\text{gyr}} = R$.

Other reference points

For the most part it is I^{cm} which is of primary interest. Other reference points are useful⑤

1. If the rigid body is hinged at a fixed point O then I^0 can be used in a slight short cut in calculation of angular momentum and energy; and

③ k **for stiffness.** The radius of gyration r_{gyr} is sometimes called k but we save k for stiffness in this book.

④ **Radius of gyration and average distance of mass.** The radius of gyration is an equivalent radius of the collection of mass. But the radius of gyration is not exactly the average radius of the distributed mass. Rather the radius of gyration is the square root of the mean square radius:

$$r_{\text{gyr}} \equiv \sqrt{\frac{\int r^2\, dm}{m_{\text{tot}}}}.$$

For the special cases where all the mass is at the same radius R (as for a circular hoop or for masses distributed on the vertices of, say, a regular pentagon) the radius of gyration is R. For all other cases the radius of gyration is larger than the average radius of the mass.

⑤ **Notation: simple vs precise.** When just the symbol I is used one assumes $I = I^{\text{cm}}_{zz}$, and if just I^{cm} is used, one assumes $I^{\text{cm}} = I^{\text{cm}}_{zz}$. Similarly I^0 and I^{C} are assumed to mean I^0_{zz} and I^{C}_{zz}, respectively.

2. If one wants to calculate the moment of inertia of a composite body about its center-of-mass then it is useful to first find the moment of inertia of each of its parts about that system's center of mass (which is generally not the center-of-mass of any of the separate parts).
3. Sometimes it is easiest to set up the integral for moment of inertia about a special point C that is not the center of mass and then use the parallel axis theorem (eqn. (13.54) to find the center of mass about a special point (see, for example the semi-circle example on 690).

The parallel axis theorem

The planar parallel axis theorem is the equation

$$I_{zz}^{C} = I_{zz}^{cm} + m_{\text{tot}} \underbrace{r_{cm/C}^{2}}_{d^2}. \tag{13.54}$$

In this equation $d = r_{cm/C}$ is the distance from the center-of-mass to a line parallel to the z-axis which passes through point C. See box 13.8 on page 694 for a derivation of the parallel axis theorem for planar objects.

Note that $I_{zz}^{C} \geq I_{zz}^{cm}$, always.

Moment of inertia of complex objects. One can calculate the moment of inertia of a composite body about its center of mass, in terms of the masses and moments of inertia of the separate parts. Say the position of the center of mass of m_i is (x_i, y_i) relative to a fixed origin, and the moment of inertia of that part about its center of mass is I_i. We can then find the moment of inertia of the composite I_{tot} about its center-of-mass (x_{cm}, y_{cm}) by the following sequence of calculations, in order:

$$\begin{array}{lrcl}
(1) & m_{\text{tot}} & = & \sum m_i \\
(2) & x_{cm} & = & \left[\sum x_i m_i\right] / m_{\text{tot}} \\
 & y_{cm} & = & \left[\sum y_i m_i\right] / m_{\text{tot}} \\
(3) & d_i^2 & = & (x_i - x_{cm})^2 + (y_i - y_{cm})^2 \\
(4) & I_{tot} & = & \sum \left[I_i^{cm} + m_i d_i^2\right].
\end{array}$$

You can reduce this recipe to one grand formula with lots of summation signs. But you would end up doing the calculations in about the order prescribed above in any case. This sequence of steps lends itself naturally to a computer spread sheet or to any program that deals easily with arrays of numbers. This is laid out for the similar center-of-mass calculation on page 128⑥.

⑥ The tidy recipe just presented is actually more commonly used, with slight modification, in strength of materials than in dynamics. The need for finding area moments of inertia of strange beam cross sections arises more frequently than the need to find the polar mass moment of inertia of a strange cutout shape.

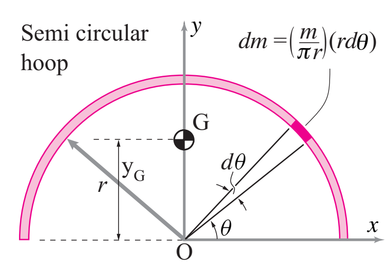


Figure 13.66: The moment of inertia of a semi-circle about the COM is calculated by calculating about point O and then using the parallel axis theorem.

Example: Inertia of a semicircle about its COM.
The moment of inertia of a semicircle with mass m and radius r about O (see fig. 13.66) is

$$I^0 = \int r_{/0}^2 \, dm = mr^2.$$

From the example on page 124 the center of mass is at $y_G = \dfrac{2}{\pi} r$. By the parallel axis theorem

$$I^0 = I^{\text{cm}} + m y_G^2$$

which we can solve for I^{cm} to get

$$I^{\mathrm{cm}} = I^{0} - my_{\mathrm{G}}^{2} = mr^{2} - \frac{4}{\pi^{2}}mr^{2} = \left(1 - \frac{4}{\pi^{2}}\right)mr^{2} \approx 0.6\,mr^{2}\,.$$

The radius of gyration is that radius ρ_{gyr} so that $m\rho_{\mathrm{gyr}}^{2} = I^{\mathrm{cm}}$ and is

$$\rho_{\mathrm{gyr}} = \sqrt{1 - \frac{4}{\pi^{2}}}\; r.$$

The perpendicular axis theorem for planar rigid bodies

The perpendicular axis theorem for planar objects is the equation

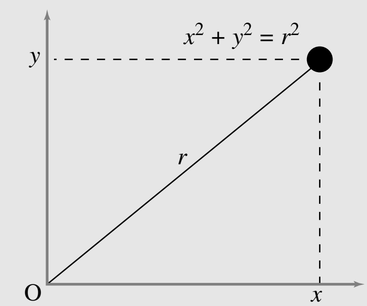
$$I_{zz} = I_{xx} + I_{yy}$$

which is derived in box 13.8 on page 694. It gives the 'polar' inertia I_{zz} in terms of the inertias I_{xx} and I_{yy}. Unlike the parallel axis theorem, the perpendicular axis theorem does *not* have a three-dimensional counterpart. The theorem is of greatest utility when one wants to study the three-dimensional mechanics of a flat object and thus are in need of its full moment of inertia matrix.

13.9 Some examples of 2-D Moment of Inertia

Here, we illustrate some simple moment of inertia calculations for two-dimensional objects. The needed formulas are summarized, in part, by the lower right corner components (that is, the elements in the third column and third row (3,3)) of the matrices in the table on the inside back cover.

One point mass



Filename:tfigure4-pointmassIo

If we assume that all mass is concentrated at one or more points, then the integral

$$I_{zz}^{o} = \int r_{/o}^{2}\, dm$$

reduces to the sum

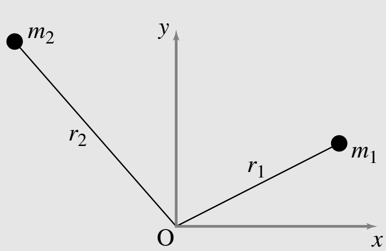
$$I_{zz}^{o} = \sum r_{i/o}^{2} m_i$$

which reduces to one term if there is only one mass,

$$I_{zz}^{o} = r^2 m = (x^2 + y^2)m.$$

So, if $x = 3\,\text{in}$, $y = 4\,\text{in}$, and $m = 0.1\,\text{lbm}$, then $I_{zz}^{o} = 2.5\,\text{lbm in}^2$. Note that, in this case, $I_{zz}^{cm} = 0$ since the radius from the center-of-mass to the center-of-mass is zero.

Two point masses



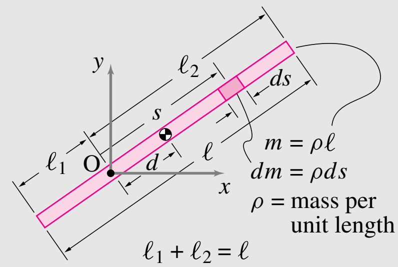
Filename:tfigure4-2pointmassesIo

In this case, the sum that defines I_{zz}^{o} reduces to two terms, so

$$I_{zz}^{o} = \sum r_{i/o}^{2} m_i = m_1 r_1^2 + m_2 r_2^2.$$

Note that, if $r_1 = r_2 = r$, then $I_{zz}^{o} = m_{tot} r^2$.

A thin uniform rod



Filename:tfigure4-2DrodIo

Consider a thin rod with uniform mass density, ρ, per unit length, and length ℓ. We calculate I_{zz}^{o} as

$$\begin{aligned}
I_{zz}^{o} &= \int r^2 \overbrace{dm}^{\rho ds} \\
&= \int_{-\ell_1}^{\ell_2} s^2 \rho ds \qquad (s = r) \\
&= \frac{1}{3}\rho s^3 \Big|_{-\ell_1}^{\ell_2} \qquad (\text{since} \quad \rho \equiv const.) \\
&= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3).
\end{aligned}$$

If either $\ell_1 = 0$ or $\ell_2 = 0$, then this expression reduces to $I_{zz}^{o} = \frac{1}{3}m\ell^2$. If $\ell_1 = \ell_2$, then O is at the center-of-mass and

$$I_{zz}^{o} = I_{zz}^{cm} = \frac{1}{3}\rho\left(\left(\frac{\ell}{2}\right)^3 + \left(\frac{\ell}{2}\right)^3\right) = \frac{ml^2}{12}.$$

We can illustrate one last point. With a little bit of algebraic histrionics of the type that only hindsight can inspire, you can verify that the expression for I_{zz}^{0} can be arranged as follows:

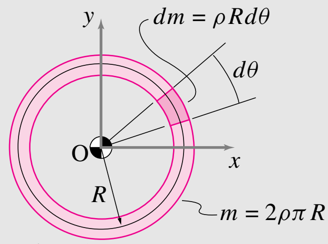
$$\begin{aligned}
I_{zz}^{0} &= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3) \\
&= \underbrace{\rho(\ell_1 + \ell_2)}_{m}\Bigg(\underbrace{\frac{\ell_2 - \ell_1}{2}}_{d}\Bigg)^2 + \underbrace{\rho\frac{(\ell_1 + \ell_2)^3}{12}}_{m\ell^2/12} \\
&= md^2 + m\frac{\ell^2}{12} \\
&= md^2 + I_{zz}^{cm}
\end{aligned}$$

That is, the moment of inertia about point O is greater than that about the center of mass by an amount equal to the mass times the distance from the center-of-mass to point O squared. This derivation of the *parallel axis theorem* is for one special case, that of a uniform thin rod.

(continued...)

13.9 Some examples of 2-D Moment of Inertia (continued)

A uniform hoop



Filename:tfigure4-2DhoopIo

For a hoop of uniform mass density, ρ, per unit length, we might consider all of the points to have the same radius R. So,

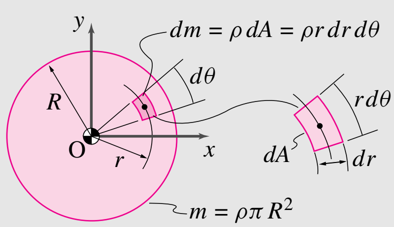
$$I_{zz}^{o} = \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 m.$$

Or, a little more tediously,

$$\begin{aligned} I_{zz}^{o} &= \int r^2 dm \\ &= \int_0^{2\pi} R^2 \rho R d\theta \\ &= \rho R^3 \int_0^{2\pi} d\theta \\ &= 2\pi\rho R^3 = \underbrace{(2\pi\rho R)}_{m} R^2 = mR^2. \end{aligned}$$

This I_{zz}^{o} is the same as for a single point mass m at a distance R from the origin O. It is also the same as for two point masses if they both are a distance R from the origin. For the hoop, however, O is at the center-of-mass so $I_{zz}^{o} = I_{zz}^{cm}$ which is not the case for a single point mass.

A uniform disk



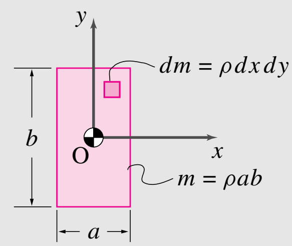
Filename:tfigure4-2DdiskIo

Assume the disk has uniform mass density, ρ, per unit area. For a uniform disk centered at the origin, the center-of-mass is at the origin so

$$\begin{aligned} I_{zz}^{o} = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_0^R \int_0^{2\pi} r^2 \rho r d\theta dr \\ &= \int_0^R 2\pi\rho r^3 dr \\ &= 2\pi\rho \left.\frac{r^4}{4}\right|_0^R = \pi\rho\frac{R^4}{2} = (\pi\rho R^2)\frac{R^2}{2} \\ &= m\frac{R^2}{2}. \end{aligned}$$

For example, a 1 kg plate of 1 m radius has the same moment of inertia as a 1 kg hoop with a 70.7 cm radius.

Uniform rectangular plate



Filename:tfigure4-2DplateIo

For the special case that the center of the plate is at point O, the center-of-mass of mass is also at O and $I_{zz}^{o} = I_{zz}^{cm}$.

$$\begin{aligned} I_{zz}^{o} = I_{zz}^{cm} &= \int r^2 dm \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2)\overbrace{\rho dx dy}^{dm} \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho \left(\frac{x^3}{3} + xy^2\right)\Bigg|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} dy \\ &= \rho\left(\frac{x^3 y}{3} + \frac{xy^3}{3}\right)\Bigg|_{x=-\frac{a}{2}}^{x=\frac{a}{2}}\Bigg|_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\ &= \rho\left(\frac{a^3 b}{12} + \frac{ab^3}{12}\right) \\ &= \frac{m}{12}(a^2+b^2). \end{aligned}$$

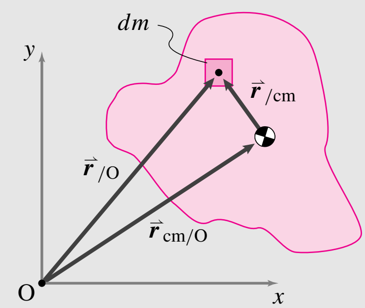
Note that $\int r^2 dm = \int x^2 dm + \int y^2 dm$ for all planar objects (the *perpendicular axis theorem*). For a uniform rectangle, $\int y^2 dm = \rho \int y^2 dA$. But the integral $y^2 dA$ is just the term often used for I, the area moment of inertia, in strength of materials calculations for the stresses and stiffnesses of beams in bending. You may recall that $\int y^2 dA = \frac{ab^3}{12} = \frac{Ab^2}{12}$ for a rectangle. Similarly, $\int x^2 dA = \frac{Aa^2}{12}$. So, the polar moment of inertia $J = I_{zz}^{o} = m\frac{1}{12}(a^2+b^2)$ can be recalled by remembering the area moment of inertia of a rectangle combined with the perpendicular axis theorem.

13.8 The 2-D parallel axis theorem and the perpendicular axis theorem

Sometimes one wants to know the moment of inertia relative to the center of mass. And, sometimes, if the object is held at a hinge joint at O, relative to that hinge point O There is a simple relation between these two moments of inertia known as *the parallel axis theorem*.

2-D parallel axis theorem

For the two-dimensional mechanics of two-dimensional objects, our only concern is I_{zz}^{o} and I_{zz}^{cm} and *not* the full moment of inertia matrix. In this case, $I_{zz}^{o} = \int r_{/O}^2 \, dm$ and $I_{zz}^{cm} = \int r_{/cm}^2 \, dm$. Now, let's prove the theorem in two dimensions referring to the figure.



Filename:tfigure4-para-axis

$$
\begin{aligned}
I_{zz}^{O} &= \int r_{/O}^2 \, dm \\
&= \int (x_{/O}^2 + y_{/O}^2) dm \\
&= \int [(\underbrace{x_{cm/O} + x_{/cm}}_{x_{/O}})^2 + (\underbrace{y_{cm/O} + y_{/cm}}_{y_{/O}})^2] dm \\
&= \int [(x_{cm/O}^2 + 2x_{cm/O}x_{/cm} + x_{/cm}^2) + \\
&\qquad (y_{cm/O}^2 + 2y_{cm/O}y_{/cm} + y_{/cm}^2)] dm \\
&= (x_{cm/O}^2 + y_{cm/O}^2) \underbrace{\int dm}_{m} + 2x_{cm/O} \underbrace{\int x_{/cm} dm}_{0} + \\
&\qquad 2y_{cm/O} \underbrace{\int y_{/cm} dm}_{0} + \int (x_{/cm}^2 + y_{/cm}^2) dm \\
&= r_{cm/O}^2 m + \underbrace{\int (x_{/cm}^2 + y_{/cm}^2) dm}_{I_{zz}^{cm}} \\
&= I_{zz}^{cm} + \underbrace{r_{cm/O}^2}_{d^2} m
\end{aligned}
$$

The cancellation $\int y_{/\mathrm{cm}} \, dm = \int x_{/\mathrm{cm}} \, dm = 0$ comes from the definition of center of mass.

Sometimes, people write the parallel axis theorem more simply as

$$I^0 = I^{cm} + md^2 \qquad \text{or} \qquad J_O = J_{cm} + md^2$$

using the symbol J to mean I_{zz}. One thing to note about the parallel axis theorem is that the moment of inertia about any point O is always *greater* than the moment of inertia about the center of mass. For a given object, the minimum moment of inertia is about the center-of-mass.

Why the name *parallel axis theorem*? We use the name because the two I's calculated are the moments of inertia about two parallel axes (both in the z direction) through the two points cm and O.

One way to think about the theorem is the following. The moment of inertia of an object about a point O not at the center-of-mass is the same as that of the object about the cm *plus* that of a point mass located at the center-of-mass. If the distance from O to the cm is larger than the outer radius of the object, then the d^2m term is larger than I_{zz}^{cm}. The distance of equality of the two terms is the radius of gyration, r_{gyr}.

Perpendicular axis theorem (applies to planar objects only)

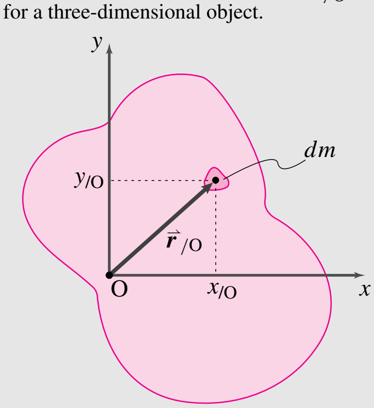
For planar objects,

$$
\begin{aligned}
I_{zz}^{O} &= \int |\vec{r}|^2 \, dm \\
&= \int (x_{/O}^2 + y_{/O}^2) dm \\
&= \int x_{/O}^2 dm + \int y_{/O}^2 dm \\
&= I_{yy}^{O} + I_{xx}^{O}
\end{aligned}
$$

Similarly,

$$I_{zz}^{cm} = I_{xx}^{cm} + I_{yy}^{cm}.$$

That is, the moment of inertia about the z-axis is the sum of the inertias about the two perpendicular axes x and y. Note that the objects must be planar ($z = 0$ everywhere) or the theorem would not be true. For example, $I_{xx}^{o} = \int (y_{/O}^2 + z_{/O}^2) dm \neq \int y_{/O}^2 dm$ for a three-dimensional object.



Filename:tfigure4-perp-axis-thm

SAMPLE 13.18 **Moment of inertia of point masses:** A pendulum is made up of two unequal point masses m and $2m$ connected by a massless rigid rod of length $4r$. The pendulum is pivoted at distance r along the rod from the small mass.

1. Find the moment of inertia I_{zz}^{cm} of the pendulum.
2. Find the moment of inertia I_{zz}^{O} of the pendulum.
3. Find the radius of gyration of the pendulum.

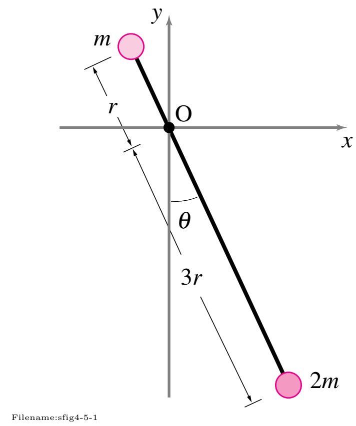


Figure 13.67:

Solution

1. First we need to find the center of mass of the system. Let the center of mass C be located at distance r_{cm} from the origin O. An easy way to find the location of C will be to consider the pendulum to lie along the x-axis with the origin at O (see fig. 13.68). Then

$$\begin{aligned} x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} \\ \Rightarrow \quad r_{cm} &= \frac{m(-r) + 2m(3r)}{m + 2m} = \frac{5}{3}r. \end{aligned}$$

So, the moment of inertia about the center of mass is

$$I_{zz}^{cm} = \sum m_i r_{i/cm}^2 = m\left(r + \frac{5}{3}r\right)^2 + 2m\left(3r - \frac{5}{3}r\right)^2 = \frac{32}{3}mr^2.$$

$$\boxed{I_{zz}^{cm} = 10.67mr^2}$$

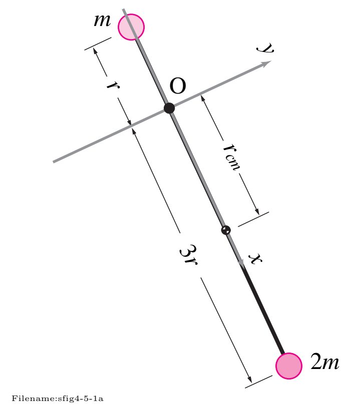


Figure 13.68:

2. We can calculate I_{zz}^{O} in two different ways, one directly by summing the contributions of each mass about point O, and the other by using I_{zz}^{cm} and the parallel axis theorem.

$$\begin{aligned} I_{zz}^{O} &= mr^2 + 2m(3r)^2 = 19mr^2 \\ \text{or} \qquad I_{zz}^{O} &= I_{zz}^{cm} + m_{tot} r_{cm/O}^2 \\ &= \frac{32}{3}mr^2 + 3m\left(\frac{5}{3}r\right)^2 = 19mr^2. \end{aligned}$$

$$\boxed{I_{zz}^{O} = 19mr^2}$$

3. The radius of gyration, by definition, is the distance that gives the desired moment of inertia if the entire mass of the system is concentrated there. Thus, if r_{gyr} is the radius of gyration for the moment of inertia about point O, then

$$\begin{aligned} I_{zz}^{O} &= (3m) \cdot r_{gyr}^2 \\ \Rightarrow \quad 19\cancel{m}r^2 &= 3\cancel{m}r_{gyr}^2 \\ \Rightarrow \quad r_{gyr} &= \sqrt{\frac{19}{3}}r = 2.52r. \end{aligned}$$

Thus the radius of gyration r_{gyr} of the given pendulum is $r_{gyr} = 2.52\,r$.

$$\boxed{r_{gyr} = 2.52\,r}$$

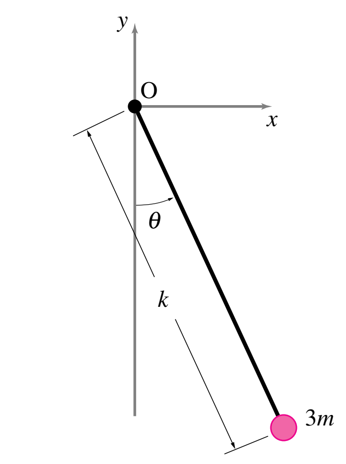


Figure 13.69:

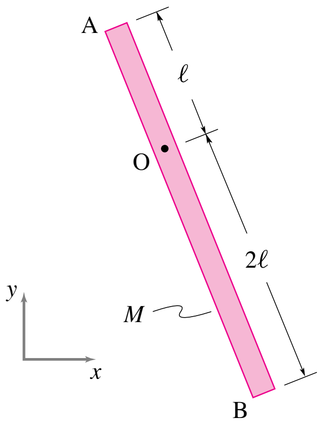


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Figure 13.70:

SAMPLE 13.19 **Moment of inertia of a rod:** A uniform rigid rod AB of mass $M = 2\,\text{kg}$ and length $3\ell = 1.5\,\text{m}$ swings about the z-axis passing through the pivot point O.

1. Find the moment of inertia I_{zz}^{O} of the bar using the fundamental definition $I_{zz}^{O} = \int_m r_{/O}^2\, dm$.
2. Find I_{zz}^{O} using the parallel axis theorem given that $I_{zz}^{\text{cm}} = \frac{1}{12} m\ell^2$ where m = total mass, and ℓ = total length of the rod. (You can find I_{zz}^{cm} for many commonly encountered objects in the table on the inside backcover of the text).

Solution

1. Since we need to carry out the integral, $I_{zz}^{O} = \int_m r_{/O}^2\, dm$, to find I_{zz}^{O}, let us consider an infinitesimal length segment $d\ell'$ of the bar at distance ℓ' from the pivot point O. (see Figure 13.71). Let the mass of the infinitesimal segment be dm.
   Now the mass of the segment may be written as

$$\begin{aligned} dm &= \text{(mass per unit length of the bar)} \cdot \text{(length of the segment)} \\ &= \frac{M}{3\ell}\, d\ell' \qquad \left(\text{Note: } \frac{\text{mass}}{\text{unit length}} = \frac{\text{total mass}}{\text{total length}} \right). \end{aligned}$$

   We also note that the distance of the segment from point O, $r_{/O} = \ell$. Substituting the values found above for $r_{/O}$ and dm in the formula we get

$$\begin{aligned} I_{zz}^{O} &= \int_{-\ell}^{2\ell} \underbrace{(\ell')^2}_{r_{/O}^2} \underbrace{\frac{M}{3\ell} d\ell'}_{dm} \\ &= \frac{M}{3\ell} \int_{-\ell}^{2\ell} (\ell')^2 d\ell' = \frac{M}{3\ell} \left[\frac{\ell'^3}{3} \right]_{-\ell}^{2\ell} \\ &= \frac{M}{3\ell} \left[\frac{8\ell^3}{3} - \left(-\frac{\ell^3}{3} \right) \right] = M\ell^2 \\ &= 2\,\text{kg} \cdot (0.5\,\text{m})^2 = 0.5\,\text{kg} \cdot \text{m}^2. \end{aligned}$$

$$\boxed{I_{zz}^{O} = 0.5\,\text{kg} \cdot \text{m}^2}$$

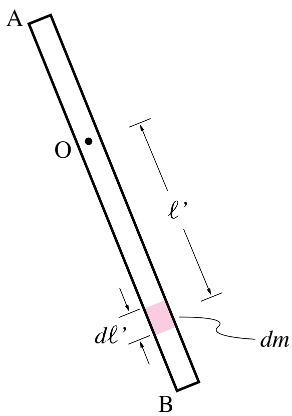

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Figure 13.71:

2. The parallel axis theorem states that

$$I_{zz}^{O} = I_{zz}^{\text{cm}} + M r_{O/\text{cm}}^2.$$

   Since the rod is uniform, its center-of-mass is at its geometric center, *i.e.,* at distance $\frac{3\ell}{2}$ from either end. From the Fig 13.72 we can see that

$$r_{O/\text{cm}} = AG - AO = \frac{3\ell}{2} - \ell = \frac{\ell}{2}$$

$$\begin{aligned} \text{Therefore,} \quad I_{zz}^{O} &= \underbrace{\frac{1}{12} M (3\ell)^2}_{I_{zz}^{\text{cm}}} + M \left(\frac{\ell}{2}\right)^2 \\ &= \frac{9}{12} M\ell^2 + M\frac{\ell^2}{4} = M\ell^2 \\ &= 0.5\,\text{kg} \cdot \text{m}^2 \quad \text{(same as in (a), of course)} \end{aligned}$$

$$\boxed{I_{zz}^{O} = 0.5\,\text{kg} \cdot \text{m}^2}$$

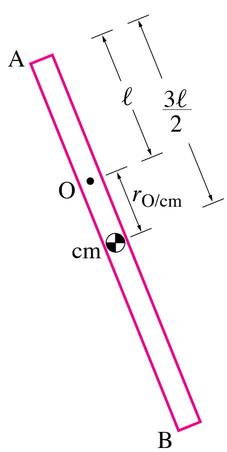

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Figure 13.72:

SAMPLE 13.20 Moment of inertia of a wheel with a cut-out: A uniform rigid wheel of radius $r = 1$ ft is made eccentric by cutting out a portion of the wheel. The center-of-mass of the eccentric wheel is at C, a distance $e = \frac{r}{3}$ from the geometric center O. The mass of the wheel (after deducting the cut-out) is 3.2 lbm. The moment of inertia of the wheel about point O, I_{zz}^{O}, is 1.8 lbm$\cdot$ ft^2. We are interested in the moment of inertia I_{zz} of the wheel about points A and B on the perimeter.

1. Without any calculations, guess which point, A or B, gives a higher moment of inertia. Why?
2. Calculate I_{zz}^{C}, I_{zz}^{A} and I_{zz}^{B} and compare with the guess in (a).

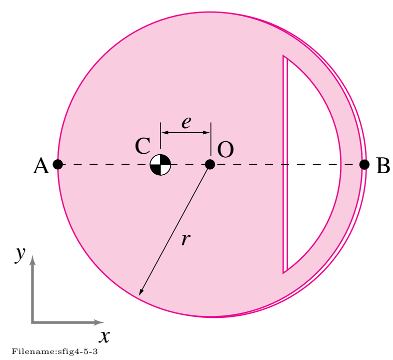

Figure 13.73:

Solution

1. The moment of inertia I_{zz}^{B} should be higher. Moment of inertia I_{zz} measures the geometric distribution of mass about the z-axis. But the distance of the mass from the axis counts more than the mass itself ($I_{zz}^{O} = \int_m r_{/O}^2 dm$). The distance $r_{/O}$ of the mass appears as a quadratic term in I_{zz}^{O}. The total mass is the same whether we take the moment of inertia about point A or about point B. However, the distribution of mass is not the same about the two points. Due to the cut-out being closer to point B there are more "dm's" at greater distances from point B than from point A. So, we guess that

$$\boxed{I_{zz}^{\mathrm{B}} > I_{zz}^{\mathrm{A}}}$$

2. If we know the moment of inertia I_{zz}^{C} (about the center-of-mass) of the wheel, we can use the parallel axis theorem to find I_{zz}^{A} and I_{zz}^{B}. In the problem, we are given I_{zz}^{O}. But,

$$
\begin{aligned}
I_{zz}^{\mathrm{O}} &= I_{zz}^{\mathrm{C}} + M r_{O/C}^2 \quad \text{(parallel axis theorem)} \\
\Rightarrow \quad I_{zz}^{\mathrm{C}} &= I_{zz}^{\mathrm{O}} - M r_{O/C}^2 \\
&= 1.8\,\text{lbm ft}^2 - 3.2\,\text{lbm} \underbrace{\left(\frac{1\,\text{ft}}{3}\right)^2}_{r_{O/C}=e=\frac{r}{3}} \\
&= 1.44\,\text{lbm}\cdot\text{ft}^2 \\
\text{Now,} \quad I_{zz}^{\mathrm{A}} &= I_{zz}^{\mathrm{C}} + M r_{A/C}^2 = I_{zz}^{\mathrm{C}} + M\left(\frac{2r}{3}\right)^2 \\
&= 1.44\,\text{lbm}\cdot\text{ft}^2 + 3.2\,\text{lbm}\left(\frac{2\,\text{ft}}{3}\right)^2 \\
&= 2.86\,\text{lbm}\cdot\text{ft}^2 \\
\text{and} \quad I_{zz}^{\mathrm{B}} &= I_{zz}^{\mathrm{C}} + M r_{B/C}^2 = I_{zz}^{\mathrm{C}} + M\left(r + \frac{r}{3}\right)^2 \\
&= 1.44\,\text{lbm}\cdot\text{ft}^2 + 3.2\,\text{lbm}\left(1\,\text{ft} + \frac{1\,\text{ft}}{3}\right)^2 \\
&= 7.13\,\text{lbm}\cdot\text{ft}^2
\end{aligned}
$$

$$\boxed{I_{zz}^{\mathrm{C}} = 1.44\,\text{lbm}\cdot\text{ft}^2,\ I_{zz}^{\mathrm{A}} = 2.86\,\text{lbm}\cdot\text{ft}^2,\ I_{zz}^{\mathrm{B}} = 7.13\,\text{lbm}\cdot\text{ft}^2}$$

Clearly, $I_{zz}^{\mathrm{B}} > I_{zz}^{\mathrm{A}}$, as guessed in (a).

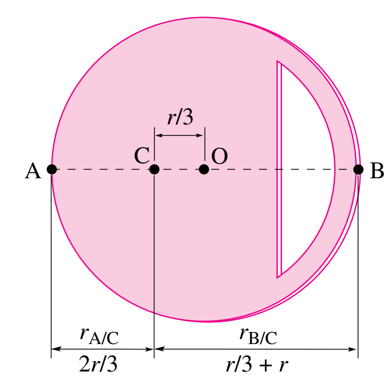

Figure 13.74:

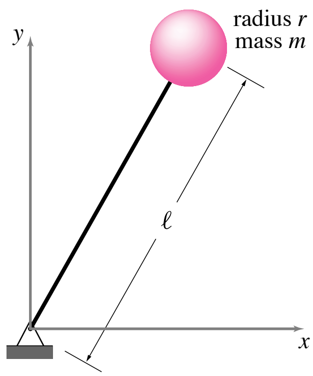

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Figure 13.75:

SAMPLE 13.21 Moment of inertia: modeling a sphere as a point mass: A uniform solid sphere of mass m and radius r is attached to a massless rigid rod of length ℓ. The sphere swings in the xy plane. Find the error in calculating I_{zz}^{O} as a function of r/ℓ if the sphere is treated as a point mass concentrated at the center-of-mass of the sphere.

Solution The exact moment of inertia of the sphere about point O can be calculated using parallel axis theorem:

$$\begin{aligned} I_{zz}^{O} &= I_{zz}^{cm} + m\ell^2 \\ &= \frac{2}{5}mr^2 + m\ell^2. \quad \text{(See Table IV on inside cover)} \end{aligned}$$

If we treat the sphere as a point mass, he moment of inertia I_{zz}^{O} is

$$\tilde{I}_{zz}^{O} = m\ell^2.$$

Therefore, the relative error in I_{zz}^{O} is

$$\begin{aligned} \text{error} &= \frac{I_{zz}^{O} - \tilde{I}_{zz}^{O}}{I_{zz}^{O}} \\ &= \frac{\frac{2}{5}mr^2 + m\ell^2 - m\ell^2}{\frac{2}{5}mr^2 + m\ell^2} \\ &= \frac{\frac{2}{5}\frac{r^2}{\ell^2}}{\frac{2}{5}\frac{r^2}{\ell^2} + 1} \end{aligned}$$

From the above expression we see that for $r \ll \ell$ the error is very small. From the graph of error in Fig. 13.76 we see that even for $r = \ell/5$, the error in I_{zz}^{O} due to approximating the sphere as a point mass is less than 2%.

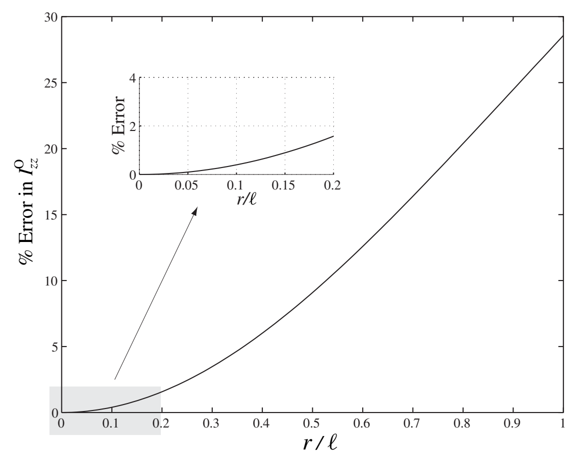

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Figure 13.76: Relative error in I_{zz}^{O} of the sphere as a function of r/ℓ.

13.6 Dynamics of a rigid object in planar circular motion

We now want to find what we can about the forces on and motions of a single planar object that is hinged at one point. The most famous examples are pendula and gears. But many things move like pendula, like people or towers falling over or boats rocking in the water. And many machine parts besides gears rotate about a bearing including propellers, lawn mower blades, and printing-press rollers. None of these things are strictly planar objects, but the planar approximation is more or less accurate for many analyses①.

① **Accuracy of planar analysis.** The planar analysis might be accurate for one of two different reasons: 1) the objects are nearly planar, or 2) the projection of the 3D equations to 2D gives the planar equations.

The basic method, as always, is to use a free body diagram and kinematics results to evaluate terms in the linear momentum, angular momentum and energy balance equations. Then these equations are used to calculate unknown forces or to find differential equations of motion. The terms involving force and moment are found from a free body diagram *exactly* as in statics. The terms involving motion, namely momentum, angular momentum and energy, are evaluated using the various tools developed earlier in this chapter.

Mechanics and the motion quantities

We can evaluate all the momentum and energy terms in the equations of motion (inside cover), namely: $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{/C}$, $\dot{\vec{H}}_{/C}$, E_K and $\dot{E}_K$ (for any reference point C of our choosing) if we can calculate the velocity and acceleration of every point in the system. For circular motion of a rigid object about point C, we know from sec. 13.4 that the velocities and accelerations of a point at $\vec{r} = \vec{r}_{/C}$ are

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r}, \\ \vec{a} &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}), \\ &= \dot{\vec{\omega}} \times \vec{r} - \omega^2 \vec{r}\end{aligned}$$

where $\vec{\omega}$ is the angular velocity of the object relative to a fixed frame. This situation is a little more complex than the special case of straight-line motion in chapter 6, where all points in a system had the same acceleration as each other, but is still quite managable. The most general case of 2D rigid-object circular motion is an arbitrarily shaped 2D rigid object with arbitrary ω and $\dot{\omega}$. The situation shown in fig. 13.77 shows the general case (but for that in that example exactly two forces are applied, as opposed to an arbitrary number).

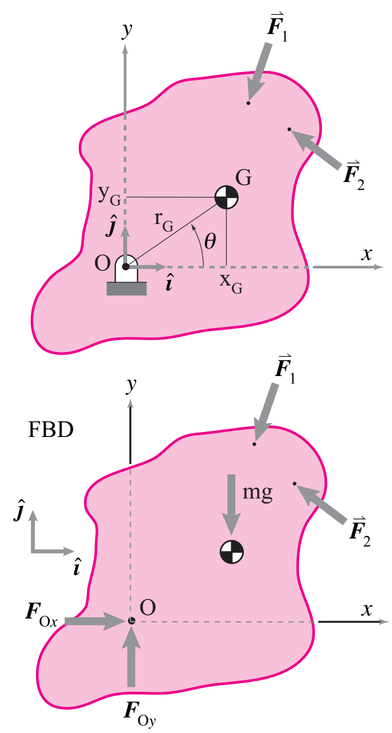

Figure 13.77: **a)** A rigid object rotates about 0. It has center of mass at G. Forces $\vec{F}_1$ and $\vec{F}_2$ are applied. **b)** The free body diagram shows all the forces acting on the object, including the reaction force ($\vec{F}_0 = F_{0x}\hat{\imath} + F_{0y}\hat{\jmath}$) at 0 and the gravity force mg at G.

Linear momentum: $\vec{L}$ and $\dot{\vec{L}}$

For any system in any motion we have the general result that

$$\vec{L} = m_{\text{tot}}\vec{v}_{\text{cm}} \quad \text{and} \quad \dot{\vec{L}} = m_{\text{tot}}\vec{a}_{\text{cm}}.$$

For a rigid object, the center-of-mass is a particular point G that is fixed relative to the object. So the velocity and acceleration of that point can be expressed the same way as for any other point. So, for an object in planar rotational motion about 0

$$\vec{L} = m_{\text{tot}}\vec{\omega} \times \vec{r}_{G/0}$$

and

$$\dot{\vec{L}} = m_{\text{tot}} \underbrace{\left(\dot{\vec{\omega}} \times \vec{r}_{G/0} - \omega^2 \vec{r}_{G/0}\right)}_{\vec{a}_G}.$$

If the center-of-mass is at 0 the momentum and its rate of change are both zero. But if the center-of-mass is off the axis of rotation, there must be a net force on the object with a component parallel to $\vec{r}_{0/G}$ (if $\omega \neq 0$) and also a component orthogonal to $\vec{r}_{0/G}$ (if $\dot{\omega} \neq 0$). This net force might be applied at 0 or G or any other place(s) on the object.

Angular momentum: $\vec{H}_{/O}$ and $\dot{\vec{H}}_{/O}$

The angular momentum itself is easy enough to calculate, using the short hand notation that $\vec{r}$ is the position vector $\vec{r}_{/0}$ of a point relative to hinge point O.

$$\begin{aligned}
\vec{H}_{/O} &= \int_{\text{all mass}} \vec{r} \times \vec{v}\, dm && \text{(a)}\\
&= \int \vec{r} \times (\vec{\omega} \times \vec{r})\, dm && \text{(b)}\\
&= \omega\hat{k} \int r^2\, dm && \text{(c)}\\
&= \underbrace{I_{zz}^{O}}\, \omega\hat{k} && (13.55)\\
\Rightarrow \quad H_0 &= \omega I_{zz}^{O} && \text{(d)}
\end{aligned}$$

I_{zz}^{O} is the 'polar' moment of inertia.

Here eqn. (13.55)c is a vector equation. But since both sides are in the $\hat{k}$ direction we can dot both sides with $\hat{k}$ to get the scalar moment equation eqn. (13.55)d, taking both M_{net} and ω as positive when counterclockwise.

The moment of inertia shortcut. Note that an integral over all the mass of $\vec{r} \times \vec{v}$ has been reduced to a simple multiplication by using that

$$\int r^2\, dm = I_{zz}^{0}.$$

This shortcut is especially useful because $\int r^2\, dm$ does not change as the object rotates about 0.

However, the moment of inertia is never needed for problem solving. One can just evaluate, for example, $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ by evaluating the integral $\int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\, dm$ for the specific problem at hand.

To get the all important term for angular momentum balance, namely $\dot{\vec{\boldsymbol{H}}}_{/0}$, for this system we can differentiate eqn. (13.55). We can also use the general expression for $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ to write the angular momentum balance equation as follows.

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \text{rate of change of angular momentum}_{/0} && \text{(a)}\\
&= \int_{\text{all mass}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}\, dm && \text{(b)}\\
&= \int \vec{\boldsymbol{r}} \times \Big(\underbrace{-\omega^2 \vec{\boldsymbol{r}} + \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}}_{\vec{\boldsymbol{a}}} \Big)\, dm && \text{(c)}\\
&= \int \vec{\boldsymbol{r}} \times \left(\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}} \right) dm \quad (\text{because } \vec{\boldsymbol{r}} \times \vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}) && \text{(d)}\\
&= \int \vec{\boldsymbol{r}} \times \left(\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}} \right) dm && \text{(e)}\\
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm \;=\; \dot{\omega} I_{zz}^{0} \hat{\boldsymbol{k}} && \text{(f)}\\
\Rightarrow \quad \dot{H}_{/0} &= \dot{\omega} \int r^2\, dm \;=\; \dot{\omega} I_{zz}^{0} && \text{(g)}
\end{aligned}
\tag{13.56}
$$

We get from eqn. (13.56)e to eqn. (13.56)f by noting that $\vec{\boldsymbol{r}}$ is perpendicular to $\hat{\boldsymbol{k}}$. Thus, using the right hand rule twice we get $\vec{\boldsymbol{r}} \times (\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}) = r^2 \hat{\boldsymbol{k}}$.

Eqn. 13.56f and eqn. (13.56)g are the vector and scalar versions for the rate of change of angular momentum with respect to point 0 for rotation of a planar object about 0. Repeating,

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm \;=\; \dot{\omega} I_{zz}^{0} \hat{\boldsymbol{k}}\\
\text{and} \qquad \dot{H}_{/0} &= \dot{\omega} \int r^2\, dm \;=\; \dot{\omega} I_{zz}^{0}.
\end{aligned}
\tag{13.57}
$$

Again, $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ can always be evaluated either of two ways: 1) as a sum or integral of $\vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{a}}$ over all the mass or 2) using the shortcut with I_{zz}^{0}.

Power and Energy

Let's assume that there are a set of point forces applied to and object②. And, to be contrary, lets assume the mass is continuously distributed (the derivation for rigidly connected point masses would be similar). The power

②Forces distributed on a curve, surface or throughout a volume could be treated similarly with sums replaced by line, surface, or volume integrals.

balance equation for one rotating rigid object is (discussed below):

$$
\begin{aligned}
\text{Net power in} &= \text{rate of change of kinetic energy} && \text{(a)}\\
P &= \dot{E}_{\mathrm{K}} && \text{(b)}\\
\sum_{\text{all applied forces}} \vec{F}_i \cdot \vec{v}_i &= \frac{d}{dt}\int_{\text{all mass}} \frac{1}{2} v^2\, dm && \text{(c)}\\
\sum \vec{F}_i \cdot (\vec{\omega} \times \vec{r}_i) &= \frac{d}{dt}\int \frac{1}{2}(\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})\, dm && \text{(d)}\\
\sum \vec{\omega} \cdot (\vec{r}_i \times \vec{F}_i) &= \frac{d}{dt}\int \frac{1}{2}\omega^2 r^2\, dm && \text{(e)}\\
\vec{\omega} \cdot \sum (\vec{r}_i \times \vec{F}_i) &= \frac{d}{dt}\left(\frac{1}{2}\omega^2\right)\int r^2\, dm && \text{(f)}\\
\vec{\omega} \cdot \sum \vec{M}_i &= \dot{\omega}\omega \int r^2\, dm && \text{(g)}\\
\vec{\omega} \cdot \vec{M}_{\mathrm{tot}} &= \dot{\vec{\omega}} \cdot \underbrace{\left(\vec{\omega}\int r^2\, dm\right)}_{\vec{H}_{/0}} && \text{(h)}
\end{aligned}
\qquad (13.58)
$$

When not directly labeled, positions and moments are assumed to be relative to the hinge at 0. Derivation 13.58 is two derivations in one. The left side about power and the right side about kinetic energy. Lets discuss one at a time.

Power. On the left side of eqn. (13.58) we note in (c) that the power of each force is the dot product of the force with the velocity of the point it touches. In (d) we use what we know about the velocities of points on rotating rigid bodies. In (e) we use the vector identity $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A}$ (see sec. 2.3, 71). In (f) we note that $\vec{\omega}$ is common to all points so factors out of the sum. In (g) we note that $\vec{r} \times \vec{F}_i$ is the moment of the force about pt O. And in (g) we sum the moments of the forces. So the power of a set of forces acting on a rigid object is the product of their net moment (about 0) and the object angular velocity,

$$P = \vec{\omega} \cdot \vec{M}_{\mathrm{tot}}. \qquad (13.59)$$

Kinetic energy. On the right side of eqn. (13.58) we note in (c) that the kinetic energy is the sum of the kinetic energy of the mass increments. In (d) we use what we know about the velocities of these bits of mass, given that they are on a common rotating object. In (e) we use that the magnitude of the cross product of orthogonal vectors is the product of the magnitudes ($|\vec{A} \times \vec{B}| = AB$) and that the dot product of a vector with itself is its magnitude squared ($\vec{A} \cdot \vec{A} = A^2$). In (f) we factor out ω^2 because it is common to all the mass increments and note that the remaining integral is constant in time for a rigid object. In (g) we carry out the derivative. In (h) we de-simplify the result from (g) in order to show a more general form that we will find later in

3D mechanics. Eqn. (h) follows from (g) because $\vec{\omega}$ is parallel to $\dot{\vec{\omega}}$ for 2D rotations.

Note that we started here with the basic power balance equation from the front inside cover. Instead, we could have derived power balance from our angular momentum balance expression (see box 13.10 on 703).

Example: Spinning disk

The round flat uniform disk in fig. 13.78 is in the xy plane spinning at the constant rate $\vec{\omega} = \omega\hat{k}$ about its center. It has mass m_{tot} and radius R_0. What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{F}_i = \dot{\vec{L}} = m_{\text{tot}}\vec{a}_{cm} = \vec{0},$$

Which we could also have calculated by evaluating the integral $\dot{\vec{L}} \equiv \int \vec{a}\, dm$ instead of using the general result that $\dot{\vec{L}} = m_{\text{tot}}\vec{a}_{cm}$. From angular momentum balance we have:

$$\begin{aligned}
\sum \vec{M}_{i/O} &= \dot{\vec{H}}_{/O} \\
\Rightarrow \quad \vec{M} &= \int \vec{r}_{/O} \times \vec{a}\, dm \\
&= \int_0^{R_0}\int_0^{2\pi} (R\hat{e}_R) \times (-R\omega^2\hat{e}_R)\, \underbrace{\overbrace{\left(\frac{m_{\text{tot}}}{\pi R_O^2}\right)}^{\rho} \overbrace{R\, d\theta\, dR}^{dA}}_{dm} \\
&= \int\int \vec{0}\, d\theta\, dR \\
&= \vec{0}.
\end{aligned}$$

So the net force and moment needed are $\vec{F} = \vec{0}$ and $\vec{M} = \vec{0}$. Like a particle that moves at constant velocity with no force, a uniform disk rotates at constant rate with no torque (at least in 2D).

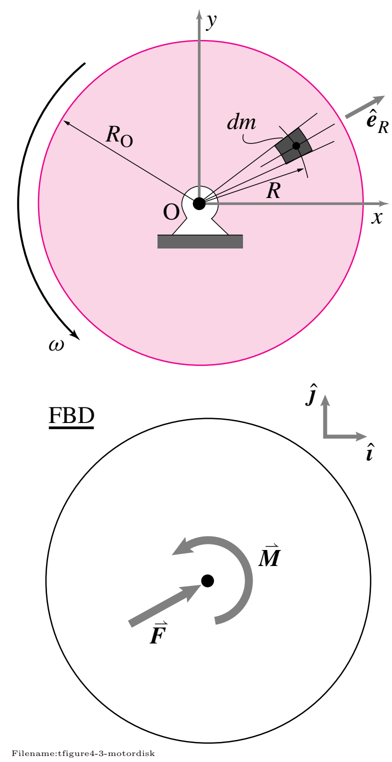

Figure 13.78: A uniform disk turned by a motor at a constant rate. The free body diagram shows a force and moment equivalent to the force system that acts including gravity, bearing forces, etc. A bit of mass dm occupies the space between the radial lines at θ and $\theta + d\theta$ and between the circles at radii R and $R = dR$.

13.10 The relation between angular momentum balance and power balance

For this system, angular momentum balance can be derived from power balance and vice versa. Thus neither is essentially more fundamental than the other and both are reliable. First we can derive power balance from angular momentum balance as follows:

$$\begin{aligned}
\vec{M}_{\text{net}} &= \dot{\omega}\hat{k}\int r^2\, dm \\
\vec{\omega}\cdot\vec{M}_{\text{net}} &= \vec{\omega}\cdot\left(\dot{\omega}\hat{k}\int r^2\, dm\right). \\
P &= \dot{E}_{\text{K}}
\end{aligned} \tag{13.60}$$

That is, when we dot both sides of the angular momentum equation with $\vec{\omega}$ we get on the left side a term which we recognize as the power of the forces and on the right side a term which is the rate of change of kinetic energy.

The opposite derivation starts with the power balance fig. 13.58(g)

$$\begin{aligned}
\vec{\omega}\cdot\sum\vec{M}_i &= \dot{\omega}\omega\int r^2\, dm \quad \text{(g)} \\
\Rightarrow \quad \omega\left(\hat{k}\cdot\sum\vec{M}_i\right) &= \dot{\omega}\omega\int r^2\, dm \\
\Rightarrow \quad \left(\hat{k}\cdot\sum\vec{M}_i\right) &= \dot{\omega}\int r^2\, dm
\end{aligned} \tag{13.61}$$

and, assuming $\omega \neq 0$, divide by ω to get the angular momentum equation for planar rotational motion.

Using moment-of-inertia in 2-D circular motion dynamics

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from chapter 1.1. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. As you recall from chapter 3.6, the linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center-of-mass of the system of interest. One wishes for something so simple for the calculation of angular momentum.

It turns out that we are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies. The luck is not so great for 3-D rigid bodies but still there is some simplification. For general motion of non-rigid bodies there is no simplification to be had. The simplification is to use the moment of inertia for the bodies rather than evaluating the momenta and energy quantities as integrals and sums. Of course one may have to do a sum or integral to evaluate $I \equiv I_{zz}^{cm}$ or $[\boldsymbol{I}^{\text{cm}}]$ but once this calculation is done, one need not work with the integrals while worrying about the dynamics. At this point we will assume that you are comfortable calculating and looking-up moments of inertia. We proceed to use it for the purposes of studying mechanics. For constant rate rotation, we can calculate the velocity and acceleration of various points on a rigid body using $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$. So we can calculate the various motion quantities of interest: linear momentum $\vec{\boldsymbol{L}}$, rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$, angular momentum $\vec{\boldsymbol{H}}$, rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}$, and kinetic energy E_{K}.

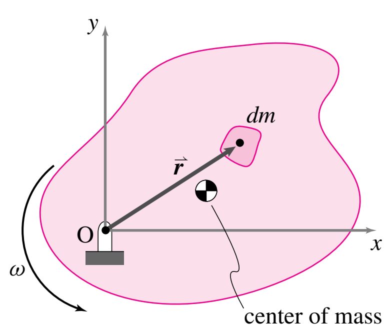

Figure 13.79: A two-dimensional body is rotating around the point O at constant rate ω. A differential bit of mass dm is shown. The center-of-mass is also shown.

Consider a two-dimensional rigid body like that shown in fig. 13.79. Now let us consider the various motion quantities in turn. First the linear momentum $\vec{\boldsymbol{L}}$. The linear momentum of any system in any motion is $\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate ω about point O (using $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$):

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}_{cm} m_{tot} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{cm/o} m_{tot}.$$

Similarly, for any system, we can calculate the rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$ as $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate,

$$\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{cm/o}) m_{tot}.$$

That is, the linear momentum is correctly calculated for this special motion, as it is for all motions, by thinking of the body as a point mass at the center-of-mass.

Unlike the calculation of linear momentum, the angular momentum turns out to be something different than would be calculated by using a point mass at the center of mass. You can remember this important fact by looking at the case when the rotation is about the center-of-mass (point O coincides with the center-of-mass). In this case one can intuitively see that the angular momentum of a rigid body *is not* zero even though the center-of-mass is not moving. Here's the calculation just to be sure:

$$
\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{O}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}}\, dm && \text{(by definition of } \vec{\boldsymbol{H}}_{/\mathrm{O}}\text{)} \\
&= \int \vec{\boldsymbol{r}}_{/O} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O})\, dm && \text{(using } \vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}\text{)} \\
&= \int (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}) \times \Big[(\omega\hat{\boldsymbol{k}}) \times (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}})\Big]\, dm && \text{(substituting } \vec{\boldsymbol{r}}_{/O} \text{ and } \vec{\boldsymbol{\omega}}\text{)} \\
&= \{\textstyle\int (x_{/O}^2 + y_{/O}^2)\, dm\}\omega\hat{\boldsymbol{k}} && \text{(doing cross products)} \\
&= \{\textstyle\int r_{/O}^2\, dm\}\omega\hat{\boldsymbol{k}} \\
&= \underbrace{I_{zz}^O}_{I_{zz}^O \text{ is the `polar' moment of inertia.}} \omega\hat{\boldsymbol{k}}
\end{aligned}
$$

We have defined the 'polar' moment of inertia as $I_{zz}^o = \int r_{/o}^2\, dm$. In order to calculate I_{zz}^o for a specific body, assuming uniform mass distribution for example, one must convert the differential quantity of mass dm into a differential of geometric quantities. For a line or curve, $dm = \rho d\ell$; for a plate or surface, $dm = \rho dA$, and for a 3-D region, $dm = \rho dV$. $d\ell$, dA, and dV are differential line, area, and volume elements, respectively. In each case, ρ is the mass density per unit length, per unit area, or per unit volume, respectively. To avoid clutter, we do not define a different symbol for the density in each geometric case. The differential elements must be further defined depending on the coordinate systems chosen for the calculation; *e.g.,* for rectangular coordinates, $dA = dxdy$ or, for polar coordinates, $dA = rdrd\theta$.

Since $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{\omega}}$ always point in the $\hat{\boldsymbol{k}}$ direction for two dimensional problems people often just think of angular momentum as a scalar and write the equation above simply as '$H = I\omega$,' the form usually seen in elementary physics courses.

The derivation above has a feature that one might not notice at first sight. The quantity called I_{zz}^{O} does not depend on the rotation of the body. That is, the value of the integral does not change with time, so I_{zz}^{O} is a constant. So, perhaps unsurprisingly, a two-dimensional body spinning about the z-axis through O has constant angular momentum about O if it spins at a constant rate. ③

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\mathbf{0}}.$$

Now, of course we could find this result about constant rate motion of 2-D bodies somewhat more cumbersomely by plugging in the general formula for

③ Note that the angular momentum about some other point than O will not be constant unless the center-of-mass does not accelerate (*i.e.,* is at point O).

rate of change of angular momentum as follows:

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/O} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm \\ &= \int \vec{\boldsymbol{r}}_{/O} \times (\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O}))\, dm \\ &= \int (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}) \times \left[\omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}))\right] dm \\ &= \vec{\boldsymbol{0}}.\end{aligned} \tag{13.62}$$

Using moment of inertia about the center of mass. Often it is easier to think of the motion as composed of two parts, motion of the center of mass and motion relative to the center of mass, as explained in box 13.11 on page 707. Thus we have two terms for angular momentum $\vec{\boldsymbol{H}}_{/O}$ and its rate of change $\dot{\vec{\boldsymbol{H}}}_{/O}$:

$$\vec{\boldsymbol{H}}_{/O} = \vec{\boldsymbol{r}}_{G/O} \times m\vec{\boldsymbol{v}}_{cm} + I_{zz}^{cm}\omega\hat{\boldsymbol{k}} \tag{13.63}$$

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \vec{\boldsymbol{r}}_{G/O} \times m\vec{\boldsymbol{a}}_{cm} + I_{zz}^{cm}\alpha\hat{\boldsymbol{k}} \tag{13.64}$$

Kinetic energy. Finally, we can calculate the kinetic energy by adding up $\frac{1}{2}m_i v_i^2$ for all the bits of mass on a 2-D body spinning about the z-axis:

$$E_K = \int \frac{1}{2}v^2\, dm = \int \frac{1}{2}(\omega r)^2\, dm = \frac{1}{2}\omega^2 \int r^2\, dm = \frac{1}{2}I_{zz}^{o}\omega^2 \quad . \tag{13.65}$$

The kinetic energy can also be written as a sum of contributions of motion of the center of mass and motion relative to the center of mass,

$$E_K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{zz}^{cm}\omega^2 \quad . \tag{13.66}$$

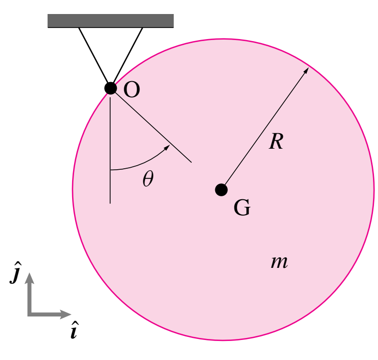

Figure 13.80:

Example: Pendulum disk

For the disk shown in fig. 13.80, we can calculate the rate of change of angular momentum about point O as

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/O} &= \vec{\boldsymbol{r}}_{G/O} \times m\vec{\boldsymbol{a}}_{cm} + I_{zz}^{cm}\alpha\hat{\boldsymbol{k}} \\ &= R^2 m\ddot{\theta}\hat{\boldsymbol{k}} + I_{zz}^{cm}\ddot{\theta}\hat{\boldsymbol{k}} \\ &= (I_{zz}^{cm} + R^2 m)\ddot{\theta}\hat{\boldsymbol{k}}.\end{aligned}$$

Alternatively, we could calculate directly

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/O} &= I_{zz}^{O}\alpha\hat{\boldsymbol{k}} \\ &= \underbrace{(I_{zz}^{cm} + R^2 m)}_{\text{by the parallel axis theorem}}\ddot{\theta}\hat{\boldsymbol{k}}.\end{aligned}$$

Note that we are using the planarity of the objects and of their motion for our calculations④.

④**2D *vs* 3D.** Beware of falling into the common misconception that the formula $M = I\alpha$ applies in three dimensions by just thinking of the scalars as vectors and matrices. In 3D the formula

$$\dot{\vec{\boldsymbol{H}}}_{/O} = [\boldsymbol{I}^O] \cdot \underbrace{\dot{\vec{\boldsymbol{\omega}}}}_{\vec{\boldsymbol{\alpha}}}$$

is only correct when $\vec{\boldsymbol{\omega}}$ is zero or when $\vec{\boldsymbol{\omega}}$ is an eigen vector of $[I_{/O}]$. That is, the vector equation

$$\sum \text{Moments about O} = [\boldsymbol{I}^O] \cdot \vec{\boldsymbol{\alpha}}$$

is generally wrong in 3D.

The equation for linear momentum balance is the same as always, we just need to calculate the acceleration of the center-of-mass of the spinning body.

$$\dot{\vec{L}} = m_{tot}\vec{a}_{cm} = m_{tot}\left[\vec{\omega}\times(\vec{\omega}\times\vec{r}_{cm/O}) + \dot{\vec{\omega}}\times\vec{r}_{cm/O}\right] \qquad (13.67)$$

Finally, the kinetic energy for a planar rigid body rotating in the plane is:

$$E_{\mathrm{K}} = \frac{1}{2}\vec{\omega}\cdot([\boldsymbol{I}^{\mathrm{cm}}]\cdot\vec{\omega}) + \frac{1}{2}m\underbrace{v_{\mathrm{cm}}^2}_{\boxed{\vec{v}_{cm} = \vec{\omega}\times\vec{r}_{\mathrm{cm/O}}}}.$$

13.11 Simplifying $\vec{H}_{/\mathrm{C}}$ using the center of mass

The definition of angular momentum relative to a point C is

$$\vec{H}_{/\mathrm{C}} = \sum \vec{r}_{i/C}\times m_i\vec{v}_i.$$

If we rewrite $\vec{v}_i$ as

$$\vec{v}_i = (\vec{v}_i - \vec{v}_{cm}) + \vec{v}_{cm} = \vec{v}_{i/cm} + \vec{v}_{cm}$$

and

$$\vec{r}_i = (\vec{r}_i - \vec{r}_{cm}) + \vec{r}_{cm} = \vec{r}_{i/cm} + \vec{r}_{cm}$$

then

$$\begin{aligned}
\vec{H}_{/\mathrm{C}} &= \sum(\vec{r}_{cm} + \vec{r}_{i/cm})\times[\vec{v}_{cm} + \vec{v}_{i/cm}]\,m_i. \\
&= \sum \vec{r}_{cm}\times\vec{v}_{cm}m_i + \sum \vec{r}_{i/cm}\times\vec{v}_{i/cm}m_i \\
&\quad + \sum \vec{r}_{cm}\times\vec{v}_{i/cm}m_i + \sum \vec{r}_{i/cm}\times\vec{v}_{cm}m_i \\
&= \vec{r}_{cm}\times\vec{v}_{cm}m_{\mathrm{tot}} + \sum \vec{r}_{i/cm}\times\vec{v}_{i/cm}\,m_i \\
&\quad + \vec{r}_{cm}\times\underbrace{\left[\sum \vec{v}_{i/cm}\,\mathrm{m}_i\right]}_{\vec{0}} + \underbrace{\left[\sum \vec{r}_{i/cm}\,m_i\right]}_{\vec{0}}\times\vec{v}_{cm}
\end{aligned}$$

So,

$$\vec{H}_{/\mathrm{C}} = \underbrace{\vec{r}_{cm}\times\vec{v}_{cm}m_{\mathrm{tot}}}_{\boxed{\text{contribution of center of mass motion}}} + \underbrace{\sum \vec{r}_{i/\mathrm{cm}}\times\vec{v}_{i/cm}\,m_i}_{\boxed{\text{contribution of motion relative to center-of-mass}}}.$$

The reason $\sum \vec{r}_{i/cm}\,m_i = \vec{\mathbf{0}}$ is somewhat intuitive. It is what you would calculate if you were looking for the center-of-mass relative to the center of mass. More formally,

$$\begin{aligned}
\sum \vec{r}_{i/cm}\,m_i &= \sum(\vec{r}_i - \vec{r}_{cm})\,m_i \\
&= \underbrace{\sum \vec{r}_i\,m_i}_{m_{\mathrm{tot}}\vec{r}_{cm}} - m_{\mathrm{tot}}\vec{r}_{cm} \\
&= \vec{\mathbf{0}}.
\end{aligned}$$

Similarly, $\sum \vec{v}_{i/cm}\,m_i = \vec{\mathbf{0}}$ because it is what you would calculate if you were looking for the velocity of the center-of-mass relative to the center of mass.

The central result of this box is that

angular momentum of any system is that due to motion *of* the center-of-mass *plus* motion *relative to* the center-of-mass.

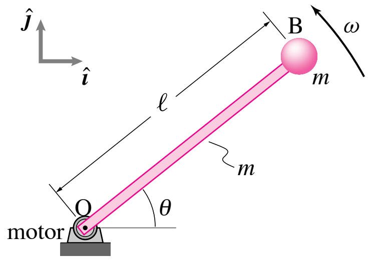

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Figure 13.81: A rod goes in circles at a constant rate.

SAMPLE 13.22 A rod in a constant rate circular motion: A uniform rod of mass m and length ℓ is connected to a motor at end O. A ball of mass m is attached to the rod at end B. The motor turns the rod in counterclockwise direction at a constant angular speed ω. There is gravity pointing in the $-\hat{\jmath}$ direction. Find the torque applied by the motor (i) at the instant shown and (ii) when $\theta = 0°, 90°, 180°$. How does the torque change if the angular speed is doubled?

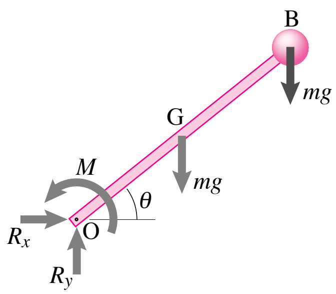
Solution The FBD of the rod and ball system is shown in Fig. 13.82(a). Since the system is undergoing circular motion at a constant speed, the acceleration of the ball as well as every point on the rod is just radial (pointing towards the center of rotation O) and is given by $\vec{a} = -\omega^2 r\hat{\boldsymbol{\lambda}}$ where r is the radial distance from the center O to the point of interest and $\hat{\boldsymbol{\lambda}}$ is a unit vector along OB pointing away from O (Fig. 13.82(b)).

Angular Momentum Balance about point O gives

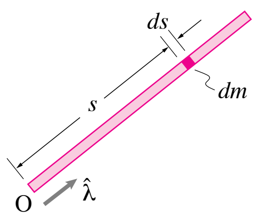
$$\sum \vec{M}_{\mathrm{O}} = \dot{\vec{H}}_{/\mathrm{O}}$$

$$\begin{aligned}
\sum \vec{M}_{\mathrm{O}} &= \vec{r}_{G/O} \times (-mg\hat{\jmath}) + \vec{r}_{B/O} \times (-mg\hat{\jmath}) + M\hat{k} \\
&= -\frac{\ell}{2}\cos\theta mg\hat{k} - \ell\cos\theta mg\hat{k} + M\hat{k} \\
&= (M - \frac{3\ell}{2}mg\cos\theta)\hat{k} \qquad (13.68)
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{H}}_{/\mathrm{O}} &= \overbrace{\vec{r}_{B/O} \times m\vec{a}_B}^{(\dot{\vec{H}}_{/\mathrm{O}})_{\mathrm{ball}}} + \overbrace{\int_m \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm}^{(\dot{\vec{H}}_{/\mathrm{O}})_{\mathrm{rod}}} \\
&= \ell\hat{\boldsymbol{\lambda}} \times (-m\omega^2\ell\hat{\boldsymbol{\lambda}}) + \int_m \overbrace{s\hat{\boldsymbol{\lambda}}}^{\vec{r}_{dm/O}} \times \overbrace{(-\omega^2 s\hat{\boldsymbol{\lambda}})}^{\vec{a}_{dm}}\, dm \\
&= \vec{0} \qquad \text{(since } \hat{\boldsymbol{\lambda}} \times \hat{\boldsymbol{\lambda}} = \vec{0}\text{)} \qquad (13.69)
\end{aligned}$$



(a) FBD of rod+ball system



(b) Calculation of $\dot{\vec{H}}_{\mathrm{O}}$ of the rod

Filename:sfig4-5-5a

Figure 13.82: A rod goes in circles at a constant rate.

(i) Equating (13.68) and (13.69) we get

$$M = \frac{3}{2}mg\ell\cos\theta.$$

$$\boxed{M = \tfrac{3}{2}mg\ell\cos\theta}$$

(ii) Substituting the given values of θ in the above expression we get

$$M(\theta = 0°) = \frac{3}{2}mg\ell, \quad M(\theta = 90°) = 0 \quad M(\theta = 180°) = -\frac{3}{2}mg\ell$$

$$\boxed{M(0°) = \tfrac{3}{2}mg\ell, \quad M(90°) = 0 \quad M(180°) = -\tfrac{3}{2}mg\ell}$$

⑤

⑤The values obtained above make sense (at least qualitatively). To make the rod and the ball go up from the 0° position, the motor has to apply some torque in the counterclockwise direction. In the 90° position no torque is required for the *dynamic balance*. In 180° position the system is accelerating downwards under gravity; therefore, the motor has to apply a clockwise torque to make the system maintain a uniform speed.

It is clear from the expression of the torque that it does not depend on the value of the angular speed ω! Therefore, the torque will not change if the speed is doubled. In fact, as long as the speed remains constant at any value, the only torque required to maintain the motion is the torque to counteract the moments due to gravity at O.

SAMPLE 13.23 At the onset of circular motion: A $2' \times 4'$ rectangular plate of mass 20 lbm is pivoted at one of its corners as shown in the figure. The plate is released from rest in the position shown. Find the force on the support immediately after release.

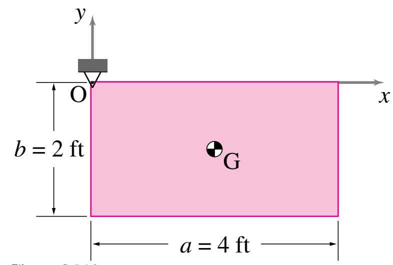


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Figure 13.83: A rectangular plate is released from rest from the position shown.

Solution The free body diagram of the plate is shown in Fig. 13.84. The force $\vec{F}$ applied on the plate by the support is unknown.

The linear momentum balance for the plate gives

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_G \\ \vec{F} - mg\hat{\jmath} &= m(\ddot{\theta}\, r_{G/O}\hat{e}_\theta - \dot{\theta}^2 R\hat{e}_R) \\ &= m\,\ddot{\theta}\, r_{G/O}\hat{e}_\theta \quad (\text{since } \dot{\theta} = 0 \text{ at } t = 0). \end{aligned} \tag{13.70}$$

Thus to find $\vec{F}$ we need to find $\ddot{\theta}$.

The angular momentum balance for the plate about the fixed support point O gives

$$\vec{M}_O = \dot{\vec{H}}_O$$

where

$$\begin{aligned} \vec{M}_O &= \vec{r}_{G/O} \times mg(-\hat{\jmath}) \\ &= \underbrace{(\frac{a}{2}\hat{\imath} - \frac{b}{2}\hat{\jmath})}_{\vec{r}_{G/O}} \times mg(-\hat{\jmath}) = -mg\frac{a}{2}\hat{k}, \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_{/O} &= \dot{\omega}\hat{k}\int_m r^2\,dm = \ddot{\theta}\hat{k}\int_0^b\int_0^a \overbrace{(x^2+y^2)}^{r^2}\overbrace{\frac{m}{ab}dx\,dy}^{dm} \\ &= \frac{m(a^2+b^2)}{3}\ddot{\theta}\hat{k}. \end{aligned}$$

Thus,

$$\begin{aligned} -mg\frac{a}{2}\hat{k} &= \frac{m(a^2+b^2)}{3}\ddot{\theta}\hat{k} \\ \Rightarrow \quad \ddot{\theta} &= -\frac{3ga}{2(a^2+b^2)} \\ &= -\frac{3 \cdot 32.2\,\text{ft/s}^2 \cdot 4\,\text{ft}}{2(16+4)\,\text{ft}^2} = -9.66\,\text{rad/s}^2. \end{aligned}$$

From eqn. (13.70), the support force is now readily calculated:

$$\begin{aligned} \vec{F} &= mg\hat{\jmath} + m\ddot{\theta}\, r_{G/O}\hat{e}_\theta \\ &= mg\hat{\jmath} + m\ddot{\theta}\frac{\sqrt{a^2+b^2}}{2}\,\frac{b\hat{\imath}+a\hat{\jmath}}{\sqrt{(a^2+b^2)}} \\ &= \frac{1}{2}m\ddot{\theta}b\hat{\imath} + (mg + \frac{1}{2}m\ddot{\theta}a)\hat{\jmath} \end{aligned}$$

Using the given numerical values of m, a, and b, $\ddot{\theta} = -9.66\,\text{rad/s}^2$, and $g = 32.2\,\text{ft/s}^2$, we get

$$\vec{F} = (-6\hat{\imath} + 8\hat{\jmath})\,\text{lbf}.$$

$$\boxed{\vec{F} = (-6\hat{\imath} + 8\hat{\jmath})\,\text{lbf}}$$

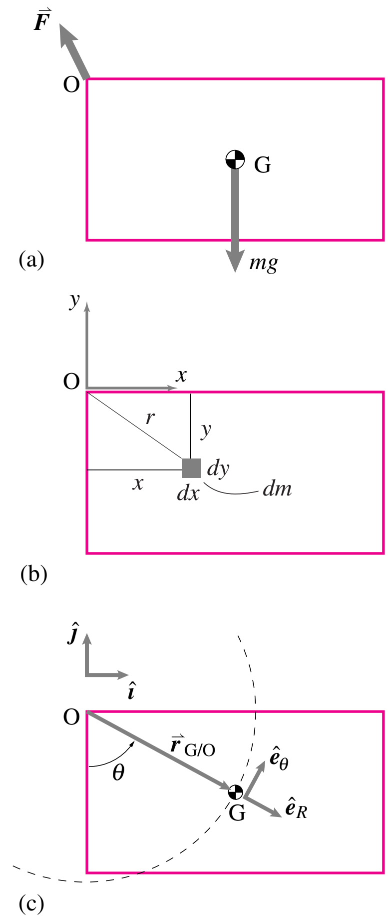

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Figure 13.84: (a) The free body diagram of the plate. (b) Computation of the integral in $\dot{\vec{H}}_{/O} = \dot{\omega}\hat{k}\int_m r^2\,dm$. (c) The geometry of motion. From the given dimensions, $\hat{e}_R = \frac{a\hat{\imath}-b\hat{\jmath}}{\sqrt{(a^2+b^2)}}$, $\hat{e}_\theta = \frac{b\hat{\imath}+a\hat{\jmath}}{\sqrt{(a^2+b^2)}}$, and $r_{G/O} = \frac{\sqrt{a^2+b^2}}{2}$.

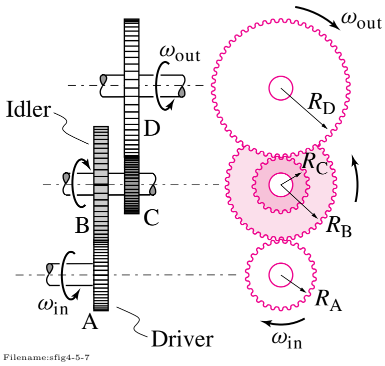

Figure 13.85: A compound gear train.

SAMPLE 13.24 A compound gear train. When the gear of an input shaft, often called the *driver* or the *pinion*, is directly meshed in with the gear of an output shaft, the motion of the output shaft is opposite to that of the input shaft. To get the output motion in the same direction as that of the input motion, an *idler* gear is used. If the idler shaft has more than one gear in mesh, then the gear train is called a *compound gear train*.

In the gear train shown in Fig. 13.85, the input shaft is rotating at 2000 rpm and the input torque is 200 N-m. The efficiency (defined as the ratio of output power to input power) of the train is 0.96 and the various radii of the gears are: $R_A = 5\,\text{cm}$, $R_B = 8\,\text{cm}$, $R_C = 4\,\text{cm}$, and $R_D = 10\,\text{cm}$. Find

1. the input power P_{in} and the output power P_{out},
2. the output speed ω_{out}, and
3. the output torque.

Solution

1. The power:

$$\begin{aligned} P_{\text{in}} &= M_{\text{in}}\omega_{\text{in}} = 200\,\text{N·m}\cdot 2000\,\text{rpm} \\ &= 400000\,\text{N·m}\cdot\frac{\cancel{\text{rev}}}{\cancel{\text{min}}}\cdot\frac{2\pi}{1\,\cancel{\text{rev}}}\cdot\frac{1\,\cancel{\text{min}}}{60\,\text{s}} \\ &= 41887.9\,\text{N m/s}\approx 42\,\text{kW}. \\ \Rightarrow\quad p_{\text{out}} &= \text{efficiency}\cdot P_{\text{in}} = 0.96\cdot 42\,\text{kW}\approx 40\,\text{kW} \end{aligned}$$

$$\boxed{P_{\text{in}} = 42\,\text{kW},\quad p_{\text{out}} = 40\,\text{kW}}$$

2. The angular speed of meshing gears can be easily calculated by realizing that the linear speed of the point of contact has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus,

$$\begin{aligned} v_P &= \omega_{\text{in}}\,R_A = \omega_B\,R_B \\ \Rightarrow\quad \omega_B &= \omega_{\text{in}}\cdot\frac{R_A}{R_B} \\ \text{and}\quad v_R &= \omega_C\,R_C = \omega_{\text{out}}\,R_D \\ \Rightarrow\quad \omega_{\text{out}} &= \omega_C\cdot\frac{R_C}{R_D} \\ \text{But}\quad \omega_C &= \omega_B \\ \Rightarrow\quad \omega_{\text{out}} &= \omega_{\text{in}}\cdot\frac{R_A}{R_B}\cdot\frac{R_C}{R_D} \\ &= 2000\,\text{rpm}\cdot\frac{5}{8}\cdot\frac{4}{10} = 500\,\text{rpm}. \end{aligned}$$

$$\boxed{\omega_{\text{out}} = 500\,\text{rpm}}$$

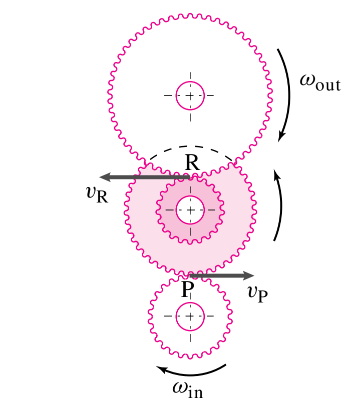

Figure 13.86:

3. The output torque,

$$\begin{aligned} M_{\text{out}} = \frac{P_{\text{out}}}{\omega_{\text{out}}} &= \frac{40\,\text{kW}}{500\,\text{rpm}} = \frac{40}{500}\cdot 1000\frac{\text{N·m}}{\cancel{\text{s}}}\cdot\frac{\cancel{\text{min}}}{\cancel{\text{rev}}}\cdot\frac{1\,\cancel{\text{rev}}}{2\pi}\cdot\frac{60\,\cancel{\text{s}}}{1\,\cancel{\text{min}}} \\ &= 764\,\text{N·m}. \end{aligned}$$

$$\boxed{M_{\text{out}} = 764\,\text{N·m}}$$

SAMPLE 13.25 An accelerating gear train. In the gear train shown in Fig. 13.87, the torque at the input shaft is $M_{\text{in}} = 200\,\text{N}{\cdot}\text{m}$ and the angular acceleration is $\alpha_{\text{in}} = 50\,\text{rad/s}^2$. The radii of the various gears are: $R_A = 5\,\text{cm}$, $R_B = 8\,\text{cm}$, $R_C = 4\,\text{cm}$, and $R_D = 10\,\text{cm}$ and the moments of inertia about the shaft axis passing through their respective centers are: $I_A = 0.1\,\text{kg m}^2$, $I_{BC} = 5I_A$, $I_D = 4I_A$. Find the output torque M_{out} of the gear train.

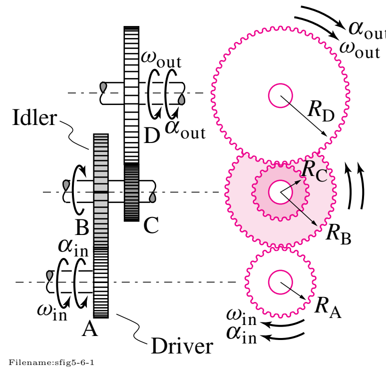

Figure 13.87: An accelerating compound gear train.

Solution Since the difference between the input power and the output power is used in accelerating the gears, we may write

$$P_{\text{in}} - P_{\text{out}} = \dot{E}_K$$

Let M_{out} be the output torque of the gear train. Then,

$$P_{\text{in}} - P_{\text{out}} = M_{\text{in}}\,\omega_{\text{in}} - M_{\text{out}}\,\omega_{\text{out}}. \tag{13.71}$$

Now,

$$\dot{E}_K = \frac{d}{dt}(E_K) \tag{13.72}$$

$$= \frac{d}{dt}\left(\frac{1}{2}I_A\,\omega_{\text{in}}^2 + \frac{1}{2}I_{BC}\,\omega_{BC}^2 + \frac{1}{2}I_D\,\omega_{\text{out}}^2\right)$$

$$= I_A\,\omega_{\text{in}}\,\dot{\omega}_{\text{in}} + I_{BC}\,\omega_{BC}\,\dot{\omega}_{BC} + I_D\,\omega_{\text{out}}\,\dot{\omega}_{\text{out}}$$

$$= I_A\,\omega_{\text{in}}\,\alpha_{\text{in}} + 5I_A\,\omega_{BC}\,\alpha_{BC} + 4I_A\,\omega_{\text{out}}\,\alpha_{\text{out}}. \tag{13.73}$$

The different ω's and the α's can be related by realizing that the linear speed or the tangential acceleration of the point of contact between any two meshing gears has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus, using the linear speed and tangential acceleration calculations for points P and R in Fig. 13.88, we can find ω_B, α_B and ω_{out}, α_{out}. Considering the linear speed of point P, we find

$$v_P = \omega_{\text{in}}\,R_A = \omega_B\,R_B$$

$$\Rightarrow \quad \omega_B = \omega_{\text{in}}\cdot\frac{R_A}{R_B},$$

and considering the tangential acceleration of point P, $(a_P)_\theta$, we find

$$(a_P)_\theta = \alpha_{\text{in}}\,R_A = \alpha_B\,R_B$$

$$\Rightarrow \quad \alpha_B = \alpha_{\text{in}}\cdot\frac{R_A}{R_B}.$$

Similarly,

$$v_R = \omega_C\,R_C = \omega_{\text{out}}\,R_D$$

$$\Rightarrow \quad \omega_{\text{out}} = \omega_C\cdot\frac{R_C}{R_D},$$

and

$$(a_R)_\theta = \alpha_C\,R_C = \alpha_{\text{out}}\,R_D$$

$$\Rightarrow \quad \alpha_{\text{out}} = \alpha_C\cdot\frac{R_C}{R_D}.$$

But

$$\omega_C = \omega_B = \omega_{BC}$$

$$\Rightarrow \quad \omega_{\text{out}} = \omega_{\text{in}}\cdot\frac{R_A}{R_B}\cdot\frac{R_C}{R_D}$$

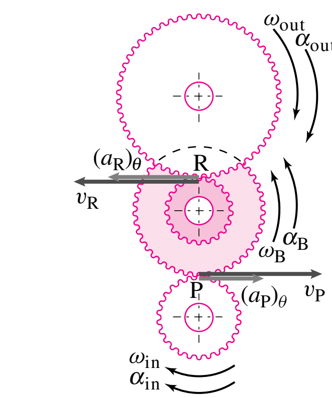

Figure 13.88: The velocity or acceleration of the point of contact between two meshing gears has to be the same irrespective of which meshing gear's geometry and motion is used to compute them.

and

$$\begin{aligned}
\alpha_C &= \alpha_B = \alpha_{BC} \\
\Rightarrow \quad \alpha_{\text{out}} &= \alpha_{\text{in}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D}.
\end{aligned}$$

Substituting these expressions for ω_{out}, α_{out}, ω_{BC} and α_{BC} in equations (13.71) and (13.73), we get

$$\begin{aligned}
P_{\text{in}} - P_{\text{out}} &= M_{\text{in}}\,\omega_{\text{in}} - M_{\text{out}}\omega_{\text{in}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \\
&= \omega_{\text{in}} \left(M_{\text{in}} - M_{\text{out}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right). \\
\dot{E}_{\text{K}} &= I_A \left[\omega_{\text{in}}\alpha_{\text{in}} + 5\omega_{\text{in}}\alpha_{\text{in}} \left(\frac{R_A}{R_B} \right)^2 + 4\omega_{\text{in}}\alpha_{\text{in}} \left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right] \\
&= I_A\omega_{\text{in}} \left[\alpha_{\text{in}} + 5\alpha_{\text{in}} \left(\frac{R_A}{R_B} \right)^2 + 4\alpha_{\text{in}} \left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right].
\end{aligned}$$

Now equating the two quantities, $P_{\text{in}} - P_{\text{out}}$ and $\dot{E}_{\text{K}}$, and canceling ω_{in} from both sides, we obtain

$$\begin{aligned}
M_{\text{out}} \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} &= M_{\text{in}} - I_A\alpha_{\text{in}} \left[1 + 5 \left(\frac{R_A}{R_B} \right)^2 + 4 \left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \right)^2 \right] \\
M_{\text{out}} \frac{5}{8} \cdot \frac{4}{10} &= 200\,\text{N·m} - 5\,\text{kg m}^2 \cdot \text{rad/s}^2 \left[1 + 5 \left(\frac{5}{8} \right)^2 + 4 \left(\frac{5}{8} \cdot \frac{4}{10} \right)^2 \right] \\
M_{\text{out}} &= 735.94\,\text{N·m} \\
&\approx 736\,\text{N·m}.
\end{aligned}$$

$$\boxed{M_{\text{out}} = 736\,\text{N·m}}$$

SAMPLE 13.26 **Drums used as pulleys.** Two drums, A and B of radii $R_o = 200\,\text{mm}$ and $R_i = 100\,\text{mm}$ are welded together. The combined mass of the drums is $m_D = 20\,\text{kg}$ and the combined moment of inertia about the z-axis passing through their common center O is $I_{zz/O} = 1.6\,\text{kg m}^2$. A string attached to and wrapped around drum B supports a mass $m = 2\,\text{kg}$. The string wrapped around drum A is pulled with a force $F = 20\,\text{N}$ as shown in Fig. 13.89. Assume there is no slip between the strings and the drums. Find

1. the angular acceleration of the drums,
2. the tension in the string supporting mass m, and
3. the acceleration of mass m.

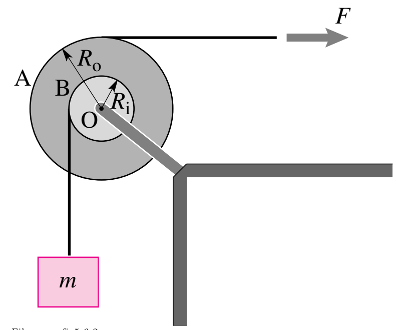

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Figure 13.89: Two drums with strings wrapped around are used to pull up a mass m.

Solution The free-body diagram of the drums and the mass are shown in Fig. 13.90 separately where T is the tension in the string supporting mass m and O_x and O_y are the support reactions at O. Since the drums can only rotate about the z-axis, let

$$\vec{\omega} = \omega\hat{k} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega}\hat{k}.$$

Now, let us do angular momentum balance about the center of rotation O:

$$\sum \vec{M}_O = \dot{\vec{H}}_{/O}$$

$$\begin{aligned} \sum \vec{M}_O &= TR_i\hat{k} - FR_o\hat{k} \\ &= (TR_i - FR_o)\hat{k}. \end{aligned}$$

Since the motion is restricted to the xy-plane (*i.e.,* 2-D motion), the rate of change of angular momentum $\dot{\vec{H}}_{/O}$ may be computed as

$$\begin{aligned} \dot{\vec{H}}_{/O} &= I_{zz/cm}\,\dot{\omega}\hat{k} + \vec{r}_{cm/O} \times \vec{a}_{cm}\; m_D \\ &= I_{zz/O}\,\dot{\omega}\hat{k} + \underbrace{\vec{r}_{O/O}}_{0} \times \underbrace{\vec{a}_{cm}}_{0}\; m_D \\ &= I_{zz/O}\,\dot{\omega}\hat{k}. \end{aligned}$$

Setting $\sum \vec{M}_O = \dot{\vec{H}}_{/O}$ we get

$$TR_i - FR_o = I_{zz/O}\,\dot{\omega}. \tag{13.74}$$

Now, let us write linear momentum balance, $\sum \vec{F} = m\vec{a}$, for mass m:

$$\underbrace{(T - mg)\hat{j}}_{\sum \vec{F}} = m\vec{a}.$$

Do we know anything about acceleration $\vec{a}$ of the mass? Yes, we know its direction ($\pm\hat{j}$) and we also know that it has to be the same as the tangential acceleration $(\vec{a}_D)_\theta$ of point D on drum B (why?). Thus,

$$\begin{aligned} \vec{a} &= (\vec{a}_D)_\theta \\ &= \dot{\omega}\hat{k} \times (-R_i\hat{\imath}) \\ &= -\dot{\omega}R_i\,\hat{\jmath}. \end{aligned} \tag{13.75}$$

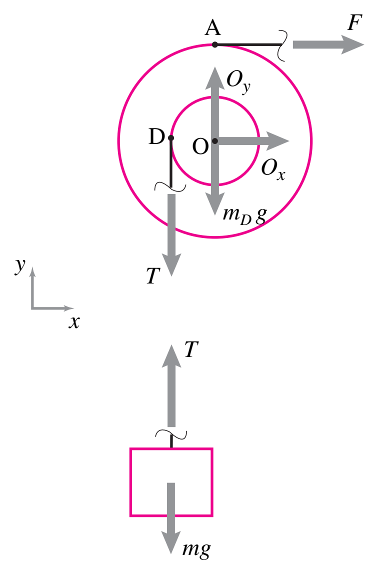

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Figure 13.90: Free-body diagram of the drums and the mass m. T is the tension in the string supporting mass m and O_x and O_y are the reactions of the support at O.

Therefore,

$$T - mg = -m\dot{\omega}R_i. \tag{13.76}$$

1. **Calculation of $\dot{\omega}$:** We now have two equations, (13.74) and (13.76), and two unknowns, $\dot{\omega}$ and T. Subtracting R_i times Eqn.(13.76) from Eqn. (13.74) we get

$$\begin{aligned}
-FR_o + mgR_i &= (I_{zz/O} + mR_i^2)\dot{\omega} \\
\Rightarrow \qquad \dot{\omega} &= \frac{-FR_o + mgR_i}{(I_{zz/O} + mR_i^2)} \\
&= \frac{-20\,\mathrm{N}\cdot 0.2\,\mathrm{m} + 2\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot 0.1m}{1.6\,\mathrm{kg\ m^2} + 2\,\mathrm{kg}\cdot(0.1\,\mathrm{m})^2} \\
&= \frac{-2.038\,\mathrm{kg\ m^2/\,s^2}}{1.62\,\mathrm{kg\ m^2}} \\
&= -1.258\frac{1}{\mathrm{s^2}}
\end{aligned}$$

$$\boxed{\dot{\vec{\omega}} = -1.26\,\mathrm{rad/s^2}\,\hat{\boldsymbol{k}}}$$

2. **Calculation of tension T:** From equation (13.76):

$$\begin{aligned}
T &= mg - m\dot{\omega}R_i \\
&= 2\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2} - 2\,\mathrm{kg}\cdot(-1.26\,\mathrm{s^{-2}})\cdot 0.1\,\mathrm{m} \\
&= 19.87\,\mathrm{N}
\end{aligned}$$

$$\boxed{T = 19.87\,\mathrm{N}}$$

3. **Calculation of acceleration of the mass:** Since the acceleration of the mass is the same as the tangential acceleration of point D on the drum, we get (from eqn. (13.75))

$$\begin{aligned}
\vec{\boldsymbol{a}} &= (\vec{\boldsymbol{a}}_D)_\theta = -\dot{\omega}R_i\,\hat{\boldsymbol{\jmath}} \\
&= -(-1.26\,\mathrm{s^{-2}})\cdot 0.1\,\mathrm{m} \\
&= 0.126\,\mathrm{m/s^2}\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = 0.13\,\mathrm{m/s^2}\,\hat{\boldsymbol{\jmath}}}$$

Comments: It is important to understand why the acceleration of the mass is the same as the tangential acceleration of point D on the drum. We have assumed (as is common practice) that the string is massless and inextensible. Therefore each point of the string supporting the mass must have the same linear displacement, velocity, and acceleration as the mass. Now think about the point on the string which is momentarily in contact with point D of the drum. Since there is no relative slip between the drum and the string, the two points must have the same vertical acceleration. This vertical acceleration for point D on the drum is the tangential acceleration $(\vec{\boldsymbol{a}}_D)_\theta$.

SAMPLE 13.27 **Energy method for pulley dynamics:** Consider the pulley problem of Sample 13.26 again. Use energy method to

1. find the angular acceleration of the pulley, and
2. the acceleration of the mass.

Solution In energy method we use speeds, not velocities. Therefore, we have to be careful in our thinking about the direction of motion. In the present problem, let us assume that the pulley rotates and accelerates clockwise. Consequently, the mass moves up against gravity.

1. The energy equation we want to use is

$$P = \dot{E}_K.$$

The power P is given by $P = \sum \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}_i$ where the sum is carried out over all external forces. For the mass and pulley system the external forces that do work are F and mg. The other external forces on the system—the reaction force of the support point O and the weight of the pulley—are acting at point O (see fig. 13.91). But, since point O is stationary, these forces do no work. Therefore,

$$\begin{aligned} P &= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_A + m\vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{v}}_m \\ &= F\hat{\boldsymbol{\imath}} \cdot v_A\hat{\boldsymbol{\imath}} + (-mg\,\hat{\boldsymbol{\jmath}}) \cdot \underbrace{v_D\,\hat{\boldsymbol{\jmath}}}_{\vec{\boldsymbol{v}}_m} \\ &= Fv_A - mgv_D. \end{aligned} \tag{13.77}$$

Now we need to calculate the rate of change of kinetic energy $\dot{E}_K$. There are two objects here that have kinetic energy—the hanging mass and the pulley. Hence,

$$\dot{E}_K = (\dot{E}_K)_m + (\dot{E}_K)_{\text{pulley}}.$$

The hanging mass has pure translational motion and hence its kinetic energy is

$$(E_K)_m = \frac{1}{2}mv^2$$

where v is the linear speed of the mass. If we assume the string to be inextensible, then the linear speed v of the mass has to be the same as the tangential speed of point D of the pulley. Thus $v = v_D$ and

$$(E_K)_m = \frac{1}{2}mv_D^2.$$

The pulley, on the other hand, has pure rotational motion about point O, and hence its kinetic energy is given by

$$(E_K)_{\text{pulley}} = \frac{1}{2}I_{zz}^{\text{O}}\omega^2.$$

Summing the two kinetic energies and differentiating with respect to time t, we get

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}m\,v_D^2 + \frac{1}{2}I_{zz}^{\text{O}}\omega^2\right) \quad (13.78) \\ &= m\,v_D\,\dot{v}_D + I_{zz}^{\text{O}}\,\omega\,\dot{\omega}. \quad (13.79) \end{aligned}$$

Now equating the power and the rate of change of kinetic energy from eqns. eqn. (13.77) and eqn. (13.79), we get

$$F\,v_A - mg\,v_D = m\,v_D\,\dot{v}_D + I_{zz}^{\text{O}}\,\omega\,\dot{\omega}.$$

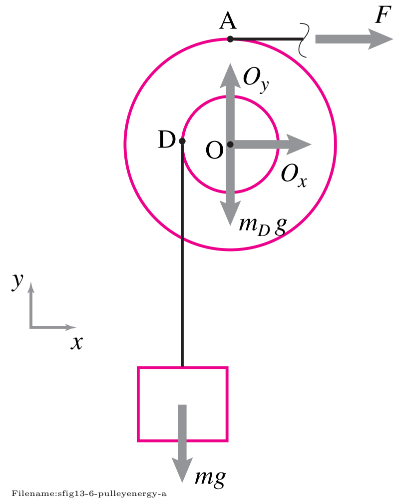

Figure 13.91: Free-body-diagram of the pulley-mass system together. Note that the forces acting at the support point O do no work since point O is stationary.

From kinematics of circular motion,

$$\begin{aligned} v_A &= \omega R_o, \\ v_D &= \omega R_i \\ \text{and} \quad \dot{v}_D \equiv (a_D)_\theta &= \dot{\omega} R_i\,. \end{aligned}$$

Substituting these values in the power balance equation above, we get

$$\begin{aligned} \omega(FR_o - mgR_i) &= \omega\,\dot{\omega}(mR_i^2 + I_{zz}^{\text{o}}) \\ \Rightarrow \quad \dot{\omega} &= \frac{FR_o - mgR_i}{(I_{zz}^{\text{o}} + mR_i^2)} \\ &= \frac{20\,\text{N}\cdot 0.2\,\text{m} - 2\,\text{kg}\cdot 9.81\,\text{m/s}^2\cdot 0.1m}{1.6\,\text{kg m}^2 + 2\,\text{kg}\cdot(0.1\,\text{m})^2} \\ &= 1.258\frac{1}{\text{s}^2}. \qquad \text{(same as the answer before.)} \end{aligned}$$

Since the sign of $\dot{\omega}$ is positive, our initial assumption of clockwise acceleration of the pulley is correct.

$$\boxed{\dot{\omega} = 1.26\,\text{rad/s}^2}$$

2. From kinematics of circular motion,

$$\begin{aligned} a_m &= (a_D)_\theta \\ &= \dot{\omega} R_i \\ &= 0.126\,\text{m/s}^2. \end{aligned}$$

$$\boxed{a_m = 0.13\,\text{m/s}^2}$$

SAMPLE 13.28 **Energy Accounting:** Consider the pulley problem of Sample 13.26 again.

1. What percentage of the input energy (work done by the applied force F) is used in raising the mass by 1 m?
2. Where does the rest of the energy go? Provide an energy-balance sheet.

Solution

1. Let W_i and W_h be the input energy and the energy used in raising the mass by 1 m, respectively. Then the percentage of energy used in raising the mass is

$$\% \text{ of input energy used} = \frac{W_h}{W_i} \times 100.$$

Thus we need to calculate W_i and W_h to find the answer. W_i is the work done by the force F on the system during the interval in which the mass moves up by 1 m. Let s be the displacement of the force F during this interval. Since the displacement is in the same direction as the force (we know it is from Sample 13.26), the input-energy is

$$W_i = F\,s.$$

So to find W_i we need to find s.
For the mass to move up by 1 m the inner drum B must rotate by an angle θ where

$$1\,\text{m} = \theta\,R_i \qquad \Rightarrow \qquad \theta = \frac{1\,\text{m}}{0.1\,\text{m}} = 10\,\text{rad}.$$

Since the two drums, A and B, are welded together, drum A must rotate by θ as well. Therefore the displacement of force F is

$$s = \theta\,R_o = 10\,\text{rad} \cdot 0.2\,\text{m} = 2\,\text{m},$$

and the energy input is

$$W_i = F\,s = 20\,\text{N} \cdot 2\,\text{m} = 40\text{J}.$$

Now, the work done in raising the mass by 1 m is

$$W_h = mgh = 2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot 1\,\text{m} = 19.62\text{J}.$$

Therefore, the percentage of input-energy used in raising the mass

$$= \frac{19.62\,\text{N·m}}{40} \times 100 = 49.05\% \approx 49\%.$$

2. The rest of the energy ($= 51\%$) goes in accelerating the mass and the pulley. Let us find out how much energy goes into each of these activities. Since the initial state of the system from which we begin energy accounting is not prescribed (that is, we are not given the height of the mass from which it is to be raised 1 m, nor do we know the velocities of the mass or the pulley at that initial height), let us assume that at the initial state, the angular speed of the pulley is ω_o and the linear speed of the mass is v_o. At the end of raising the mass by 1 m from this state, let the angular speed of the pulley be ω_f and the linear speed of the mass be v_f. Then, the energy used in accelerating

the pulley is

$$
\begin{aligned}
(\Delta E_{\mathrm{K}})_{\text{pulley}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
&= \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_o^2 \\
&= \frac{1}{2} I \underbrace{(\omega_f^2 - \omega_o^2)}_{\boxed{\text{assuming constant acceleration, } \omega_f^2 = \omega_o^2 + 2\alpha\theta, \text{ or } \omega_f^2 - \omega_o^2 = 2\alpha\theta.}} \\
&= I\alpha\,\theta \qquad \text{(from Sample 13.27, } \alpha = 1.258\,\mathrm{rad/s^2}.\text{)} \\
&= 1.6\,\mathrm{kg\,m^2} \cdot 1.258\,\mathrm{rad/s^2} \cdot 10\,\mathrm{rad} \\
&= 20.13\,\mathrm{N{\cdot}m} = 20.13\,\mathrm{J}.
\end{aligned}
$$

Similarly, the energy used in accelerating the mass is

$$
\begin{aligned}
(\Delta E_{\mathrm{K}})_{\text{mass}} &= \text{final kinetic energy} - \text{initial kinetic energy} \\
&= \frac{1}{2} m v_f^2 - \frac{1}{2} m v_o^2 \\
&= \frac{1}{2} m \underbrace{(v_f^2 - v_o^2)}_{2ah} \\
&= mah \\
&= 2\,\mathrm{kg} \cdot 0.126\,\mathrm{m/s^2} \cdot 1\,\mathrm{m} \\
&= 0.25\,\mathrm{J}.
\end{aligned}
$$

We can calculate the percentage of input energy used in these activities to get a better idea of energy allocation. Here is the summary table:

Activities	**Energy Spent**	
	in Joule	as % of input energy
In raising the mass by 1 m	19.62	49.05%
In accelerating the mass	0.25	0.62 %
In accelerating the pulley	20.13	50.33 %
Total	40.00	100 %

So, what would you change in the set-up so that more of the input energy is used in raising the mass? Think about what aspects of the motion would change due to your proposed design.

SAMPLE 13.29 Equation of motion of a swinging stick: A uniform bar of mass m and length ℓ is pinned at one of its ends O. The bar is displaced from its vertical position by an angle θ and released (Fig. 13.93).

1. Find the equation of motion using momentum balance.
2. Find the reaction at O as a function of $(\theta,\ \dot{\theta},\ g,\ m,\ \ell)$.

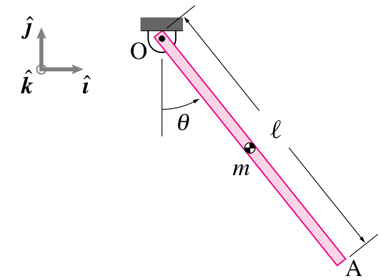

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Figure 13.93: A uniform rod swings in the plane about its pinned end O.

Solution First we draw a simple sketch of the given problem showing relevant geometry (Fig. 13.93(a)), and then a free-body diagram of the bar (Fig. 13.93(b)).

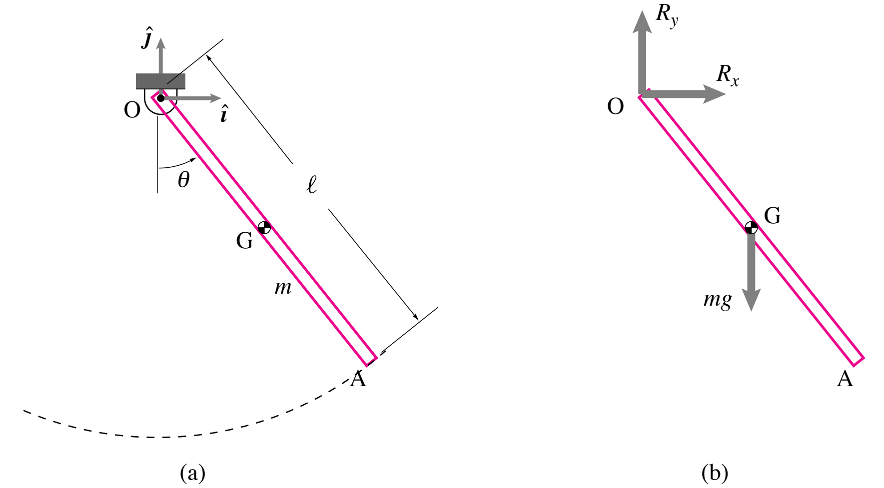

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Figure 13.92: (a) A line sketch of the swinging rod and (b) free-body diagram of the rod.

We should note for future reference that

$$\begin{aligned}\vec{\omega} &= \omega\hat{k} \equiv \dot{\theta}\hat{k}\\ \dot{\vec{\omega}} &= \dot{\omega}\hat{k} \equiv \ddot{\theta}\hat{k}\end{aligned}$$

1. **Equation of motion using momentum balance:** We can write angular momentum balance about point O as

$$\sum \vec{M}_{\mathrm{O}} = \dot{\vec{H}}_{/\mathrm{O}}.$$

Let us now calculate both sides of this equation:

$$\begin{aligned}\sum \vec{M}_{\mathrm{O}} &= \vec{r}_{G/O} \times mg(-\hat{\jmath})\\ &= \frac{\ell}{2}(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \times mg(-\hat{\jmath})\\ &= -\frac{\ell}{2}mg\ \sin\theta\hat{k}. \qquad (13.80)\end{aligned}$$

$$\begin{aligned}\dot{\vec{H}}_{/\mathrm{O}} &= \dot{\omega}\hat{k}\int_m r^2 dm\\ &= \ddot{\theta}\hat{k}\int_0^{\ell} s^2\frac{m}{\ell}ds \quad (dm = m/\ell\cdot ds)\\ &= \frac{m\ddot{\theta}}{\ell}\left[\left.\frac{s^3}{3}\right|_0^{\ell}\right] = \frac{m\ell^2}{3}\ddot{\theta}\hat{k} \qquad (13.81)\end{aligned}$$

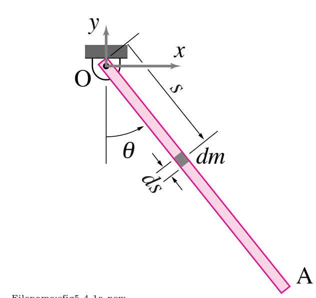

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Figure 13.94: Computation of $\dot{\vec{H}}_{/\mathrm{O}}$ by integration over the rod.

Equating (13.80) and (13.81) we get

$$
\begin{aligned}
-\frac{\ell}{2}\cancel{m}\, g\, \sin\theta &= \cancel{m}\frac{\ell^2}{3}\ddot{\theta} \\
\text{or} \qquad \dot{\omega} + \frac{3g}{2\ell}\sin\theta &= 0 \\
\text{or} \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \qquad (13.82)
\end{aligned}
$$

$$\boxed{\ddot{\theta} + \tfrac{3g}{2\ell}\sin\theta = 0}$$

2. **Reaction at O:** Using linear momentum balance

$$
\begin{aligned}
\sum \vec{F} &= m\vec{a}_G, \\
\text{where} \quad \sum \vec{F} &= R_x\hat{\imath} + (R_y - mg)\hat{\jmath}, \\
\text{and} \quad \vec{a}_G &= \frac{\ell}{2}\dot{\omega}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + \frac{\ell}{2}\omega^2(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) \\
&= \frac{\ell}{2}[(\dot{\omega}\cos\theta - \omega^2\sin\theta)\hat{\imath} + (\dot{\omega}\sin\theta + \omega^2\cos\theta)\hat{\jmath}].
\end{aligned}
$$

Dotting both sides of $\sum \vec{F} = m\vec{a}_G$ with $\hat{\imath}$ and $\hat{\jmath}$ and rearranging, we get

$$
\begin{aligned}
R_x &= m\frac{\ell}{2}(\dot{\omega}\cos\theta - \omega^2\sin\theta) \\
&\equiv m\frac{\ell}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \\
R_y &= mg + m\frac{\ell}{2}(\dot{\omega}\sin\theta + \omega^2\cos\theta) \\
&\equiv mg + m\frac{\ell}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta).
\end{aligned}
$$

Now substituting the expression for $\ddot{\theta}$ from (13.82) in R_x and R_y, we get

$$
\begin{aligned}
R_x &= -m\sin\theta\left(\frac{3}{4}g\cos\theta + \frac{\ell}{2}\dot{\theta}^2\right), \qquad (13.83) \\
R_y &= mg\left(1 - \frac{3}{4}\sin^2\theta\right) + m\frac{\ell}{2}\dot{\theta}^2\cos\theta. \qquad (13.84)
\end{aligned}
$$

$$\boxed{\vec{R} = -m(\tfrac{3}{4}g\cos\theta + \tfrac{\ell}{2}\dot{\theta}^2)\sin\theta\hat{\imath} + [mg(1 - \tfrac{3}{4}\sin^2\theta) + m\tfrac{\ell}{2}\dot{\theta}^2\cos\theta]\hat{\jmath}}$$

Check: We can check the reaction force in the special case when the rod does not swing but just hangs from point O. The forces on the bar in this case have to satisfy static equilibrium. Therefore, the reaction at O must be equal to mg and directed vertically upwards. Plugging $\theta = 0$ and $\dot{\theta} = 0$ (no motion) in Eqn. (13.83) and (13.84) we get $R_x = 0$ and $R_y = mg$, the values we expect.

SAMPLE 13.30 Swinging stick dynamics using moment of inertia: A uniform bar of mass m and length ℓ is pinned at one of its ends O. The bar is displaced from its vertical position by an angle θ and released (Fig. 13.95). Find the equation of motion of the stick.

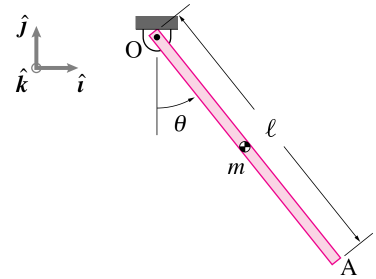

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Figure 13.95: A uniform rod swings in the plane about its pinned end O.

Solution We repeat the problem solved in Sample 13.29 here with just one different step of finding the rate of change of angular momentum with the help of moment of inertia formula. As usual, we first draw a free-body diagram of the bar (Fig. 13.96). We assume, $\vec{\omega} = \omega\hat{k} \equiv \dot{\theta}\hat{k}$, and $\dot{\vec{\omega}} = \dot{\omega}\hat{k} \equiv \ddot{\theta}\hat{k}$ We can write angular momentum balance about point O as

$$\sum \vec{M}_{\mathrm{O}} = \dot{\vec{H}}_{/\mathrm{O}}.$$

Let us now calculate both sides of this equation:

$$\begin{aligned}
\sum \vec{M}_{\mathrm{O}} &= \vec{r}_{G/O} \times mg(-\hat{\jmath}) \\
&= \frac{\ell}{2}(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \times mg(-\hat{\jmath}) \\
&= -\frac{\ell}{2}mg\,\sin\theta\hat{k}. && (13.85) \\
\dot{\vec{H}}_{/\mathrm{O}} &= I_{zz/G}\,\dot{\vec{\omega}} + \vec{r}_G \times m\vec{a}_G \\
&= \frac{m\ell^2}{12}\dot{\omega}\hat{k} + \vec{r}_G \times m\overbrace{(\dot{\omega}\hat{k} \times \vec{r}_G - \omega^2\vec{r}_G)}^{\vec{a}_G} \\
&= \frac{m\ell^2}{12}\dot{\omega}\hat{k} + \frac{m\ell^2}{4}\dot{\omega}\hat{k} && (13.86) \\
&= \frac{m\ell^2}{3}\dot{\omega}\hat{k} && (13.87)
\end{aligned}$$

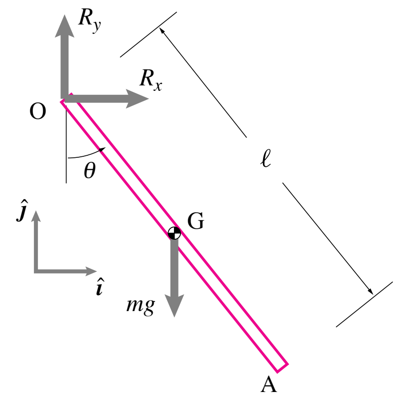

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Figure 13.96: The free-body diagram of the rod.

where the last step, $\vec{r}_G \times m\vec{a}_G = \frac{m\ell^2}{4}\dot{\omega}\hat{k}$, should be clear from Fig. 13.97. Equating (13.80) and (13.81) we get

$$\begin{aligned}
-\frac{\ell}{2}\cancel{m}\,g\,\sin\theta &= \cancel{m}\frac{\ell^2}{3}\dot{\omega} \\
\text{or} \qquad \dot{\omega} + \frac{3g}{2\ell}\sin\theta &= 0 \\
\text{or} \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. && (13.88)
\end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0}$$

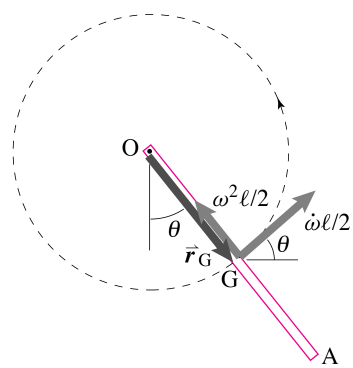

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Figure 13.97: Radial and tangential components of $\vec{a}_G$. Since the radial component is parallel to $\vec{r}_G$, $\vec{r}_G \times \vec{a}_G = \frac{\ell^2}{4}\dot{\omega}\hat{k}$.

SAMPLE 13.31 Numerical solution of the swinging stick motion: For the swinging stick considered in Samples 13.29 or 13.32, find the time that the rod takes to fall from $\theta = \pi/2$ to $\theta = 0$ if it is released from rest at $\theta = \pi/2$?

Solution The given initial angle $\pi/2$ is a big value of θ – big in that we cannot assume $\sin\theta \approx \theta$ (obviously $1 \neq 1.5708$). Therefore we may not use the linearized equation (13.91) to solve for t explicitly. We have to solve the full nonlinear equation (13.88) to find the required time. Unfortunately, we cannot get a *closed form* solution of this equation using mathematical skills you have at this level. Therefore, we resort to numerical integration of this equation.

For numerical integration, we need to first write the given differential equation as a set of first order ordinary differential equations. To do so, we introduce ω as a new variable and rewrite eqn. (13.88) as

$$\dot{\theta} = \omega \qquad (13.89)$$

$$\dot{\omega} = -\frac{3g}{2\ell}\sin\theta \qquad (13.90)$$

Now we need to specify the initial conditions and the time duration for integration, and solve the equations using some ODE solver program. Here is a pseudo-code that lists the steps:

```
g = 9.81,    L = 1       % define constants
ODES = { thetadot = omega
         omegadot = -3*g/(2*L) * sin(theta) }
 ICs = { theta_0 = pi/2
         omega_0 = 0 }
solve ODES with ICs for t = 0 to 4 s
plot theta vs t and plot omega vs t
```

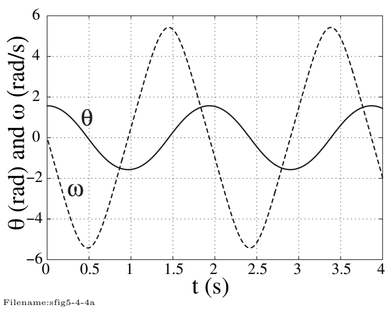


Figure 13.98: Numerical solution is shown by plotting θ and ω against time.

The results obtained from the numerical solution are shown in Fig. 13.98.

The problem of finding the time taken by the bar to fall from $\theta = \pi/2$ to $\theta = 0$ numerically is nontrivial. It is called a *boundary value problem*. We have only illustrated how to solve *initial value problems.* However, we can get fairly good estimate of the time from the solution obtained.

We first plot θ against time t as shown in fig. 13.99. We find the values of t and the corresponding values of θ that bracket $\theta = 0$. Now, we can use linear interpolation to find the value of t at $\theta = 0$. Proceeding this way, we get $t = 0.48$ (seconds), a little more than we get from the linear ODE in sample 13.32 ($t = 0.41$).

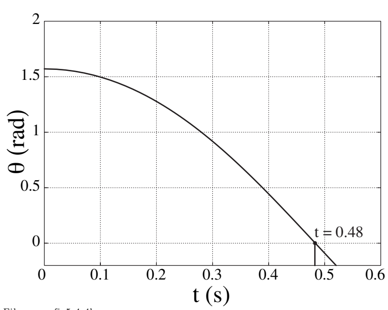


Figure 13.99: Time t corresponding to $\theta = 0$.

Comments: Additionally, we can also get the value of $\dot{\theta} \equiv \omega$ when $\theta = 0$ using similar interpolation. In fact, from the ω vs t plot, we find that at $t = 0.48\,\mathrm{s}$, $\omega = -5.42\,\mathrm{rad/s}$. How does this result compare with the analytical value of ω from sample 13.32 (which did not depend on the small angle approximation)? Well, we found that

$$\omega = -\sqrt{\frac{3g}{\ell}} = -\sqrt{\frac{3 \cdot 9.81\,\mathrm{m/s^2}}{1\,\mathrm{m}}} = -5.4249\,\mathrm{s}^{-1}.$$

Thus, we get a fairly accurate value from numerical integration.

SAMPLE 13.32 The swinging stick dynamics with energy balance: Consider the same swinging stick as in Sample 13.29. The stick is, again, displaced from its vertical position by an angle θ and released (See Fig. 13.93).

1. Find the equation of motion using energy balance.
2. What is $\dot{\theta}$ at $\theta = 0$ if $\theta(t = 0) = \pi/2$?
3. Find the period of small oscillations about $\theta = 0$.

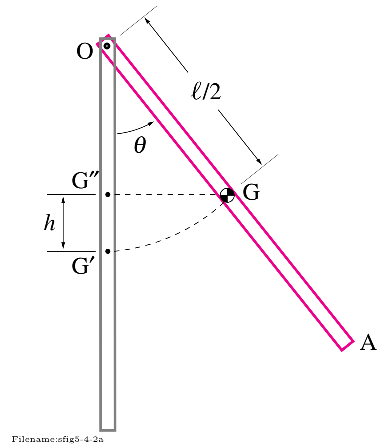


Figure 13.100: Work done by the force of gravity in moving from G′ to G $\int \vec{F} \cdot d\vec{r} = -mg\hat{\jmath} \cdot h\hat{\jmath} = -mgh$.

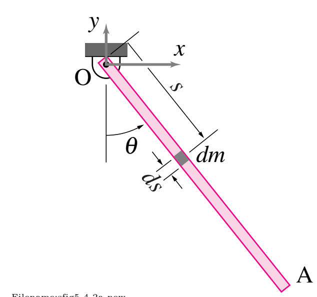


Figure 13.101: The infinitesimal mass dm considered in the calculation of E_K.

Solution

1. **Equation of motion using energy balance:** We use the power equation, $\dot{E}_K = P$, to derive the equation of motion of the bar. Now, the kinetic energy is given by

$$E_K = \frac{1}{2}\int_m v^2 dm$$

where v is the speed of the infinitesimal mass element dm. Refering to fig. 13.101, we can write, $dm = (m/\ell)ds$, and $v = \omega s \equiv \dot{\theta}s$. Thus,

$$\begin{aligned} E_K &= \frac{1}{2}\int_0^\ell \dot{\theta}^2 s^2 \frac{m}{\ell} ds \\ &= \frac{m\dot{\theta}^2}{2\ell}\int_0^\ell s^2 ds \\ &= \frac{1}{6}m\ell^2\dot{\theta}^2 \end{aligned}$$

and, therefore,

$$\dot{E}_K = \frac{d}{dt}(\frac{1}{6}m\ell^2\omega^2) = \frac{1}{3}m\ell^2\omega\,\dot{\omega} = \frac{1}{3}m\ell^2\dot{\theta}\ddot{\theta}.$$

Calculation of power (P): There are only two forces acting on the bar, the reaction force, $\vec{R}(= R_x\hat{\imath} + R_y\hat{\jmath})$ and the force due to gravity, $-mg\hat{\jmath}$. Since the support point O does not move, no work is done by $\vec{R}$. Therefore,

$$W \quad = \quad \text{Work done by gravity force in moving from G}' \text{ to G.} = -mgh$$

Note that the negative sign stands for the work done *against* gravity. Now,

$$h = OG' - OG'' = \frac{\ell}{2} - \frac{\ell}{2}\cos\theta = \frac{\ell}{2}(1 - \cos\theta).$$

Therefore,

$$\begin{aligned} W &= -mg\frac{\ell}{2}(1 - \cos\theta) \\ \text{and} \qquad P &= \dot{W} = \frac{dW}{dt} = -mg\frac{\ell}{2}\sin\theta\dot{\theta}. \end{aligned}$$

Equating $\dot{E}_K$ and P we get

$$\begin{aligned} -\cancel{m}g\frac{\ell}{2}\sin\theta\cancel{\dot{\theta}} &= \frac{1}{3}\cancel{m}\ell^2\cancel{\dot{\theta}}\ddot{\theta} \\ \text{or} \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0}$$

This equation is, of course, the same as we obtained using balance of angular momentum in Sample 13.29.

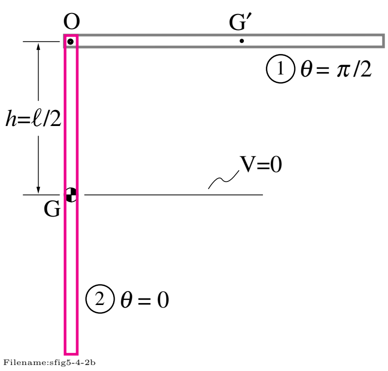


Figure 13.102: The total energy between positions (1) and (2) is constant.

2. **Find ω at $\theta = 0$:** We are given that at $t = 0$, $\theta = \pi/2$ and $\dot{\theta} \equiv \omega = 0$ (released from rest). This position is (1) shown in Fig. 13.102. In position (2) $\theta = 0$, i.e., the rod is vertical. Since there are no dissipative forces, the total energy of the system remains constant. Therefore, taking datum for potential energy as shown in Fig. 13.102, we may write

$$
\begin{aligned}
\underbrace{E_{K1}}_{0} + V_1 &= E_{K2} + \underbrace{V_2}_{0} \\
\text{or} \qquad mg\frac{\ell}{2} &= \frac{1}{2}\int_m v^2 dm \\
&= \frac{1}{6}m\ell^2\omega^2 \quad \text{(see part (a))} \\
\Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}}
\end{aligned}
$$

$$\boxed{\omega = \pm\sqrt{\frac{3g}{\ell}}}$$

3. **Period of small oscillations:** The equation of motion is

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0.$$

For small θ, $\sin\theta \approx \theta$

$$
\begin{aligned}
\Rightarrow \qquad & \ddot{\theta} + \tfrac{3g}{2\ell}\theta = 0 \qquad (13.91) \\
\text{or} \qquad & \ddot{\theta} + \lambda^2\theta = 0 \\
\text{where} \qquad & \lambda^2 = \tfrac{3g}{2\ell}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\text{the circular frequency} &= \lambda = \sqrt{\frac{3g}{2\ell}}, \\
\text{and the time period T} &= \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{2\ell}{3g}}.
\end{aligned}
$$

$$\boxed{T = 2\pi\sqrt{\frac{2\ell}{3g}}}$$

[Say for $g = 9.81\,\text{m/s}^2, \ell = 1\,\text{m}$ we get $\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{2}{3}\frac{1}{9.81}}\,\text{s} = 0.4097\,\text{s}$]

SAMPLE 13.33 The swinging stick with a destabilizing torque. Consider the swinging stick of Sample 13.29 once again.

1. Find the equation of motion of the stick, if a torque $\vec{M} = M\hat{k}$ is applied at end O and a force $\vec{F} = F\hat{\imath}$ is applied at the other end A.
2. Take $F = 0$ and $M = C\theta$. For $C = 0$ you get the equation of free oscillations obtained in Sample 13.29 or 13.32 For small C, does the period of the pendulum increase or decrease?
3. What happens if C is big?

Solution

1. A free body diagram of the bar is shown in Fig. 13.103. Once again, we can use $\sum \vec{M}_{\text{O}} = \dot{\vec{H}}_{/\text{O}}$ to derive the equation of motion as in Sample 13.29. We calculated $\sum \vec{M}_{\text{O}}$ and $\dot{\vec{H}}_{/\text{O}}$ in Sample 13.29. Calculation of $\dot{\vec{H}}_{/\text{O}}$ remains the same in the present problem. We only need to recalculate $\sum \vec{M}_{\text{O}}$.

$$\begin{aligned}\sum \vec{M}_{\text{O}} &= M\hat{k} + \vec{r}_{G/O} \times mg(-\hat{\jmath}) + \vec{r}_{A/O} \times \vec{F}\\ &= M\hat{k} - \frac{\ell}{2} mg\sin\theta\hat{k} + F\ell\cos\theta\hat{k}\\ &= (M + F\ell\cos\theta - \frac{\ell}{2}mg\sin\theta)\hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_{/\text{O}} = m\ddot{\theta}\frac{\ell^2}{3}\hat{k} \quad \text{(see Sample 13.29)}$$

Therefore, from $\sum \vec{M}_{\text{O}} = \dot{\vec{H}}_{/\text{O}}$

$$\begin{aligned} M + F\ell\cos\theta - \frac{\ell}{2}mg\sin\theta &= m\ddot{\theta}\frac{\ell^2}{3}\\ \Rightarrow \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3F}{m\ell}\cos\theta - \frac{3M}{m\ell^2} &= 0.\end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3F}{m\ell}\cos\theta - \frac{3M}{m\ell^2} = 0}$$

2. Now, setting $F = 0$ and $M = C\,\theta$ we get

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3C\theta}{m\ell^2} = 0 \qquad (13.92)$$

Numerical Solution: We can numerically integrate (13.92) just as in the previous Sample to find $\theta(t)$. Here is the pseudo-code that can be used for this purpose.

```
g = 9.81, L = 1       % specify parameters
m = 1, C = 4
ODES = { thetadot = omega
         omegadot = -(3*g/(2*L)) * sin(theta)
                    + 3*C/(m*L^2) * theta      }
ICs =  { thetazero = pi/20
         omegazero = 0  }
solve ODES with ICs until t = 10
```

Using this pseudo-code, we find the response of the pendulum. Figure 13.104 shows different responses for various values of C. Note that for $C = 0$, it is the same case as unforced bar pendulum considered above. From Fig. 13.104 it is clear that the bar

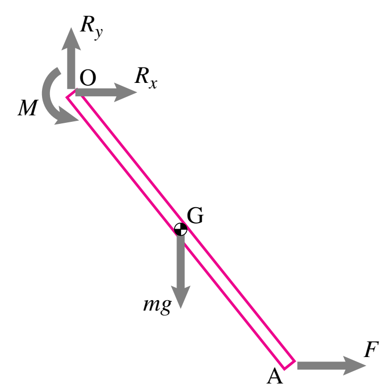


Figure 13.103: Free-body diagram of the bar with applied torque $\vec{M}$ and force $\vec{F}$

has periodic motion for small C, with the period of motion increasing with increasing values of C. It makes sense if you look at Eqn. (13.92) carefully. Gravity acts as a restoring force while the applied torque acts as a destabilizing force. Thus, with the resistance of the applied torque, the stick swings more sluggishly making its period of oscillation bigger.

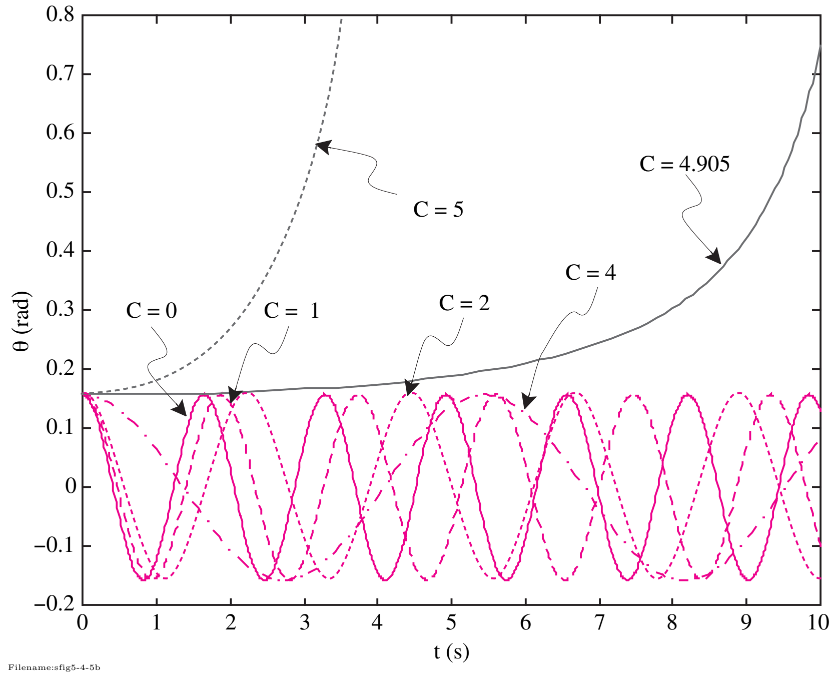


Figure 13.104: $\theta(t)$ with applied torque $M = C\,\theta$ for $C = 0,\ 1,\ 2,\ 4,\ 4.905,\ 5.$ Note that for small C the motion is periodic but for large C ($C \geq 4.4$) the motion becomes aperiodic.

3. From Fig. 13.104, we see that at about $C \approx 4.9$ the stability of the system changes completely. $\theta(t)$ is not periodic anymore. It keeps on increasing at faster and faster rate, that is, the bar makes complete loops about point O with ever increasing speed. Does it make physical sense? Yes, it does. As the value of C is increased beyond a certain value (can you guess the value?), the applied torque overcomes any restoring torque due to gravity. Consequently, the bar is forced to rotate continuously in the direction of the applied force.

SAMPLE 13.34 **A torsional pendulum with linear springs:** A uniform rigid bar of mass $m = 2\,\text{kg}$ and length $\ell = 1\,\text{m}$ is pinned at one end and connected to two springs, each with spring constant k, at the other end. The bar is tweaked slightly from its vertical position. It then oscillates about its original position. The bar is timed for 20 full oscillations which take 12.5 seconds. Ignore gravity.

1. Find the equation of motion of the rod.
2. Find the spring constant k.
3. What should be the spring constant of a torsional spring if the bar is attached to one at the bottom and has the same oscillating motion characteristics?

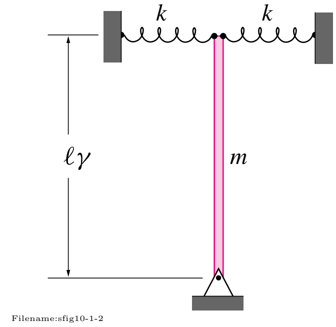


Figure 13.105:

Solution

1. Refer to the free-body diagram in figure 13.106. Angular momentum balance for the rod about point O gives

$$\sum \vec{M}_O = \dot{\vec{H}}_{/O}$$

$$\begin{aligned} \text{where} \quad \vec{M}_O &= -2k \overbrace{x}^{\ell \sin\theta} \cdot \ell \cos\theta \hat{k} \\ &= -2k\ell^2 \sin\theta \, \cos\theta \hat{k}, \\ \text{and} \quad \dot{\vec{H}}_{/O} &= I_{zz}^O \ddot{\theta} \hat{k} = \underbrace{\frac{1}{3} m\ell^2}_{I_{zz}^O} \ddot{\theta} \hat{k}. \end{aligned}$$

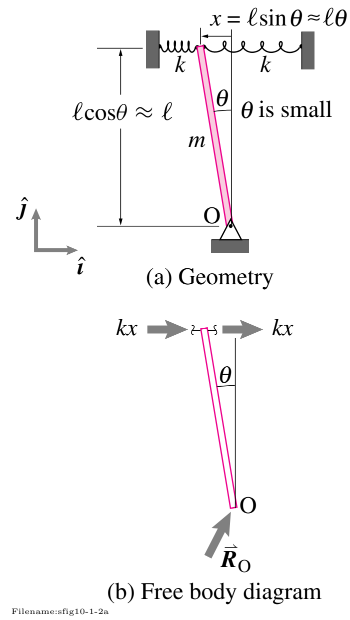
Thus

$$\frac{1}{3} m\ell^2 \, \ddot{\theta} = -2k\ell^2 \sin\theta \cos\theta.$$

However, for small θ, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$,

$$\Rightarrow \quad \ddot{\theta} + \frac{6k\cancel{\ell^2}}{m\cancel{\ell^2}} \theta = 0. \tag{13.93}$$

$$\boxed{\ddot{\theta} + \frac{6k}{m}\theta = 0}$$



(a) Geometry

(b) Free body diagram

Figure 13.106:

2. Comparing Eqn. (13.93) with the standard harmonic oscillator equation $\ddot{x} + \lambda^2 x = 0$, we get

$$\begin{aligned} \text{angular frequency} \quad \lambda &= \sqrt{\frac{6k}{m}}, \\ \text{and the time period} \quad T &= \frac{2\pi}{\lambda} \\ &= 2\pi \sqrt{\frac{m}{6k}}. \end{aligned}$$

From the measured time for 20 oscillations, the time period (time for one oscillation) is

$$T = \frac{12.5}{20}\,\text{s} = 0.625\,\text{s}$$

Now equating the measured T with the derived expression for T we get

$$\begin{aligned} 2\pi\sqrt{\frac{m}{6k}} &= 0.625\,\mathrm{s} \\ \Rightarrow \qquad k &= 4\pi^2 \cdot \frac{m}{6(0.625\,\mathrm{s})^2} \\ &= \frac{4\pi^2 \cdot 2\,\mathrm{kg}}{6(0.625\,\mathrm{s})^2} \\ &= 33.7\,\mathrm{N/\,m}. \end{aligned}$$

$\boxed{k = 33.7\,\mathrm{N/\,m}}$

3. If the two linear springs are to be replaced by a torsional spring at the bottom, we can find the spring constant of the torsional spring by comparison. Let k_{tor} be the spring constant of the torsional spring. Then, as shown in the free body diagram (see figure 13.107), the restoring torque applied by the spring at an angular displacement θ is $k_{\mathrm{tor}}\theta$. Now, writing the angular momentum balance about point O, we get

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \\ -k_{\mathrm{tor}}\theta\hat{\boldsymbol{k}} &= I_{zz}^{O}(\ddot{\theta}\hat{\boldsymbol{k}}) \\ \Rightarrow \qquad \ddot{\theta} + \frac{k_{\mathrm{tor}}}{I_{zz}^{O}}\theta &= 0. \end{aligned}$$

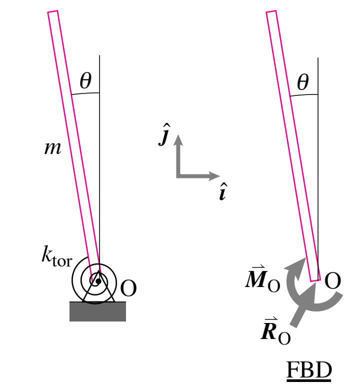


Figure 13.107:

Comparing with the standard harmonic equation, we find the angular frequency

$$\lambda = \sqrt{\frac{k_{\mathrm{tor}}}{I_{zz}^{O}}} = \sqrt{\frac{k_{\mathrm{tor}}}{\frac{1}{3}m\ell^2}}.$$

If this system has to have the same period of oscillation as the first system, the two angular frequencies must be equal, *i.e.,*

$$\begin{aligned} \sqrt{\frac{k_{\mathrm{tor}}}{\frac{1}{3}m\ell^2}} &= \sqrt{\frac{6k}{m}} \\ \Rightarrow \qquad k_{\mathrm{tor}} &= 6k \cdot \frac{1}{3}\ell^2 = 2k\ell^2 \\ &= 2 \cdot (33.7\,\mathrm{N/\,m}) \cdot (1\,\mathrm{m})^2 \\ &= 67.4\,\mathrm{N{\cdot}m}. \end{aligned}$$

$\boxed{k_{\mathrm{tor}} = 67.4\,\mathrm{N{\cdot}m}}$

SAMPLE 13.35 **A spring loaded seesaw:** A kid, modelled as a point mass with $m = 10\,\text{kg}$, is sitting at end B of a rigid rod AB of negligible mass. The rod is supported by a spring at end A and a pin at point O. The system is in static equilibrium when the rod is horizontal. Someone pushes the kid vertically downwards by a small distance y and lets go. Given that $AB = 3\,\text{m}$, $AC = 0.5\,\text{m}$, $k = 1\,\text{kN/m}$; find

1. the unstretched (relaxed) length of the spring,
2. the equation of motion (a differential equation relating the position of the mass to its acceleration) of the system, and
3. the natural frequency of the system.

If the rod is pinned at the midpoint instead of at O, what is the natural frequency of the system? How does the new natural frequency compare with that of a mass m simply suspended by a spring with the same spring constant?

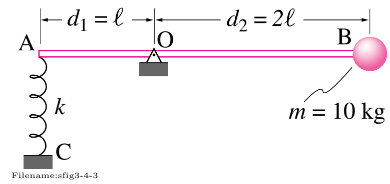


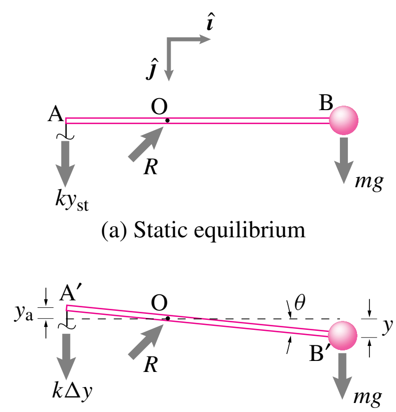
Figure 13.108:

Solution

1. **Static Equilibrium:** The FBD of the (rod + mass) system is shown in Fig. 13.109. Let the stretch in the spring in this position be y_{st} and the relaxed length of the spring be ℓ_0. The balance of angular momentum about point O gives:

$$
\begin{aligned}
\sum \vec{M}_{/\mathrm{o}} = \dot{\vec{H}}_{/\mathrm{o}} &= \vec{0} \qquad \text{(no motion)} \\
\Rightarrow \qquad (k y_{st}) d_1 - (mg) d_2 &= 0 \\
\Rightarrow \qquad y_{st} &= \frac{mg}{k} \cdot \frac{d_2}{d_1} \\
&= \frac{10\,\text{kg} \cdot 9.8\,\text{m/s}^2 \cdot 2\ell}{1000\,\text{N/m} \cdot \ell} = 0.196\,\text{m} \\
\text{Therefore,} \qquad \ell_0 &= AC - y_{st} \\
&= 0.5\,\text{m} - 0.196\,\text{m} = 0.304\,\text{m}.
\end{aligned}
$$

$$\boxed{\ell_0 = 30.4\,\text{cm}}$$



(a) Static equilibrium

(b) Mass m is displaced downwards

Figure 13.109: Free body diagrams

2. **Equation of motion:** As point B gets displaced downwards by a distance y, point A moves up by a proportionate distance y_a. From geometry, ⑥

$$
\begin{aligned}
y &\approx d_2\theta \qquad \Rightarrow \qquad \theta = \frac{y}{d_2} \\
y_a &\approx d_1\theta = \frac{d_1}{d_2} y
\end{aligned}
$$

Therefore, the total stretch in the spring, in this position,

$$\Delta y = y_a + y_{st} = \frac{d_1}{d_2} y + \frac{d_2}{d_1}\frac{mg}{k}$$

Now, Angular Momentum Balance about point O gives:

$$
\begin{aligned}
\sum \vec{M}_{/\mathrm{o}} &= \dot{\vec{H}}_{/\mathrm{o}} \\
\sum \vec{M}_{/\mathrm{o}} &= \vec{r}_B \times mg\,\hat{\boldsymbol{\jmath}} + \vec{r}_A \times k\Delta y\,\hat{\boldsymbol{\jmath}} \\
&= (d_2 mg - d_1 k \Delta y)\hat{\boldsymbol{k}} \qquad (13.94) \\
\dot{\vec{H}}_{/\mathrm{o}} &= \vec{r}_B \times m\vec{a} = \vec{r}_B \times m\ddot{y}\,\hat{\boldsymbol{\jmath}} \qquad (13.95) \\
&= d_2 m \ddot{y}\,\hat{\boldsymbol{k}} \qquad (13.96)
\end{aligned}
$$

⑥ Here, we are considering a very small y so that we can ignore the arc the point mass B moves on and take its motion to be just vertical (i.e., $\sin\theta \approx \theta$ for small θ).

Equating (13.94) and (13.96) we get

$$
\begin{aligned}
d_2 mg - d_1 k \Delta y &= d_2 m\ddot{y} \\
\text{or} \qquad d_2 mg - d_1 k \left(\frac{d_1}{d_2} y + \frac{d_2 mg}{d_1 k} \right) &= d_2 m\ddot{y} \\
\text{or} \qquad d_2 \cancel{m} g - k \frac{d_1^2}{d_2} y - d_2 \cancel{m} g &= d_2 m\ddot{y} \\
\text{or} \qquad \ddot{y} + \frac{k}{m} \left(\frac{d_1}{d_2} \right)^2 y &= 0
\end{aligned}
$$

$$\boxed{\ddot{y} + \frac{k}{m} \left(\frac{d_1}{d_2} \right)^2 y = 0}$$

3. **The natural frequency of the system:** We may also write the previous equation as

$$\ddot{y} + \lambda^2 y = 0 \qquad \text{where} \qquad \lambda^2 = \frac{k}{m} \frac{d_1^2}{d_2^2}. \tag{13.97}$$

Substituting $d_1 = \ell$ and $d_2 = 2\ell$ in the expression for λ we get the natural frequency of the system

$$\lambda = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}\sqrt{\frac{1000\,\mathrm{N/\,m}}{10\,\mathrm{kg}}} = 5\,\mathrm{s}^{-1}$$

$$\boxed{\lambda = 5\,\mathrm{s}^{-1}}$$

4. **Comparison with a simple spring mass system:**

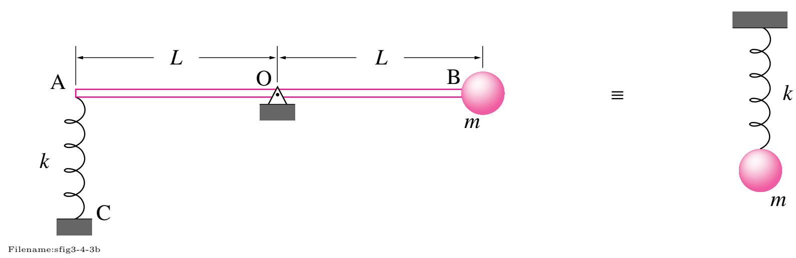


Figure 13.110:

When $d_1 = d_2$, the equation of motion (13.97) becomes

$$\ddot{y} + \frac{k}{m} y = 0$$

and the natural frequency of the system is simply

$$\lambda = \sqrt{\frac{k}{m}}$$

which corresponds to the natural frequency of a simple spring mass system shown in Fig. 13.110.

In our system (with $d_1 = d_2$) any vertical displacement of the mass at B induces an equal amount of stretch or compression in the spring which is exactly the case in the simple spring-mass system. Therefore, the two systems are mechanically equivalent. Such equivalences are widely used in modeling complex physical systems with simpler mechanical models.

Problems for Chapter 13

Circular motion

13.1 Kinematics of a particle in circular motion

Preparatory Problems

13.1.1 A particle goes on a circular path with radius R making the angle $\theta = ct$ measured counter clockwise from the positive x axis. Assume $R = 5\,\text{cm}$ and $c = 2\pi\ \text{s}^{-1}$.

a) Plot the path.

b) What is the angular rate in revolutions per second?

c) Put a dot on the path for the location of the particle at $t = t^* = 1/6$ s.

d) What are the x and y coordinates of the particle position at $t = t^*$? Mark them on your plot.

e) Draw the vectors $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_R$ at $t = t^*$.

f) What are the x and y components of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ at $t = t^*$?

g) What are the R and θ components of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ at $t = t^*$?

h) Draw an arrow representing both the velocity and the acceleration at $t = t^*$.

i) Find the $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ components of position $\vec{\boldsymbol{r}}$, velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ at $t = t^*$.

j) Find the x and y components of position $\vec{\boldsymbol{r}}$, velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ at $t = t^*$. Find the velocity and acceleration two ways:

   1. Differentiate the position given as $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$.
   2. Differentiate the position give as $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{e}}_r$ and then convert the results to Cartesian coordinates.

13.1.2 A bead goes around a circular track of radius 1 ft at a constant speed. It makes it around the track in exactly 1 s.

a) Find the speed of the bead. Does this vary in time?

b) Find the magnitude of acceleration of the bead. Does this vary in time?

c) Is the magnitude of the acceleration the derivative of the speed (*i.e.,* $|\vec{\boldsymbol{a}}| \stackrel{?}{=} \frac{d}{dt}|\vec{\boldsymbol{v}}|$)

13.1.3 If a particle moves along a circle at constant rate (constant $\dot{\theta}$) following the equation

$$\vec{\boldsymbol{r}}(t) = R\cos(\dot{\theta}t)\hat{\boldsymbol{\imath}} + R\sin(\dot{\theta}t)\hat{\boldsymbol{\jmath}}$$

which of these things are true and why? If not true, explain why.

a) $\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$

b) $\vec{\boldsymbol{v}} =$ constant

c) $|\vec{\boldsymbol{v}}| =$ constant

d) $\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$

e) $\vec{\boldsymbol{a}} =$ constant

f) $|\vec{\boldsymbol{a}}| =$ constant

g) $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{a}}$

13.1.4 A particle moves according to:

$$\begin{aligned} x(t) &= R\cos(ct), \\ y(t) &= R\sin(ct). \end{aligned}$$

where $R = 1\,\text{m}$ and $c = 5\,\text{rad/s}$.

a) Show that the speed of the particle is constant.

b) How much time does the particle take to go from P at $(0, 1\,\text{m})$ to Q at $(1\,\text{m}, 0)$?

c) What is the acceleration of the particle at point Q?

More-Involved Problems

13.1.5 A 200 mm diameter gear rotates at a constant speed of 100 rpm.

a) What is the speed of a peripheral point on the gear?

b) If no point on the gear is to exceed the centripetal acceleration of $25\,\text{m/s}^2$, find the maximum allowable angular speed (in rpm) of the gear.

13.1.6 A particle is in circular motion in the xy-plane at the constant angular speed of $\dot{\theta} = 2\,\text{rad/s}$ at radius 0.5 m. At $t = 0$ the particle is at $\theta = 0$.

a) Draw the path and mark the position of the particle at $t = 0.5\,\text{s}$ and $t = 15\,\text{s}$.

b) Find the velocity and acceleration of the particle at $t = 0.5\,\text{s}$ and $t = 15\,\text{s}$.

.

13.1.7 A particle undergoes constant rate circular motion in the xy-plane. At some instant t_0, its velocity is $\vec{\boldsymbol{v}}(t_0) = -3\,\text{m/s}\hat{\boldsymbol{\imath}} + 4\,\text{m/s}\hat{\boldsymbol{\jmath}}$ and after 5 s the velocity is $v(t_0 + 5\,\text{s}) = (5/\sqrt{2})\,\text{m/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$. If the particle has not yet completed one revolution between the two instants, find

a) the angular speed of the particle,

b) the distance traveled by the particle in 5 s, and

c) the acceleration of the particle at the two instants.

13.1.8 A bead on a circular path of radius R in the xy-plane has rate of change of angular speed $\alpha = bt^2$. The bead starts from rest at $\theta = 0$.

a) What is the bead's angular position θ (measured from the positive x-axis) and angular speed $\dot{\theta}$ as a function of time ?

b) What is the angular speed as function of angular position?

13.1.9 A bead on a circular wire has an angular speed given by $\dot{\theta} = c\theta^{1/2}$. The bead starts from rest at $\theta = 0$. What is the angular position and speed of the bead as a function of time? [This problem is subtle because it has multiple solutions. One answer you can find with a quick guess. Another you can find by separation of variables. The full general solution is an appropriate mixture of these two.)] *

13.1.10 Solve $\ddot{\theta} = C$, given $\dot{\theta}(0) = \dot{\theta}_0$, $\theta(0) = \theta_0$ and that C is a constant. That is, find θ in terms of some or all of C, $\dot{\theta}_0$, θ_0 and t.

13.1.11 Given that $\ddot{\theta} - \lambda^2\theta = 0$, $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$, and $\lambda = 3/\,\text{s}$ find the value of θ at $t = 1\,\text{s}$.

13.1.12 Two runners run on a circular track side-by-side at the same constant angular rate $\dot{\theta} = 0.25\,\text{rad/s}$ about the center of the track. The inside runner is in a lane of radius $r_i = 35\,\text{m}$ and the outside runner is in a lane of radius $r_o = 37\,\text{m}$. What is the velocity of the outside runner relative to the inside runner?

13.1.13 A particle oscillates on the arc of a circle with radius R according to the equation $\theta = \theta_0 \cos(\lambda t)$. What are the conditions on R, θ_0, and λ so that the maximum acceleration in this motion occurs at $\theta = 0$. "Acceleration" here means the magnitude of the acceleration vector.

13.1.14 A particle moves on a circular arc starting from rest at $\theta_0 = 0$. As θ increases, the magnitude of the acceleration is constant. Assume, all in consistent units, that $R = 1$ and $|\vec{a}| = 1$.

a) Write the statement 'the magnitude of acceleration is constant' as an equation in terms of $\dot{\theta}$ and $\ddot{\theta}$.

b) Find a solution to the equation with the given initial conditions (analytically or numerically).

c) Find and plot $\dot{\theta}$ vs t and θ vs t.

d) In circular motion does $|\vec{a}|$=constant necessarily mean that the motion is at or is gradually approaching constant rate circular motion? Is so, why? If not show a counter-example.

13.1.15 A particle moves in circles so that its acceleration $\vec{a}$ always makes a fixed angle ϕ with the position vector $-\vec{r}$, with $0 \le \phi \le \pi/2$. For example, $\phi = 0$ would be constant rate circular motion. Assume $\phi = \pi/4$, $R = 1\,\text{m}$ and $\dot{\theta}_0 = 1\,\text{rad/s}$. How long does it take the particle to reach

a) the speed of sound ($\approx 300\,\text{m/s}$)?

b) the speed of light ($\approx 3 \cdot 10^8\,\text{m/s}$)?

c) ∞?

13.2 Dynamics of a particle in circular motion

Preparatory Problems

13.2.1 Force on a person standing on the equator. Find the magnitude of the total force acting on a 150 lbm person standing on the equator. The total force is the gravity force plus the force of the ground on the person (note that these two do not exactly cancel). Neglect the motion of the earth around the sun and of the sun around the solar system, etc. The radius of the earth is 3963 mi. Give your solution in both pounds (lbf) and Newtons (N). *

13.2.2 Consider a mass m in circular motion. Let $\sum \vec{F} = \sum F_r \hat{e}_r + \sum F_\theta \hat{e}_\theta$. Using $\sum \vec{F} = m\vec{a}$, express $\sum F_r$ and $\sum F_\theta$ in terms of some or all of $\theta, \dot{\theta}, \ddot{\theta}, , r$, and m.

13.2.3 Using $\sum \vec{F} = m\vec{a}$, find the expressions for $\sum F_x$ and $\sum F_y$ in terms of $\ddot{\theta}$, $\dot{\theta}$, r, and θ. [Hint $\hat{e}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$].

13.2.4 A bead of mass m goes around a circular path of radius R in the xy-plane with angular acceleration $\ddot{\theta} = ct^3$. The bead starts from rest at $\theta = 0$.

a) What is the angular momentum of the bead about the origin at $t = t_1$?

b) what is the rate of change of angular momentum about the origin at $t = t_1$?

c) What is the kinetic energy of the bead at $t = t_1$?

d) Does the kinetic energy increase, decrease, or remain constant with time? Why?

13.2.5 A 200 gm particle goes in circles about a fixed center at a constant speed $v = 1.5\,\text{m/s}$. It takes 7.5 s to go around the circle once.

a) Find the angular speed of the particle.

b) Find the magnitude of acceleration of the particle.

c) Take center of the circle to be the origin of a xy-coordinate system. Find the net force on the particle when it is at $\theta = 30^\circ$ from the x-axis.

13.2.6 A race car cruises on a circular track at a constant speed of 120 mph. It goes around the track once in three minutes. Find the magnitude of the centripetal force on the car. What applies this force on the car? Does the driver have any control over this force?

13.2.7 A particle moves on a counterclockwise, origin-centered circular path in the xy-plane at a constant rate. The radius of the circle is r, the mass of the particle is m, and the particle completes one revolution in time τ.

a) Neatly draw the following things:
   1. The path of the particle.
   2. A dot on the path when the particle is at $\theta = 0^\circ$, 90°, and 210°, where θ is measured from the x-axis (positive counter-clockwise).
   3. Arrows representing $\hat{e}_R$, $\hat{e}_\theta$, $\vec{v}$, and $\vec{a}$ at each of these points.

b) Calculate all of the quantities in part (3) above at the points defined in part (2), (represent vector quantities in terms of the cartesian base vectors $\hat{\imath}$ and $\hat{\jmath}$). *

c) If this motion was imposed by the tension in a string, what would that tension be? *

d) Is radial tension enough to maintain this motion or is another force needed to keep the motion going (assuming no friction) ? *

e) Again, if this motion was imposed by the tension in a string, what is F_x, the x component of the force in the string, when $\theta = 210^\circ$? Ignore gravity.

13.2.8 The velocity and acceleration of a 1 kg particle, undergoing constant rate circular motion, are known at some instant t:

$$\vec{v} = -10\,\text{m/s}(\hat{\imath}+\hat{\jmath}), \quad \vec{a} = 2\,\text{m/s}^2(\hat{\imath}-\hat{\jmath}).$$

a) Write the position of the particle at time t using $\hat{e}_R$ and $\hat{e}_\theta$ base vectors.

b) Find the net force on the particle at time t.

c) At some later time t^*, the net force on the particle is in the $-\hat{\jmath}$ direction. Find the elapsed time $t - t^*$.

d) After how much time does the force on the particle reverse its direction.

13.2.9 A particle of mass 3 kg moves in the xy-plane so that its position is given by

$$\vec{r}(t) = 4\,\text{m}\left[\cos\left(\frac{2\pi t}{s}\right)\hat{\imath} + \sin\left(\frac{2\pi t}{s}\right)\hat{\jmath}\right]$$

with respect to point O, the origin of a fixed cartesian coordinate system.

a) What is the path of the particle? Show how you know what the path is.

b) What is the angular velocity of the particle? Is it constant? Show how you know if it is constant or not.

c) What is the velocity of the particle in polar coordinates?

d) What is the speed of the particle at $t = 3\,\text{s}$?

e) What net force does it exert on its surroundings at $t = 0\,\text{s}$? Assume the x and y axes are fixed.

f) What is the angular momentum of the particle at $t = 3\,\text{s}$ about point O?

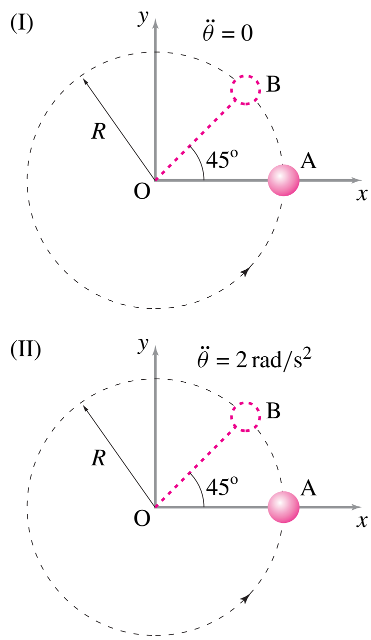
13.2.10 A comparison of constant and nonconstant rate circular motion. A 100 gm mass is going in circles of radius $R = 20\,\text{cm}$ at a constant rate $\dot{\theta} = 3\,\text{rad/s}$. Another identical mass is going in circles of the same radius but at a non-constant rate. The second mass is accelerating at $\ddot{\theta} = 2\,\text{rad/s}^2$ and at position A, it happens to have the same angular speed as the first mass.

a) Find and draw the accelerations of the two masses (call them I and II) at position A.

b) Find $\dot{\vec{H}}_{/O}$ for both masses at position A. *

c) Find $\vec{H}_{/O}$ for both masses at positions A and B. Do the changes in $\vec{H}_{/O}$ between the two positions reflect (qualitatively) the results obtained in (b)? *

d) If the masses are pinned to the center O by massless rigid rods, is tension in the rods enough to keep the two motions going? Explain.



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Problem 13.2.10

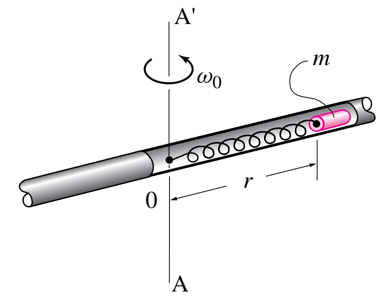
More-Involved Problems

13.2.11 A small mass m is connected to one end of a spring. The other end of the spring is fixed to the center of a circular track. The radius of the track is R, the unstretched length of the spring is ℓ_0 (with $\ell_0 < R$), and the spring constant is k.

a) With what speed should the mass be launched in the track so that it keeps going at a constant speed?

b) If the spring is replaced by another spring of same relaxed length but twice the stiffness, what will be the new required launch speed of the particle?

13.2.12 A bead of mass m is attached to a spring of stiffness k. The bead slides without friction in the tube shown. The tube is driven at a constant angular rate $\dot{\theta}_0$ about axis AA' by a motor (not pictured). There is no gravity. The unstretched spring length is r_0. Find the radial position r of the bead if it is stationary with respect to the rotating tube. *



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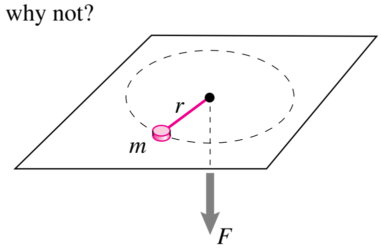
Problem 13.2.12

13.2.13 A particle of mass m is restrained by a string to move with a constant angular speed ω around a circle of radius R on a horizontal frictionless table. If the radius of the circle is reduced slowly to r, by pulling the string with a slowly varying force F through a hole in the table, what will the particle's angular velocity be in the final circular motion? Is kinetic energy changed in moving from circular motion at R to circular motion at r? Why or why not?



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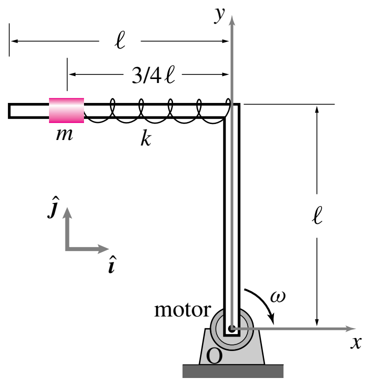
Problem 13.2.13

13.2.14 An 'L' shaped rigid, massless, and frictionless bar is made up of two uniform segments of length $\ell = 0.4\,\text{m}$ each. A collar of mass $m = 0.5\,\text{kg}$, attached to a spring at one end, slides frictionlessly on one of the arms of the 'L'. The spring is fixed to the elbow of the 'L' and has a spring constant $k = 6\,\text{N/m}$. The structure rotates clockwise at a constant rate $\omega = 2\,\text{rad/s}$. If the collar is steady at a distance $\frac{3}{4}\ell = 0.3\,\text{m}$ away from the elbow of the 'L', find the relaxed length of the spring, ℓ_0. Neglect gravity. *



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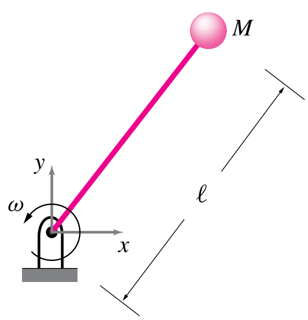
Problem 13.2.14: **Forces in constant rate circular motion.**

13.2.15 A massless rigid rod with length ℓ attached to a ball of mass M spins at a constant angular rate ω which is maintained by a motor (not shown) at the hinge point. The rod can only withstand a tension of T_{cr} before breaking. Find the maximum angular speed of the ball so that the rod does not break assuming

a) there is no gravity, and

b) there is gravity (neglect bending stresses).



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Problem 13.2.15

13.2.16 A 1 m long massless string has a particle of 10 grams mass at one end and is tied to a stationary point O at the other end. The particle rotates counter-clockwise in circles on a frictionless horizontal plane. The rotation rate is 2π rev/sec. Assume an xy-coordinate system in the plane with its origin at O.

a) Make a clear sketch of the system.

b) What is the tension in the string (in Newtons)? *

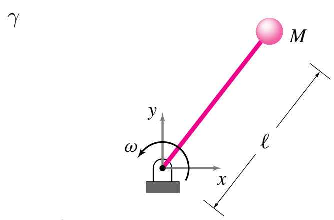
c) What is the angular momentum of the mass about O? *

d) When the string makes a 45° angle with the positive x and y axis on the plane, the string is quickly and cleanly cut. What is the position of the mass 1 sec later? Make a sketch of the particle's trajectory. *

13.2.17 A ball of mass M fixed to an inextensible rod of length ℓ and negligible mass rotates about a frictionless hinge as shown in the figure. A motor (not shown) at the hinge point accelerates the mass-rod system from rest by applying a constant torque M_O. The rod is initially lined up with the positive x-axis. The rod can only withstand a tension of T_{cr} before breaking. At what time will the rod break and after how many revolutions? Neglect bending stresses.

a) Neglect gravity.

b) Include gravity.



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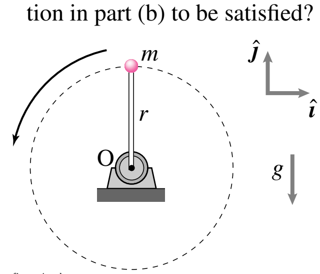
Problem 13.2.17

13.2.18 A particle of mass m, tied to one end of a rod whose other end is fixed at point O to a motor, moves in a circular path in the vertical plane at a constant rate. Gravity acts in the $-\hat{\jmath}$ direction.

a) Find the difference between the maximum and minimum tension in the rod. *

b) Find the ratio $\frac{\Delta T}{T_{max}}$ where $\Delta T = T_{max} - T_{min}$. A criterion for ignoring gravity might be if the variation in tension is less than 2% of the maximum tension; i.e., when $\frac{\Delta T}{T_{max}} < 0.02$. For a given length r of the rod, find the rotation rate ω for which this condition is met. *.

c) For $\omega = 300$ rpm, what would be the length of the rod for the condition in part (b) to be satisfied? *.



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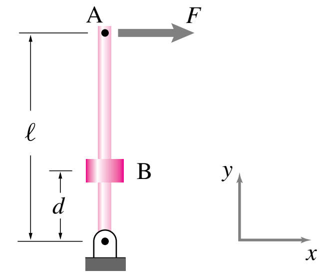
Problem 13.2.18

13.2.19 A massless rigid bar of length L is hinged at the bottom. A force F is applied at point A at the end of the bar. A mass m is glued to the bar at point B, a distance d from the hinge. There is no gravity. What is the acceleration of point A at the instant shown? Assume the angular velocity is initially zero.



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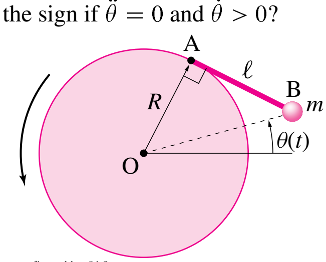
Problem 13.2.19

13.2.20 The mass m is attached rigidly to the rotating disk by the light rod AB of length ℓ. Neglect gravity. Find M_A (the moment on the rod AB from its support point at A) in terms of $\dot{\theta}$ and $\ddot{\theta}$. What is the sign of M_A if $\dot{\theta} = 0$ and $\ddot{\theta} > 0$? What is the sign if $\ddot{\theta} = 0$ and $\dot{\theta} > 0$?



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Problem 13.2.20

Pendulum problems

13.2.21 Simple pendulum, comprehensive version. This problem covers many aspects of a simple pendulum. A point mass M hangs on a massless string or rod of length L. The gravitational force is Mg. The pendulum is in a vertical plane. At any time t, the angle between the straight down line and the pendulum, measured counter clockwise, is $\theta(t)$. Neglect air friction. When numbers are called for use $M = 1\,\mathrm{kg}$, $L = 1\,\mathrm{m}$ and $g = 10\,\mathrm{m/s^2}$.

a) **Find the equations of motion.** That is, assume that you know both θ and $\dot{\theta}$, find $\ddot{\theta}$. There are several ways to do this problem.* Find the equations using

1. Linear momentum balance
2. Angular momentum balance

3. Conservation of energy

b) **Tension.** Assuming that you know θ and $\dot{\theta}$, find the tension T in the string.

c) **Reaction components.** Assuming you know θ and $\dot{\theta}$, find the x and y components of the force that the hinge support causes on the pendulum. Clearly define the directions of positive x and y with a sketch.

d) **Reduction to first order equations.** The equation that you found in (a) is a nonlinear second order ordinary differential equation. It can be changed to a pair of first order equations by defining a new variable $\omega \equiv \dot{\theta}$. Write the equation from (a) as a pair of first order equations. *

e) **Numerical solution.** Given the initial conditions $\theta(t = 0) = \pi/2$ and $\omega(t = 0) = \dot{\theta}(t = 0) = 0$. Using numerical integration, find: $\theta(t)$, $\dot{\theta}(t)$ & $T(t)$. Make a single plot, or three vertically aligned plots, of these variables for one full oscillation of the pendulum.

f) **Maximum tension.** Using your numerical solutions, find the maximum value of the tension in the rod as the mass swings. *

g) Plot the x and y reaction components as a function of time.

h) **Period of oscillation.** How long does it take to make one oscillation?

i) **Other observations.** Some questions:

1. Does the solution to (f) depend on the length of the string?
2. Is the solution to (f) exactly 30 or just a number near 30? If it is exact can you find the result analytically?
3. Is the period found in (h) longer or shorter than the period found by solving the linear equation $\ddot{\theta} + (g/l)\theta = 0$, based on the (inappropriate-to-use in this case) small angle approximation $\sin\theta = \theta$? Explain intuitively why you expect the period to be longer or shorter?

13.2.22 Tension in a simple pendulum string. A simple pendulum of length 2 m with mass 3 kg is released from rest at an initial angle of 60° from the vertically down position.

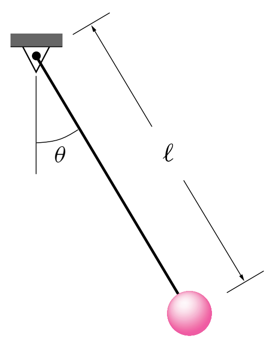
a) What is the tension in the string just after the pendulum is released?

b) What is the tension in the string when the pendulum has reached 30° from the vertical?

13.2.23 Cartesian coordinates Find the nonlinear governing differential equation for a simple pendulum

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

using linear momentum in Cartesian coordinates and without using the polar coordinate formulas for velocity and acceleration. Of course you can use that $x = \ell\cos\theta$ and $y = \ell\sin\theta$ for x pointing down.



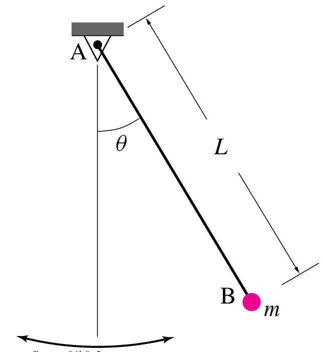
Filename:spend

Problem 13.2.23

13.2.24 Tension in a rope-swing rope. Model a swinging person as a point mass. The swing starts from rest at an angle $\theta = 90°$. When the rope passes through vertical the tension in the rope is higher (it is hard to hang on). A person wants to know ahead of time if she is strong enough to hold on. How hard does she have to hang on compared, say, to her own weight? You are to find the solution two ways. Use the same m, g, and L for both solutions.

a) Find $\ddot{\theta}$ as a function of g, L, θ, and m. This equation is the governing differential equation. Write it as a system of first order equations. Solve them numerically. Once you know $\dot{\theta}$ at the time the rope is vertical you can use other mechanics relations to find the tension. If you like, you can plot the tension as a function of time as the mass falls.

b) Use conservation of energy to find $\dot{\theta}$ at $\theta = 0$. Then use other mechanics relations to find the tension. *



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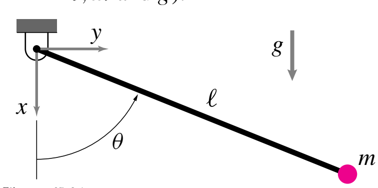
Problem 13.2.24

13.2.25 Pendulum. A pendulum with a negligible-mass rod and point mass m is released from rest at the horizontal position $\theta = \pi/2$.

a) Find the acceleration (a vector) of the mass just after it is released at $\theta = \pi/2$ in terms of ℓ, m, g and any base vectors you define clearly.

b) Find the acceleration (a vector) of the mass when the pendulum passes through the vertical at $\theta = 0$ in terms of ℓ, m, g and any base vectors you define clearly

c) Find the string tension when the pendulum passes through the vertical at $\theta = 0$ (in terms of ℓ, m and g).



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Problem 13.2.25

13.2.26 Write a computer program to solve the nonlinear pendulum equation, $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$, over a given time interval $(0, t)$, and initial conditions $\theta(0)$, and $\dot{\theta}(0)$. The output should be a vector of time instants, t_i, in the given time interval and the corresponding θ_i and $\dot{\theta}_i$.

Now use your computer program to find the solution of

$$\ddot{\theta} = -\sin\theta, \qquad \theta(0) = \pi/4,\ \dot{\theta}(0) = 0.$$

Compare the solution obtained with the analytical solution of the corresponding simple pendulum equation, $\ddot{\theta} = -\theta$ with the same initial conditions. In particular,

a) Find the difference in the time period of oscillations of the two systems.

b) Plot $\dot{\theta}$ obtained from the two solutions against time and comment on the differences.

c) Plot $\dot{\theta}$ against θ from the two solutions on the same plot and compare the two phase portraits. Comment on the differences.

13.2.27 Solve the nonlinear pendulum equation numerically taking 20 different initial angular positions between $\theta = 0$ and $\theta = \pi$, each time releasing the pendulum gently from rest. Find the time period of oscillation, T, from each solution and plot it against the amplitude of motion, *i.e.,* $\theta(0)$.

a) How does the period of oscillation depend on the amplitude for small amplitudes?

b) What is the limiting value of the time period for large amplitudes, *i.e.,* $\theta(0) \to \pi$?

c) How does T depend on the amplitude over the entire range?

13.2.28 A pendulum of mass m and length ℓ is released from rest at $\theta(0) = 60°$. It executes oscillatory motion. If the pendulum were to be released from two different positions, $\theta(0) = 45°$ and $\theta(0) = 0°$, with some corresponding initial angular speed such that the ensuing motion were exactly the same as that with $\theta(0) = 60°$ and $\dot{\theta}(0) = 0$, find the required initial angular speeds.

a) First, find the corresponding $\dot{\theta}(0)$ without any computer simulation.

b) Verify your answer by plotting computer generated solutions for the three different initial conditions.

c) What is the general relationship between $\theta(0)$ and $\dot{\theta}(0)$ that produces a predefined motion generated by, say, a given set of $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$.

13.2.29 Use a computer program to solve the nonlinear pendulum equation, $\ddot{\theta} + \lambda^2 \sin\theta = 0$, where $\lambda^2 = 1.56/\mathrm{s}^2$, with the following 11 initial conditions: $[\theta(0), \dot{\theta}(0)] =$ [1, 0], [2, 0], [3, 0], [4, -1], [-4, 1], [4, -1.02], [-4, 1.02], [4, -1.1], [-4, 1.1], [4, 1.4], and [-4, 1.4], where θ is in radians and $\dot{\theta}$ in rad/s. Obtain each solution over the time interval $t = 0$ to $t = 20$ s.

a) Plot all solutions in the phase space (*i.e.,* θ vs $\dot{\theta}$) in a single graph.

b) What does the extension of the plot beyond $\theta = \pm\pi$ mean?

c) Which initial conditions give solutions outside the separatrix? What do these solutions mean? Are these solutions periodic?

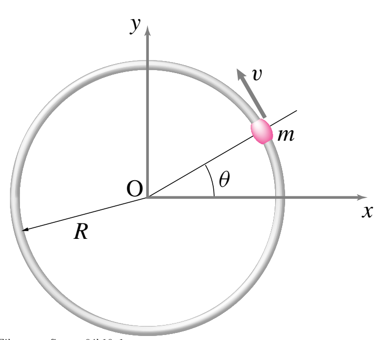
d) If you added a little bit of viscous damping to the pendulum motion, can you guess what will happen to the solutions inside the separatrix? [Hint: think about energy associated with these solutions.]

More circular motion problems

13.2.30 Bead on a hoop with friction. A bead slides on a rigid, stationary, circular wire. The coefficient of friction between the bead and the wire is μ. The bead is loose on the wire (not a tight fit but not so loose that you have to worry about rattling). Assume gravity is negligible.

a) Given v, m, R, & μ; what is $\dot{v}$? *

b) If $v(\theta = 0) = v_0$, how does v depend on θ, μ, v_0 and m? *



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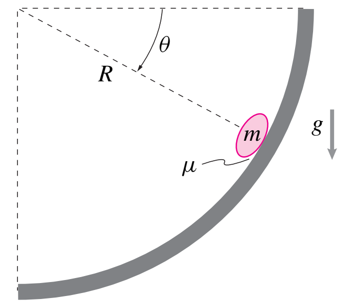
Problem 13.2.30

13.2.31 Particle in a chute. One of a million non-interacting rice grains is sliding in a circular chute with radius R. Its mass is m and it slides with coefficient of friction μ (Actually it slides, rolls and tumbles — μ is just the effective coefficient of friction from all of these interactions.) Gravity g acts downwards.

a) Find a differential equation that is satisfied by θ that governs the speed of the rice as it slides down the hoop. Parameters in this equation can be m, g, R and μ [Hint: Draw FBD, write eqs of mechanics, express as ODE.]

b) Find the particle speed at the bottom of the chute if $R = 0.5m$, $m = 0.1$ grams, $g = 10\,\mathrm{m/s^2}$, and $\mu = 0.2$ as well as the initial values of $\theta_0 = 0$ and its initial downward speed is $v_0 = 10\,\mathrm{m/s}$. [Hint: you are probably best off using a numerical solution.]



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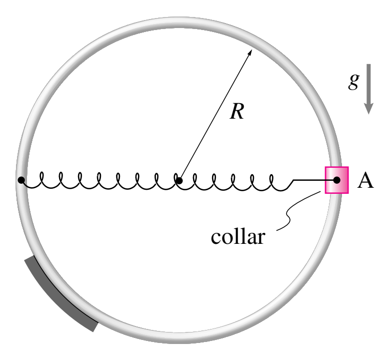
Problem 13.2.31

13.2.32 Due to a push which happened in the past, the collar with mass m is sliding up at speed v_0 on the circular ring when it passes through the point A. The ring is frictionless. A spring of constant k and unstretched length R is also pulling on the collar.

a) What is the acceleration of the collar at A. Solve in terms of R, v_0, m, k, g and any base vectors you define.

b) What is the force on the collar from the ring when it passes point A? Solve in terms of R, v_0, m, k, g and any base vectors you define.



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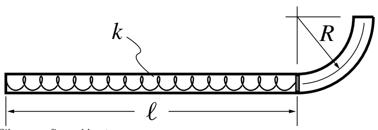
Problem 13.2.32

13.2.33 A toy used to shoot pellets is made out of a thin tube which has a spring of spring constant k on one end. The spring is placed in a straight section of length ℓ; it is unstretched when its length is ℓ. The straight part is attached to a (quarter) circular tube of radius R, which points up in the air.

a) A pellet of mass m is placed in the device and the spring is pulled to the left by an amount $\Delta\ell$. Ignoring friction along the travel path, what is the pellet's velocity $\vec{v}$ as it leaves the tube? *

b) What force acts on the pellet just prior to its departure from the tube? What about just after? *

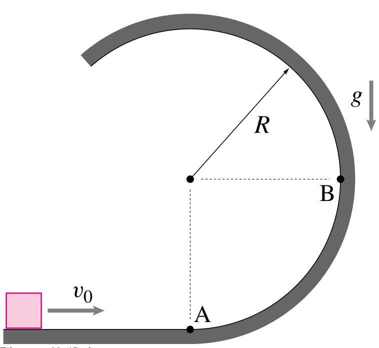


Filename:pfigure-blue-toygun
Problem 13.2.33

13.2.34 A block with mass m is moving to the right at speed v_0 when it reaches a circular frictionless portion of the ramp.

a) What is the speed of the block when it reaches point B? Solve in terms of R, v_0, m and g.

b) What is the force on the block from the ramp just after it gets onto the ramp at point A? Solve in terms of R, v_0, m and g. Remember, force is a vector.



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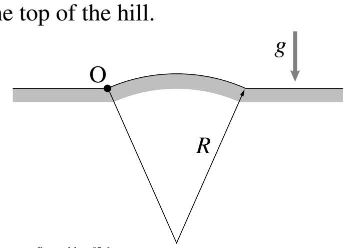
Problem 13.2.34

13.2.35 A car moves with speed v along the surface of the hill shown which can be approximated as a circle of radius R. The car starts at a point on the hill at point O. Compute the magnitude of the speed v such that the car just leaves the ground at the top of the hill.



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Problem 13.2.35

13.3 2D rigid-object rotation kinematics

Preparatory Problems

13.3.1 Give the definition of each term below in words and, if possible, with an equation.

a) rotation angle

b) (x', y')

c) (x, y)

d) rotation matrix

13.3.2 Assume that in the reference configuration, when $\theta = 0$, that the $x'y'$ axes attached to an object are aligned with the xy axes. Consider a point P attached to the moving frame (object) that has coordinates (x', y'). Find each of the quantities below in terms of some or all of $x', y', \hat{\imath}, \hat{\jmath}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

a) $\vec{r}_{\mathrm{P}}$

b) The rotation matrix $[R]$.

13.3.3 Assume that in the reference configuration, when $\theta = 0$, that the $x'y'$ axes are aligned with the xy axes. Consider two points P_1 and P_2 attached to the moving frame. P_1 and P_2 have coordinates (x'_1, y'_1) (x'_2, y'_2). Find each of the quantities below in terms of some or all of $x'_1, y'_1, x'_2, y'_2, \hat{\imath}, \hat{\jmath}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

a) $\vec{r}_{\mathrm{P}_1/\mathrm{P}_2}$

b) The rotation matrix $[R]$.

13.3.4 The square plate OABC shown in the figure measures $\ell = 1$ m on a side. The coordinate axes $x'y'$ are fixed to the plate and the axes xy are fixed in space. The plate is rotated by an angle $\theta = 60°$ as shown.

a) Find the coordinates x' and y' of the corner point B in the rotated frame.

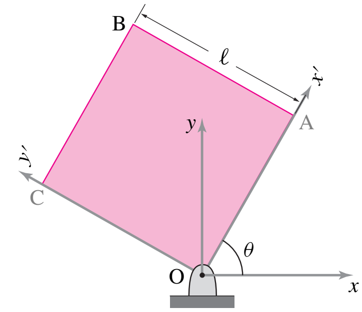
b) Find the coordinates x and y of the same point in the fixed frame without using the rotation matrix.

c) Write the rotation matrix R for to the given rotation.

d) Find the coordinates of point B in the fixed frame using its (x', y') coordinates and the rotation matrix R.

e) Find the coordinates of point B in the rotated frame using its (x, y) coordinates and the rotation matrix R in its appropriate form.

*



Filename:pfigure13-3-rectangle1
Problem 13.3.4

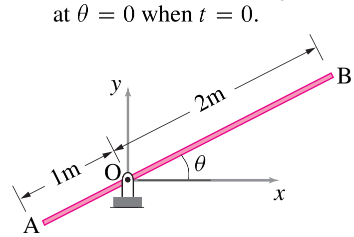
13.3.5 The rod shown in the figure rotates counterclockwise, starting from $\theta = 0$. Find the following quantities.

a) Write the position vector of points A and B as a list of numbers in an array, *e.g.*, $[\vec{r}_{\mathrm{A}}]_{xy} = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$, at $\theta = 0$.

b) Write the rotation matrix R for $\theta = 45°$.

c) Find the coordinates of points A and B using the rotation matrix R for $\theta = 45°$.

d) If the rod rotates at a constant angular speed $\omega = \pi/3\,\mathrm{rad/s}$, find the coordinates of points A and B after 2 seconds, assuming the rod is at $\theta = 0$ when $t = 0$.



Filename:pfigure13-3-rodrotation
Problem 13.3.5

13.3.6 Find the angle of rotation corresponding to the following rotation matrices:

a) $\begin{bmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{bmatrix}$.

b) $\begin{bmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{bmatrix}$.

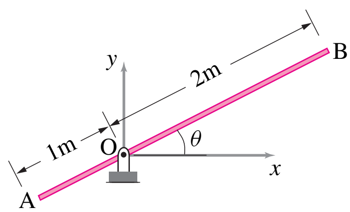
c) $\begin{bmatrix} 0.7071 & 0.7071 \\ -0.7071 & 0.7071 \end{bmatrix}$.

More-Involved Problems

13.3.7 The rod shown in the figure rotates with constant angular speed $\omega = 10\,\mathrm{rad/s}$.

a) Find the position vector $\vec{r}_B|_{t_1}$ and the velocity $\vec{v}_B|_{t_1}$ of point B at $t_1 = 1\,\mathrm{s}$.

b) Find the position vector $\vec{r}_B|_{t_2}$ of point B at $t_2 = 1.2\,\mathrm{s}$. What is the net displacement of point B during the time interval $\Delta t = t_2 - t1$? Is this net displacement equal to $\vec{v}_B|_{t_1}\Delta t$? Why not?



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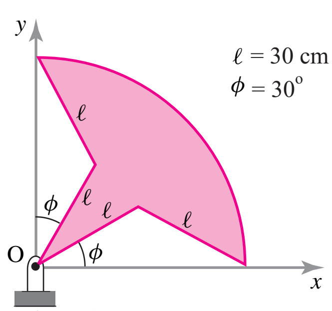
Problem 13.3.7

13.3.8 Write a computer program to animate the rotation of an object. Your input should be a set of x and y coordinates defining the object (such that `plot y vs x` draws the object on the screen) and the rotation angle θ. The output should be the rotated coordinates of the object.

a) From the geometric information given in the figure, generate coordinates of enough points to define the given object.

b) Using your program, plot the object at $\theta = 20°, 60°, 100°, 160°$, and $270°$.

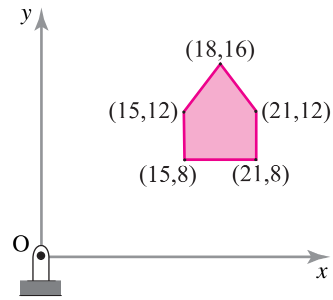
c) Assume that the object rotates with constant angular speed $\omega = 2\,\mathrm{rad/s}$. Find and plot the position of the object at $t = 1\,\mathrm{s},\ 2\,\mathrm{s},$ and3 s.



Filename:pfigure13-3-objectrotation

Problem 13.3.8

13.3.9 The arrow shaped object shown in the figure is defined by the five points whose coordinates are given in some normalized units. The position shown is at $t = 0$. The time dependent angular position of the object is given by $\theta(t) = C_1 t^2 + C_2 t$ where $C_1 = 0.25\,\mathrm{rad/s^2}$ and $C_2 = 0.1\,\mathrm{rad/s}$. Animate the motion of the object and show its position from $t = 0$ to $t = 5\,\mathrm{s}$ at every second.



Filename:pfigure13-3-boatrotation

Problem 13.3.9

13.4 2D rigid-object angular velocity

Preparatory Problems

13.4.1 Give the definition of each term below in words and, but for the first, with an equation.

a) angular velocity

b) angular acceleration

13.4.2 Assume that in the reference configuration, when $\theta = 0$, that the $x'y'$ axes attached to an object are aligned with the xy axes. Consider a point P attached to the moving frame (object) that has coordinates (x', y'). Find each of the quantities below in terms of some or all of $x', y', \hat{\imath}, \hat{\jmath}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

a) $\vec{r}_P$

b) $\vec{v}_P$

c) $\vec{a}_P$

13.4.3 Assume that in the reference configuration, when $\theta = 0$, that the $x'y'$ axes are aligned with the xy axes. Consider two points P_1 and P_2 attached to the moving frame. P_1 and P_2 have coordinates (x'_1, y'_1) (x'_2, y'_2). Find each of the quantities below in terms of some or all of $x'_1, y'_1, x'_2, y'_2, \hat{\imath}, \hat{\jmath}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

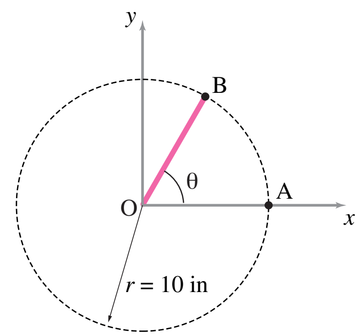
a) $\vec{r}_{P_1/P_2}$

b) $\vec{v}_{P_1/P_2}$

c) $\vec{a}_{P_1/P_2}$

13.4.4 Find $\vec{v} = \vec{\omega} \times \vec{r}$, if $\vec{\omega} = 1.5\,\mathrm{rad/s}\hat{k}$ and $\vec{r} = 2\,\mathrm{m}\hat{\imath} - 3\,\mathrm{m}\hat{\jmath}$.

13.4.5 A rod OB rotates with its end O fixed as shown in the figure with angular velocity $\vec{\omega} = 5\,\mathrm{rad/s}\hat{k}$ and angular acceleration $\vec{\alpha} = 2\,\mathrm{rad/s^2}\hat{k}$ at the moment of interest. Find, draw, and label the tangential and normal acceleration of point B at $\theta = 60°$.



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Problem 13.4.5

13.4.6 A disc rotates at 15 rpm. How many seconds does it take to rotate by 180 degrees? What is the angular speed of the disc in rad/s?

13.4.7 A motor turns a uniform disc of radius R counter-clockwise about its mass center at a constant rate ω. The disc lies in the xy-plane and its angular displacement θ is measured (positive counter-clockwise) from the x-axis. What is the angular displacement $\theta(t)$ of the disc if it starts at $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \omega$? What are the velocity and acceleration of a point P at position $\vec{r} = x\hat{\imath} + y\hat{\jmath}$?

13.4.8 Find the angular velocities of the second, minute, and hour hands of a clock.

13.4.9 A disc $\mathcal{C}$ spins at a constant rate of two revolutions per second counter-clockwise about its geometric center, G, which is fixed. A point P is marked on the disk at a radius of one meter. At the moment of interest, point P is on the x-axis of an xy-coordinate system centered at point G.

a) Draw a neat diagram showing the disk, the particle, and the coordinate axes.

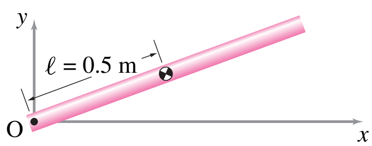
b) What is the angular velocity of the disk, $\vec{\omega}_C$?

c) What is the angular acceleration of the disk, $\dot{\vec{\omega}}_C$?

d) What is the velocity $\vec{v}_P$ of point P?

e) What is the acceleration $\vec{a}_P$ of point P?

13.4.10 A uniform rigid rod rotates at constant speed in the xy-plane about a peg at point O. The center of mass of the rod may not exceed a specified acceleration $a_{max} = 0.5\,\mathrm{m/s^2}$. Find the maximum angular velocity of the rod.

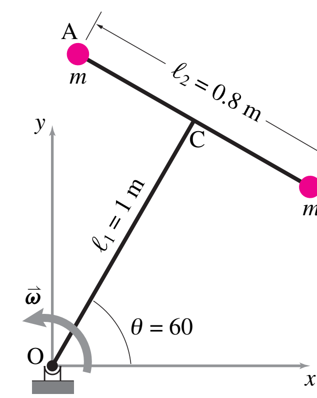


Filename:pfigure4-3-rp3
Problem 13.4.10

13.4.11 A dumbbell AB is welded to a rigid arm OC such that OC is perpendicular to AB. Arm OC rotates about O at a constant angular velocity $\vec{\omega} = 10\,\mathrm{rad/s}\hat{k}$. At the instant when $\theta = 60^\circ$,

a) Find the relative velocity of B with respect to A. *

b) Find the acceleration of point B relative to point A. *



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Problem 13.4.11

More-Involved Problems

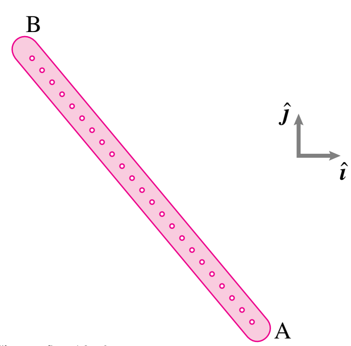
13.4.12 Two discs $\mathcal{A}$ and $\mathcal{B}$ rotate at constant speeds about their centers. Disc $\mathcal{A}$ rotates at 100 rpm and disc $\mathcal{B}$ rotates at 10 rad/s. Which is rotating faster?

13.4.13 A motor turns a uniform disc of radius R counter-clockwise about its mass center at a constant rate ω. The disc lies in the xy-plane and its angular displacement θ is measured (positive counter-clockwise) from the x-axis. What are the velocity and acceleration of a point P at position $\vec{r}_P = c\hat{\imath} + d\hat{\jmath}$ relative to the velocity and acceleration of a point Q at position $\vec{r}_Q = 0.5(-d\hat{\imath} + c\hat{\jmath})$ on the disk? ($c^2 + d^2 < R^2$.)

13.4.14 A 0.4 m long rod AB has many holes along its length such that it can be pegged at any of the various locations. It rotates counter-clockwise at a constant angular speed about a peg whose location is not known. At some instant t, the velocity of end B is $\vec{v}_B = -3\,\mathrm{m/s}\hat{\jmath}$. After $\frac{\pi}{20}$ s, the velocity of end B is $\vec{v}_B = -3\,\mathrm{m/s}\hat{\imath}$. If the rod has not completed one revolution during this period,

a) find the angular velocity of the rod, and

b) find the location of the peg along the length of the rod.



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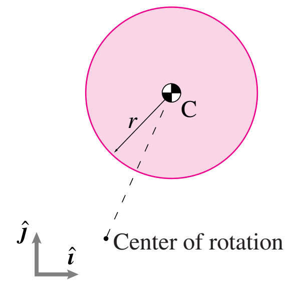
Problem 13.4.14

13.4.15 A circular disc of radius $r = 250\,\mathrm{mm}$ rotates in the xy-plane about a point which is at a distance $d = 2r$ away from the center of the disk. At the instant of interest, the linear speed of the center C is 0.60 m/s and the magnitude of its centripetal acceleration is $0.72\,\mathrm{m/s^2}$.

a) Find the rotational speed of the disk.

b) Is the given information enough to locate the center of rotation of the disk?

c) If the acceleration of the center has no component in the $\hat{\jmath}$ direction at the moment of interest, can you locate the center of rotation? If yes, is the point you locate unique? If not, what other information is required to make the point unique?

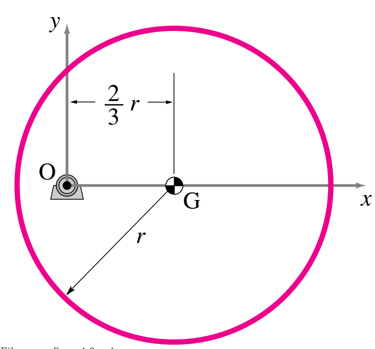


Filename:pfigure4-3-rp8
Problem 13.4.15

13.4.16 A uniform disc of radius $r = 200\,\mathrm{mm}$ is mounted eccentrically on a motor shaft at point O. The motor rotates the disc at a constant angular speed. At the instant shown, the velocity of the center of mass is $\vec{v}_G = -1.5\,\mathrm{m/s}\hat{\jmath}$.

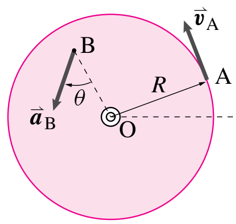
a) Find the angular velocity of the disc.

b) Find the point with the highest linear speed on the disc. What is its velocity?



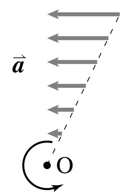
Filename:pfigure4-3-rp4
Problem 13.4.16

13.4.17 The circular disc of radius $R = 100\,\mathrm{mm}$ rotates about its center O. At a given instant, point A on the disk has a velocity $v_A = 0.8\,\mathrm{m/s}$ in the direction shown. At the same instant, the tangent of the angle θ made by the total acceleration vector of any point B with its radial line to O is 0.6. Compute the angular acceleration α of the disc.



Filename:pfigure-blue-43-2
Problem 13.4.17

13.4.18 Show that, for non-constant rate circular motion, the acceleration of all points in a given radial line are parallel.



Filename:pfigure-DH-5-1a
Problem 13.4.18

13.4.19 A motor turns a uniform disc of radius R about its mass center at a variable angular rate ω with rate of change $\dot{\omega}$, counter-clockwise. The disc lies in the xy-plane and its angular displacement θ is measured from the x-axis, positive counter-clockwise. What are the velocity and acceleration of a point P at position $\vec{r}_P = c\hat{\imath} + d\hat{\jmath}$ relative to the velocity and acceleration of a point Q at position $\vec{r}_Q = 0.5(-d\hat{\imath} + y\hat{\jmath})$ on the disk? ($c^2 + d^2 < R^2$.)

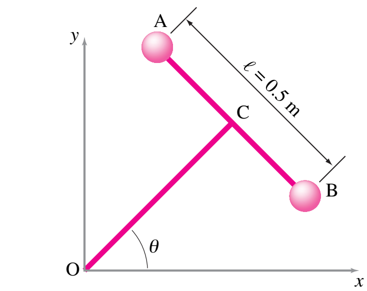
13.4.20 The dumbbell AB shown in the figure rotates counterclockwise about point O with angular acceleration $3\,\text{rad/s}^2$. Bar AB is perpendicular to bar OC. At the instant of interest, $\theta = 45^\circ$ and the angular speed is 2 rad/s.

a) Find the velocity of point B relative to point A. Will this relative velocity be different if the dumbbell were rotating at a *constant* rate of 2 rad/s?

b) Without calculations, draw a vector approximately representing the acceleration of B relative to A.

c) Find the acceleration of point B relative to A. What can you say about the direction of this vector as the motion progresses in time?

*



Filename:pfigure-sfig5-3-3
Problem 13.4.20

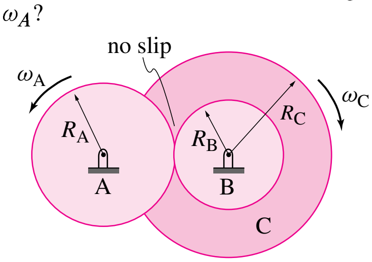
13.4.21 Bit-stream kinematics of a CD. A Compact Disk (CD) has bits of data etched on concentric circular tracks. The data from a track is read by a beam of light from a head that is positioned under the track. The angular speed of the disk remains constant as long as the head is positioned over a particular track. As the head moves to the next track, the angular speed of the disk changes, so that the *linear speed* at any track is always the same. The data stream comes out at a constant rate 4.32×10^6 bits/second. When the head is positioned on the outermost track, for which $r = 56$ mm, the disk rotates at 200 rpm.

a) What is the number of bits of data on the outermost track.

b) find the angular speed of the disk when the head is on the innermost track ($r = 22$ mm), and

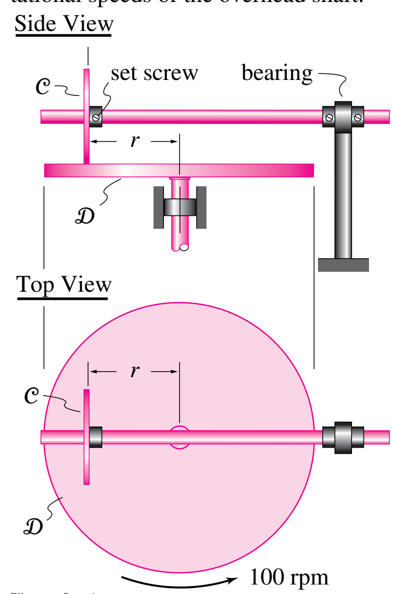
c) find the numbers of bits on the innermost track.

13.4.22 2-D constant rate gear train. The angular velocity of the input shaft (driven by a motor not shown) is a constant, $\omega_{\text{input}} = \omega_A$. What is the angular velocity $\omega_{\text{output}} = \omega_C$ of the output shaft and the speed of a point on the outer edge of disc C, in terms of R_A, R_B, R_C, and ω_A?



Filename:ch4-3
Problem 13.4.22: Gear B is welded to C and engages with A.

13.4.23 A horizontal disk $\mathcal{D}$ of diameter $d = 500$ mm is driven at a constant speed of 100 rpm. A small disk $\mathcal{C}$ can be positioned anywhere between $r = 10$ mm and $r = 240$ mm on disk $\mathcal{D}$ by sliding it along the overhead shaft and then fixing it at the desired position with a set screw (see the figure). Disk $\mathcal{C}$ rolls without slip on disk $\mathcal{D}$. The overhead shaft rotates with disk $\mathcal{C}$ and, therefore, its rotational speed can be varied by varying the position of disk $\mathcal{C}$. This gear system is called *brush gearing*. Find the maximum and minimum rotational speeds of the overhead shaft. *



Filename:pfigure4-rpc
Problem 13.4.23

13.4.24 Two points A and B are on the same machine part that is hinged at an as yet unknown location C. Assume you are given that points at positions $\vec{r}_A$ and $\vec{r}_B$ are supposed to move in given directions, indicated by unit vectors $\hat{\boldsymbol{\lambda}}_A$ and $\hat{\boldsymbol{\lambda}}_B$. For each of the problem parts below, illustrate your results with two numerical examples (in consistent units): i) $\vec{r}_A = 1\hat{\imath}$, $\vec{r}_B = 1\hat{\jmath}$, $\hat{\boldsymbol{\lambda}}_A = 1\hat{\jmath}$, and $\hat{\boldsymbol{\lambda}}_B = -1\hat{\imath}$ (thus $\vec{r}_C = \vec{\mathbf{0}}$), and ii) a more complex example of your choosing.

a) Describe in detail what equations must be satisfied by the point $\vec{r}_C$.

b) Write a computer program that takes as input the 4 pairs of numbers $[\vec{r}_A]$, $[\vec{r}_B]$, $[\hat{\boldsymbol{\lambda}}_A]$ and $[\hat{\boldsymbol{\lambda}}_B]$ and gives as output the pair of numbers $[\vec{r}_C]$.

c) Find a formula of the form $\vec{r}_C = \ldots$ that explicitly gives the position vector for point C in terms of the 4 given vectors.

13.5 Polar moment of inertia

Preparatory Problems

13.5.1 Find an expression for the polar moment of inertia for each of the following systems about the specified points. Answer in terms of some or all of the variables given. If a position vector $\vec{r}$ is given then you can assume the coordinates (x, y) are given so that $\vec{r} = x\hat{\imath} + y\hat{\jmath}$.

a) A **particle** with mass m at $\vec{r}$ about
   1. Its center of mass
   2. The origin

b) **Two particles** with masses m_1 and m_2 a distance d apart located at $\vec{r}_1$ and $\vec{r}_2$, respectively, about
   i) Mass 1
   ii) Mass 2
   i) The center of mass of the system
   iv) The origin

c) A collection of n **particles** m_i at locations $\vec{r}_{i/0}$ about
   i) The origin
   ii) The center of mass G of the system

d) A uniform **line segment** with length ℓ and mass m with center of mass at location $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) One end
   iii) The origin

e) A **uniform hoop** with mass m and radius R and center at $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) A point on the hoop
   iiii) The origin

f) A **uniform disk** with mass m and radius R and center at $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) A point on the perimeter
   iii) The origin

g) A **uniform rectangle** with mass m and sides ℓ and w and with center at $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) A corner
   iii) The origin

13.5.2 Two objects have masses m_1 and m_2 and polar moments of inertia I_1 and I_2 about their respective centers of mass G_1 and G_2 which are a distance d apart. What is the moment of inertia of the system about its center of mass?

13.5.3 A point mass $m = 0.5\,\text{kg}$ is located at $x = 0.3\,\text{m}$ and $y = 0.4\,\text{m}$ in the xy-plane. Find the moment of inertia of the mass about the z-axis. *

13.5.4 A small ball of mass 0.2 kg is attached to a 1 m massless rod.

a) What is value of I_{zz} of the ball about the other end of the rod? *

b) How much must you shorten the rod to reduce the moment of inertia of the ball by half? *

More-Involved Problems

13.5.5 Two identical point masses are attached to the two ends of a rigid massless bar of length ℓ (one mass at each end). Locate a point along the length of the bar about which the polar moment of inertia of the system is 20% more than that calculated about the mid point of the bar. *

13.5.6 A dumbbell consists of a rigid massless bar of length ℓ and two identical point masses m and m, one at each end of the bar.

a) About which point on the dumbbell is its polar moment of inertia I_{zz} a minimum and what is this minimum value? *

b) About which point on the dumbbell is its polar moment of inertia I_{zz} a maximum and what is this maximum value? *

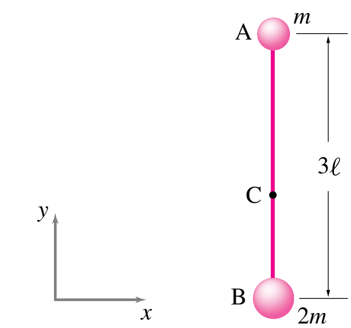
13.5.7 A light rigid rod AB of length 3ℓ has a point mass m at end A and a point mass $2m$ at end B. Point C is the center of mass of the system. First, answer the following questions without any calculations and then do calculations to verify your guesses.

a) About which point A, B, or C, is the polar moment of inertia I_{zz} of the system a minimum? *

b) About which point is I_{zz} a maximum? *

c) What is the ratio of I_{zz}^A and I_{zz}^B? *

d) Is the radius of gyration of the system greater, smaller, or equal to the length of the rod? *



Filename:pfigure4-4-rp1
Problem 13.5.7

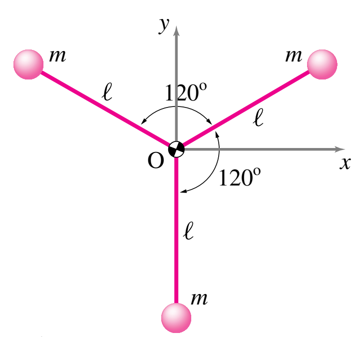
13.5.8 Three identical particles of mass m are connected to three identical massless rods of length ℓ and welded together at point O as shown in the figure.

a) Guess (no calculations) which of the three moment of inertia terms I_{xx}^O, I_{yy}^O, I_{zz}^O is the smallest and which is the biggest. *

b) Calculate the three moments of inertia to check your guess. *

c) If the orientation of the system is changed, so that one mass is along the x-axis, will your answer to part (a) change?

d) Find the radius of gyration of the system for the polar moment of inertia. *



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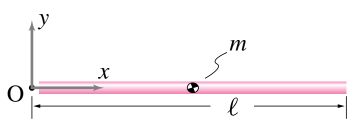
Problem 13.5.8

13.5.9 Show that the polar moment of inertia I_{zz}^O of the uniform bar of length ℓ and mass m, shown in the figure, is $\frac{1}{3}m\ell^2$, in two different ways:

a) by using the basic definition of polar moment of inertia $I_{zz}^{O} = \int r^2\, dm$, and

b) by computing I_{zz}^{cm} first and then using the parallel axis theorem.



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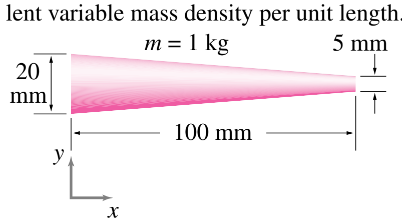
Problem 13.5.9

13.5.10 Approximately locate the center of mass of the tapered rod shown in the figure and compute the polar moment of inertia I_{zz}^{cm}. [Hint: use the variable thickness of the rod to define an approximate equivalent variable mass density per unit length.]



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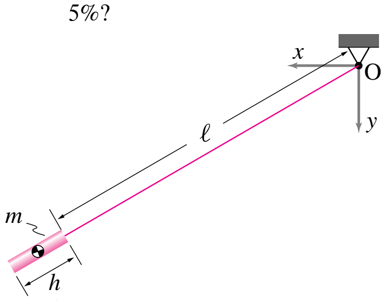
Problem 13.5.10

13.5.11 A short rod of mass m and length h hangs from an inextensible and massless rod of length ℓ.

a) Find the moment of inertia I_{zz}^{O} of the rod.

b) Find the moment of inertia of the rod I_{zz}^{O} by considering it as a point mass located at its center of mass.

c) Find the percent error in I_{zz}^{O} in treating the bar as a point mass by comparing the expressions in parts (a) and (b). Plot the percent error versus h/ℓ. For what values of h/ℓ is the percentage error less than 5%?



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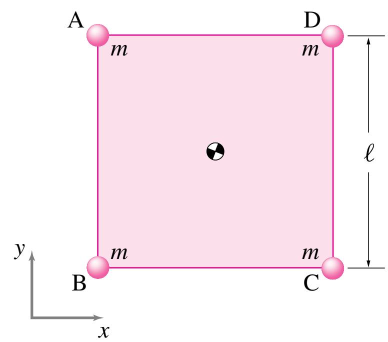
Problem 13.5.11

13.5.12 Parallel axis theorem? A massless square plate $ABCD$ has four identical point masses located at its corners.

a) Find the polar moment of inertia I_{zz}^{cm}. *

b) Find a point P on the plate about which the system's moment of inertia I_{zz} is maximum? *

c) Find the radius of gyration of the system about the rectangle center. *



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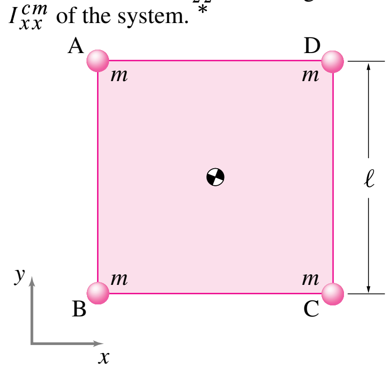
Problem 13.5.12

13.5.13 For the massless square plate with four point masses on the corners, the polar moment of inertia $I_{zz}^{cm} = 0.6\,\mathrm{kg\cdot m^2}$. Find I_{xx}^{cm} of the system. *



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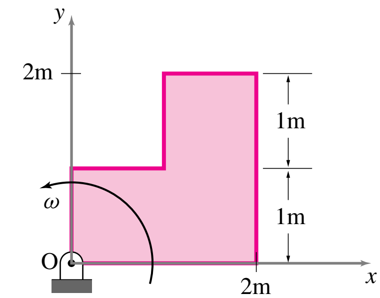
Problem 13.5.13

13.5.14 A uniform square plate (2 m on edge) has a corner cut out. The total mass of the remaining plate is 3 kg.

a) Where is the center of mass of the plate at the instant shown?

b) What is the moment of inertia of the plate about point O?

c) What is the moment of inertia of the plate about its center of mass?



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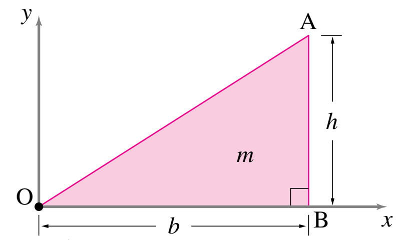
Problem 13.5.14

13.5.15 A uniform thin triangular plate of mass m, height h , and base b lies in the xy-plane.

a) Set up the integral to find the polar moment of inertia I_{zz}^{O} of the plate. *

b) Show that $I_{zz}^{O} = \frac{m}{6}(h^2 + 3b^2)$ by evaluating the integral in part (a).

c) Locate the center of mass of the plate and calculate I_{zz}^{cm}.



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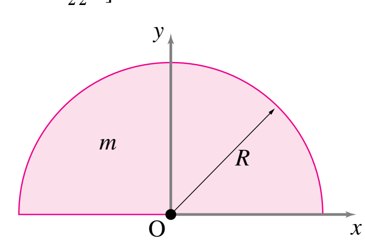
Problem 13.5.15

13.5.16 A uniform thin plate of mass m is cast in the shape of a semi-circular disk of radius R as shown in the figure.

a) Find the location of the center of mass of the plate

b) Find the polar moment of inertia of the plate, I_{zz}^{cm}. [Hint: It may be easier to set up and evaluate the integral for I_{zz}^{O} and then use the parallel axis theorem to calculate I_{zz}^{cm}.]



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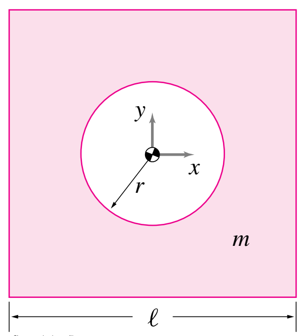
Problem 13.5.16

13.5.17 A uniform square plate of side $\ell = 250\,\text{mm}$ has a circular cut-out of radius $r = 50\,\text{mm}$. The mass of the plate is $m = \frac{1}{2}\,\text{kg}$.

a) Find the polar moment of inertia of the plate about its center.

b) Plot I_{zz}^{cm} versus r/ℓ.

c) Find the limiting values of I_{zz}^{cm} for $r = 0$ and $r = \ell$.



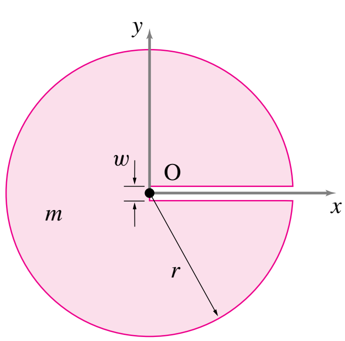
Filename:pfigure4-4-rp7
Problem 13.5.17

13.5.18 A uniform thin circular disk of radius $r = 100\,\text{mm}$ and mass $m = 2\,\text{kg}$ has a rectangular slot of width $w = 10\,\text{mm}$ cut into it as shown in the figure. You can approximate the right edge of the cutout as a straight line (thus your calculation would be exact for a system that has a very thin sliver of material closing the gap at the right).

a) Find the polar moment of inertia I_{zz}^{O} of the disk.

b) Locate the center of mass of the disk and calculate I_{zz}^{cm}.

c) estimate the error in your calculation of the mass, center of mass location and moment of inertia due to not subtracting out the sliver of material at the right edge of the slot.



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Problem 13.5.18

13.5.19 Calculate the polar moment of inertia of a uniform square plate with mass m and side ℓ various ways. Choose numerical values for m and ℓ if you want to address this as a numerical calculation rather than a theoretical one.

a) As a single uniform plate

b) As a composite of 4 plates, each with sides $\ell/2$

c) As a composite of 100 plates, each with sides $\ell/10$ (These first three methods should give exactly the same answer).

d) For the composite of 100 plates now neglect the terms which concern the moments of inertia of each small square about its center of mass. What is the error in making this neglect? *

13.6 Dynamics of rigid-object planar circular motion

Preparatory Problems

13.6.1 An object consists of a massless bar with two attached masses m_1 and m_2. The object is hinged at O.

a) What is the moment of inertia of the object about point O (I_{zz}^{O})?

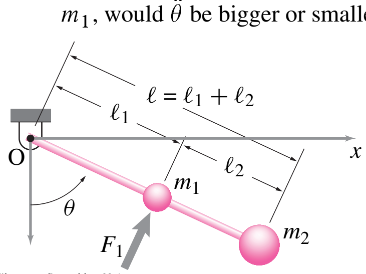
b) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is $\vec{\boldsymbol{H}}_{/O}$, the angular momentum about point O?

c) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is $\dot{\vec{\boldsymbol{H}}}_{/O}$, the rate of change of angular momentum about point O?

d) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is T, the total kinetic energy?

e) Assume that you don't know θ, $\dot{\theta}$ or $\ddot{\theta}$ but you do know that F_1 is applied to the rod, perpendicular to the rod at m_1. What is $\ddot{\theta}$? (Neglect gravity.)

f) If F_1 were applied to m_2 instead of m_1, would $\ddot{\theta}$ be bigger or smaller?



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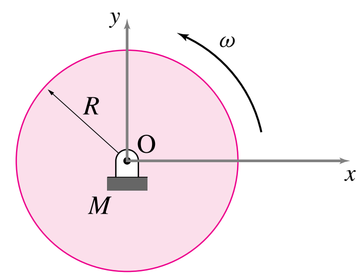
Problem 13.6.1

13.6.2 A uniform circular disc rotates at constant angular speed ω about the origin, which is also the center of the disc. It's radius is R. It's total mass is M.

a) What is the total force and moment required to hold it in place (use the origin as the reference point of angular momentum and torque).

b) What is the total kinetic energy of the disk?



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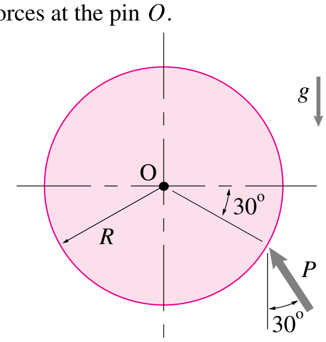
Problem 13.6.2

13.6.3 The hinged disk of mass m (uniformly distributed) is acted upon by a force P shown in the figure. Determine the *initial* angular acceleration and the reaction forces at the pin O.



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Problem 13.6.3

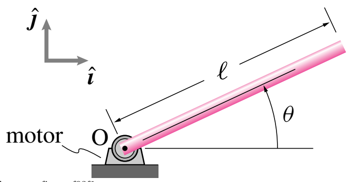
13.6.4 **A motor turns a bar.** A uniform bar of length ℓ and mass m is turned by a motor whose shaft is attached to the end of the bar at O. The angle that the bar makes (measured counter-clockwise) from the positive x axis is $\theta = 2\pi\ t^2/\text{s}^2$. Neglect gravity.

a) Draw a free body diagram of the bar.

b) Find the force acting on the bar from the motor and hinge at $t = 1\,\text{s}$.

c) Find the torque applied to the bar from the motor at $t = 1\,\text{s}$.

d) What is the power produced by the motor at $t = 1\,\text{s}$?



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Problem 13.6.4

13.6.5 A physical pendulum. A swinging stick is sometimes called a 'physical' pendulum. Take the 'body', the system of interest, to be the whole stick.

a) Draw a free body diagram of the system.

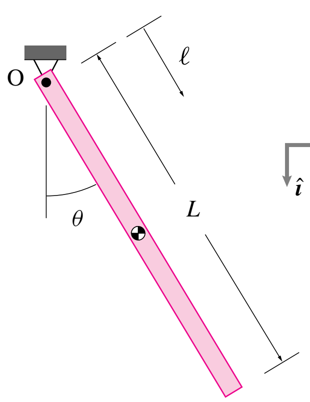
b) Write the equation of angular momentum balance for this system about point O.

c) Evaluate the left-hand-side as explicitly as possible in terms of the forces showing on your Free Body Diagram.

d) Evaluate the right hand side as completely as possible. You may use the following facts:

$$
\begin{aligned}
\vec{v} &= l\dot{\theta}\cos\theta\,\hat{\jmath} + -l\dot{\theta}\sin\theta\,\hat{\imath} \\
\vec{a} &= -l\dot{\theta}^2\left[\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right] \\
&\quad + l\ddot{\theta}\left[\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath}\right]
\end{aligned}
$$

where ℓ is the distance along the pendulum from the top, θ is the angle by which the pendulum is displaced counter-clockwise from the vertically down position, $\hat{\imath}$ is vertically down, and $\hat{\jmath}$ is to the right. You will have to set up and evaluate an integral.



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Problem 13.6.5

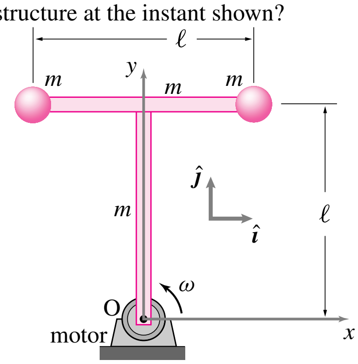
13.6.6 A uniform one meter bar is hung from a hinge that is at the end. It is allowed to swing freely. $g = 10\,\mathrm{m/s^2}$.

a) What is the period of small oscillations for this pendulum?

b) Suppose the rod is hung 0.4 m from one end. What is the period of small oscillations for this pendulum? Can you explain why it is longer or shorter than when it is hung by its end?

More-Involved Problems

13.6.7 Motor turns a dumbbell. Two uniform bars of length ℓ and mass m are welded at right angles. At the ends of the horizontal bar are two more masses m. The bottom end of the vertical rod is attached to a hinge at O where a motor keeps the structure rotating at constant rate ω (counterclockwise). What is the net force and moment that the motor and hinge cause on the structure at the instant shown?



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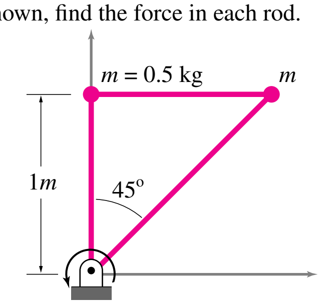
Problem 13.6.7

13.6.8 The structure shown in the figure consists of two point masses connected by three rigid, massless rods such that the whole structure behaves like a rigid body. The structure rotates counterclockwise at a constant rate of 60 rpm. At the instant shown, find the force in each rod.



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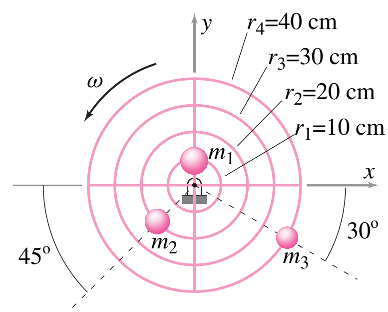
Problem 13.6.8

13.6.9 Balancing a system of rotating particles. A wire frame structure is made of four concentric loops of massless and rigid wires, connected to each other by four rigid wires presently coincident with the x and y axes. Three masses, $m_1 = 200$ grams, $m_2 = 150$ grams and $m_3 = 100$ grams, are glued to the structure as shown in the figure. The structure rotates counter-clockwise at a constant rate $\dot{\theta} = 5\,\mathrm{rad/s}$. There is no gravity.

a) Find the net force exerted by the structure on the support at the instant shown. *

b) You are to put a mass m at an appropriate location on the third loop so that the net force on the support is zero. Find the appropriate mass and the location on the loop.

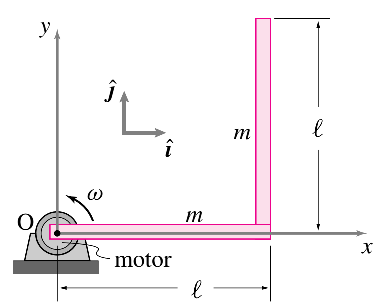


Filename:pfigure-s94h6p2
Problem 13.6.9

13.6.10 Motor turns a bent bar. Two uniform bars of length ℓ and uniform mass m are welded at right angles. One end is attached to a hinge at O where a motor keeps the structure rotating at a constant rate ω (counterclockwise). What is the net force and moment that the motor and hinge cause on the structure at the instant shown.

a) neglecting gravity *

b) including gravity. *



Filename:pg85-3
Problem 13.6.10: A bent bar is rotated by a motor.

13.6.11 A uniform disk of mass M and radius R rotates about a hinge O in the xy-plane. A point mass m is fixed to the disk at a distance $R/2$ from the hinge. A motor at the hinge drives the disk/point mass assembly with constant angular acceleration α. What torque at the hinge does the motor supply to the system?

13.6.12 A rigid rod of length ℓ and total mass m is held fixed at one end and whirled around in circular motion at a constant rate ω in the horizontal plane. Ignore gravity.

a) Find the tension in the rod as a function of r, the radial distance from the center of rotation to any desired location on the rod. *

b) Where does the maximum tension occur in the rod? *.

c) At what distance from the center of rotation does the tension drop to half its maximum value? *.

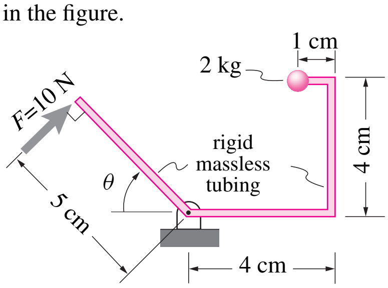
13.6.13 A thin uniform circular disc of mass M and radius R rotates in the xy plane about its center of mass point O. Driven by a motor, it has rate of change of angular speed proportional to angular position, $\alpha = k\theta^{5/2}$. The disc starts from rest at $\theta = 0$.

a) What is the rate of change of angular momentum about the origin at $\theta = \frac{\pi}{3}$ rad?

b) What is the torque of the motor at $\theta = \frac{\pi}{3}$ rad?

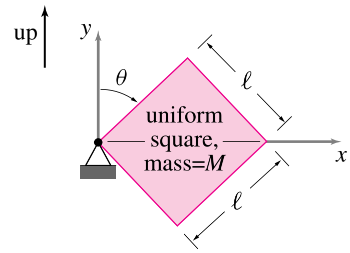
c) What is the total kinetic energy of the disk at $\theta = \frac{\pi}{3}$ rad?

13.6.14 Neglecting gravity, calculate $\alpha = \dot{\omega} = \ddot{\theta}$ at the instant shown for the system in the figure.



Filename:pfigure-blue-20-1
Problem 13.6.14

13.6.15 The uniform square shown is released from rest at $t = 0$. What is $\alpha = \dot{\omega} = \ddot{\theta}$ immediately after release?



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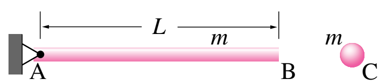
Problem 13.6.15

13.6.16 Acceleration of a trap door. A uniform bar AB of mass m and a ball of the same mass are released from rest from the same horizontal position. The bar is hinged at end A. There is gravity.

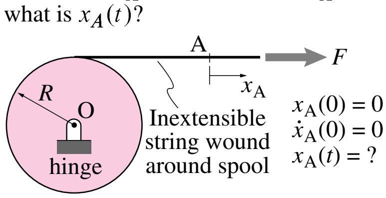
a) Which point on the rod has the same acceleration as the ball, immediately after release. *

b) What is the reaction force on the bar at end A just after release? *



Filename:pfigure-s94h8p3
Problem 13.6.16

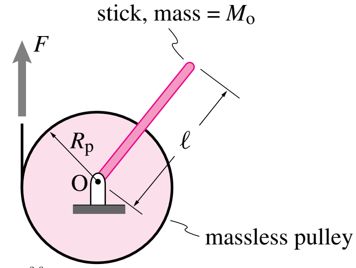
13.6.17 A disk with radius R has a string wrapped around it which is pulled with a force F. The disk is free to rotate about the axis through O normal to the page. The moment of inertia of the disk about O is I_O. A point A is marked on the string. Given that $x_A(0) = 0$ and that $\dot{x}_A(0) = 0$, what is $x_A(t)$?



Spool has mass moment of inertia, I_O

Filename:pfigure-blue-42-1
Problem 13.6.17

13.6.18 A uniform stick with length ℓ and mass M_o is welded to a pulley hinged at the center O. The pulley has negligible mass and radius R_p. A string is wrapped many times around the pulley. At time $t = 0$, the pulley, stick, and string are at rest and a force F is suddenly applied to the string. How long does it take for the pulley to make one full revolution? Ignore gravity. *



Filename:p2-3
Problem 13.6.18: String wraps around a pulley with a stick glued to it.

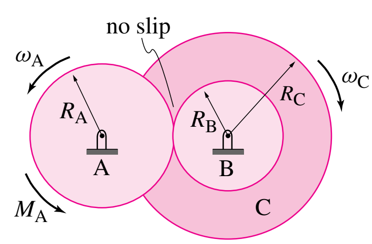
Gears, Belt Drives, and Gear Trains

13.6.19 Constant speed gear train. Gear A is connected to a motor (not shown) and gear B, which is welded to gear C, is connected to a taffy-pulling mechanism. Assume you know the torque $M_{\text{input}} = M_A$ and angular velocity $\omega_{\text{input}} = \omega_A$ of the input shaft. Assume the bearings and contacts are frictionless.

a) What is the input power?

b) What is the output power?

c) What is the output torque $M_{\text{output}} = M_C$, the torque that gear C applies to its surroundings in the clockwise direction?



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Problem 13.6.19

13.6.20 At the input to a gear box a 100 lbf force is applied to gear A. At the output, the machinery (not shown) applies a force of F_B to the output gear. Gear A rotates at constant angular rate $\omega = 2\,\text{rad/s}$, clockwise.

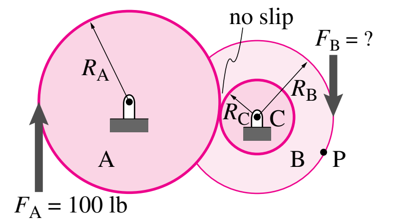
a) What is the angular speed of the right gear?

b) What is the velocity of point P?

c) What is F_B?

d) If the gear bearings had friction, would F_B have to be larger or smaller in order to achieve the same constant velocity?

e) If instead of applying a 100 lbf to the left gear it is driven by a motor (not shown) at constant angular speed ω, what is the angular speed of the right gear?



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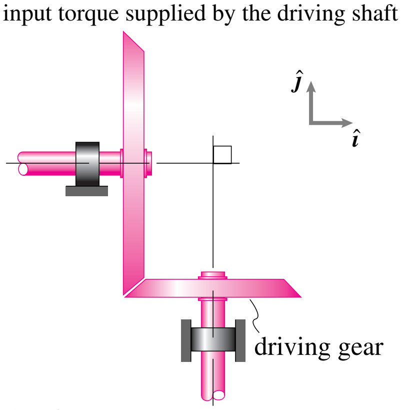
Problem 13.6.20: Two gears with end loads.

13.6.21 A bevel gear system. A bevel-type gear system, shown in the figure, is used to transmit power between two shafts that are perpendicular to each other. The driving gear has a mean radius of 50 mm and rotates at a constant speed $\omega = 150$ rpm. The mean radius of the driven gear is 80 mm and the driven shaft is expected to deliver a torque of $M_{out} = 25$ N·m. Assuming no power loss, find the input torque supplied by the driving shaft



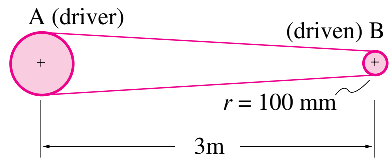
Filename:pfigure4-rpa
Problem 13.6.21: A bevel gear

13.6.22 *Belt drives* are used to transmit power between parallel shafts. Two parallel shafts, 3 m apart, are connected by a belt passing over the pulleys A and B fixed to the two shafts. The driver pulley A rotates at a constant 200 rpm. The speed ratio between the pulleys A and B is 1:2.5. The input torque is 350 N·m. Assume no loss of power between the two shafts.

a) Find the input power. *

b) Find the rotational speed of the driven pulley B *.

c) Find the output torque at B. *



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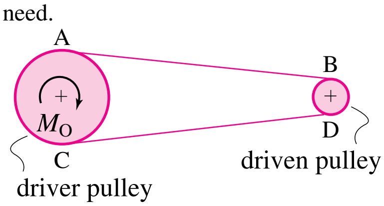
Problem 13.6.22

13.6.23 In the belt drive system shown, assume that the driver pulley rotates at a constant angular speed ω. If the motor applies a constant torque M_O on the driver pulley, show that the tensions in the two parts, AB and CD, of the belt must be different. Which part has a greater tension? Does your conclusion about unequal tension depend on whether the pulley is massless or not? Assume any dimensions you need.



Filename:pfigure4-rpe
Problem 13.6.23

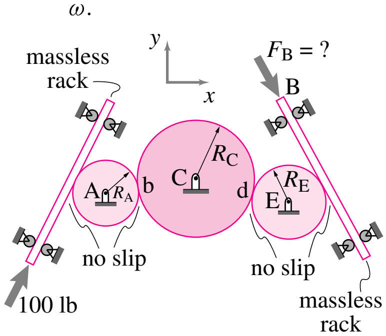
13.6.24 Two racks connected by three constant rate gears. A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of F_B to the other rack.

a) Assume the gear-train is spinning at constant rate and is frictionless. What is F_B? *

b) If the gear bearings had friction would that increase or decrease F_B to achieve the same constant rate?

c) If instead of applying a 100 lbf to the left rack it is driven by a motor (not shown) at constant speed v, what is the speed of the right rack? *

d) If the angular velocity of the gear is increasing at rate α does this increase or decrease F_B at the given ω.



Filename:ch4-5
Problem 13.6.24: Two racks connected by three gears.

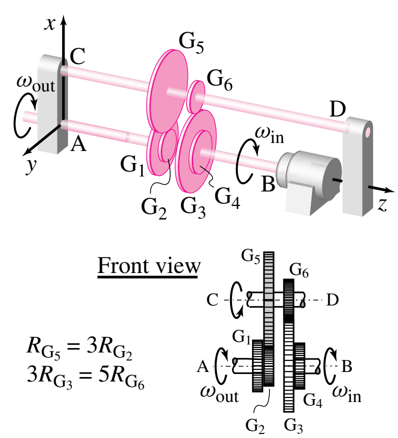
13.6.25 -3-D accelerating gear train. This is really a 2-D problem; each gear turns in a different parallel plane. Shaft B is rigidly connected to gears G_4 and G_5. G_3 meshes with gear G_6. Gears G_6 and G_5 are both rigidly attached to shaft AD. Gear G_5 meshes with G_2 which is welded to shaft A. Shaft A and shaft B spin independently. Assume you know the torque M_{input}, angular velocity ω_{input} and the angular acceleration α_{input} of the input shaft. Assume the bearings and contacts are frictionless.

a) What is the input power?

b) What is the output power?

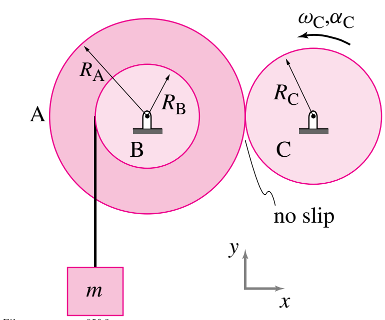
c) What is the angular velocity ω_{output} of the output shaft?

d) What is the output torque M_{output}?



Filename:3dgeartrain
Problem 13.6.25: A 3-D gear train.

13.6.26 Gear A with radius $R_A = 400$ mm is rigidly connected to a drum B with radius $R_B = 200$ mm. The combined moment of inertia of the gear and the drum about the axis of rotation is $I_{zz} = 0.5\ \text{kg}\cdot\text{m}^2$. Gear A is driven by gear C which has radius $R_C = 300$ mm. As the drum rotates, a 5 kg mass m is pulled up by a string wrapped around the drum. At the instant of interest, The angular speed and angular acceleration of the driving gear are 60 rpm and 12 rpm/s, respectively. Find the acceleration of the mass m. *

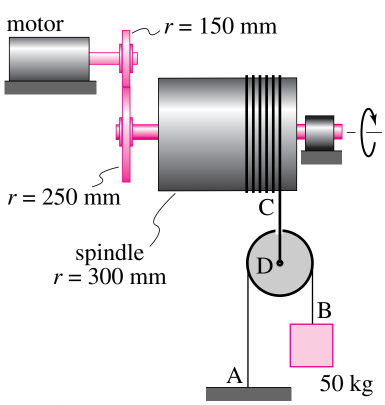


Filename:summer95f-2
Problem 13.6.26

13.6.27 A spindle and pulley arrangement is used to hoist a 50 kg mass as shown in the figure. Assume that the pulley is to be of negligible mass. When the motor is running at a constant 100 rpm,

a) Find the velocity of the mass at B.

b) Find the tension in strings AB and CD.



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Problem 13.6.27

13.6.28 Accelerating rack and pinion. The two gears shown are welded together and spin on a frictionless bearing. The inner gear has radius 0.5 m and negligible mass. The outer disk has 1 m radius and a uniformly distributed mass of 0.2 kg. They are loaded as shown with the force $F = 20$ N on the massless rack which is held in place by massless frictionless rollers. At the time of interest the angular velocity is $\omega = 2$ rad/s (though ω is not constant). The point P is on the disk a distance 1 m from the center. At the time of interest, point P is on the positive y axis.

a) What is the speed of point P?

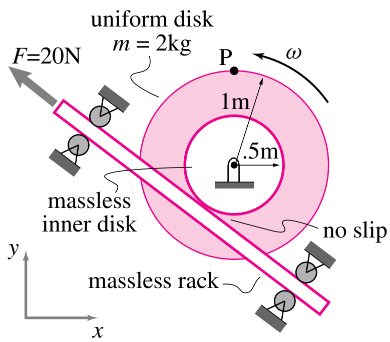
b) What is the velocity of point P?

c) What is the angular acceleration α of the gear?

d) What is the acceleration of point P?

e) What is the magnitude of the acceleration of point P?

f) What is the rate of increase of the speed of point P?



Filename:pg45-2
Problem 13.6.28: Accelerating rack and pinion

13.6.29 Two racks connected by a gear. A 100 lbf force is applied to one rack. At the output the machinery (not shown) applies a force F_B to the other rack.

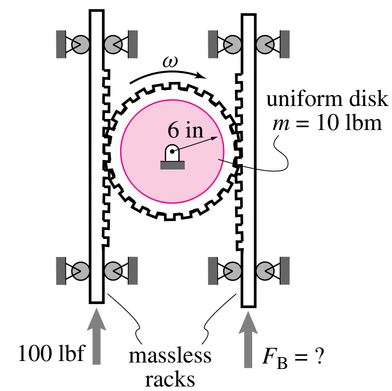
a) Assume the gear is spinning at constant rate and is frictionless. What is F_B?

b) If the gear bearing had friction, would that increase or decrease F_B to achieve the same constant rate?

c) If the angular velocity of the gear is increasing at rate α, does this increase or decrease F_B at the given ω.

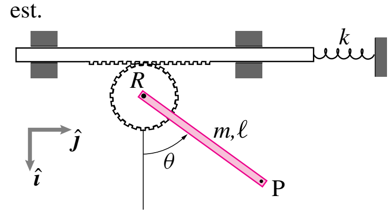
d) If the output load F_B is given then the motion of the machine can be found from the input load. Assume that the machine starts from rest with a given output load. So long as rack B moves in the opposite direction of the output force F_B the output power is positive.

   1. For what values of F_B is the output power positive?
   2. For what values of F_B is the output work maximum if the machine starts from rest and runs for a fixed amount of time?



Filename:pg131-1
Problem 13.6.29: Two racks connected by a gear.

13.6.30 A rack and pinion constrained by a linear spring. Neglect gravity. The spring is relaxed when the angle $\theta = 0$. Assume the system is released from rest at $\theta = \theta_0$. What is the acceleration of the point P at the end of the stick when $\theta = 0$? Answer in terms of any or all of m, R, ℓ, θ_0, k, $\hat{\imath}$, and $\hat{\jmath}$. [Hint: There are several steps of reasoning required. You might want to draw FBD(s), use angular momentum balance, set up a differential equation, solve it, plug values into this solution, and use the result to find the quantities of interest.



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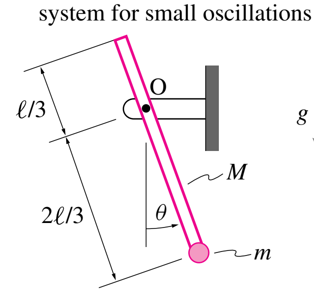
Problem 13.6.30

Pendulum Problems

13.6.31 A small particle of mass m is attached to the end of a thin rod of mass M (uniformly distributed), which is pinned at hinge O, as depicted in the figure.

a) Obtain the equation of motion governing the rotation θ of the rod.

b) What is the natural frequency of the system for small oscillations θ?



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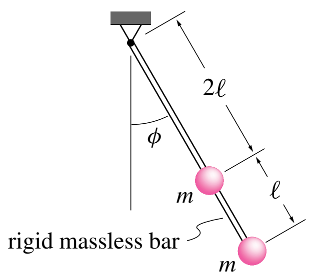
Problem 13.6.31

13.6.32 A rigid massless rod has two equal masses m_B and m_C ($m_B = m_C = m$) attached to it at distances 2ℓ and 3ℓ, respectively, measured along the rod from a frictionless hinge located at a point A. The rod swings freely from the hinge. There is gravity. Let ϕ denote the angle of the rod measured from the vertical. Assume that ϕ and $\dot{\phi}$ are known at the moment of interest.

a) What is $\ddot{\phi}$? Find $\ddot{\phi}$ in terms of m, ℓ, g, ϕ and $\dot{\phi}$.

b) What is the force of the hinge on the rod? Solve in terms of m, ℓ, g, ϕ, $\dot{\phi}$, $\ddot{\phi}$ and any unit vectors you may need to define.

c) Would you get the same answers if you put a mass $2m$ at 2.5ℓ? Why or why not?



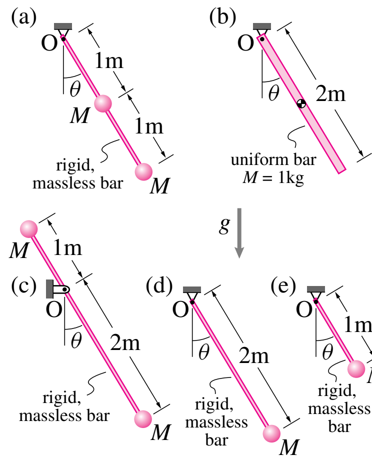
Filename:pfigure-twomasspend
Problem 13.6.32

13.6.33 For the pendula in the figure :

a) Without doing any calculations, try to figure out the relative durations of the periods of oscillation for the five pendula (i.e. the order, slowest to fastest) Assume small angles of oscillation.

b) Calculate the period of small oscillations. [Hint: use balance of angular momentum about the point 0]. *

c) Rank the relative duration of oscillations and compare to your intuitive solution in part (a), and explain in words why things work the way they do.



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Problem 13.6.33

13.6.34 A pegged compound pendulum. A uniform bar of mass m and length ℓ hangs from a peg at point C and swings in the vertical plane about an axis passing through the peg. The distance d from the center of mass of the rod to the peg can be changed by putting the peg at some other point along the length of the rod.

a) Find the angular momentum of the rod about point C.

b) Find the rate of change of angular momentum of the rod about C.

c) How does the period of the pendulum vary with d? Show the variation by plotting the period against $\frac{d}{\ell}$. [Hint, you must first find the equations of motion, linearize for small θ, and then solve.] *

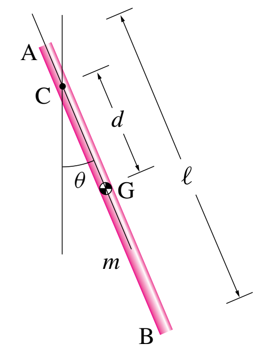
d) Find the total energy of the rod (using point C as a datum for potential energy).

e) Find $\ddot{\theta}$ when $\theta = \pi/6$.

f) Find the reaction force on the rod at C, as a function of m, d, ℓ, θ, and $\dot{\theta}$.

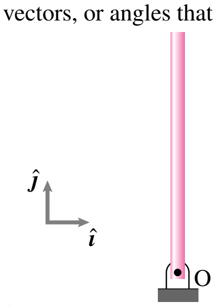
g) For the given rod, what should be the value of d (in terms of ℓ) in order to have the fastest pendulum? *

h) **Test of Schuler's pendulum.** The pendulum with the value of d obtained in (g) is called the Schuler's pendulum. It is not only the fastest pendulum but also the "most accurate pendulum". The claim is that even if d changes slightly over time due to wear at the support point, the period of the pendulum does not change much. Verify this claim by calculating the percent error in the time period of a pendulum of length $\ell = 1\,\text{m}$ under the following three conditions: (i) initial $d = 0.15\,\text{m}$ and after some wear $d = 0.16\,\text{m}$, (ii) initial $d = 0.29\,\text{m}$ and after some wear $d = 0.30\,\text{m}$, and (iii) initial $d = 0.45\,\text{m}$ and after some wear $d = 0.46\,\text{m}$. Which pendulum shows the least error in its time period? What is the connection between this result and the plot obtained in (c)?



Filename:pfigure-s94h8p6
Problem 13.6.34

13.6.35 A uniform stick of length ℓ and mass m is a hair away from vertically up position when it is released with no angular velocity (a 'hair' is a technical word that means 'very small amount, zero for some purposes'). It falls to the right. What is the force on the stick at point O when the stick is horizontal. Solve in terms of ℓ, m, g, $\hat{\imath}$, and $\hat{\jmath}$. Carefully define any coordinates, base vectors, or angles that you use.



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Problem 13.6.35

13.6.36 A massless 10 meter long bar is supported by a frictionless hinge at one end and has a 3.759 kg point mass at the other end. It is released at $t = 0$ from a tip angle of $\phi = .02$ radians measured from vertically upright position (hinge at the bottom). Use $g = 10\,\text{m}\,\text{s}^{-2}$.

a) Using a small angle approximation and the solution to the resulting linear differential equation, find the angle of tip at $t = 1$s and $t = 7$s. Use a calculator, not a numerical integrator.

b) Using numerical integration of the non-linear differential equation for an inverted pendulum find ϕ at $t = 1$s and $t = 7$s.

c) Make a plot of the angle versus time for your numerical solution. Include on the same plot the angle versus time from the approximate linear solution from part (a).

d) Comment on the similarities and differences in your plots.

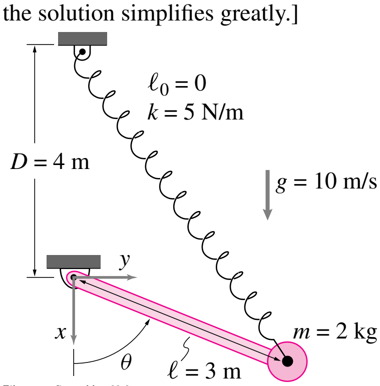
13.6.37 A zero length spring (relaxed length $\ell_0 = 0$) with stiffness $k = 5\,\mathrm{N/m}$ supports the pendulum shown.

a) Find $\ddot{\theta}$ assuming $\dot{\theta} = 2\,\mathrm{rad/s}$, $\theta = \pi/2$. *

b) Find $\ddot{\theta}$ as a function of $\dot{\theta}$ and θ (and k, ℓ, m, and g.) *

[Hint: use vectors (otherwise it's hard)]

[Hint: For the special case, $kD = mg$, the solution simplifies greatly.]



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Problem 13.6.37

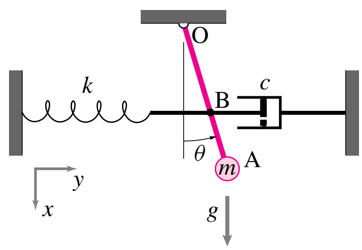
13.6.38 A spring-mass-damper system is depicted in the figure. The horizontal damping force applied at B is given by $F_D = -c\,\dot{y}_B$
The dimensions and parameters are as follows:

$$\begin{aligned} r_{B/0} &= 2\,ft \\ r_{A/0} &= \ell = 3\,ft \\ k &= 2\,\mathrm{lbf/\,ft} \\ c &= 0.3\,\mathrm{lbf \cdot s/\,ft} \end{aligned}$$

For small θ, assume that $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$.

a) Determine the natural circular frequency of small oscillations about equilibrium for the pendulum shown. The static equilibrium position is $\theta = 0$ (pendulum hanging vertically), so the spring is at its rest point in this position. Idealize the pendulum as a point mass attached to a rigid massless rod of length 1, so $I^O = m\ell^2$. Also use the "small angle approximation" where appropriate.

b) Sketch a graph of θ as a function of $t (t \geq 0)$ if the pendulum is released from rest at position $\theta = 0.2$ rad when $t = 0$. Your graph should show the correct qualitative behavior, but calculations are not necessary.



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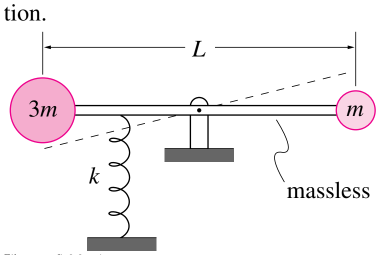
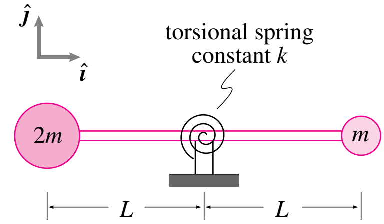
Problem 13.6.38

13.6.39 The asymmetric dumbbell shown in the figure is pivoted in the center and also attached to a spring at one quarter of its length from the bigger mass. When the bar is horizontal, the compression in the spring is y_s. At the instant of interest, the bar is at an angle θ from the horizontal; θ is small enough so that $y \approx \frac{L}{2}\theta$. If, at this position, the velocity of mass 'm' is $v\,\hat{\jmath}$ and that of mass $3m$ is $-v\,\hat{\jmath}$, evaluate the power term ($\sum \vec{F} \cdot \vec{v}$) in the energy balance equation.



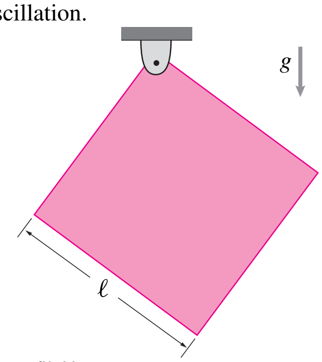
Filename:pfig2-3-rp4
Problem 13.6.39

13.6.40 The dumbbell shown in the figure has a torsional spring with spring constant k (torsional stiffness units are $\frac{\mathrm{kN \cdot m}}{\mathrm{rad}}$). The dumbbell oscillates about the horizontal position with small amplitude θ. At an instant when the angular velocity of the bar is $\dot{\theta}\hat{k}$, the velocity of the left mass is $-L\dot{\theta}\,\hat{\jmath}$ and that of the right mass is $L\dot{\theta}\,\hat{\jmath}$. Find the expression for the power P of the spring on the dumbbell at the instant of interest.



Filename:pfig2-3-rp10
Problem 13.6.40

13.6.41 A square plate with side ℓ and mass m is hinged at one corner in a gravitational field g. Find the period of small oscillation.

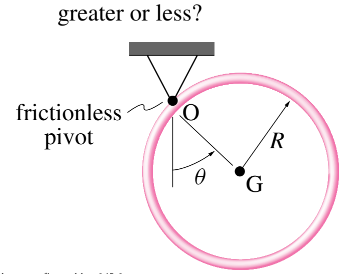


Filename:p-f96-p3-2
Problem 13.6.41

13.6.42 A thin hoop of radius R and mass M is hung from a point on its edge and swings in its plane. Assuming it swings near to the position where its center of mass G is below the hinge:

a) What is the period of its swinging oscillations?

b) If, instead, the hoop was set to swinging in and out of the plane would the period of oscillations be greater or less?



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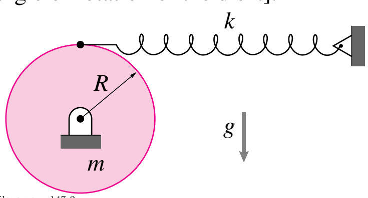
Problem 13.6.42

13.6.43 Oscillating disk. A uniform disk with mass m and radius R pivots around a frictionless hinge at its center. It is attached to a massless spring which is horizontal and relaxed when the attachment point is directly above the center of the disk. Assume small rotations and the consequent geometrical simplifications. Assume the spring can carry compression. What is the period of oscillation of the disk if it is disturbed from its equilibrium configuration? [You may use the fact that, for the disk shown, $\vec{\dot{H}}_{/O} = \frac{1}{2}mR^2\ddot{\theta}\hat{k}$, where θ is the angle of rotation of the disk.]. *

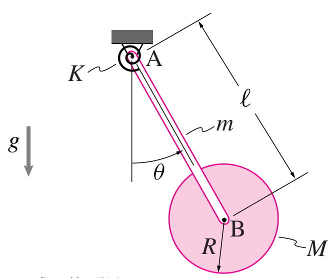


Filename:pg147-2
Problem 13.6.43

13.6.44 A thin rod of mass m and length ℓ is hinged with a torsional spring of stiffness K at A, and is connected to a thin disk of mass M and radius R at B. The spring is uncoiled when $\theta = 0$. Determine the natural frequency ω_n of the system for small oscillations θ, assuming that the disk is:

a) welded to the rod, and *

b) pinned frictionlessly to the rod. *



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Problem 13.6.44

13.6.45 Robotics problem: Simplest balancing of an inverted pendulum. You are holding a stick upside down, one end is in your hand, the other end sticking up. To simplify things, think of the stick as massless but with a point mass at the upper end. Also, imagine that it is only a two-dimensional problem (either you can ignore one direction of falling for simplicity or imagine wire guides that keep the stick from moving in and out of the plane of the paper on which you draw the problem).

You note that if you model your holding the stick as just having a stationary hinge then you get $\ddot{\phi} = \frac{g}{l}\sin\phi$. Assuming small angles, this hinge leads to exponentially growing solutions. Upside-down sticks fall over. How can you prevent this falling?

One way to do keep the stick from falling over is to firmly grab it with your hand, and if the stick tips, apply a torque in order to right it. This corrective torque is (roughly) how your ankles keep you balanced when you stand upright. Your task in this assignment is to design a robot that keeps an inverted pendulum balanced by applying appropriate torque.

Your model is: Inverted pendulum, length ℓ, point mass m, and a hinge at the bottom with a motor that can apply a torque T_m. The stick might be tipped an angle ϕ from the vertical. A *horizontal* disturbing force $F(t)$ is applied to the mass (representing wind, annoying friends, etc).

a) Draw a picture and a FBD

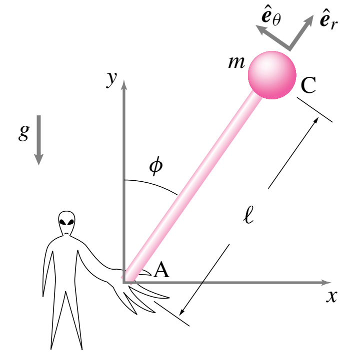
b) Write the equation for angular momentum balance about the hinge point. *

c) Imagine that your robot can sense the angle of tip ϕ and its rate of change $\dot{\phi}$ and can apply a torque in response to that sensing. That is you can make T_m any function of ϕ and $\dot{\phi}$ that you want. Can you find a function that will make the pendulum stay upright? Make a guess (you will test it below).

d) Test your guess the following way: plug it into the equation of motion from part (b), linearize the equation, assume the disturbing force is zero, and see if the solution of the differential equation has exponentially growing (i.e. unstable) solutions. Go back to (c) if it does and find a control strategy that works.

e) Pick numbers and model your system on a computer using the full non-linear equations. Use initial conditions both close to and far from the upright position and plot ϕ versus time.

f) If you are ambitious, pick a non-zero forcing function $F(t)$ (say a sine wave of some frequency and amplitude) and see how that affects the stability of the solution in your simulations.



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Problem 13.6.45

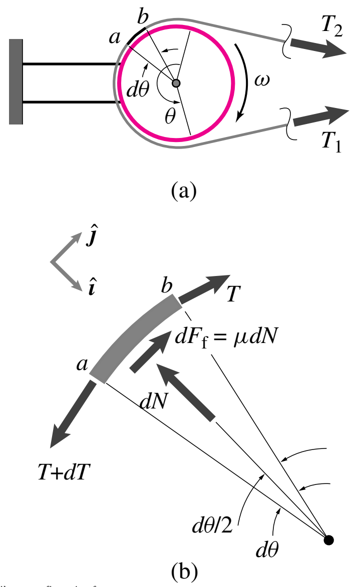
Mixed Problems

13.6.46 Assume that the pulley shown in figure(a) rotates at a constant speed ω. Let the angle of contact between the belt and pulley surface be θ. Assume that the belt is massless and that the condition of impending slip exists between the pulley and the belt. The free body diagram of an infinitesimal section ab of the belt is shown in figure(b).

a) Write the equations of linear momentum balance for section ab of the belt in the $\hat{\imath}$ and $\hat{\jmath}$ directions.

b) Eliminate the normal force N from the two equations in part (a) and get a differential equation for the tension T in terms of the coefficient of friction μ and The contact angle θ.

c) Show that the solution to the equation in part (b) satisfies $\frac{T_1}{T_2} = e^{\mu\theta}$, where T_1 and T_2 are the tensions in the lower and the upper segments of the belt, respectively.



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Problem 13.6.46

13.6.47 A belt drive is required to transmit 15 kW power from a 750 mm diameter pulley rotating at a constant 300 rpm to a 500 mm diameter pulley. The centers of the pulleys are located 2.5 m apart. The coefficient of friction between the belt and pulleys is $\mu = 0.2$.

a) (See problem 13.6.46.) Draw a neat diagram of the pulleys and the belt-drive system and find the *angle of lap*, the contact angle θ, of the belt on the driver pulley.

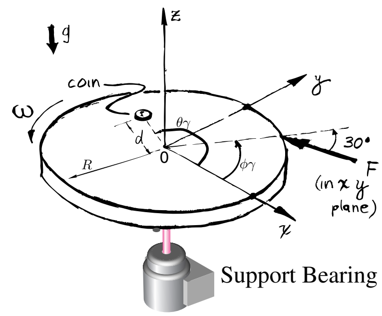
b) Find the rotational speed of the driven pulley.

c) (See the figure in problem 13.6.46.) The power transmitted by the belt is given by *power = net tension* × *belt speed*, i.e., $P = (T_1 - T_2)v$, where v is the linear speed of the belt. Find the maximum tension in the belt. [Hint: $\frac{T_1}{T_2} = e^{\mu\theta}$ (see problem 13.6.46).]

d) The belt in use has a 15 mm × 5 mm rectangular cross-section. Find the maximum tensile stress in the belt.

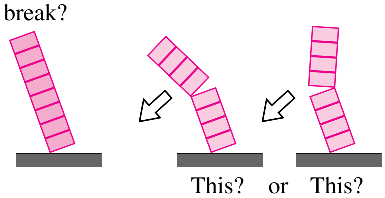
13.6.48 Slippery money A round uniform flat horizontal platform with radius R and mass m is mounted on frictionless bearings with a vertical axis at 0. At the moment of interest it is rotating counter clockwise (looking down) with angular velocity $\vec{\omega} = \omega\hat{k}$. A force in the xy plane with magnitude F is applied at the perimeter at an angle of 30° from the radial direction. The force is applied at a location that is ϕ from the fixed positive x axis. At the moment of interest a small coin sits on a radial line that is an angle θ from the fixed positive x axis (with mass much much smaller than m). Gravity presses it down, the platform holds it up, and friction (coefficient=μ) keeps it from sliding.

Find the biggest value of d for which the coin does not slide in terms of some or all of $F, m, g, R, \omega, \theta, \phi$, and μ.



Filename:pfigure-slipperymoney
Problem 13.6.48

13.6.49 Frequently parents will build a tower of blocks for their children. Just as frequently, kids knock them down. In falling (even when they start to topple aligned), these towers invariably break in two (or more) pieces at some point along their length. Why does this breaking occur? What condition is satisfied at the point of the break? Will the stack bend towards or away from the floor after the break?



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Problem 13.6.49

CHAPTER 14

General planar motion of a single rigid object

The main goal here is to generate equations of motion for general planar motion of a (planar) rigid object that may roll, slide or be in free flight. Multi-object systems are also considered so long as they do not involve other kinematic constraints between the bodies. Features of the solution that can be obtained from analysis are discussed, as are numerical solutions.

Contents

Many machine and structural parts move in straight-lines (Chapter 12) or circles (Chapters 13). But other things have with more general motions, like a plane in unsteady flight or a connecting rod in a car engine. Keeping track of such motion is a bit more difficult.

In this chapter we will use these two modeling approximations:

- The objects are planar, or symmetric with respect to a plane; and
- They have planar motions in that plane.

A *planar object* is one where the whole object is flat and all its matter is confined to one plane, say the xy plane. This is a palatable approximation for a piece cut out of flat sheet metal. For more substantial real objects, like a full car, the approximation seems at a glance to be terrible. But it turns out that so long as the *motion* is planar and the car is reasonably idealized as symmetrical (left to right) that treating the car as equivalent to it being squished into a plane does not introduce any more approximation. Thus, even in this 3-D world we live in with 3-D objects, it is fruitful to do 2-D analysis of the type you will learn in this chapter.

A *planar motion* is one where the velocities of all points are in the same constant plane, say a fixed xy plane, at all times and where points with, say, the same z coordinate have the same velocity. The positions of the points do not have to be in same plane for a planar motion. Each point stays in a plane, but different points can be in different planes, with each plane parallel to the others.

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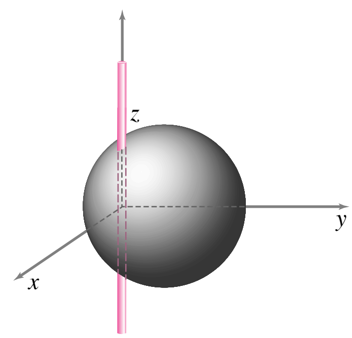
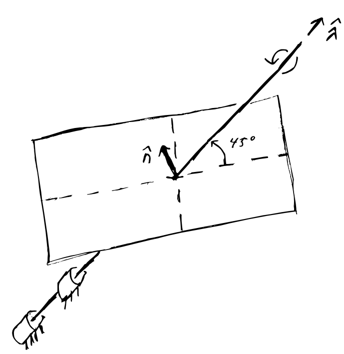
Figure 14.1: Planar motion of a 3D car. If the car is symmetrical it can be studied by the means of this chapter.



Filename:tfigure-ball

Figure 14.2: Planar motion of a skewered sphere. This can be studied by the means in this chapter.



Filename:tfigure-crookedplate

Figure 14.3: Planar motion of a planar object. But the plane of the motion is *not* the plane of the object. This is not a natural topic for this chapter.

Example: A car going over a hill

Assume the road is straight in map view in, say, the x direction. Assume the whole width of the road has the same hump. Although the car is clearly not planar, the car motion is probably close to planar, with the velocities of all points in the car in the xy plane (see fig. 14.1)

Example: Skewered sphere

A sphere skewered and rotating about a fixed axes in the $\hat{\boldsymbol{k}}$ direction has a planar motion (see fig. 14.2). The points on the object do not all lie in a common plane. But all of the velocities are orthogonal to $\hat{\boldsymbol{k}}$ and thus in the xy plane. This problem does fit in with the methods of this chapter. The symmetry of the sphere with respect to the xy plane makes it so that the three-dimensional mass distribution does not invalidate the two-dimensional analysis.

Example: Skewered plate
A flat rectangular plate with normal $\hat{\boldsymbol{n}}$ has a fixed axis of rotation in the direction $\hat{\boldsymbol{\lambda}}$ that makes a 45° to $\hat{\boldsymbol{n}}$ (see fig. 14.3). This is a planar object (a plane normal to $\hat{\boldsymbol{n}}$) in planar motion (all velocities are in the plane normal to $\hat{\boldsymbol{\lambda}}$). But the plane of motion is not the plane of the mass distribution, the object is not symmetric with respect to a motion plane, so this example does not fit into the discussion of this chapter①.

① Actually, a two-dimensional analysis of the plate in this example we would be legitimate in this sense. Project all the plate's mass into the plane normal to the $\hat{\boldsymbol{\lambda}}$ direction. The projections of the forces on this plane would be correctly predicted, but three dimensional effects, like those associated with dynamic imbalance, would be lost in this projection.

No real object is exactly planar and no real motion is exactly a planar motion. But many objects are relatively flat and thin or symmetrical and many motions are approximately planar motions. Thus many, if not most, simple engineering analyses assume planar motion. For bodies that are approximately symmetric about the xy plane of motion (such as a car, if the asymmetrically placed driver's mass *etc.* is neglected), there is no loss in doing a two-dimensional planar rather than full three dimensional analysis.

The plan of this chapter. We start with planar kinematics. Then we evaluate and use expressions for the rates of change of linear and angular momentum for planar bodies. Finally we discuss rolling, sliding and collisions.

14.1 Description of motion: planar rigid-object kinematics

As a rigid object moves, how do the points on it move? There are two reasons to ask this question. First, velocities and accelerations of mass points are needed to apply the momentum-balance equations. Second, formulas for positions, velocities and accelerations of points are useful to understand *mechanisms*, machines where various parts (each one usually idealized as a rigid object) are connected to each other with hinges and bearings of one type of another.

The central observation in all rigid-object kinematics is the definition of a rigid object:

> All pairs of points on a single rigid object keep constant distance from each other as the object moves.

In this section you will learn how to use rigidity to calculate positions, velocities and accelerations of all points (millions and billions of them) on a rigid object given only a few numbers (9 of them). This goal is achieved by putting together the ideas from Chapter 11 (arbitrary motion of one particle), Chapter 12 (straight-line motion), and Chapter 13 (circular motion of a rigid object in a plane).

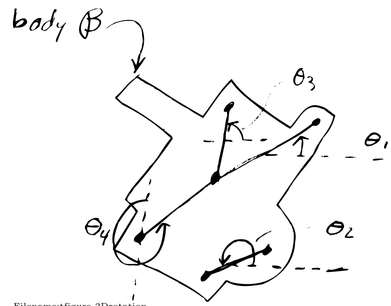
Displacement and rotation

When a planar object (say a machine part) $\mathcal{B}$ moves from one configuration in the plane to another it has a displacement and a rotation. It starts in some reference configuration *. We mark a reference point on the body that, in the reference configuration, coincides with a fixed reference point, say 0①. We also mark a (directed) line on the body that, in the reference configuration, coincides with a fixed reference line, say the positive x axis.

We could measure rotation by measuring the rotation of any line segment connecting any pair of points fixed to the object. For each line we keep track of the angle that line makes with a line fixed in space, say the positive x or y axis. Its simplest to stick to the convention that counter-clockwise rotations are positive (fig. 14.4). The angles θ_1, $\theta_2 \ldots$, all change with time and are all different from each other. But all the angles change the same amount, just like in section 13.3. We can pick any one line we like for definiteness and measure the object rotation by the rotation of that line. So

> The net motion of a rigid planar object is described by *translation*, the vector displacement of a reference point from a reference position $\vec{r}_{0'/0} = \vec{r}_{00'}$, and a scalar *rotation* θ of the reference line from its reference orientation.

①The body never has to pass through this reference position, however. For example, the position of an airplane flying from New York to Mumbai is measured relative to a point in the Gulf of Guinea 1000 miles west of Gabon (the location of O° longitude and O° latitude) even though the airplane never goes there (nor does anyone want it to).



Filename:tfigure-2Drotation

Figure 14.4: Rotation of object $\mathcal{B}$ is measured by the rotation of real or imagined lines marked on the object. The lines make different angles: $\theta_1 \neq \theta_2$, $\theta_2 \neq \theta_3$ etc, but $\dot\theta_1 = \dot\theta_2 = \dot\theta_3 = \ldots$. Angular velocity is defined as $\vec{\omega} = \omega\hat{k}$ with $\omega \equiv \dot\theta_1 = \dot\theta_2 = \dot\theta_3 = \ldots$.

The position of a point on a moving rigid object.

Given that P on the object is at $\vec{r}_{P/0}{}^*$ in the reference (*) configuration, where is it (What is $\vec{r}_{P/0}$?) after the object has been displaced by $\vec{r}_{0'/0}$ and rotated an angle θ?

The base-independent or direct vector notation. An easy way to treat this is to write the new position of P as (see fig. 14.5), in the *base-independent* or *direct vector* representation of the position of P,

$$\vec{r}_{P/0} = \vec{r}_{0'/0} + \vec{r}_{P/0'}. \tag{14.1}$$

The vector $\vec{r}_{0'/0}$ describes *translation*, that's half the story. The other term $\vec{r}_{P/0'}$ we find by rotating $\vec{r}_{P/0}{}^*$ as we did in Section 13.4. This 'base-independent' formula is correct no matter what base vectors are used to represent the vectors in the formula (*e.g.,* $\hat{\imath}$ and $\hat{\jmath}$ or $\hat{\imath}'$ and $\hat{\jmath}'$).

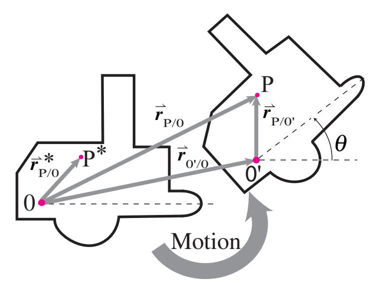


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Figure 14.5: The displacement and rotation of a planar object relative to a reference configuration.

Fixed basis or component representation. For a given basis, say $\hat{\imath}$ and $\hat{\jmath}$ associated with x and y, we can write the coordinates of a point as,

$$[\vec{r}_{P/0}]_{xy} = \underbrace{[\vec{r}_{0'/0}]_{xy}}_{\text{displacement}} + \underbrace{[R(\theta)]\,[\vec{r}_{P/0'}]_{x'y'}}_{\text{rotation}} \tag{14.2}$$

or, writing out all the components of the vectors and matrices, We get the *fixed basis* or *component* representation of the motion:

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x^*_{P/0} \\ y^*_{P/0} \end{bmatrix}. \tag{14.3}$$

As the motion progresses the displacement $\begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix}$ changes with time as does the rotation angle θ. In this *fixed basis* or *component* representation of the motion of eqn. (14.3) we get the components of the position in terms of base vectors that are fixed in space.

Example:
If in the reference position a particle on a rigid object is at $\vec{r}_{P/0} = (1\hat{\imath} + 2\hat{\jmath})$ m and the object displaces by $\vec{r}_{0'/0} = (3\hat{\imath} + 4\hat{\jmath})$ m and rotates by $\theta = \pi/3\,\text{rad} = 60°$ relative to that configuration, then its new position is:

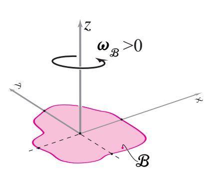
$$\begin{aligned}
[\vec{r}_{P/0}]_{xy} &= [\vec{r}_{0'/0}]_{xy} + [R(\theta)]\,[\vec{r}_{P/0'}]_{x'y'} \\
&= \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x^*_{P/0} \\ y^*_{P/0} \end{bmatrix} \\
&= \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \text{m} \\
&= \begin{bmatrix} 3.5 - \sqrt{3} \\ 5 + \sqrt{3}/2 \end{bmatrix} \text{m} \\
\Rightarrow \vec{r}_{P/0} &= \left((3.5 - \sqrt{3})\hat{\imath} + (5 + \sqrt{3}/2)\hat{\jmath}\right) \text{m}
\end{aligned}$$

Changing-base representation. Finally, the *changing base* representation uses base vectors $\hat{\imath}'$, $\hat{\jmath}'$ that are aligned with $\hat{\imath}$, $\hat{\jmath}$ in the reference configuration but which are glued to the rotating object. If we define x' and y' as the x and y components of P in the reference (*) configuration we have that

$$\begin{aligned}
[\vec{r}_{P/0}{}^*]_{xy} &= [\vec{r}_{P/0'}{}^*]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix} \\
\text{so} \quad \vec{r}_{P/0} &= \left(x_{0'/0}\hat{\imath} + y_{0'/0}\hat{\jmath}\right) + \left(x'\hat{\imath}' + y'\hat{\jmath}'\right). \qquad (14.4)
\end{aligned}$$

The second equation above is the *changing base* representation.

Although all three notations are in some sense equivalent, they have their best uses depending on context. The base-independent or direct-vector notation eqn. (14.1) is the most direct, least ornate and shortest to write. The component or fixed base representation eqn. (14.3) is the best for computer calculations. And the changing-base notation eqn. (14.4) is the most explicit for pencil and paper work.



Filename:tfigure-posomega

Figure 14.6: It is generally best to take positive ω to be counterclockwise when viewed from the positive z axis.

Angular velocity

Because all lines object $\mathcal{B}$ rotate at the same rate (at a given instant) $\mathcal{B}$'s rotation rate is the single number we call $\omega_{\mathcal{B}}$ ('omega b'). As for the case of pure rotation, we define a vector angular velocity with magnitude $\omega_{\mathcal{B}}$ which is perpendicular to the xy plane:

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \underbrace{\omega_{\mathcal{B}}}_{\dot{\theta}}\,\hat{\boldsymbol{k}}$$

where $\dot{\theta}$ is the rate of change of the angle of any line marked on object $\mathcal{B}$.

So long as you are careful to define angular velocity by the rotation of line segments and not by the motion of individual particles, the concept of angular velocity in general motion is defined exactly as for an object rotating about a fixed axis. The key kinematic fact is, in words:

> On a rigid object in 2D any given point is moving in circles about any other given point (relative to that point).

So you can think of the general planar motion of a rigid object is the general motion of a point plus uniform circular motion about that point. When a rigid object moves it always has an angular velocity (possibly zero). If we call the object $\mathcal{B}$ (script B for body), we then call the object's angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$. Again, it is generally best to use the CCW sign convention that when $\omega_{\mathcal{B}} > 0$ the object is rotating counterclockwise when viewed looking in from a point outside the object on the positive z axis (that is, 'looking down', see fig. 14.6).

> The angular velocity vector $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ of a object $\mathcal{B}$ describes it's rate and direction of rotation. For planar motions $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega_{\mathcal{B}}\hat{\boldsymbol{k}}$.

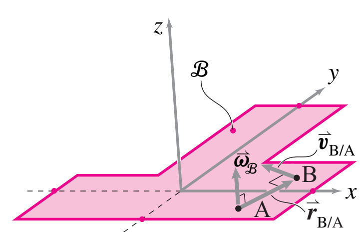
Relative velocity of two points on a rigid object

For any two points A and B glued to a rigid object $\mathcal{B}$ the relative velocity $\vec{\boldsymbol{v}}_{\mathrm{B/A}}$ of the points ('the velocity of B relative to A') is given by the cross product of the angular velocity of the object with the relative position of the two points (fig. 14.7):

$$\vec{\boldsymbol{v}}_{B/A} \equiv \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}. \tag{14.5}$$

That is, the relative velocity of two points on a rigid object is the same as would be predicted for one of the points if the other were stationary. The derivation of this formula is the same as for planar circular motion.

Note that we use three-dimensional cross products even though we are doing planar 2D kinematics. Generally the plane of motion is the xy plane and $\vec{\boldsymbol{\omega}}$ will be in the z direction. Because $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ must be perpendicular to $\vec{\boldsymbol{\omega}}$ it is perpendicular to the z axis. So the three dimensional cross product $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ gives a vector that is perpendicular to $\vec{\boldsymbol{r}}$ and is in the xy plane .



Filename:tfigure-vperptoomega

Figure 14.7: The relative velocity of points A and B is in the xy plane and perpendicular to the line segment AB.

We can also represent the relative velocity in the changing base notation as

$$\begin{aligned}
\vec{\boldsymbol{v}}_{B/A} &= \frac{d}{dt}\left(x'_{\mathrm{B/A}}\hat{\boldsymbol{\imath}}' + y'_{\mathrm{B/A}}\hat{\boldsymbol{\jmath}}'\right)\\
&= x'_{\mathrm{B/A}}\frac{d}{dt}\hat{\boldsymbol{\imath}}' + y'_{\mathrm{B/A}}\frac{d}{dt}\hat{\boldsymbol{\jmath}}'\\
&= x'_{\mathrm{B/A}}\left(\vec{\boldsymbol{\omega}}_{\mathcal{B}}\times\hat{\boldsymbol{\imath}}'\right) + y'_{\mathrm{B/A}}\left(\vec{\boldsymbol{\omega}}_{\mathcal{B}}\times\hat{\boldsymbol{\jmath}}'\right).
\end{aligned}$$

That is, the relative velocity is the same as the relative position, but with the base vectors substituted with their rates of change.

Finally, we can use the fixed-base or component notation for position and differentiate the terms 13.7 (See also box 13.7 on page 682):

$$\begin{aligned}
\left[\vec{\boldsymbol{v}}_{\mathrm{B/A}}\right]_{xy} &= \frac{d}{dt}\begin{bmatrix} x_{B/A} \\ y_{B/A}\end{bmatrix}\\
&= \frac{d}{dt}\left\{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x^*_{\mathrm{B/A}}\\ y^*_{\mathrm{B/A}}\end{bmatrix}\right\}\\
&= \begin{bmatrix}-\dot\theta\sin\theta & -\dot\theta\cos\theta\\ \dot\theta\cos\theta & -\dot\theta\sin\theta\end{bmatrix}\begin{bmatrix}x^*_{\mathrm{B/A}}\\ y^*_{\mathrm{B/A}}\end{bmatrix}\\
&= \begin{bmatrix}0 & -\omega\\ \omega & 0\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x^*_{\mathrm{B/A}}\\ y^*_{\mathrm{B/A}}\end{bmatrix}
\end{aligned}$$

where $x^*_{B/A}$ and $y^*_{B/A}$ are the components of the position of B with respect to A in the reference configuration and hence do not change with time.

Absolute velocity of a point on a rigid object

If one knows the velocity of one point on a rigid object and one also knows the angular velocity of the object, then one can find the velocity of any other point. How? By addition. Say we know the velocity of point A, the angular velocity of the object, and the relative position of A and B, then

$$\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + (\vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A)\\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A}\\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}}\times\underbrace{\vec{\boldsymbol{r}}_{B/A}}_{\vec{\boldsymbol{r}}_B-\vec{\boldsymbol{r}}_A}.
\end{aligned}\qquad(14.6)$$

That is, the absolute velocity of the point B is the absolute velocity of the point A plus the velocity of the point B relative to the point A.

Equation (14.6) is a 2D vector equation, and thus equivalent to 2 scalar equations, in 4 vectors which have 7 independent components ($\vec{\boldsymbol{\omega}}$ only has one independent component). Generally one could use eqn. (14.6) to solve for any 2 components in terms of the other 5.

Instantaneous Center of Rotation. Interestingly, for any planar object with non-zero ω there is some point C 'on' the object that has exactly zero velocity. By 'on' the object we mean a point that moves with the object in a rigid way. Imagine a giant sheet of clear plexiglass glued to the object and extending way beyond the object. By 'on' the object we mean a point on the plexiglass. Why must there be such a point? Consider a point A on the object with some velocity. Relative to A the other points are going in circles. Each radial line extending from A has a relative velocity orthogonal to that line. Thus, considering all radial lines, every velocity direction is represented. And the relative velocities have magnitudes from 0 to infinity as points on a radial line are scanned from A outwards. Thus each possible relative velocity is had by some point 'on' the body. And one of those exactly cancels the velocity of A. Thus some point C has no velocity and all other points on the body rotate around it. This point is called the *center of rotation*. For other than purely circular motion, as an object moves, its center of rotation changes with time. See Sample 14.2 on page 764 for more details.

When trying to understand a mechanism it is often useful to think about the locus of locations of the center of rotation of each of the parts.

Angular acceleration

The angular acceleration $\vec{\boldsymbol{\alpha}}$ ('alpha') of a rigid object is the rate of change of angular velocity, $\vec{\boldsymbol{\alpha}} = \dot{\vec{\boldsymbol{\omega}}}$. The angular acceleration of a object $\mathcal{B}$ is $\vec{\boldsymbol{\alpha}}_{\mathcal{B}}$. As for angular velocity, in 2D angular acceleration is perpendicular to the plane of motion:

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} \qquad \text{and} \qquad \vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{k}} = \dot{\omega}\hat{\boldsymbol{k}}.$$

In 2D the angular acceleration is only due to the speeding up or slowing down of the rotation rate; i.e., $\alpha = \dot{\omega} = \ddot{\theta}$.

Relative acceleration of two points on a rigid object

For any two points A and B glued to a rigid object $\mathcal{B}$, the acceleration of B relative to A is

$$\begin{aligned}
\vec{\boldsymbol{a}}_{B/A} &= \frac{d}{dt}\vec{\boldsymbol{v}}_{\mathrm{B/A}} \\
&= \frac{d}{dt}\left\{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}\right\} \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{v}}_{\mathrm{B/A}}), \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}), \\
&= \alpha_{\mathcal{B}}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A} + (-\omega_{\mathcal{B}}^2\vec{\boldsymbol{r}}_{\mathrm{B/A}}),
\end{aligned} \tag{14.7}$$

This is the base-independent or direct-vector expression for relative acceleration. If point A has no acceleration, this formula is the same as that for the acceleration of a point going in circles from chapter 7.

Equation (14.7) could also be derived, with some algebra, by taking two time derivatives of the relative position coordinate expression

$$[\vec{r}_{B/A}]_{xy} = [R(\theta)]\,[\vec{r}_{B/A}{}^{*}]_{x'y'}$$

or by taking two time derivatives of the changing base vector expression

$$\vec{r}_{B/A} = x'_{B/A}\hat{\imath}' + y'_{B/A}\hat{\jmath}'.$$

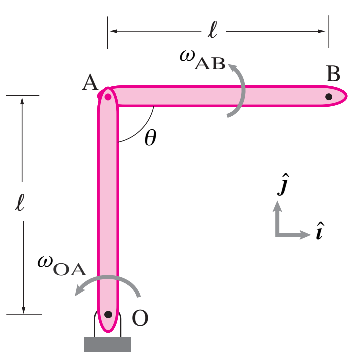
Absolute acceleration of a point on a rigid object

If one knows the acceleration of one point on a rigid body *and* the angular velocity and acceleration of the body, then one can find the acceleration of any other point. How? Add the absolute acceleration of one point to the acceleration of the second relative to the first.

$$\begin{aligned}\vec{a}_B &= \vec{a}_A + (\vec{a}_B - \vec{a}_A) = \vec{a}_A + \vec{a}_{B/A}\\ &= \vec{a}_A + \vec{\omega}_\mathcal{B} \times (\vec{\omega}_\mathcal{B} \times \vec{r}_{B/A}) + \dot{\vec{\omega}}_\mathcal{B} \times \vec{r}_{B/A}\\ &= \vec{a}_A - \omega_\mathcal{B}^2 \vec{r}_{B/A} + \alpha_\mathcal{B}\hat{k} \times \vec{r}_{B/A} \qquad (14.8)\end{aligned}$$

This is the base-independent or direct-vector expression for acceleration. The fixed-base (component) and changing-base notations are somewhat more complex.

Equation 14.8 is often called 'the three term acceleration formula' because the acceleration of B is the sum of three terms. First is $\vec{a}_A$, the acceleration of some point A on the object. Second is $\vec{\omega}_\mathcal{B} \times (\vec{\omega}_\mathcal{B} \times \vec{r}_{B/A})$, the centripetal acceleration of B going in circles relative to A. It is directed from B towards A. Third is $\dot{\vec{\omega}}_\mathcal{B} \times \vec{r}_{B/A}$, due to the change in the magnitude of the angular velocity. The third term is in the direction normal to the line from A to B (tangent to the circle of B going around A).



Filename:tfigure-robotarm

Figure 14.8: A two link robot arm.

Example: Robot arm

Given the configuration shown in fig. 14.8 the acceleration of point B can be found by thinking of link AB as the object $\mathcal{B}$ in eqn. (14.8) and using what you know about circular motion to find the acceleration of A:

$$\begin{aligned}\vec{a}_B &= \vec{a}_A - \omega_\mathcal{B}^2 \vec{r}_{B/A} + \alpha_\mathcal{B}\hat{k} \times \vec{r}_{B/A}\\ &= \left(-\omega_{OA}^2 \ell\hat{\jmath} - \dot{\omega}_{OA}\ell\hat{\imath}\right) - \left(\omega_{AB}^2 \ell\hat{\imath}\right) + \left(\dot{\omega}_{AB}\hat{k} \times (\ell\hat{\imath})\right)\\ &= -\left(\dot{\omega}_{OA}\ell + \omega_{AB}^2\ell\right)\hat{\imath} + \left(-\omega_{OA}^2\ell + \dot{\omega}_{AB}\ell\right)\hat{\jmath}\end{aligned}$$

[Note that $\omega_{AB} \neq \dot{\theta}$ where θ is the angle between the links. Rather $\omega_{AB} = \omega_{0A} + \dot{\theta}$.]

Computer graphics

Given one point given by the xy pair $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ we can find out what happens to it by rotation $[R]$ as

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R]\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

For example the point $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ gets changed by a 45 deg rotation to

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} = [R]\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \\ &\approx \begin{bmatrix} .7 & -.7 \\ .7 & .7 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \approx \begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix}.\end{aligned}$$

A translation is just a vector addition. For example the point $\begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix}$ gets translated a distance 2 in the y direction by the addition of $\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ like this

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{translated}} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1.4 \\ 3.4 \end{bmatrix}.$$

Putting these together the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ gets rotated and translated by first multiplying by the rotation matrix and then adding the translation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R]\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & -.7 \\ .7 & .7 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \approx \begin{bmatrix} -1.4 \\ 3.4 \end{bmatrix}.$$

A collection of points all rotated the same amount and then all translated the same amount keep their relative distances.

A picture is a set of points on a plane. If all the points are rotated and translated the same amount the picture is rotated and translated. Thus a picture of a rigid object described by points is rigidly rotated and translated. On a computer line drawings are often represented as a connect-the-dots picture. The picture is represented by the x and y coordinates of the reference dots at the corners. These can be stored in an array with the first row being the x coordinates and the second row the y coordinates as explained on page 672. Each column of this matrix represents one point of the connect-the-dots picture. Thus a primitive picture of a house at the origin is given by the array

$$[P_0] \equiv [xy \text{ points originally}] = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \end{bmatrix}$$

with the lower left corner of the house at the origin.

To rotate this picture we rotate each of the columns of the matrix P_0]. But this is exactly what is accomplished by the matrix multiplication $[R][P_0]$. To

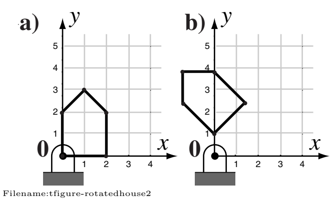


Figure 14.9: (a) A house drawn as 6 dots connected by line segments. The first and last point are the same. (b) The same house but rigidly rotated and translated.

translate the points you add the translation vector to each of the columns of the resulting matrix. Thus the whole picture rotated by 45° and translated up by 1 is given by

$$[P_{\text{new}}] = [R][P_0] + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix} [P_0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives a new array of points that, when connected give the picture shown. We have allowed the informal notation of adding a column matrix to a rectangular matrix, by which we mean adding to each column of the rectangular matrix.

To animate the motion of a house flying in, say, the Wizard of Oz, you would first define the house as the set of points $[P_0]$. Then define, maybe by means of numerical solution of differential equations, a set of rotations and translations. Then for each rotation and translation the picture of the house should be drawn, one after the other. The sequence of such pictures is an animation of a flying and spinning house.

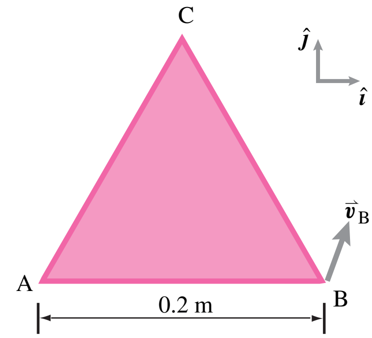
Summary of the kinematics of one rigid object in general 2D motion

You can use the position of one reference point and the rotation of the object as simple kinematic measures of the entire motion of the object. If you know the position, velocity, and acceleration of one point on a rigid object (represented by 6 scalars, say) , and you know the angular rate and angular acceleration (3 scalars) then you can find the position, velocity and acceleration of any point on the object (also given its initial position). In 2D, just 9 numbers tell you the position, velocity, and acceleration of any of the billions of points whose initial positions you know②.

②In 1D it takes just 3 numbers and in 3D just 18. The unusual pattern (3,9,18) comes from rotation being characterized by 0, 1, and 3 numbers in 1, 2, and 3 dimensions, respectively.

SAMPLE 14.1 **Velocity of a point on a rigid body in planar motion.** An equilateral triangular plate ABC is in motion in the x-y plane. At the instant shown in the figure, point B has velocity $\vec{v}_B = 0.3\,\text{m/s}\hat{\imath} + 0.6\,\text{m/s}\hat{\jmath}$ and the plate has angular velocity $\vec{\omega} = 2\,\text{rad/s}\hat{k}$.

1. Find the velocity of point A using vector calculations.
2. Find the velocity of point C using matrix calculations.



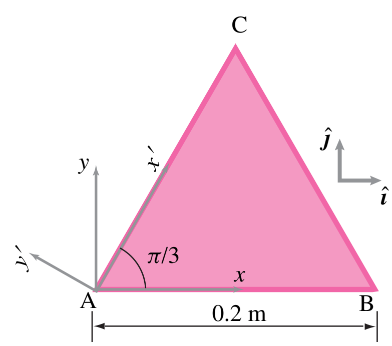
Filename:sfig9-1-triang-1

Figure 14.10: The rigid triangular plate ABC rotates counterclockwise at 5 rad/s. Velocity of point B is given. We have to find the velocity of points A and C.

Solution

1. We are given $\vec{v}_B$ and ω, and we need to find $\vec{v}_A$, the velocity of point A on the same rigid body. We know that,
$$\vec{v}_A = \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}$$
Thus, to find $\vec{v}_A$, we need to find $\vec{r}_{A/B}$. Let us take an x-y coordinate system whose origin coincides with point A of the plate at the instant of interest and the x-axis is along AB. Then,
$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B = \vec{0} - (0.2\,\text{m}\hat{\imath}) = -0.2\,\text{m}\hat{\imath}$$
Thus,
$$\begin{aligned}\vec{v}_A &= \vec{v}_B + \vec{\omega} \times \vec{r}_{A/B}\\ &= (0.3\hat{\imath} + 0.6\hat{\jmath})\,\text{m/s} + 2\,\text{rad/s}\hat{k} \times (-0.2\hat{\imath})\,\text{m}\\ &= (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}.\end{aligned}$$
$$\boxed{\vec{v}_A = (0.3\hat{\imath} + 0.2\hat{\jmath})\,\text{m/s}}$$

2. Let xy axes with origin at A be fixed in space with x-axis along AB at the instant of interest. Let $x'y'$ axes be fixed to the plate with origin at O and the x'-axis along AC as shown in fig. 14.11. Thus the coordinates of point C in the rotating axes are
$$\begin{bmatrix} x'_C \\ y'_C \end{bmatrix} = \begin{bmatrix} 0.2\,\text{m} \\ 0 \end{bmatrix}$$
Clearly, at the instant of interest, the rotating axes make an angle of $\theta = \pi/3$ with the fixed axes. Now, we can calculate the velocity of point C as follows.
$$[\vec{v}_C]_{xy} = [\vec{v}_A]_{xy} + \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'_{C/A} \\ y'_{C/A} \end{bmatrix}.$$
Using $\theta = \pi/3$, the angle that the rotating axes make with the fixed axes at the instant of interest, and $\omega = 2\,\text{rad/s}$, we get
$$\begin{aligned}[\vec{v}_C]_{xy} &= \begin{bmatrix} 0.3\,\text{m/s} \\ 0.2\,\text{m/s} \end{bmatrix} + \begin{bmatrix} 0 & -2\,\text{rad/s} \\ 2\,\text{rad/s} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.2\,\text{m} \\ 0 \end{bmatrix}\\ &= \begin{bmatrix} 0.3\,\text{m/s} \\ 0.2\,\text{m/s} \end{bmatrix} + \begin{bmatrix} -0.35\,\text{m/s} \\ 0.2\,\text{m/s} \end{bmatrix} = \begin{bmatrix} -0.05\,\text{m/s} \\ 0.4\,\text{m/s} \end{bmatrix}.\end{aligned}$$
Thus, $\vec{v}_C = -0.05\,\text{m/s}\hat{\imath} + 0.4\,\text{m/s}\hat{\jmath}$. We can also verify this answer by computing $\vec{v}_C = \vec{v}_B + \vec{\omega} \times \vec{r}_{C/B}$.
$$\boxed{\vec{v}_C = -0.05\,\text{m/s}\hat{\imath} + 0.4\,\text{m/s}\hat{\jmath}}$$



Filename:sfig9-1-triang-2

Figure 14.11: Velocity of point A is known. The axes xy are fixed in space and the axes $x'y'$ are fixed to the rotating plate in the orientation shown. Thus in the $x'y'$ axes, the coordinates of point C will always be $x'_C = 0.2\,\text{m}$, $y'_C = 0$.

SAMPLE 14.2 The instantaneous center of rotation. A rigid body is in planar motion. At some instant t, the angular velocity of the body is $\vec{\omega} = 5\,\text{rad/s}\hat{\boldsymbol{k}}$ and the linear velocity of a point C on the body is $\vec{v}_C = 2\,\text{m/s}\hat{\boldsymbol{\imath}} - 5\,\text{m/s}\hat{\boldsymbol{\jmath}}$. Find the instantaneous center of rotation.

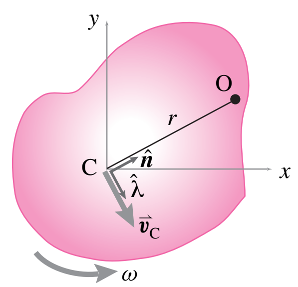


Figure 14.12:

Solution Let a point O be the instantaneous center of rotation. We need to find the location of O using the given information. Whether O lies inside the object or outside is irrelevant here since the object boundary is not specified. So, let us consider an abstract rigid object as shown in fig. 14.12. What we know about O is that it has zero velocity (since it is the center of rotation). Let point O be a distance r away from C in some unknown direction $\hat{\boldsymbol{n}}$ (to be found). Thus, $\vec{r}_{O/C} = r\hat{\boldsymbol{n}}$ is the position vector of point O with respect to point C. Now,

$$\begin{aligned}
\vec{v}_C &= \underbrace{\vec{v}_O}_{\vec{0}} + \vec{\omega} \times \underbrace{\vec{r}_{C/O}}_{-\vec{r}_{O/C}} \\
&= -\omega\hat{\boldsymbol{k}} \times r\hat{\boldsymbol{n}} = -\omega r(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}).
\end{aligned}$$

Thus, $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}}$ must be parallel to and in the opposite direction of $\vec{v}_C$. Let $\vec{v}_C = v_C\hat{\boldsymbol{\lambda}}$. Then, we must have $\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}} = -\hat{\boldsymbol{\lambda}}$, which says that $\hat{\boldsymbol{n}}$ must be normal to $\hat{\boldsymbol{\lambda}}$. Since $\hat{\boldsymbol{\lambda}} = \vec{r}_C/|\vec{r}_C| = (2\,\text{m/s}\hat{\boldsymbol{\imath}} - 5\,\text{m/s}\hat{\boldsymbol{\jmath}})/(\sqrt{2^2+5^2}\,\text{m/s}) = 0.37\hat{\boldsymbol{\imath}} - 0.93\hat{\boldsymbol{\jmath}}$, we can immediately write $\hat{\boldsymbol{n}} = 0.93\hat{\boldsymbol{\imath}} + 0.37\hat{\boldsymbol{\jmath}}$. So, now,

$$\begin{aligned}
v_C\hat{\boldsymbol{\lambda}} &= -\omega r\overbrace{(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{n}})}^{-\hat{\boldsymbol{\lambda}}} \\
\Rightarrow \quad v_C &= \omega r \\
\Rightarrow \quad r &= v_C/\omega \\
&= \sqrt{29}\,\text{m/s}/(5\,\text{rad/s}) = 1.08m.
\end{aligned}$$

Thus

$$\vec{r}_{O/C} = r\hat{\boldsymbol{n}} = 1.08\,\text{m}(0.93\hat{\boldsymbol{\imath}} + 0.37\hat{\boldsymbol{\jmath}}) = 1\,\text{m}\hat{\boldsymbol{\imath}} + 0.4\,\text{m}\hat{\boldsymbol{\jmath}}.$$

$$\boxed{\vec{r}_{O/C} = 1\,\text{m}\hat{\boldsymbol{\imath}} + 0.4\,\text{m}\hat{\boldsymbol{\jmath}}}$$

Alternatively,

We can find $\vec{r}_{O/C}$ purely by vector algebra. Let $\vec{r}_{O/C} = (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})\,\text{m}$. So,

$$\begin{aligned}
\vec{v}_O &= \vec{v}_C + \vec{\omega} \times \vec{r}_{O/C} \\
\text{or} \quad \vec{0} &= 2\,\text{m/s}\hat{\boldsymbol{\imath}} - 5\,\text{m/s}\hat{\boldsymbol{\jmath}} + (5\,\text{rad/s})\hat{\boldsymbol{k}} \times (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}})\,\text{m} \\
&= (2-5y)\,\text{m/s}\hat{\boldsymbol{\imath}} + (-5+5x)\,\text{m/s}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

Dotting this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ respectively, we get

$$\begin{aligned}
2 - 5y &= 0 \\
-5 + 5x &= 0.
\end{aligned}$$

Solving these two equations simultaneously, we get, $x = 1$ and $y = 0.4$. Thus,

$$\vec{r}_{O/C} = 1\,\text{m}\hat{\boldsymbol{\imath}} + 0.4\,\text{m}\hat{\boldsymbol{\jmath}}$$

as obtained above.

SAMPLE 14.3 A cheerleader throws her baton up in the air in the vertical xy-plane. At an instant when the baton axis is at $\theta = 60^\circ$ from the horizontal, the velocity of end A of the baton is $\vec{\boldsymbol{v}}_A = 2\,\mathrm{m/s}\hat{\boldsymbol{\imath}} + \sqrt{3}\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}$. At the same instant, end B of the baton has velocity in the negative x-direction (but $|\vec{\boldsymbol{v}}_B|$ is not known). If the length of the baton is $\ell = \frac{1}{2}$ m and the center-of-mass is in the middle of the baton, find the velocity of the center-of-mass.

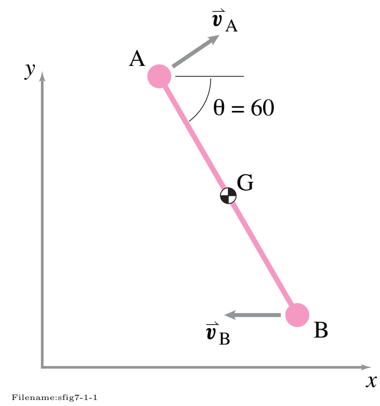


Figure 14.13:

Solution

$$\begin{aligned}\text{We are given:}\quad \vec{\boldsymbol{v}}_A &= (2\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}\\ \text{and}\quad \vec{\boldsymbol{v}}_B &= -v_B\hat{\boldsymbol{\imath}}\end{aligned}$$

where $v_B = |\vec{\boldsymbol{v}}_B|$ is unknown. We need to find $\vec{\boldsymbol{v}}_G$. Using the relative velocity formula for two points on a rigid body, we can write:

$$\vec{\boldsymbol{v}}_G = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{G/A} \tag{14.9}$$

Here, $\vec{\boldsymbol{v}}_A$ and $\vec{\boldsymbol{r}}_{G/A}$ are known. Thus, to find $\vec{\boldsymbol{v}}_G$, we need to find $\vec{\boldsymbol{\omega}}$, the angular velocity of the baton. Since the motion is in the vertical xy-plane, let $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. Then,

$$\begin{aligned}\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{A/B} = \vec{\boldsymbol{v}}_A + \omega\hat{\boldsymbol{k}} \times \underbrace{\ell(-\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_{A/B}}\\ \text{or}\quad -v_B\hat{\boldsymbol{\imath}} &= (2\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s} - \omega\ell(\cos\theta\hat{\boldsymbol{\jmath}} + \sin\theta\hat{\boldsymbol{\imath}})\\ &= (2\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s} - \omega\cdot\frac{1}{2}\,\mathrm{m}\cdot(\frac{1}{2}\hat{\boldsymbol{\jmath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\imath}})\end{aligned}$$

Dotting both sides of this equation with $\hat{\boldsymbol{\jmath}}$ we get:

$$\begin{aligned}0 &= \sqrt{3}\,\mathrm{m/s} - \frac{\omega}{2}\,\mathrm{m}\cdot\frac{1}{2}\\ \Rightarrow\quad \omega &= \sqrt{3}\frac{\cancel{\mathrm{m}}}{\mathrm{s}}\cdot\frac{4}{1\,\cancel{\mathrm{m}}} = 4\sqrt{3}\,\mathrm{rad/s}.\end{aligned}$$

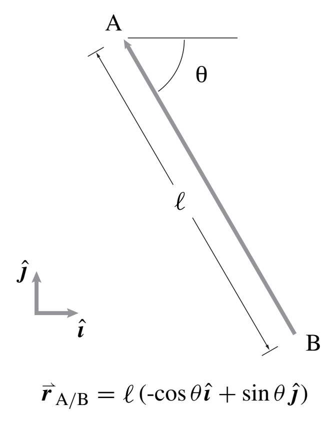


Figure 14.14:

Now substituting the appropriate values in Eqn 14.9 we get:

$$\begin{aligned}\vec{\boldsymbol{v}}_G &= \vec{\boldsymbol{v}}_A + \omega\hat{\boldsymbol{k}} \times \underbrace{\frac{\ell}{2}(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_{G/A}}\\ &= \vec{\boldsymbol{v}}_A + \frac{\omega\ell}{2}(\cos\theta\hat{\boldsymbol{\jmath}} + \sin\theta\hat{\boldsymbol{\imath}})\\ &= (2\hat{\boldsymbol{\imath}} + \sqrt{3}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s} + \sqrt{3}\,\mathrm{m/s}\cdot(\frac{1}{2}\hat{\boldsymbol{\jmath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\imath}})\\ &= (2 + \frac{3}{2})\,\mathrm{m/s}\hat{\boldsymbol{\imath}} + (\sqrt{3} + \frac{\sqrt{3}}{2})\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}\\ &= 3.5\,\mathrm{m/s}\hat{\boldsymbol{\imath}} + 2.6\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_G = (3.5\hat{\boldsymbol{\imath}} + 2.6\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}}$$

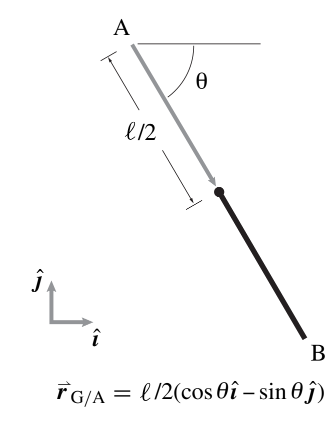


Figure 14.15:

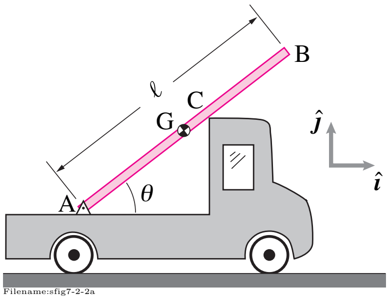


Figure 14.16: A board in the back of an accelerating truck.

SAMPLE 14.4 A board in the back of an accelerating truck. A 10 ft long uniform board AB rests in the back of a flat-bed truck as shown in Fig. 14.16. End A of the board is hinged to the bed of the truck. The truck is going on a level road at 55 mph. In preparation for overtaking a vehicle in the front, the trucker accelerates at a rate of 3 mph/s. At the instant when the speed of the truck is 60 mph, the magnitude of the relative velocity and relative acceleration of end B with respect to the bed of the truck are 10 ft/s and 12 ft/s^2, respectively. There is wind and at this instant, the board has lost contact with point C. If the angle θ between the board and the bed is 45° at the instant mentioned, find

1. the angular velocity and angular acceleration of the board,
2. the absolute velocity and absolute acceleration of point B, and
3. the acceleration of the center-of-mass G of the board.

Solution At the instant of interest, the quantities given are:

$$\begin{aligned}
\vec{v}_A &= \text{velocity of the truck } = 60\,\text{mph}\,\hat{\imath} = 88\,\text{ft/s}\,\hat{\imath} \\
\vec{a}_A &= \text{acceleration of the truck } = 3\,\text{mph}/s = 4.4\,\text{ft/s}^2\,\hat{\imath} \\
|\vec{v}_{B/A}| &= v_{B/A} = \text{magnitude of relative velocity of B } = 10\,\text{ft/s} \\
|\vec{a}_{B/A}| &= a_{B/A} = \text{magnitude of relative acceleration of B } = 12\,\text{ft/s}^2.
\end{aligned}$$

Let $\vec{\omega} = \omega\hat{k}$ be the angular velocity and $\dot{\vec{\omega}} = \dot{\omega}\hat{k}$ be the angular acceleration of the board.

1. The relative velocity of end B of the board with respect to end A is

$$\begin{aligned}
\vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} = \omega\hat{k} \times \ell(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
&= \omega\ell(\cos\theta\,\hat{\jmath} - \sin\theta\,\hat{\imath}) \\
\Rightarrow \quad \omega &= \frac{|\vec{v}_{B/A}|}{\ell} = \frac{v_{B/A}}{\ell} = \frac{10\,\text{ft/s}}{10\,\text{ft}} = 1\,\text{rad/s}.
\end{aligned}$$

Note that we have taken the positive value for ω because the board is rotating counter-clockwise at the instant of interest (since it is given that the board has lost contact with point C).

Similarly, we can compute the angular acceleration:

$$\begin{aligned}
\vec{a}_{B/A} &= \dot{\vec{\omega}} \times \vec{r}_{B/A} - \omega^2\vec{r}_{B/A} \\
&= \dot{\omega}\hat{k} \times \ell(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) - \omega^2\ell(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) \\
&= \dot{\omega}\ell(\cos\theta\,\hat{\jmath} - \sin\theta\hat{\imath}) - \omega^2\ell(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) \\
\Rightarrow \quad |\vec{a}_{B/A}| &= \sqrt{(\dot{\omega}\ell)^2 + (\omega^2\ell)^2} = a_{B/A} \text{ (given)} \\
\Rightarrow \quad a^2_{B/A} &= (\dot{\omega}\ell)^2 + (\omega^2\ell)^2 \\
\Rightarrow \quad \dot{\omega} &= \sqrt{\frac{a^2_{B/A}}{\ell^2} - \omega^4} = \sqrt{\left(\frac{12\,\text{ft/s}^2}{10\,\text{ft}}\right)^2 - (1\,\text{rad/s})^4} \\
&= \pm 0.663\,\text{rad/s}^2.
\end{aligned}$$

Once again, we select the positive value for $\dot{\omega}$ since we assume that the board accelerates counterclockwise.

$$\boxed{\vec{\omega} = 1\,\text{rad/s}\hat{k},\ \dot{\vec{\omega}} = 0.663\,\text{rad/s}^2\hat{k}}$$

2. The absolute velocity and the absolute acceleration of the end point B can be found as follows.

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\
&= v_A \hat{\imath} + v_{B/A}(\cos\theta\,\hat{\jmath} - \sin\theta\hat{\imath}) \\
&= 88\,\text{ft/s}\hat{\imath} + 10\,\text{ft/s}(\frac{1}{\sqrt{2}}\hat{\jmath} - \frac{1}{\sqrt{2}}\hat{\imath}) \\
&= 80.93\,\text{ft/s}\hat{\imath} + 7.07\,\text{ft/s}\hat{\jmath}.
\end{aligned}$$

$$\begin{aligned}
\vec{a}_B &= \vec{a}_A + \vec{a}_{B/A} \\
&= \vec{a}_A + \dot{\vec{\omega}} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \\
&= a_A \hat{\imath} + \dot{\omega}\hat{k} \times \ell(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) - \omega^2\ell(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) \\
&= (a_A - \dot{\omega}\ell\sin\theta - \omega^2\ell\cos\theta)\hat{\imath} + (\dot{\omega}\ell\cos\theta - \omega^2\ell\sin\theta)\hat{\jmath} \\
&= \left(4.4\,\text{ft/s}^2 - \frac{0.66}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}}\right)\hat{\imath} \\
&\quad + \left(\frac{0.66}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}} - \frac{1}{\text{s}^2}\cdot 10\,\text{ft}\cdot\frac{1}{\sqrt{2}}\right)\hat{\jmath} \\
&= -7.34\,\text{ft/s}^2\hat{\imath} - 2.40\,\text{ft/s}^2\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{v}_B = (80.93\hat{\imath} + 7.07\hat{\jmath})\,\text{ft/s}, \quad \vec{a}_B = -(7.34\hat{\imath} + 2.40\hat{\jmath})\,\text{ft/s}^2.}$$

3. Now, we can easily calculate the acceleration of the center-of-mass as follows.

$$\begin{aligned}
\vec{a}_G &= \vec{a}_A + \vec{a}_{G/A} \\
&= a_A\hat{\imath} + \dot{\vec{\omega}} \times \vec{r}_{G/A} - \omega^2\vec{r}_{G/A} \\
&= a_A\hat{\imath} + \dot{\omega}\hat{k} \times \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) - \omega^2\frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) \\
&= a_A\hat{\imath} + \dot{\omega}\frac{\ell}{2}(\cos\theta\,\hat{\jmath} - \sin\theta\hat{\imath}) - \omega^2\frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\,\hat{\jmath}) \\
&= 4.4\,\text{ft/s}^2\hat{\imath} + 0.663\,\text{rad/s}^2\cdot\frac{10\,\text{ft}}{2}\cdot(\frac{1}{\sqrt{2}}\hat{\jmath} - \frac{1}{\sqrt{2}}\hat{\imath}) \\
&\quad -(1\,\text{rad/s})^2\cdot\frac{10\,\text{ft}}{2}\cdot(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}) \\
&= -1.48\,\text{ft/s}^2\hat{\imath} - 1.19\,\text{ft/s}^2\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{a}_G = -(1.48\hat{\imath} + 1.19\hat{\jmath})\,\text{ft/s}^2}$$

Comments: This problem is admittedly artificial. We, however, solve this problem to show kinematic calculations.

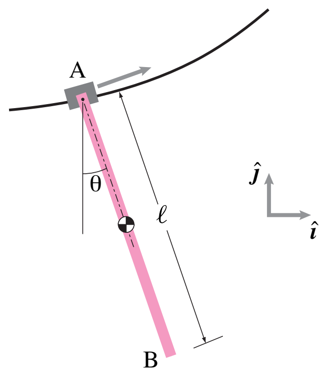


Figure 14.17:

SAMPLE 14.5 Tracking motion. A cart moves along a suspended curved path. A rod AB of length $\ell = 1\,\mathrm{m}$ hangs from the cart. End A of the rod is attached to a motor on the cart. The other end, B, hangs freely. The motor rotates the rod such that $\theta(t) = \theta_0 \sin(\lambda t)$ (θ is measured with respect to a fixed vertical line) while the cart moves along the path such that $\vec{r}_A = t\hat{\imath} + \frac{t^3}{18}\hat{\jmath}$, where all variables ($r$, t, etc.) are nondimensional.

1. Find the velocity and acceleration of point B as a function of nondimensional time t.
2. Take $\theta_0 = \pi/3$ and $\lambda = 6$. Find and plot the position of the bar at $t = 0, 0.1, 0.3, 0.9, 1, 1.1, 1.2$, and 1.5. Find and draw $\vec{v}_\mathrm{B}$ and $\vec{a}_\mathrm{B}$ at the specified t.

Solution

1. The velocity and acceleration of point B are given by

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} = \vec{v}_A + \vec{\omega}\times\vec{r}_{B/A}\\ \vec{a}_B &= \vec{a}_A + \dot{\vec{\omega}}\times\vec{r}_{B/A} - \omega^2\vec{r}_{B/A}.\end{aligned}$$

Thus, in order to find the velocity and acceleration of point B, we need to find the velocity and acceleration of point A and the angular velocity and angular acceleration of the bar. We are given the position of point A and the angular position of the bar as functions of t. We can, therefore, find $\vec{v}_A$, $\vec{a}_A$, $\vec{\omega}$, and $\dot{\vec{\omega}}$ by differentiating the given functions with respect to t.

$$\begin{aligned}\vec{r}_A &= t\hat{\imath} + \frac{t^3}{18}\hat{\jmath}\\ \Rightarrow\quad \vec{v}_A &\equiv \dot{\vec{r}}_A = \hat{\imath} + (t^2/6)\hat{\jmath} &(14.10)\\ \Rightarrow\quad \vec{a}_A &\equiv \dot{\vec{v}}_A = (t/3)\hat{\jmath} &(14.11)\end{aligned}$$

and

$$\begin{aligned}\theta\hat{k} &= \theta_0\sin(\lambda t)\hat{k}\\ \Rightarrow\quad \vec{\omega} &\equiv \dot{\theta}\hat{k} = \theta_0\lambda\cos(\lambda t)\hat{k} &(14.12)\\ \Rightarrow\quad \dot{\vec{\omega}} &\equiv \ddot{\theta}\hat{k} = -\theta_0\lambda^2\sin(\lambda t)\hat{k}. &(14.13)\end{aligned}$$

So,

$$\begin{aligned}\vec{v}_B &= \vec{v}_A + \vec{\omega}\times\ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath})\\ &= \hat{\imath} + (t^2/6)\hat{\jmath} + \ell\dot{\theta}(\sin\theta\hat{\jmath} + \cos\theta\hat{\imath})\\ &= (1 + \ell\dot{\theta}\cos\theta)\hat{\imath} + (t^2/6 + \ell\dot{\theta}\sin\theta)\hat{\jmath} &(14.14)\\ \vec{a}_B &= \vec{a}_A + \ddot{\theta}\hat{k}\times\ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) - \dot{\theta}^2\ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath})\\ &= (t/3)\hat{\jmath} + \ell\ddot{\theta}\sin\theta\hat{\jmath} + \ell\ddot{\theta}\cos\theta\hat{\imath} - \ell\dot{\theta}^2\sin\theta\hat{\imath} + \ell\dot{\theta}^2\cos\theta\hat{\jmath}\\ &= \ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\imath} + [t/3 + \ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)]\hat{\jmath}\end{aligned}$$

(14.15)

where $\theta = \theta_0\sin(\lambda t)$, $\dot{\theta} = \theta_0\lambda\cos(\lambda t)$, and $\ddot{\theta} = -\theta_0\lambda^2\sin(\lambda t) = -\lambda^2\theta$. Thus $\vec{v}_B$ and $\vec{a}_B$ are functions of t.

2. The position of the rod at any time t is specified by the position of the two end points A and B (or alternatively, the position of A and the angle of the rod θ). The position of point A is easily determined by substituting the value of t in the given expression for $\vec{r}_A$. The position of end B is given by

$$\begin{aligned}\vec{r}_B &= \vec{r}_A + \vec{r}_{B/A} = t\hat{\imath} + (t^3/18)\hat{\jmath} + \ell(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath})\\ &= (t + \ell\sin\theta)\hat{\imath} + (t^3/18 - \ell\cos\theta)\hat{\jmath}.\end{aligned}$$

To compute the positions, velocities, and accelerations of end points A and B at the given instants, we first compute θ, $\dot{\theta}$, and $\ddot{\theta}$, and then substitute them in the expressions for $\vec{r}_A$, $\vec{r}_B$, $\vec{v}_A$, $\vec{v}_B$, $\vec{a}_A$, and $\vec{a}_B$. A pseudocode for computer calculation is given below.

```
t = [0 0.1 0.3 0.9 1.0 1.1 1.2 1.5]
theta0=pi/3, L=.5, lam=6
for each t, compute
  theta = theta0*sin(lam*t)
  w = lam*theta0*cos(lam*t)
  wdot = -lam^2*theta
       % Position of A and B
  xA=t, yA=t^3/18
  xB = xA + L*sin(theta)
  yB = yA - L*cos(theta)
       % Velocity of A and B
  uA = 1,  vA = t^2/6
  uB = uA + L*w.*cos(theta)
  vB = vA + L*w.*sin(theta)
       % Acceleration of A and B
  axA = 0,  ayA = t/3
  axB = L*wdot*cos(theta) - L*w^2*sin(theta)
  ayB = ayA + L*wdot*sin(theta) + L*w^2*cos(theta)
```

From the above calculation, we get the desired quantities at each t. For example, at $t = 0$ we get,

```
xA = 0, yA = 0, xB = 0, yB = -0.5
uA = 1, vA = 0, uB = 4.14, vB = 0, axB = 0, ayB = 19.74
```

which means,

$$\vec{r}_A = \vec{0}, \quad \vec{r}_B = -0.5\hat{\jmath}, \quad \vec{v}_A = \hat{\imath}, \quad \vec{v}_B = 4.14\hat{\imath}, \quad \vec{a}_B = 19.74\hat{\jmath}.$$

The position of the bar, the velocity vectors at points A and B, and the acceleration vector at B, thus obtained, are shown in fig. 14.18 graphically. ③

③ We can take several values of t, say 400 equally spaced values between $t = 0$ and $t = 4$, and draw the bar at each t to see its motion and the trajectory of its end points. fig. 14.19 shows such a graph.

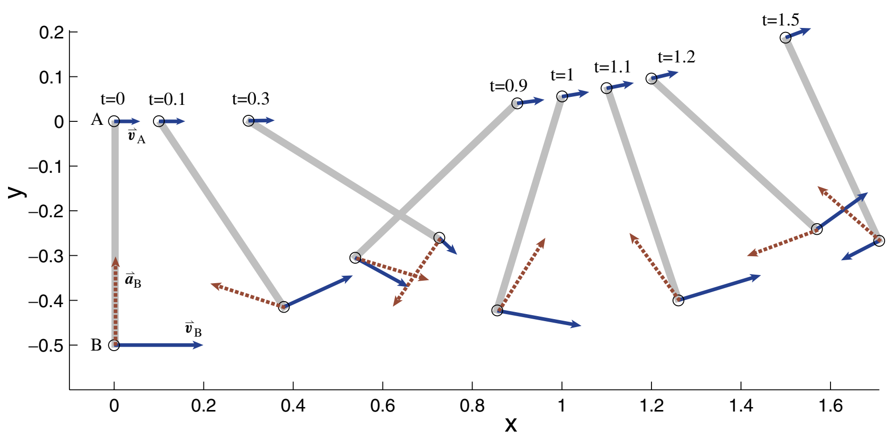


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Figure 14.18: Position, velocity of the end points A and B, and acceleration of point B at various time instants.

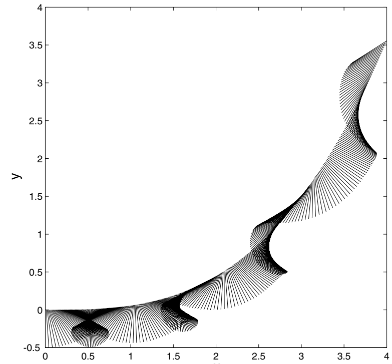

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Figure 14.19: Graph of closely spaced configuration of the bar between $t = 0$ to $t = 4$.

14.2 General planar mechanics of a rigid-object

We now apply the kinematics ideas of the last section to the general mechanics principles in Table I in the inside cover. The goal is to understand the relation between forces and motion for a planar object in general 2-D motion. The simple measures① of motion will be the displacement of one reference point $0'$ on the object and the rotation the object. We also need the first and second time derivatives of the displacement and rotation, altogether we use $\vec{\boldsymbol{r}}_{0'}$, $\vec{\boldsymbol{v}}_{0'}$, $\vec{\boldsymbol{a}}_{0'}$, θ, $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$.

①**Advanced aside:** What we call "simple measures" are examples of "generalized coordinates" in more advanced books. The idea sounds intimidating, but is simply this: If something can only move in a few ways, you should only keep track of the motion with that many variables. The kinematics of a rigid object (Sect. 14.1) allow us to "evaluate" the motion quantities, namely linear momentum, angular momentum, kinetic energy, and their rates of change in terms of these "simple measures". By "evaluate" we mean express the motion quantities in terms of these measures. The alternative is as sums over Avogadro's number of particles (There are on the order of 10^{23} atoms in a typical engineering part.). Even neglecting atoms and viewing matter as continuous we would still be stuck with integrals over complicated regions if we did not describe the motion with as few variables as possible. In the case of 2-D rigid object motion, the position of a reference point (x and y) with the rotation θ is called a set of *minimal coordinates*. These, and their time derivatives are the minimal information needed to describe all important mechanics motion quantities.

We will treat all bodies as if they are squished into the plane (planar) and moving in the plane (planar motion). But the analysis is sensible for a object that is symmetric with respect to the plane containing the velocities (see Box 14.1 on page 776).

The balance laws for a rigid object

As always, once you have defined the system and the forces acting on it by drawing a free body diagram, the basic momentum balance equations are applicable (and exact for engineering purposes). Namely,

$$\text{Linear momentum balance:} \qquad \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \qquad \text{and}$$

$$\text{Angular momentum balance:} \qquad \sum \vec{\boldsymbol{M}}_{i/O} = \dot{\vec{\boldsymbol{H}}}_{/\text{O}}.$$

The same point 0, any point, is used on both sides of the angular momentum balance equation. We also have power balance which, for a system with no internal energy or dissipation, is

$$\text{Power balance:} \qquad P = \dot{E}_\text{K}.$$

The left hand sides of the momentum balance equations are evaluated the same way, whether the system is composed of one object or many, whether the bodies are deformable or not, and whether the points move in straight lines, circles, hither and thither, or not at all. It is the right hand sides of the momentum equations that involve the motion. Similarly, in the energy balance equations the applied power P only depends on the position of the forces and the motions of the material points at those positions. But the kinetic energy E_K and its rate of change depend on the motion of the whole system. You already know how to evaluate the momenta and energy, and their rates of change, for a variety of special cases, namely

- Systems composed of particles where all the positions and accelerations are known (chapter 11);
- Systems where all points have the same acceleration. That is, the system moves like a rigid object that does not rotate (chapter 12); and
- Systems where all points move in circles about the same fixed axis in 2D (chapter 13).

Now we go on to consider the general 2-D motions of a planar rigid object. Its now a little harder to evaluate $\vec{\boldsymbol{L}}$, $\dot{\vec{\boldsymbol{L}}}$, $\vec{\boldsymbol{H}}_{/\mathrm{O}}$, $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$, E_K and $\dot{E}_\mathrm{K}$. But not much.

Summary of the motion quantities

Table I in the back of the book describes the motion quantities for various special cases, including the planar motions we consider in this chapter. Most relevant is row (7).

The basic idea is that the momenta for general motion, which involves translation and rotation, is the sum of the momenta (both linear and angular, and their rates of change too) from those two effects. Namely, the linear momentum is described, as for any system with any motion, by the motion of the center-of-mass

$$\vec{\boldsymbol{L}} = m_\mathrm{tot}\vec{\boldsymbol{v}}_\mathrm{cm} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_\mathrm{tot}\vec{\boldsymbol{a}}_\mathrm{cm}, \tag{14.16}$$

and the angular momentum has two contributions, one from the motion of the center-of-mass and one from rotation of the object about the center of mass,

$$\begin{aligned} \vec{\boldsymbol{H}}_{/\mathrm{O}} &= \overbrace{\vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{O}} \times (m_\mathrm{tot}\vec{\boldsymbol{v}}_\mathrm{cm})}^{\text{Angular momentum due to motion of the center-of-mass}} + \overbrace{I_{zz}^\mathrm{cm}\vec{\boldsymbol{\omega}}}^{\text{Angular momentum due to motion relative to the center-of-mass}} & (14.17) \\ \text{and} \quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{O}} \times (m_\mathrm{tot}\vec{\boldsymbol{a}}_\mathrm{cm}) + I_{zz}^\mathrm{cm}\dot{\vec{\boldsymbol{\omega}}}. & (14.18) \end{aligned}$$

These simplifications, for 2D objects moving in the plane, are discussed in box 14.2 on page 777. An important special case for the angular momentum evaluation is when the reference point is coincident with the center-of-mass. Then the angular momentum and its rate of change simplify to

$$\vec{\boldsymbol{H}}_\mathrm{cm} = I_{zz}^\mathrm{cm}\vec{\boldsymbol{\omega}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{H}}}_\mathrm{cm} = I_{zz}^\mathrm{cm}\dot{\vec{\boldsymbol{\omega}}}. \tag{14.19}$$

The kinetic energy and its rate of change are given by

$$E_K = \overbrace{\frac{1}{2} m_{\text{tot}} \underbrace{v_{cm}^2}_{\vec{v}_{\text{cm}} \cdot \vec{v}_{\text{cm}}}}^{\boxed{\text{kinetic energy from center-of-mass motion}}} + \overbrace{\frac{1}{2} I_{zz}^{\text{cm}} \omega^2}^{\boxed{\text{kinetic energy relative to the center-of-mass}}} \tag{14.20}$$

$$\text{and} \quad \dot{E}_K = m_{\text{tot}} \underbrace{v_{cm} \dot{v}_{cm}}_{\vec{v}_{\text{cm}} \cdot \vec{a}_{\text{cm}}} + I_{zz}^{\text{cm}} \omega \dot{\omega} \tag{14.21}$$

The relations above are easily derived from the general center of mass theorems (see box 14.2 on page 777 for some of these derivations).

Equations of motion

Putting together the general balance equations and the expressions for the motion quantities we can now write linear momentum balance, angular momentum balance and power balance as:

$$
\begin{array}{lrcll}
\text{LMB :} & \sum \vec{F}_i & = & m_{\text{tot}} \vec{a}_{\text{cm}}, & \text{(a)} \\
\text{AMB :} & \sum \vec{M}_{\text{O}} & = & \vec{r}_{\text{cm/O}} \times (m_{\text{tot}} \vec{a}_{\text{cm}}) + I_{zz}^{\text{cm}} \dot{\vec{\omega}} & \text{(b)} \\
\text{or} & \sum \vec{M}_{\text{cm}} & = & I_{zz}^{\text{cm}} \dot{\vec{\omega}}, \quad \text{and} & \\
\text{Power :} & \vec{F}_{\text{tot}} \cdot \vec{v}_{\text{cm}} + \vec{\omega} \cdot \vec{M}_{\text{cm}} & = & m_{\text{tot}} v \dot{v} + I_{zz}^{\text{cm}} \omega \dot{\omega}. & \text{(c)}
\end{array}
\tag{14.22}
$$

Independent equations?

Equations are only independent if no one of them can be derived from the others. When counting equations and unknowns one needs to make sure one is writing independent equations. How many independent equations are in the set eqns. (14.22)abc applied to one free object diagram? The short answer is 3.

The linear momentum balance equation 14.22a yields two independent equations by dotting with any two non-parallel vectors (say, $\hat{\imath}$ and $\hat{\jmath}$). Dotting with a third vector yields a dependent equation.

For any one reference point the angular momentum equation 14.22c yields one scalar equation. It is a vector equation but always has zero components in the $\hat{\imath}$ and $\hat{\jmath}$ directions. But angular momentum equation can yield up to three independent equations by being applied to any set of three non-colinear points.

The power balance equation is one scalar equation.

In total, however, the full set of equations above only makes up a set of three independent equations.

To avoid thinking about what is or is not an independent set of equations some people prefer to stick with one of the canonical sets of independent equations:

- The coordinate based set ("old standard")
  - {LMB}$\cdot\hat{\imath}$ or, equivalently, $\sum F_x = m_{\text{tot}} a_{\text{cm}_x}$,
  - {LMB}$\cdot\hat{\jmath}$ or, equivalently, $\sum F_y = m_{\text{tot}} a_{\text{cm}_y}$, and
  - {AMB}$\cdot\hat{\boldsymbol{k}}$ or, equivalently, $\sum M_{\text{cm}} = I_{zz}^{\text{cm}} \dot{\omega}$.
- Moment only (good for eliminating unknown reaction forces)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {AMB about pt C}$\cdot\hat{\boldsymbol{k}}$ (C is a third point not on the line AB)
- Two moments and a force component
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}$ (where $\hat{\boldsymbol{\lambda}}$ is not perpendicular to the line AB)
- Two force components and a moment (also good for eliminating unknown forces)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_1$ (where $\hat{\boldsymbol{\lambda}}_1$ is any unit vector)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_2$ (where $\hat{\boldsymbol{\lambda}}_2$ is any other unit vector)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)

Any of these will always do the job. The power balance equation is often used as a consistency check rather than an independent equation.

From a theoretical point of view one might ask the related question of which of the equations of motion can be derived from the others. There are many answers. Here are some of them:

- Power balance follows from LMB and AMB,
- AMB about three non-colinear points implies LMB, and
- LMB and power balance yield AMB

Interestingly, there is no way to derive angular momentum balance from linear momentum balance without the questionable microscopic assumptions discussed in box C.3 on page 1004.

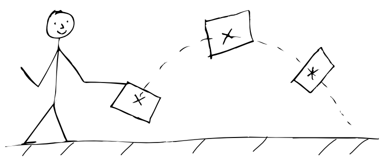
Some simple examples

Here we consider some simple examples of unconstrained motion of a rigid object.

Example: The simplest case: no force and no moment.
If the net force and moment applied to a object are zero we have:

$$\begin{aligned} \text{LMB} \quad &\Rightarrow \vec{\mathbf{0}} &= \; & m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}} \quad \text{and} \\ \text{AMB} \quad &\Rightarrow \vec{\mathbf{0}} &= \; & I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \end{aligned}$$

so $\vec{\boldsymbol{a}}_{\text{cm}} = \vec{\mathbf{0}}$ and $\dot{\omega} = 0$ and the object moves at constant speed in a constant direction with a constant rate of rotation, all determined by the initial conditions. Throw an object in space and its center-of-mass goes in a straight line and it spins at constant rate (subject to the 2-D restrictions of this chapter).



Filename:tfigure-clipboard

Figure 14.20: The X marked at the center-of-mass of a thrown spinning clipboard follows a parabolic trajectory.

Example: Constant force applied to the center-of-mass.
In this case angular momentum balance about the center-of-mass again yields that the rotation rate is constant. Linear momentum balance is now the same as for a particle at the center-of-mass, *i.e.,* the center-of-mass has a parabolic trajectory.

Near-earth (constant) gravity provides a simple example. An 'X' marked at the center-of-mass of a clipboard tossed across a room follows a parabolic trajectory (see fig. 14.20).

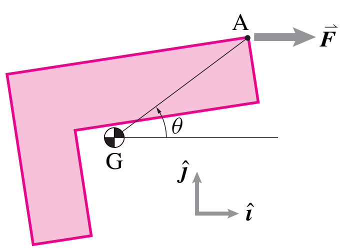
Example: Constant force not at the center-of-mass.
Assume the only force applied to an object is a constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$ at A (see fig. 14.21). Then linear momentum balance gives us that

$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \quad \Rightarrow \quad F\hat{\boldsymbol{\imath}} = m\vec{\boldsymbol{a}}_{\text{G}} \quad \Rightarrow \quad \vec{\boldsymbol{a}}_{\text{G}} = F/m\hat{\boldsymbol{\imath}} = \text{constant.}$$

So if the object starts at rest, the point G will move in a straight line in the $\hat{\boldsymbol{\imath}}$ direction (The common intuition that point G will be pulled up is incorrect). Angular momentum about the center-of-mass gives

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\text{cm}\,\text{i}} = \dot{\vec{\boldsymbol{H}}}_{\text{cm}} \quad &\Rightarrow \quad \left\{ \vec{\boldsymbol{r}}_{\text{A/G}} \times F\hat{\boldsymbol{\imath}} = I_{zz}^{\text{cm}}\ddot{\theta}\hat{\boldsymbol{k}} \right\} \\ \{\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad \ddot{\theta} + \frac{F\ell}{I_{zz}^{\text{cm}}} \sin\theta = 0, \end{aligned}$$

with $\ell = |\vec{\boldsymbol{r}}_{\text{A/G}}|$, which is the pendulum equation. That is, the object can swing back and forth about $\theta = 0$ just like a pendulum, approximately sinusoidally if the angle θ starts small and with $\dot{\theta}$ initially also small. [One might wonder how to do this experiment. One way would be with a jet on a space craft that keeps re-orienting itself to keep in a constant spatial direction as the object changes orientation. Another would be with a string tied to A and pulled from a great distance.]



Filename:tfigure-constforce

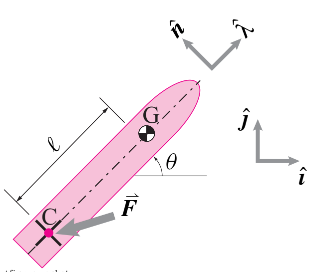
Figure 14.21: The only force applied to the object is the constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$ applied at point A. The resulting motion is a constant acceleration of the center-of-mass $\vec{\boldsymbol{a}}_{\text{G}} = (F/m)\hat{\boldsymbol{\imath}}$ and an oscillatory motion of θ identical to that of a pendulum hinged at G. Note that the center of mass goes in a straight line.

Example: The flight of an arrow or rocket.
As a primitive model of an arrow or rocket assume that the only force is from drag on the fins at C and that this force opposes motion according to

$$\vec{\boldsymbol{F}} = -c\vec{\boldsymbol{v}}_{\text{C}}$$

where c is a drag coefficient (see fig. 14.22). From linear momentum balance we have:

$$\begin{aligned} \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \quad \Rightarrow \quad \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -c\vec{\boldsymbol{v}}_{\text{C}} &= m\dot{\vec{\boldsymbol{v}}} \\ m\dot{\vec{\boldsymbol{v}}} &= -c\left(\vec{\boldsymbol{v}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{C/G}}\right) \\ &= -c\left(\vec{\boldsymbol{v}} + \dot{\theta}\hat{\boldsymbol{k}} \times (-\ell\hat{\boldsymbol{\lambda}})\right) \\ (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{n}}) \quad \Rightarrow \quad \dot{\vec{\boldsymbol{v}}} &= \frac{c}{m}\left(\dot{\theta}\ell\hat{\boldsymbol{n}} - \vec{\boldsymbol{v}}\right). \end{aligned}$$



Filename:tfigure-rocket

Figure 14.22: A rocket is pointed in the direction $\hat{\boldsymbol{\lambda}}$ which makes and angle θ with the positive x axis. The position and velocity of the center-of-mass at G are called $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$. The velocity of the tail is $\vec{\boldsymbol{v}}_{\text{C}}$

So if $\vec{v}$, θ and $\dot{\theta}$ are known the acceleration $\dot{\vec{v}}$ is calculated by the formula above.

Similarly angular momentum balance about G gives

$$\begin{array}{rclcrcl} \sum \vec{M}_{\mathrm{G}} = \dot{\vec{H}}_{\mathrm{G}} & \Rightarrow & \{\vec{r}_{\mathrm{C/G}} \times \vec{F} & = & I_{zz}^{\mathrm{cm}} \dot{\omega} \hat{k}\} \\ \{\} \cdot \hat{k} & \Rightarrow & I_{zz}^{\mathrm{cm}} \dot{\omega} & = & \vec{r}_{\mathrm{C/G}} \times \vec{F} \cdot \hat{k}. \end{array}$$

Then, making the same substitutions as before for $\vec{r}_{\mathrm{C/G}}$ and $\vec{F}$ we get

$$\dot{\omega} = \frac{c\ell}{I_{zz}^{\mathrm{cm}}} \left(\hat{\lambda} \times \vec{v} \cdot \hat{k} - \dot{\theta}\ell \right)$$

which determines the rate of change of ω if the present values of $\vec{v}$, θ and $\dot{\theta}$ are known.

Setting up differential equations for solution

If one knows the forces and torques on a object in terms of its position, velocity, orientation and angular velocity one then has a 'closed' set of differential equations. That is, one has enough information to define the equations for a mathematician or a computer to solve them.

The full set of differential equations is gathered from linear and angular momentum balance and also from simple kinematics. Namely, one has the following set of 6 first order differential equations:

$$\begin{array}{rcl} \dot{x} & = & v_x, \\ \dot{v}_x & = & F_x/m, \\ \dot{y} & = & v_y, \\ \dot{v}_y & = & F_y/m, \\ \dot{\theta} & = & \omega, \text{ and} \\ \dot{\omega} & = & M_{\mathrm{cm}}/I_{zz}^{\mathrm{cm}}, \end{array}$$

where the positions and velocities are the positions and velocities of the center-of-mass. The expressions for F_x, F_y, and M_{cm} may well be complicated, as in the rocket example above.

14.1 2-D mechanics makes sense in a 3-D world

The math for two-dimensional mechanics analysis is simpler than the math for three-dimensional analysis. And thus easier to learn first. But we do actually live in a three-dimensional world you might wonder at the utility of learning something that is not right. There are three answers.

1. Two dimensional analysis can give partial information about the three-dimensional system that is exactly the same as the three-dimensional analysis would give by *projection*, no matter what the motion;
2. if the motion is planar the 2-D kinematics can be used; and
3. if the object is planar or symmetric about the motion plane, and any constraints that hold the object are also symmetric about the motion plane, the 2-D analysis is not only correct, but complete.

Of course no machine is exactly planar or exactly symmetric, but if the approximation seems reasonable most engineers will accept a small loss in accuracy for great gain in simplicity.

a) Projection

First lets relax our assumption of 2-D motion. Consider arbitrary 3-D motions of an arbitrarily complex system. If we take the dot product of the linear momentum equations with $\hat{\imath}$ and $\hat{\jmath}$ and the angular momentum balance equation with $\hat{k}$ we get

$$\begin{aligned}
&\left\{\sum \vec{F}_i = \sum m_i \vec{a}_i\right\} \cdot \hat{\imath} \;\Rightarrow \sum F_{ix} = \sum m_i a_{ix}, && \text{(a)}\\
&\left\{\sum \vec{F}_i = \sum m_i \vec{a}_i\right\} \cdot \hat{\jmath} \Rightarrow \sum F_{iy} = \sum m_i a_{iy}, \qquad \text{and} && \text{(b)}\\
&\left\{\sum \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i \times m_i \vec{a}_i\right\} \cdot \hat{k} && \\
&\qquad \Rightarrow \sum r_{ix} F_{iy} - r_{iy} F_{ix} = \sum m_i (r_{ix} a_{iy} - r_{iy} a_{ix}). && \text{(c)}
\end{aligned} \tag{14.23}$$

These are exactly the equations of 2-D mechanics. That is, if we only consider the planar components of the forces, the planar components of the positions, and the planar components of the motions, we get a correct but partial set of the 3-D equations. In this sense 2-D analysis is correct but incomplete.

b) Planar motion

If all the velocities of the parts of a 3-D system have no z component the motion is planar (in the xy plane). Thus the right-hand sides of eqns. (14.23) are not just projections, but the whole story. Further, in the case of rigid-object motion, the 2-D kinematics equation

$$\vec{v}_P = \vec{v}_G + \omega \hat{k} \times \vec{r}_{P/G} = \vec{v}_G + \omega \hat{k} \times \left(r_{P/G_x} \hat{\imath} + r_{P/G_y} \hat{\jmath}\right) \tag{14.24}$$

also applies (the z component of the position drops out of the cross product) and the expression for, say, the z component of the angular momentum of a object about its center-of-mass is

$$H_{\text{cm}z} = I_{zz}^{\text{cm}} \omega.$$

Differentiating, or adding up the $m_i \vec{a}_i$ terms we get,

$$H_{\text{cm}z} = I_{zz}^{\text{cm}} \omega.$$

Similarly, the z component of the full angular momentum balance equation for a 3-D rigid object in planar motion is the same as the z component of eqn. (14.22)b.

$$\sum \vec{M}_O \cdot \hat{k} = (\vec{r}_{\text{cm/O}} \times (m_{\text{tot}} \vec{a}_{\text{cm}})) \cdot \hat{k} + I_{zz}^{\text{cm}} \dot{\omega}$$

So for planar motion of 3-D rigid bodies one can do a correct 2-D analysis with the full ease of analyzing a planar object.

But this result is deceptively simple. The free object diagram in 3-D most likely shows forces in the z direction, pairs of forces in the x or y directions that are applied at points with the same x and y coordinates but different z values, or moments with components in the x or y directions. Full information about these force and moment components can't be found from 2-D analysis. That is,

> the nature of the forces that it takes to *keep* a system in planar motion can't be found from a planar analysis.

For example, a system rotating about the z axis which is statically balanced but is dynamically imbalanced has no *net* x or y reaction force, as a planar analysis would reveal, yet the bearing reaction forces are not zero.

Another example would be a plan view of a car in a turn (assuming a stiff suspension). A 2-D analysis could be accurate, but would no be complete enough to describe the tire reaction forces needed to keep the car flat.

c) Symmetric bodies and planar bodies

If the rigid object has all its mass in the xy plane, or its mass is symmetrically distributed about the xy plane, and it is in planar motion in the xy plane then

$$\sum m_i a_{iz} = 0 \quad \text{and}$$
$$\left\{\sum \vec{r}_i \times m_i \vec{a}_i\right\} \cdot \hat{\imath} = 0 \quad \text{and} \quad \left\{\sum \vec{r}_i \times m_i \vec{a}_i\right\} \cdot \hat{\jmath} = 0$$

where $\vec{r}$ is measured relative to any point in the plane. Thus, by linear and angular momentum balance,

$$\sum F_z = 0 \quad \text{and}$$
$$\left\{\sum \vec{r}_i \times \vec{F}_i\right\} \cdot \hat{\imath} = 0 \quad \text{and} \quad \left\{\sum \vec{r}_i \times \vec{F}_i\right\} \cdot \hat{\jmath} = 0$$

so

> A planar object or a symmetric object in planar motion needs no force in the z direction and no moment in the x or y direction to keep it in the plane.

Systems that are symmetric or flat and moving in an approximately planar manner, are thus both accurately and completely modeled with a 2-D analysis. A slight generalization of the result is to any object or collection of objects whose center's of mass are on the plane and each of which is dynamically balanced for rotation about a $\hat{k}$ axis through its center-of-mass.

14.2 The center-of-mass theorems for 2-D rigid bodies

That all the particles in a system are part of one planar object in planar motion (in that plane) allows highly useful simplification of the expressions for the motion quantities, namely Eqns. 14.16 to 14.20. We can derive these expressions from the center-of-mass theorems from chapter C. For completeness, we repeat some of those derivations here. To save space, we only use the integral ($\int$) forms for the general expressions; the derivations with sums ($\sum$) are similar. In all cases position, velocity, and acceleration are relative to a fixed point in space (that is $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{a}}$ mean $\vec{\boldsymbol{r}}_{/0}$, $\vec{\boldsymbol{v}}_{/0}$, and $\vec{\boldsymbol{a}}_{/0}$ respectively).

Linear momentum.

Here we show that to evaluate linear momentum and its rate of change you only need to know the motion of the center of mass.

$$\begin{aligned}\vec{\boldsymbol{L}} &\equiv \int \vec{\boldsymbol{v}}\,dm = \int \frac{d}{dt}\vec{\boldsymbol{r}}\,dm = \frac{d}{dt}\int \vec{\boldsymbol{r}}\,dm = \frac{d}{dt}(m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}}) \\ &= m_{\text{tot}}\frac{d}{dt}\vec{\boldsymbol{r}}_{\text{cm}} = m_{\text{tot}}\vec{\boldsymbol{v}}_{\text{cm}}\end{aligned}$$

By identical reasoning, or by differentiating the expression above with respect to time,

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}$$

Thus for linear momentum balance one need not pay heed to rotation, only the center-of-mass motion matters.

Angular momentum.

Here we attempt a derivation like the one above but get slightly more complicated results. For simplicity we evaluate angular momentum and its rate of change relative to the origin, but a very similar derivation would hold relative to any fixed point C.

$$\begin{aligned}\vec{\boldsymbol{H}}_{/\text{O}} &\equiv \int \vec{\boldsymbol{r}}\times\vec{\boldsymbol{v}}\,dm \\ &= \int \left(\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_{\text{cm}}+\vec{\boldsymbol{r}}_{\text{cm}}\right)\times\left(\vec{\boldsymbol{v}}-\vec{\boldsymbol{v}}_{\text{cm}}+\vec{\boldsymbol{v}}_{\text{cm}}\right)\,dm \\ &= \int \left(\vec{\boldsymbol{r}}_{/\text{cm}}+\vec{\boldsymbol{r}}_{\text{cm}}\right)\times\left(\vec{\boldsymbol{v}}_{/\text{cm}}+\vec{\boldsymbol{v}}_{\text{cm}}\right)\,dm \\ &= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \int \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\,dm \\ &\quad + \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\,dm + \int \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm \\ &= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\int dm \\ &\quad + \underbrace{\left(\int \vec{\boldsymbol{r}}_{/\text{cm}}\,dm\right)}_{\vec{\boldsymbol{0}}}\times\vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{r}}_{\text{cm}}\times\underbrace{\left(\int \vec{\boldsymbol{v}}_{/\text{cm}}\,dm\right)}_{\vec{\boldsymbol{0}}} \\ &= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.\end{aligned}$$

This much is true for any system in any motion. For a rigid object we know about the motions of the parts. Using the center-of-mass as a reference point we know that for all points on the object $\vec{\boldsymbol{v}}_{/\text{cm}} = \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{/\text{cm}}$. Thus we can continue the derivation above, following the same reasoning as was used in chapter 7 for circular motion of rigid bodies:

$$\vec{\boldsymbol{H}}_{/\text{O}} = \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\left(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{/\text{cm}}\right)\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.$$

Using the identity for the triple cross product (see box 14.4 on page 779) or using the geometry of the cross product directly with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ as in chapters 7 and 8 we get

$$\vec{\boldsymbol{H}}_{/\text{O}} = \omega\hat{\boldsymbol{k}}\int r^2_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.$$

Then defining $I^{\text{cm}}_{zz} \equiv \int r^2_{/\text{cm}}\,dm$ we get the desired result:

$$\vec{\boldsymbol{H}}_{/\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}} + I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}.$$

A similar derivation, or differentiation of the result above (and using that $(\frac{d}{dt}\vec{\boldsymbol{r}})\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$) gives

$$\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{a}}_{\text{cm}}m_{\text{tot}} + I^{\text{cm}}_{zz}\dot{\omega}\hat{\boldsymbol{k}}.$$

The results above hold for any reference point, not just the origin of the fixed coordinate system. Thus, relative to a point instantaneously coinciding with the center-of-mass

$$\vec{\boldsymbol{H}}_{\text{cm}} = \underbrace{\vec{\boldsymbol{r}}_{\text{cm}/\text{cm}}}_{\vec{\boldsymbol{0}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}+I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}} = I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}.$$

and similarly

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = I^{\text{cm}}_{zz}\dot{\omega}\hat{\boldsymbol{k}}.$$

Kinetic energy.

Unsurprisingly the expression for kinetic energy and its rate of change are also simplified by derivations very similar to those above. Skipping the details (or leaving them as an exercise for the peppy reader):

$$\begin{aligned}E_{\text{K}} &\equiv \int \frac{1}{2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}\,dm \\ &= \frac{1}{2}m_{\text{tot}}v^2_{cm} + \frac{1}{2}I^{\text{cm}}_{zz}\omega^2\end{aligned}$$

and

$$\begin{aligned}\dot{E}_{\text{K}} &\equiv \frac{d}{dt}E_{\text{K}} \\ &= m_{\text{tot}}v\dot{v} + I^{\text{cm}}_{zz}\omega\dot{\omega}.\end{aligned}$$

14.3 The work of a moving force and of a couple

The work of a force acting on a object from state one to state two is

$$W_{12} = \int_{t_1}^{t_2} P dt.$$

But sometimes we like to think not of the time integral of the power, but of the path integral of the moving force. So we rearrange this integral as follows.

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} P dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \underbrace{\vec{v} dt}_{d\vec{r}} \\ &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \end{aligned} \quad (14.25)$$

The validity of equation 14.25 depends on the force acting on the same material point of the moving object as it moves from position 1 to position 2; i.e., the force moves with the object. If the material point of force application changes with time, eqn. (14.25) is senseless and should be replaced with the following more generally applicable equation:

$$W_{12} \equiv \int_{t_1}^{t_2} P\, dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt \quad (14.26)$$

where $\vec{v}$ is the velocity of the material point at the instantaneous location of the applied force.

Hand drags on a passing train: a subtlety

There is a subtle distinction between eqn. (14.25) and eqn. (14.26). As an example think of standing still and dragging your hand on a passing train. Your hand slows down the train with the force

$$\vec{F}_{\text{hand on train}}.$$

It might seem that the work of the hand on the train is zero because your hand doesn't move; work is force times distance and the distance is zero and eqn. (14.25) superficially evaluates to

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = 0.$$

But we have violated the condition for the validity of eqn. (14.25): the force be applied to a fixed material point as time progresses. Whereas on the train your hand smears a whole line of material points.

Clearly your hand does slow the train, so it must do (negative) work on the train, as eqn. (14.26) correctly shows because

$$P_{\text{force on train}} = \vec{F}_{\text{hand on train}} \cdot \vec{v}_{\text{train}} \neq 0.$$

The power of the hand force on the train is the force on the train dotted with the velocity of the train (*not* with the velocity of your hand. Thus, your hand does negative work on the train. eqn. (14.26) applies to the train and eqn. (14.25) does not.

On the other hand (so to speak) if one looks at the power of the force on the hand we have:

$$\vec{F}_{\text{train on hand}} = -\vec{F}_{\text{hand on train}}$$

while the velocity of the hand is zero so

$$P_{\text{force on hand}} = \vec{F}_{\text{train on hand}} \cdot \vec{v}_{\text{hand}} = 0.$$

So the train does *no* work on your hand since while your hand does (negative) work on the train. The difference, of course, is mechanical energy lost to heat.

Work of an applied torque

By thinking of an applied torque as really a distribution of forces, the work of an applied torque is the sum of the contributions of the applied forces. If a collection of forces equivalent to a torque is applied to one rigid object the power of these forces turns out to be $\vec{M} \cdot \vec{\omega}$. At a given angular velocity a bigger torque applies more power. And a given torque applies more power to a faster spinning object.

Here's a quick derivation for a collection of forces $\vec{F}_i$ that add to zero acting at points with positions $\vec{r}_i$ relative to a reference point on the object o$'$.

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= \sum \vec{F}_i \cdot (\vec{v}_{o'} + \vec{\omega} \times \vec{r}_{i/o'}) \\ &= \vec{v}_{o'} \cdot \underbrace{\sum \vec{F}_i}_{\vec{0}} + \sum \vec{F}_i \cdot (\vec{\omega} \times \vec{r}_{i/o'}) \\ &= \sum \vec{\omega} \cdot \left(\vec{r}_{i/o'} \times \vec{F}_i\right) \\ &= \vec{\omega} \cdot \sum \vec{r}_{i/o'} \times \vec{F}_i \\ &= \vec{\omega} \cdot \vec{M}_{o'} \end{aligned} \quad (14.27)$$

Work of a general force distribution

A general force distribution has, by reasoning close to that above, a power of:

$$P = \vec{F}_{\text{tot}} \cdot \vec{v}_{o'} + \vec{\omega} \cdot \vec{M}_{o'}. \quad (14.28)$$

For a given force system applied to a given object in a given motion any point o$'$ can be used. The terms in the formula above will depend on o$'$, but the sum does not. Besides the center-of-mass, another convenient locations for o$'$ is a fixed hinge, in which case the applied force has no power.

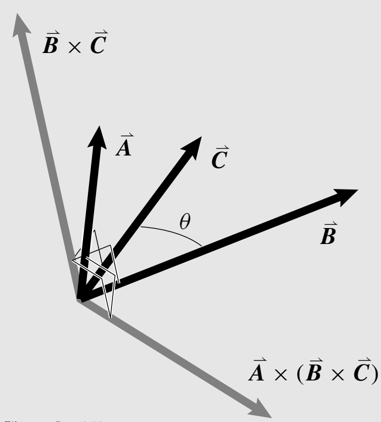
14.4 The vector triple product $\vec{A} \times (\vec{B} \times \vec{C})$

The formula

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \tag{14.29}$$

can be verified by writing each of the vectors in terms of its orthogonal components (*e.g.*, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$) and checking equality of the 27 terms on the two sides of the equations (only 12 are non-zero). If this 20 minute proof seems tedious it can be replaced by a more abstract geometric argument partly presented below that surely takes more than 20 minutes to grasp.

Geometry of the vector triple product



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Because $\vec{B} \times \vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ it is perpendicular to the plane of $\vec{B}$ and $\vec{C}$, that is, it is 'normal' to the plane BC. $\vec{A} \times (\vec{B} \times \vec{C})$ is perpendicular to both $\vec{A}$ and $\vec{B} \times \vec{C}$, so it is perpendicular to the normal to the plane of BC. That is, it must be in the plane of $\vec{B}$ and $\vec{C}$. But any vector in the plane of $\vec{B}$ and $\vec{C}$ must be a combination of $\vec{B}$ and $\vec{C}$. Also, the vector triple product must be proportional in magnitude to each of $\vec{A}$, $\vec{B}$ and $\vec{C}$. Finally, the triple cross product of $\vec{A} \times (\vec{B} \times \vec{C})$ must be the negative of $\vec{A} \times (\vec{C} \times \vec{B})$ because $\vec{B} \times \vec{C} = -\vec{C} \times \vec{B}$. So the identity

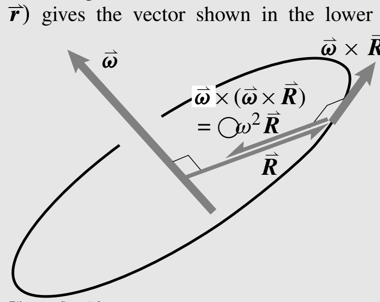
$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

is almost natural: The expression above is almost the only expression that is a linear combination of $\vec{A}$ and $\vec{B}$ that is linear in both, also linear in $\vec{C}$ and switches sign if $\vec{B}$ and $\vec{C}$ are interchanged. These properties would be true if the whole expression were multiplied by any constant scalar. But a test of the equation with three unit vectors shows that such a multiplicative constant must be one. This reasoning constitutes an informal derivation of the identity 14.29.

Using the triple cross product in dynamics equations

We will use identity 14.29 for two purposes in the development of dynamics equations:

1. In the 2D expression for acceleration, the centripetal acceleration is given by $\vec{\omega} \times (\vec{\omega} \times \vec{R})$ simplifies to $-\omega^2\vec{R}$ if $\vec{\omega} \perp \vec{R}$. This equation follows by setting $\vec{A} = \vec{\omega}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{R}$ in equation 14.29 and using $\vec{R} \cdot \vec{\omega} = 0$ if $\vec{\omega} \perp \vec{R}$. In 3D $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ gives the vector shown in the lower figure below.



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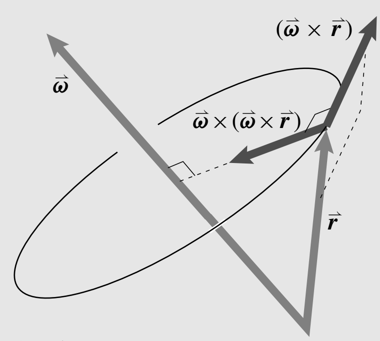

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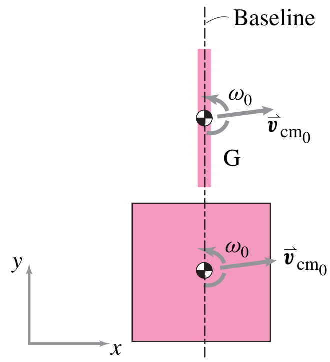
2. The term $\vec{r} \times (\vec{\omega} \times \vec{r})$ will appear in the calculation of the angular momentum of a rigid body. By setting $\vec{A} = \vec{r}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{r}$, in equation 14.29 and use $\vec{r} \cdot \vec{r} = r^2$ because $\vec{r} \parallel \vec{r}$ we get the useful result that $\vec{r} \times (\vec{\omega} \times \vec{r}) = r^2\vec{\omega} - (\vec{r} \cdot \vec{\omega})\vec{r}$.



Filename:sfig9-2-rodandplate

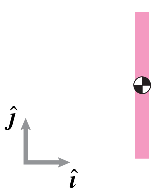
Figure 14.23:

SAMPLE 14.6 Free planar motion. A rigid rod of length $\ell = 1\,\mathrm{m}$ and mass $m_{\mathrm{r}} = 1\,\mathrm{kg}$, and a rigid square plate of side 1 m and mass $m_{\mathrm{p}} = 10\,\mathrm{kg}$ are launched in motion on a frictionless plane (*e.g.,* an ice hockey rink) with exactly the same initial velocities $\vec{v}_{\mathrm{cm}}(0) = 10\,\mathrm{m/s}\hat{\imath} + 1\,\mathrm{m/s}\hat{\jmath}$ and $\vec{\omega}(0) = 1\,\mathrm{rad/s}\hat{k}$. Both the rod and the plate have their center-of-mass at the baseline at $t = 0$.

1. Which of the two is farther from the base line in 3 seconds and which one has undergone more number of revolutions?
2. Find and draw the position of the bar at $t = 1$ sec and at $t = 3$ sec.

Solution

1. The free-body diagram of the rod is shown in fig. 14.24. There are no forces acting on the rod in the xy-plane. Although there is force of gravity and the normal reaction of the surface acting on the rod, these forces are inconsequential since they act normal to the xy-plane. Therefore, we do not include these forces in our free-body diagram . The linear momentum balance for the rod gives



Filename:sfig9-2-rodfbd

Figure 14.24:

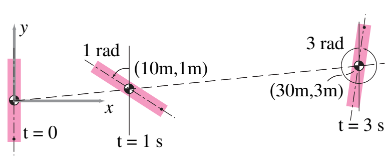
$$
\begin{aligned}
\sum \vec{F} &= m_{\mathrm{r}}\vec{a}_{\mathrm{cm}} \\
\vec{0} &= m_{\mathrm{r}}\dot{\vec{v}}_{\mathrm{cm}} \\
\Rightarrow \quad \vec{v}_{\mathrm{cm}} &= \int \vec{0}\,dt = \text{constant} = \vec{v}_{\mathrm{cm}0} \\
\Rightarrow \quad \vec{r}_{\mathrm{cm}} &= \int \vec{v}_{\mathrm{cm}0}\,dt = \vec{r}_{\mathrm{cm}0} + \vec{v}_{\mathrm{cm}0}t
\end{aligned}
\tag{14.30}
$$

It is clear from the analysis above that in the absence of any applied forces, the position of the body depends only on the initial position and the initial velocity. Since both the plate and the rod start from the same base line with the same initial velocity, they travel the same distance from the base line during any given time period; mass or its geometric distribution play no role in the motion. Thus the center-of-mass of the rod and the plate will be exactly the same distance ($|\vec{r}_{\mathrm{cm}}(t) - \vec{r}_{\mathrm{cm}0}| = |\vec{v}_{\mathrm{cm}0}t|$) at time t. Similarly, the angular momentum balance about the center-of-mass of the rod gives

$$
\begin{aligned}
\sum \vec{M}_{\mathrm{cm}} &= \dot{\vec{H}}_{\mathrm{cm}} \\
\vec{0} &= I_{zz}^{\mathrm{cm}}\dot{\vec{\omega}} \\
\Rightarrow \quad \vec{\omega} &= \int \vec{0}\,dt = \text{constant} = \vec{\omega}_0 = \dot{\theta}_0\hat{k} \\
\Rightarrow \quad \theta &= \int \dot{\theta}_0\,dt = \theta_0 + \dot{\theta}_0 t
\end{aligned}
\tag{14.31}
$$

Thus the angular position of the body is also, as expected, independent of the mass and mass distribution of the body, and depends entirely on the initial position and the initial angular velocity. Therefore, both the rod and the plate undergo exactly the same amount of rotation $\left(\theta(t) - \theta_0 = \dot{\theta}_0 t\right)$ during any given time.

2. We can find the position of the rod at $t = 1$ s and $t = 3$ s by substituting these values of t in eqns. (14.30) and 14.31. For convenience, let us assume that $\vec{r}_{\mathrm{cm}0} = \vec{0}$. From the initial configuration of the rod, we also know that $\theta_0 = 0$.



Filename:sfig9-2-rodposition

Figure 14.25:

$$
\begin{aligned}
\vec{r}_{\mathrm{cm}}(t = 1\,\mathrm{s}) &= \vec{v}_{\mathrm{cm}0}\cdot(1\,\mathrm{s}) = (10\,\mathrm{m/s}\hat{\imath} + 1\,\mathrm{m/s}\hat{\jmath})\cdot(1\,\mathrm{s})10\,\mathrm{m}\hat{\imath} + 1\,\mathrm{m}\hat{\jmath} \\
\vec{r}_{\mathrm{cm}}(t = 3\,\mathrm{s}) &= \vec{v}_{\mathrm{cm}0}\cdot(3\,\mathrm{s}) = 30\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}. \\
\theta(t = 1\,\mathrm{s}) &= \dot{\theta}_0\cdot(1\,\mathrm{s}) = (1\,\mathrm{rad/s})\cdot(1\,\mathrm{s}) = 1\,\mathrm{rad} \\
\theta(t = 3\,\mathrm{s}) &= \dot{\theta}_0\cdot(3\,\mathrm{s}) = 3\,\mathrm{rad}.
\end{aligned}
$$

Accordingly, we show the position of the rod in fig. 14.25.

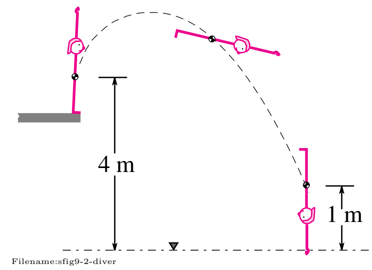


Figure 14.26:

SAMPLE 14.7 A passive rigid diver. An experimental model of a rigid diver is to be launched from a diving board that is 3 m above the water level. Say that the initial velocity of the center-of-mass and the initial angular velocity of the diver can be controlled at launch. The diver is launched into the dive in almost vertical position, and it is required to be as vertical as possible at the very end of the dive (which is marked by the position of the diver's center-of-mass at 1 m above the water level). If the initial vertical velocity of the diver's center-of-mass is 3 m/s, find the required initial angular velocity for the vertical entry of the diver into the water.

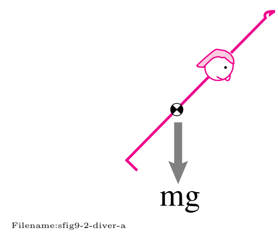


Figure 14.27:

Solution Once the diver leaves the diving board, it is in free flight under gravity, *i.e.,* the only force acting on it is the force due to gravity. The free-body diagram of the diver is shown in fig. 14.27. The linear momentum balance for the diver gives

$$
\begin{aligned}
\sum \vec{F} &= m\vec{a}_{\text{cm}} \\
-mg\hat{\jmath} &= m\ddot{y}\hat{\jmath} \\
\Rightarrow \quad \ddot{y} &= -g \\
\sum \vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\
\vec{0} &= I_{zz}^{\text{cm}}\ddot{\theta}\hat{k} \\
\Rightarrow \quad \ddot{\theta} &= 0.
\end{aligned}
$$

From these equations of motion, it is clear that the linear and the angular motions of the diver are uncoupled. We can easily solve the equations of motion to get

$$
\begin{aligned}
y(t) &= y_0 + \dot{y}_0 t - \frac{1}{2}gt^2 \\
\theta(t) &= \theta_0 + \dot{\theta}_0 t.
\end{aligned}
$$

We need to find the initial angular speed $\dot{\theta}_0$ such that $\theta = \pi$ when $y = 1\,\text{m}$ (the center-of-mass position at the water entry). From the expression for $\theta(t)$, we get, $\dot{\theta}_0 = \pi/t$. Thus we need to find the value of t at the instant of water entry. We can find t from the expression for $y(t)$ since we know that $y = 1\,\text{m}$ at that instant, and that $y_0 = 3\,\text{m}$ and $\dot{y}_0 = 3\,\text{m/s}$. We have,

$$
\begin{aligned}
y &= y_0 + \dot{y}_0 - \frac{1}{2}gt^2 \\
\Rightarrow \quad t &= \frac{\dot{y}_0 \pm \sqrt{\dot{y}_0^2 + 2g(y_0 - y)}}{g} \\
&= \frac{3\,\text{m/s} \pm \sqrt{(3\,\text{m/s})^2 + 2\cdot 9.8\,\text{m/s}^2 \cdot (3\,\text{m} - 1\,\text{m})}}{9.8\,\text{m/s}^2} \\
&= 1.15 \text{ or } -0.53\,\text{s}.
\end{aligned}
$$

We reject the negative value of time as meaningless in this context. Thus it takes the diver 1.15 s to complete the dive. Since, the diver must rotate by π during this time, we have

$$\dot{\theta}_0 = \pi/t = \pi/(1.15\,\text{s}) = 2.73\,\text{rad/s}.$$

$$\boxed{\dot{\theta}_0 = 2.73\,\text{rad/s}}$$

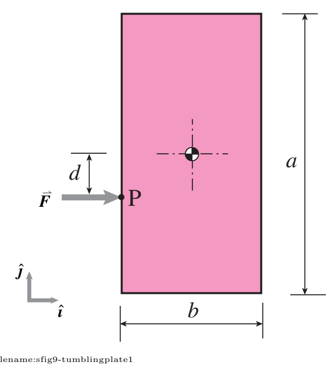


Figure 14.28:

SAMPLE 14.8 A plate tumbling in space. A rectangular plate of mass $m = 0.5\,\text{kg}$, $I_{zz}^{\text{cm}} = 2.08 \times 10^{-3}\,\text{kg}\cdot\text{m}^2$, and dimensions $a = 200\,\text{mm}$ and $b = 100\,\text{mm}$ is pushed by a force $\vec{F} = 0.5\,\text{N}\hat{\imath}$, acting at $d = 30\,\text{mm}$ away from the mass-center, as shown in the figure. Assume that the force remains constant in magnitude and direction but remains attached to the material point P of the plate. There is no gravity.

1. Find the initial acceleration of the mass-center.
2. Find the initial angular acceleration of the plate.
3. Write the equations of motion of the plate (for both linear and angular motion).

Solution The only force acting on the plate is the applied force $\vec{F}$. Thus, fig. 14.28 is also the free-body diagram of the plate at the start of motion.

1. From the linear momentum balance we get,

$$\begin{aligned}\sum \vec{F} &= m\vec{a}_{\text{cm}}\\ \Rightarrow \quad \vec{a}_{\text{cm}} &= \frac{\sum \vec{F}}{m} = \frac{0.5\,\text{N}\hat{\imath}}{0.5\,\text{kg}} = 1\,\text{m/s}^2\hat{\imath}.\end{aligned}$$

which is the initial acceleration of the mass-center.

$$\boxed{\vec{a}_{\text{cm}} = 1\,\text{m/s}^2\hat{\imath}}$$

2. From the angular momentum balance about the mass-center, we get

$$\begin{aligned}\vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}}\\ Fd\,\hat{k} &= I_{zz}^{\text{cm}}\dot{\vec{\omega}}\\ \Rightarrow \quad \dot{\vec{\omega}} &= \frac{Fd}{I_{zz}^{\text{cm}}}\hat{k} = \frac{0.5\,\text{N}\cdot 0.03\,\text{m}}{2.08\,\text{kg}\cdot\text{m}^2} = 7.2\,\text{rad/s}^2\hat{k}\end{aligned}$$

which is the initial angular acceleration of the plate.

$$\boxed{\dot{\vec{\omega}} = 7.2\,\text{rad/s}^2\hat{k}}$$

3. To find the equations of motion, we can use the linear momentum balance and the angular momentum balance as we have done above. So, why aren't the equations obtained above for the linear acceleration, $\vec{a}_{\text{cm}} = F/m\hat{\imath}$, and the angular acceleration, $\dot{\vec{\omega}} = Fd/I_{zz}^{\text{cm}}\hat{k}$, qualified to be called equations of motion? Because, they are not valid for a general configuration of the plate during its motion. The expressions for the accelerations are valid only in the initial configuration (and hence those are initial accelerations).

Let us first draw a free-body diagram of the plate in a general configuration during its motion (see fig. 14.29). Assume the center-of-mass to be displaced by $x\hat{\imath}$ and $y\hat{\jmath}$, and the longitudinal axis of the plate to be rotated by $\theta\hat{k}$ with respect to the vertical (inertial y-axis). The applied force remains horizontal and attached to the material point P, as stated in the problem. The linear momentum balance gives

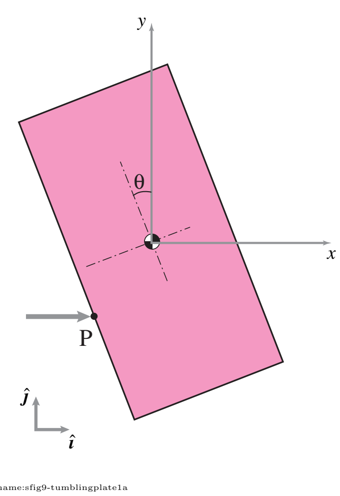


Figure 14.29: Free body diagram of the plate at some instant t when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed vertical axis.

$$\begin{aligned}&\sum \vec{F} = m\vec{a}_{\text{cm}} &&\Rightarrow &&\vec{a}_{\text{cm}} = \frac{\sum \vec{F}}{m}\\ \text{or}\quad &\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \frac{F}{m}\hat{\imath} &&\Rightarrow &&\ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0.\end{aligned}$$

Since F/m is constant, the equations of motion of the center-of-mass indicate that the acceleration is constant and that the mass-center moves in the x-direction.

Similarly, we now use angular momentum balance to determine the rotation (angular motion) of the plate. The angular momentum balance about the mass-center give

$$\begin{aligned}\vec{\boldsymbol{M}}_{\mathrm{cm}} &= \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}\\ \vec{r}_{\mathrm{P/cm}}\times\vec{\boldsymbol{F}} &= I_{zz}^{\mathrm{cm}}\ddot{\theta}\hat{\boldsymbol{k}}.\end{aligned}$$

Now,

$$\begin{aligned}\vec{r}_{\mathrm{P/cm}} &= -r[\cos(\theta+\alpha)\hat{\imath}+\sin(\theta+\alpha)\hat{\jmath}]\\ \vec{\boldsymbol{F}} &= F\hat{\imath}\\ \Rightarrow\quad \vec{r}_{\mathrm{P/cm}}\times\vec{\boldsymbol{F}} &= Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}}.\end{aligned}$$

Thus,

$$\ddot{\theta} = \frac{Fr}{I_{zz}^{\mathrm{cm}}}\sin(\theta+\alpha)$$

where $r=\sqrt{d^2+(b/2)^2}$ and $\alpha=\tan^{-1}(2d/b)$.

Thus, we have got the equations of motion for both the linear and the angular motion.

$$\boxed{\ddot{x}=\frac{F}{m},\quad \ddot{y}=0,\quad \ddot{\theta}=\frac{Fr}{I_{zz}^{\mathrm{cm}}}\sin(\theta+\alpha)}$$

4. The equations of linear motion of the plate are very simple and we can solve them at once to get

$$\begin{aligned}x(t) &= x_0+\dot{x}_0t+\frac{1}{2}\frac{F}{m}t^2\\ y(t) &= y_0+\dot{y}_0t.\end{aligned}$$

If the plate starts from rest ($\dot{x}_0=0,\ \dot{y}_0=0$) with the center-of-mass at the origin ($x_0=0,\ y_0=0$), then we have

$$x(t)=\frac{F}{2m}t^2,\quad\text{and}\quad y(t)=0.$$

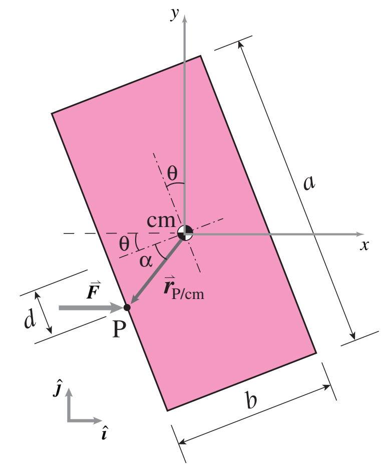
Thus the center-of-mass moves along the x-axis with acceleration F/m.

The equation of angular motion of the plate is not so simple. In fact, it is a nonlinear ODE. It is difficult to get an analytical solution of this equation. However, we can solve it numerically using, say, a Runge-Kutta ODE solver:

```
ODEs = {thetadot = w, wdot = (F*r/Icm)*sin(theta+a)}
IC   = {theta(0) = 0, w(0) = 0}
Set  F=.5, d=0.03; b=0.1; Icm=2.08e-03
compute r = sqrt(d^2+.25*b^2),   a = atan(2*d/b)
Solve ODEs with IC for t=0 to t=10
Plot theta(t)
```

The plot obtained from this calculation is shown in fig. 14.32 which shows that the plate oscillates in its plane as the center of mass races along the x-axis.

The trajectory of point P can now be obtained from $\vec{r}_{\mathrm{P}}=\vec{r}_{\mathrm{cm}}-r\cos(\theta+\alpha)\hat{\imath}-r\sin(\theta+\alpha)\hat{\jmath}$. Using this relationship, the trajectory of point P, along with the trajectory of the center of mass, is shown in fig. 14.30 for the first 3 seconds.



Filename:sfig9-tumblingplate1b

Figure 14.31: Geometry of the plate at the instant t when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed y-axis. The position of point P is $\vec{r}_{P/cm}$ which makes a fixed angle $\alpha(=\tan^{-1}\frac{d}{b/2})$ with the transverse axis of the plate. This angle is shown here merely for ease of calculation.

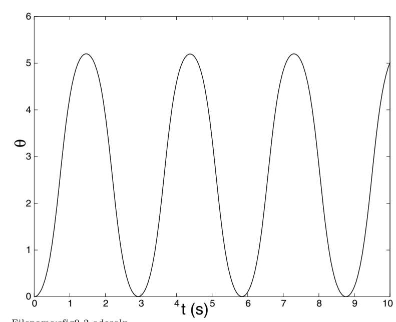


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Figure 14.32: Variation of plate orientation θ with time.

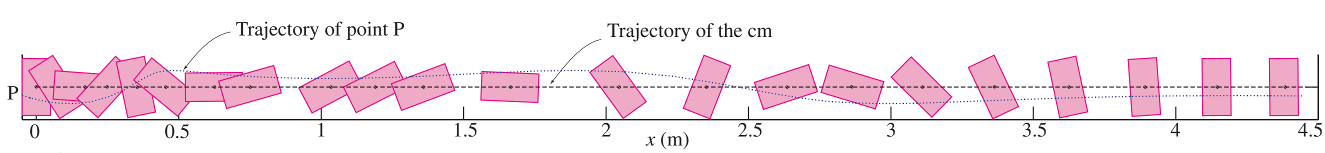

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Figure 14.30: Motion of the plate during the first 3 seconds: Position of the plate at various time instants along with the trajectories of the center of mass and point P.

SAMPLE 14.9 Impulse-momentum. Consider the plate problem of Sample 14.8 (page 782) again. Assume that the plate is at rest at $t = 0$ in the vertical upright position and that the force acts on the plate for 2 seconds.

1. Find the velocity of the center-of-mass of the plate at the end of 2 seconds.
2. Can you also find the angular velocity of the plate at the end of 2 seconds?

Solution

1. Since we are interested in finding the velocity at a particular instant t, given the velocity at another instant $t = 0$, we can use the impulse-momentum equations to find the desired velocity.

$$\begin{aligned}
\vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{F}}\, dt \\
m\vec{\boldsymbol{v}}_{\text{cm}}(t) - m\vec{\boldsymbol{v}}_{\text{cm}}(0) &= \int_0^t \vec{\boldsymbol{F}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{v}}_{\text{cm}}(t) &= \vec{\boldsymbol{v}}_{\text{cm}}(0) + \frac{1}{m}\int_0^2 \vec{\boldsymbol{F}}\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{0.5\,\text{kg}}\int_0^2 (0.5\,\text{N}\hat{\boldsymbol{\imath}})\, dt \\
&= 2\,\text{m/s}\hat{\boldsymbol{\imath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_{\text{cm}}(2\,\text{s}) = 2\,\text{m/s}\hat{\boldsymbol{\imath}}}$$

2. Now, let us try to find the angular velocity the same way, using angular impulse-momentum relation. We have,

$$\begin{aligned}
(\vec{\boldsymbol{H}}_{\text{cm}})_2 - (\vec{\boldsymbol{H}}_{\text{cm}})_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt \\
I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}(t) - I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}(0) &= \int_0^t \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{\omega}}(t) &= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\text{cm}}}\int_0^t \sum \vec{\boldsymbol{M}}_{\text{cm}}\, dt \\
&= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\text{cm}}}\int_0^t (\vec{\boldsymbol{r}}_{\text{P/cm}} \times \vec{\boldsymbol{F}})\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{I_{zz}^{\text{cm}}}\int_0^t (Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}})\, dt \\
&= \frac{Fr}{I_{zz}^{\text{cm}}}\left(\int_0^t \sin(\theta+\alpha)\, dt\right)\hat{\boldsymbol{k}}.
\end{aligned}$$

Now, we are in trouble; how do we evaluate the integral? In the integrand, we have θ which is an implicit function of t. Unless we know how θ depends on t we cannot evaluate the integral. To find $\theta(t)$ we have to solve the equation of angular motion we derived in the previous sample. However, we were not able to solve for $\theta(t)$ analytically, we had to resort to numerical solution. Thus, it is not possible to evaluate the integral above and, therefore, we cannot find the angular velocity of the plate at the end of 2 seconds using impulse-momentum equations. We could, however, find the desired velocity easily from the numerical solution.

14.3 Kinematics of rolling and sliding

Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, and rolling of round things on other round things like gears and cams.

2-D rolling of a round wheel on level ground

The simplest case, the no-slip rolling of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in fig. 14.33. For convenience, we pick a point D which was at $x_D = 0$ at the start of rolling, when $x_C = 0$. The key to the kinematics is that:

> *The arc length traversed on the wheel is the distance traveled by the wheel center.*

That is,

$$\begin{aligned} x_C &= s_D \\ &= R\phi \\ \Rightarrow v_C = \dot{x}_C &= R\dot{\phi} \\ \Rightarrow a_C = \dot{v}_C = \ddot{x}_C &= R\ddot{\phi} \end{aligned}$$

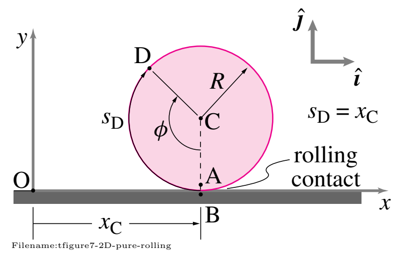


Figure 14.33: Pure rolling of a round wheel on a level support.

So the rolling condition amounts to the following set of restrictions on the position of C, $\vec{r}_C$, and the rotations of the wheel ϕ:

$$\vec{r}_C = R\phi\hat{\imath} + R\hat{\jmath}, \quad \vec{v}_C = R\dot{\phi}\hat{\imath}, \quad \vec{a}_C = R\ddot{\phi}\hat{\imath}, \quad \vec{\omega} = -\dot{\phi}\hat{k}, \text{ and } \quad \vec{\alpha} = \dot{\vec{\omega}} = -\ddot{\phi}\hat{k}.$$

If we want to track the motion of a particular point, say D, we could do so by using the following parametric formula:

$$\begin{aligned} \vec{r}_D &= \vec{r}_C + \vec{r}_{D/C} \\ &= R(\phi\hat{\imath} + \hat{\jmath}) + R(-\sin\phi\hat{\imath} - \cos\phi\hat{\jmath}) \\ &= R\left[(\phi - \sin\phi)\hat{\imath} + (1 - \cos\phi)\hat{\jmath})\right] \\ \Rightarrow \vec{v}_D &= R\left[(\dot{\phi}(1 - \cos\phi)\hat{\imath} + \dot{\phi}\sin\phi\hat{\jmath})\right] \qquad (14.32) \\ \Rightarrow \vec{a}_D &= R\dot{\phi}^2(\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}). \end{aligned}$$

assuming $\dot{\phi}$= constant

Note that if $\phi = 0$ or 2π or 4π, etc., then the point D is on the ground and eqn. (14.32) correctly gives that

$$\vec{v}_D = R\left[\dot{\phi}\underbrace{(1 - \overbrace{\cos(2n\pi)}^{1})}_{0}\right]\hat{\imath} + \dot{\phi}\underbrace{\sin(2n\pi)}_{0}\hat{\jmath} = \vec{0}.$$

Instantaneous kinematics

Instead of tracking the wheel from its start, we could analyze the kinematics at the instant of interest. Here, we make the observation that the wheel rolls without slip. Therefore, the point on the wheel touching the ground has no velocity relative to the ground.

$$\overbrace{\vec{v}_A}^{\text{Velocity of point on the wheel touching the ground}} = \overbrace{\vec{v}_B}^{\text{Velocity of ground} = \vec{0}} \tag{14.33}$$

Now, we know how to calculate the velocity of points on a rigid body. So,

$$\vec{v}_A = \vec{v}_C + \vec{v}_{A/C},$$

where, since A and C are on the same rigid body (fig. 14.33), we have from eqn. (13.42) that

$$\vec{v}_{A/C} = \vec{\omega} \times \vec{r}_{A/C}.$$

Putting this equation together with eqn. (14.33), we get

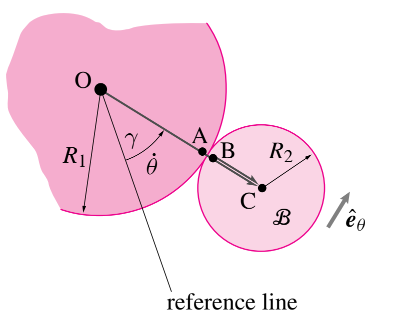
$$\begin{aligned} \vec{v}_A &= \vec{v}_B \\ \Rightarrow \underbrace{\vec{v}_C}_{v_C\hat{\imath}} + \underbrace{\vec{\omega}}_{\omega\hat{k}} \times \underbrace{\vec{r}_{A/C}}_{-R\hat{\jmath}} &= \vec{0} \\ \Rightarrow v_C\hat{\imath} + \omega R\hat{\imath} &= \vec{0} \end{aligned}$$

$$\Rightarrow \qquad v_C = -\omega R. \tag{14.34}$$

We use $\vec{v}_C = v\hat{\imath}$ since the center of the wheel goes neither up nor down. Note that if you measure the angle by ϕ, like we did before, then $\vec{\omega} = -\dot{\phi}\hat{k}$ so that positive rotation rate is in the counter-clockwise direction. Thus, $v_C = -\omega R = -(-\dot{\phi})R = \dot{\phi}R$.

Since there is always *some* point of the wheel touching the ground, we know that $v_C = -\omega R$ for all time. Therefore,

$$\vec{a}_C = \dot{v}_C\hat{\imath} = -\dot{\omega}R\hat{\imath}.$$



Filename:tfigure7-rolling-on-another

Figure 14.34: One round object rolling on another round object.

Rolling of round objects on round surfaces

For round objects rolling on or in another round object, the analysis is similar to that for rolling on a flat surface. Common applications are the so-called epicyclic, hypo-cyclic, or planetary gears (See Box 14.5 on planetary gears on page 788). Referring to fig. 14.34, we can calculate the velocity of C with respect to a fixed frame two ways and compare:

$$
\begin{aligned}
\vec{v}_C &= \vec{v}_B + \vec{v}_{C/B} \\
\vec{v}_C &= \overbrace{\underbrace{\vec{v}_A}_{\vec{0}} + \underbrace{\vec{v}_{B/A}}_{\vec{0}}}^{\vec{v}_B} + \vec{v}_{C/B}. \\
\dot{\theta}(R_1 + R_2)\hat{e}_\theta &= \omega_{\mathcal{B}} R_2 \hat{e}_\theta \\
\Rightarrow \omega_{\mathcal{B}} &= \frac{\dot{\theta}(R_1 + R_2)}{R_2} = \dot{\theta}(1 + \frac{R_1}{R_2}).
\end{aligned}
$$

Example: Two quarters.
The formula above can be tested in the case of $R_1 = R_2$ by using two quarters or two dimes on a table. Roll one quarter, call it $\mathcal{B}$, around another quarter pressed fast to the table. You will see that as the rolling quarter $\mathcal{B}$ travels around the stationary quarter one time, it makes two full revolutions. That is, the orientation of $\mathcal{B}$ changes twice as fast as the angle of the line from the center of the stationary quarter to its center. Or, in the language of the calculation above, $\omega_{\mathcal{B}} = 2\dot{\theta}$.

Sliding

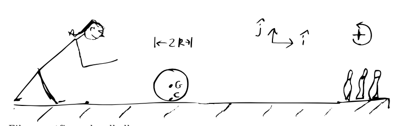
Although wheels and balls are known for rolling, they do sometimes slide such as when a car screeches at fast acceleration or sudden braking or when a bowling ball is released on the lane.

The *sliding velocity* is the velocity of the material point on the wheel (or ball) relative to its contacting substrate. In the case of pure rolling, the sliding velocity is zero. In the case of a ball or wheel moving against a stationary support surface, whether round or curved, the sliding velocity is

$$\vec{v}_{\text{sliding}} = \vec{v}_{\text{circle center}} + \vec{\omega} \times \vec{r}_{\text{contact/center}} \tag{14.35}$$

Example: Bowling ball
The velocity of that point on the bowling ball which is instantaneously in contact with the alley (ground) is $\vec{v}_C = v_G\hat{\imath} + \omega\hat{k} \times \vec{r}_{C/G} = (v_G + \omega R)\hat{\imath}$. So unless $\omega = -v_G/R$ the ball is sliding.

Note that, if sliding, the friction force on the ball opposes the slip of the ball and tends to accelerate the balls rotation towards rolling. That is, for example, if the ball is not rotating the sliding velocity is $v_G\hat{\imath}$, the friction force is in the $-\hat{\imath}$ direction and angular momentum balance about the center-of-mass implies $\dot{\omega} < 0$ and a clockwise rotational acceleration. No matter what the initial velocity or rotational rate the ball will eventually roll.

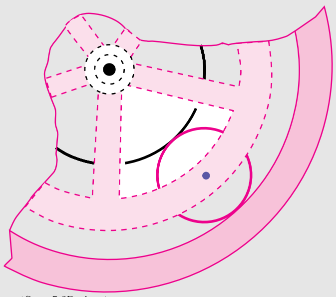


Filename:tfigure-bowlball

Figure 14.35: The bowling ball is sliding so long as $v_G \neq -\omega R$

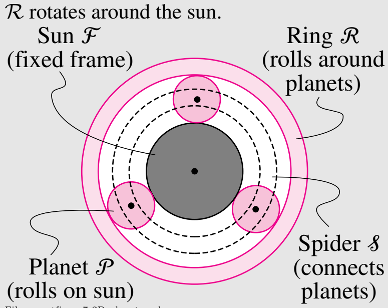
14.5 The Sturmey-Archer hub

In 1903, the year the Wright Brothers first flew powered airplanes, the Sturmey-Archer company patented the internal-hub three-speed bicycle transmission. This marvel of engineering was sold on the best bikes until finicky but fast racing bicycles using derailleurs started to push them out of the market in the 1960's. Now, a hundred years later, internal bicycle hubs (now made by Shimano and Sachs) are having something of a revival, particularly in Europe. These internal-hub transmissions utilize a system called *planetary gears*, gears which roll around other gears. See the figure below.



Filename:tfigure7-2D-planetary

In order to understand this gear system, we need to understand its kinematics—the motion of its parts. The central 'sun' gear $\mathcal{F}$ is stationary, at least we treat it as stationary in this discussion since it is fixed to the bike frame, so it is fixed in body $\mathcal{F}$. The 'planet' gears roll around the sun gear. Let's call one of these planets $\mathcal{P}$. The spider $\mathcal{S}$ connects the centers of the rolling planets. Finally, the ring gear $\mathcal{R}$ rotates around the sun.



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The gear transmission steps up the angular velocity when the spider $\mathcal{S}$ is driven and ring $\mathcal{R}$, which moves faster, is connected to the wheel. The transmission steps down the angular velocity when the ring gear is driven and the slower spider is connected to the wheel. The third 'speed' in the three-speed gear transmission is direct drive (the wheel is driven directly).

What are the 'gear ratios' in the planetary gear system? The 'trick' is to recognize that for rolling contact that the contacting points have the same velocity, $\vec{v}_A = \vec{v}_B$ and $\vec{v}_D = \vec{v}_E$. Let's define some terms.

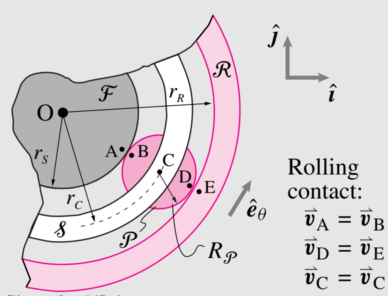
$$\begin{aligned}
\vec{\omega}_\mathcal{S} &= \omega_\mathcal{S}\hat{k} && \text{angular velocity of the spider}\\
\vec{\omega}_\mathcal{P} &= \omega_\mathcal{P}\hat{k} && \text{angular velocity of the planet}\\
\vec{\omega}_\mathcal{R} &= \omega_\mathcal{R}\hat{k} && \text{angular velocity of the ring}
\end{aligned}$$

Now, we can find the relation of these angular velocities as follows. Look at the velocity of point C in two ways. First,

$$\begin{aligned}
\overbrace{\vec{v}_C}^{\boxed{\text{A point on the spider}}} &= \overbrace{\vec{v}_C}^{\boxed{\text{A point on the planetary gear}}}\\
\Rightarrow \vec{\omega}_\mathcal{S}\times\vec{r}_C &= \underbrace{\vec{v}_B}_{\vec{0}} + \vec{\omega}_\mathcal{P}\times\vec{r}_{C/B}\\
\Rightarrow \omega_\mathcal{S} r_C &= \omega_\mathcal{P} R_P\\
\Rightarrow \omega_\mathcal{P} &= \frac{r_C}{R_P}\omega_\mathcal{S}
\end{aligned} \tag{14.36}$$

Next, let's look at point D and E:

$$\begin{aligned}
\vec{v}_D &= \vec{v}_E\\
\vec{v}_A + \vec{v}_{D/A} &= \vec{\omega}_\mathcal{R}\times\vec{r}_R\\
\vec{0} + \vec{\omega}_\mathcal{P}\times\vec{r}_{D/A} &= \omega_\mathcal{R}\hat{k}\times\vec{r}_R\\
\omega_\mathcal{P}(2R_P)\hat{e}_\theta &= \omega_\mathcal{R} r_R \hat{e}_\theta\\
\Rightarrow \omega_\mathcal{R} &= \frac{2R_P}{r_R}\underbrace{\omega_\mathcal{P}}_{\boxed{\omega_\mathcal{P} = \frac{r_C}{R_P}\omega_\mathcal{S}}}\\
\omega_\mathcal{R} &= \frac{2R_P}{r_R}\frac{\overbrace{r_C}^{\boxed{r_C = r_S + R_P}}}{R_P}\omega_\mathcal{S}\\
&= \frac{2(r_S+R_P)}{\underbrace{r_R}_{\boxed{r_R = r_S + 2R_P}}}\omega_\mathcal{S}\\
\Rightarrow \frac{\omega_\mathcal{R}}{\omega_\mathcal{S}} &= 2\frac{1+\frac{R_P}{r_S}}{1+\frac{2R_P}{r_S}} = \text{angular velocity step-up.}
\end{aligned}$$



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Typically, the gears have radius ratio of $\frac{R_P}{r_S} = \frac{3}{2}$ which gives a gear ratio of $\frac{5}{4}$. Thus, the ratio of the highest gear to the lowest gear on a Sturmey-Archer hub is $\frac{5}{4}/\frac{4}{5} = \frac{25}{16} = 1.5625$. You might compare this ratio to that of a modern mountain bike, with eighteen or twenty-one gears, where the ratio of the highest gear to the lowest is about 4:1.

SAMPLE 14.10 **Falling ladder:** The ends of a ladder of length $L = 3\,\text{m}$ slip along the frictionless wall and floor shown in Figure 14.36. At the instant shown, when $\theta = 60^{\circ}$, the angular speed $\dot{\theta} = 1.15\,\text{rad/s}$ and the angular acceleration $\ddot{\theta} = 2.5\,\text{rad/s}^2$. Find the absolute velocity and acceleration of end B of the ladder.

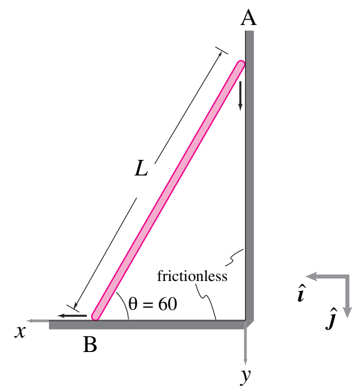

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Figure 14.36:

Solution Since the ladder is falling, it is rotating clockwise. From the given information:

$$\begin{aligned}
\vec{\omega} &= \dot{\theta}\hat{\boldsymbol{k}} = -1.15\,\text{rad/s}\hat{\boldsymbol{k}} \\
\dot{\vec{\omega}} &= \ddot{\theta}\hat{\boldsymbol{k}} = -2.5\,\text{rad/s}^2\hat{\boldsymbol{k}}.
\end{aligned}$$

We need to find $\vec{v}_B$, the absolute velocity of end B, and $\vec{a}_B$, the absolute acceleration of end B.

Since the end A slides along the wall and end the B slides along the floor, we know the directions of $\vec{v}_A$, $\vec{v}_B$, $\vec{a}_A$ and $\vec{a}_B$.

Let $\vec{v}_A = v_A\hat{\boldsymbol{j}}$, $\vec{a}_A = a_A\hat{\boldsymbol{j}}$, $\vec{v}_B = v_B\hat{\boldsymbol{\imath}}$ and $\vec{a}_B = a_B\hat{\boldsymbol{\imath}}$ where the scalar quantities v_A, a_A, v_B and a_B are unknown.

$$\begin{aligned}
\text{Now,}\quad \vec{v}_A &= \vec{v}_B + \vec{v}_{A/B} = \vec{v}_B + \vec{\omega}\times\vec{r}_{A/B} \\
\text{or}\quad v_A\hat{\boldsymbol{j}} &= v_B\hat{\boldsymbol{\imath}} + \dot{\theta}\hat{\boldsymbol{k}}\times\underbrace{L(-\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{j}})}_{\vec{r}_{A/B}} \\
&= (v_B + \dot{\theta}L\sin\theta)\hat{\boldsymbol{\imath}} - \dot{\theta}L\cos\theta\hat{\boldsymbol{j}}.
\end{aligned}$$

Dotting both sides of the equation with $\hat{\boldsymbol{\imath}}$, we get:

$$\begin{aligned}
v_A\underbrace{\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{\imath}}}_{0} &= (v_B + \dot{\theta}L\sin\theta)\underbrace{\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{\imath}}}_{1} + \dot{\theta}L\cos\theta\underbrace{\hat{\boldsymbol{j}}\cdot\hat{\boldsymbol{\imath}}}_{0} \\
\Rightarrow\quad 0 &= v_B + \dot{\theta}L\sin\theta \\
\Rightarrow\quad v_B &= -\dot{\theta}L\sin\theta = -(-1.15\,\text{rad/s})\cdot 3\,\text{m}\cdot\frac{\sqrt{3}}{2} \\
&= 2.99\,\text{m/s}.
\end{aligned}$$

$$\boxed{\vec{v}_B = 2.9\,\text{m/s}\hat{\boldsymbol{\imath}}}$$

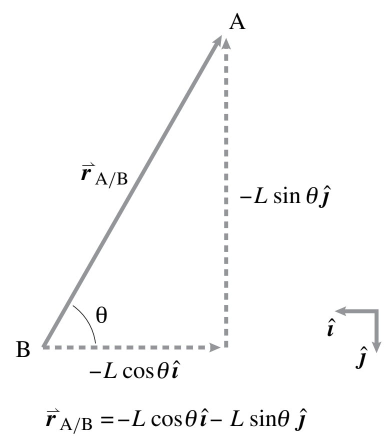

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Figure 14.37:

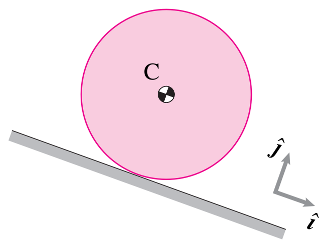
Similarly,

$$\begin{aligned}
\vec{a}_A &= \vec{a}_B + \dot{\vec{\omega}}\times\vec{r}_{A/B} + \overbrace{\vec{\omega}\times(\vec{\omega}\times\vec{r}_{A/B})}^{-\omega^2\vec{r}_{A/B}} \\
a_A\hat{\boldsymbol{j}} &= a_B\hat{\boldsymbol{\imath}} + \ddot{\theta}\hat{\boldsymbol{k}}\times L(-\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{j}}) - \dot{\theta}^2L(-\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\hat{\boldsymbol{j}}) \\
&= (a_B + \ddot{\theta}L\sin\theta + \dot{\theta}^2L\cos\theta)\hat{\boldsymbol{\imath}} + (-\ddot{\theta}L\cos\theta + \dot{\theta}^2L\sin\theta)\hat{\boldsymbol{j}}.
\end{aligned}$$

Dotting both sides of this equation with $\hat{\boldsymbol{\imath}}$ (as we did for velocity) we get:

$$\begin{aligned}
0 &= a_B + \ddot{\theta}L\sin\theta + \dot{\theta}^2L\cos\theta \\
\Rightarrow\quad a_B &= -\ddot{\theta}L\sin\theta - \dot{\theta}^2L\cos\theta \\
&= -(-2.5\,\text{rad/s}^2\cdot 3\,\text{m}\cdot\frac{\sqrt{3}}{2}) - (-1.15\,\text{rad/s})^2\cdot 3\,\text{m}\cdot\frac{1}{2} \\
&= 4.51\,\text{m/s}^2.
\end{aligned}$$

$$\boxed{\vec{a}_B = 4.51\,\text{m/s}^2}$$



Filename:sfig9-rolling-may00

Figure 14.38:

SAMPLE 14.11 A cylinder of diameter 500 mm rolls down an inclined plane with uniform acceleration (of the center-of-mass) $a = 0.1\,\mathrm{m/s^2}$. At an instant t_0, the mass-center has speed $v_0 = 0.5\,\mathrm{m/s}$.

1. Find the angular speed ω and the angular acceleration $\dot{\omega}$ at t_0.
2. How many revolutions does the cylinder make in the next 2 seconds?
3. What is the distance travelled by the center-of-mass in those 2 seconds?

Solution This problem is about simple kinematic calculations. We are given the velocity, $\dot{x}$, and the acceleration, $\ddot{x}$, of the center-of-mass. We are supposed to find angular velocity ω, angular acceleration $\dot{\omega}$, angular displacement θ in 2 seconds, and the corresponding linear distance x along the incline. The radius of the cylinder $R = \text{diameter}/2 = 0.25\,\mathrm{m}$.

1. From the kinematics of pure rolling,

$$\begin{aligned} \omega &= \frac{\dot{x}}{R} = \frac{0.5\,\mathrm{m/s}}{0.25\,\mathrm{m}} = 2\,\mathrm{rad/s}, \\ \dot{\omega} &= \frac{\ddot{x}}{R} = \frac{0.1\,\mathrm{m/s^2}}{0.25\,\mathrm{m}} = 0.4\,\mathrm{rad/s^2}. \end{aligned}$$

$$\boxed{\omega = 2\,\mathrm{rad/s}, \quad \dot{\omega} = 0.4\,\mathrm{rad/s^2}}$$

2. We can find the number of revolutions the cylinder makes in 2 seconds by solving for the angular displacement θ in this time period. Since,

$$\ddot{\theta} \equiv \dot{\omega} = \text{constant},$$

we integrate this equation twice and substitute the initial conditions, $\dot{\theta}(t=0) = \omega = 2\,\mathrm{rad/s}$ and $\theta(t=0) = 0$, to get

$$\begin{aligned} \theta(t) &= \omega t + \frac{1}{2}\dot{\omega}t^2 \\ \Rightarrow \quad \theta(t = 2\,\mathrm{s}) &= (2\,\mathrm{rad/s})\cdot(2\,\mathrm{s}) + \frac{1}{2}(0.4\,\mathrm{rad/s})\cdot(4\,\mathrm{s}^2) \\ &= 4.8\,\mathrm{rad} = \frac{4.8}{2\pi}\,\mathrm{rev} = 0.76\,\mathrm{rev}. \end{aligned}$$

$$\boxed{\theta = 0.76\,\mathrm{rev}}$$

3. Now that we know the angular displacement θ, the distance travelled by the mass-center is the arc-length corresponding to θ, *i.e.,*

$$x = R\theta = (0.25\,\mathrm{m})\cdot(4.8) = 1.2\,\mathrm{m}.$$

$$\boxed{x = 1.2\,\mathrm{m}}$$

Note that we could have found the distance travelled by the mass-center by integrating the equation $\ddot{x} = 0.1\,\mathrm{m/s^2}$ twice.

SAMPLE 14.12 Condition of pure rolling. A cylinder of radius $R = 20\,\text{cm}$ rolls on a flat surface with absolute angular speed $\omega = 12\,\text{rad/s}$ under the conditions shown in the figure (In cases (ii) and (iii), you may think of the 'flat surface' as a conveyor belt). In each case,

1. Write the condition for pure rolling.
2. Find the velocity of the center C of the cylinder.

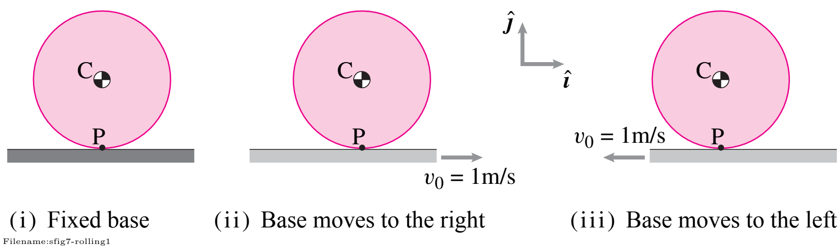


Figure 14.39:

Solution At any instant during rolling, the cylinder makes a point-contact with the flat surface. Let the point of instantaneous contact on the cylinder be P, and let the corresponding point on the flat surface be Q. The condition of pure rolling, in each case, is $\vec{v}_P = \vec{v}_Q$, that is, there is no relative motion between the two contacting points (a relative motion will imply slip). Now, we analyze each case.

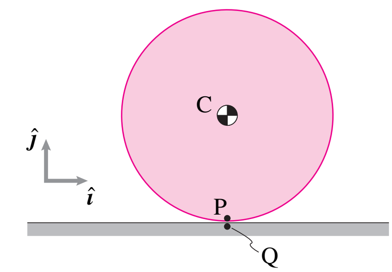


Figure 14.40: The cylinder rolls on the flat surface. Instantaneously, point P on the cylinder is in contact with point Q on the flat surface. For pure rolling, points P and Q must have the same velocity.

Case(i) In this case, the bottom surface is fixed. Therefore,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = \vec{0}$.
2. Velocity of the center:

$$
\begin{aligned}
\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = \vec{0} + (-\omega\hat{k}) \times R\hat{\jmath} \\
&= \omega R\hat{\imath} = (12\,\text{rad/s}) \cdot (0.2\,\text{m})\hat{\imath} = 2.4\,\text{m/s}\hat{\imath}.
\end{aligned}
$$

Case(ii) In this case, the bottom surface moves with velocity $\vec{v} = 1\,\text{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = 1\,\text{m/s}\hat{\imath}$. Thus,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = 1\,\text{m/s}\hat{\imath}$.
2. Velocity of the center:

$$
\begin{aligned}
\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = v_0\hat{\imath} + \omega R\hat{\imath} \\
&= 1\,\text{m/s}\hat{\imath} + 2.4\,\text{m/s}\hat{\imath} = 3.4\,\text{m/s}\hat{\imath}.
\end{aligned}
$$

Case(iii) In this case, the bottom surface moves with velocity $\vec{v} = -1\,\text{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = -1\,\text{m/s}\hat{\imath}$. Thus,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = -1\,\text{m/s}\hat{\imath}$.
2. Velocity of the center:

$$
\begin{aligned}
\vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = -v_0\hat{\imath} + \omega R\hat{\imath} \\
&= -1\,\text{m/s}\hat{\imath} + 2.4\,\text{m/s}\hat{\imath} = 1.4\,\text{m/s}\hat{\imath}.
\end{aligned}
$$

(a) :	(i)$\vec{v}_P = \vec{0}$,	(ii)$\vec{v}_P = 1\,\text{m/s}\hat{\imath}$,	(iii)$\vec{v}_P = -1\,\text{m/s}\hat{\imath}$,
(b) :	(i)$\vec{v}_C = 2.4\,\text{m/s}\hat{\imath}$,	(ii)$\vec{v}_C = 3.4\,\text{m/s}\hat{\imath}$,	(iii)$\vec{v}_C = 1.4\,\text{m/s}\hat{\imath}$

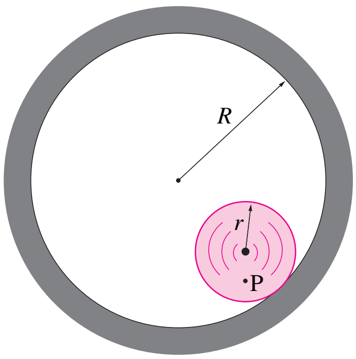


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Figure 14.41: A uniform disk of radius r rolls without slipping inside a fixed cylinder.

SAMPLE 14.13 Motion of a point on a disk rolling inside a cylinder. A uniform disk of radius r rolls without slipping with constant angular speed ω inside a fixed cylinder of radius R. A point P is marked on the disk at a distance ℓ ($\ell < r$) from the center of the disk. At a general time t during rolling, find

1. the position of point P,
2. the velocity of point P, and
3. the acceleration of point P

Solution Let the disk be vertically below the center of the cylinder at $t = 0$ s such that point P is vertically above the center of the disk (Fig. 14.42). At this instant, Q is the point of contact between the disk and the cylinder. Let the disk roll for time t such that at instant t the line joining the two centers (line OC) makes an angle ϕ with its vertical position at $t = 0$ s. Since the disk has rolled for time t at a constant angular speed ω, point P has rotated counter-clockwise by an angle $\theta = \omega t$ from its original vertical position P'.

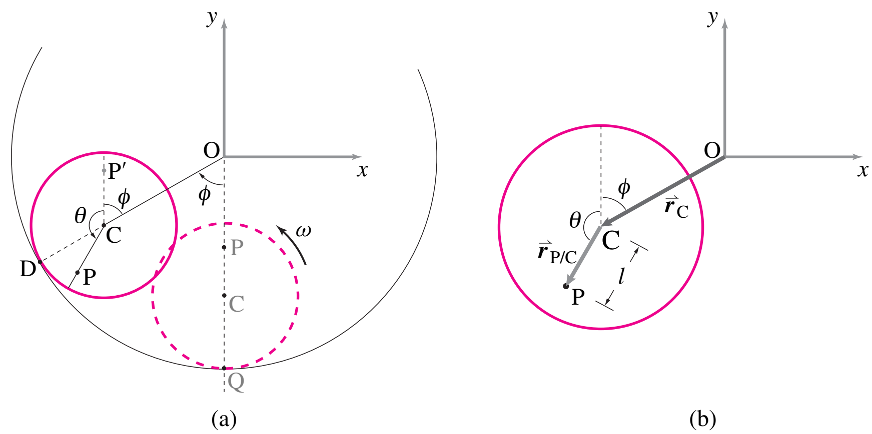

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Figure 14.42: Geometry of motion: keeping track of point P while the disk rolls for time t, rotating by angle $\theta = \omega t$ inside the cylinder.

1. **Position of point P:** From Fig. 14.42(b) we can write

$$\vec{r}_P = \vec{r}_C + \vec{r}_{P/C} = (R-r)\hat{\boldsymbol{\lambda}}_{OC} + \ell\hat{\boldsymbol{\lambda}}_{CP}$$

where

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OC} &= \text{a unit vector along OC } = -\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}, \\ \hat{\boldsymbol{\lambda}}_{CP} &= \text{a unit vector along CP } = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}.\end{aligned}$$

Thus,

$$\vec{r}_P = [-(R-r)\sin\phi - \ell\sin\theta]\hat{\imath} + [-(R-r)\cos\phi + \ell\cos\theta]\hat{\jmath}.$$

We have thus obtained an expression for the position vector of point P as a function of ϕ and θ. Since we also want to find velocity and acceleration of point P, it will be nice to express $\vec{r}_P$ as a function of t. As noted above, $\theta = \omega t$; but how do we find ϕ as a function of t? Note that the center of the disk C is going around point O in circles with angular velocity $-\dot{\phi}\hat{\boldsymbol{k}}$. The disk, however, is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$

about the instantaneous center of rotation, point D. Therefore, we can calculate the velocity of point C in two ways:

$$\begin{array}{rrcl} & \vec{v}_C & = & \vec{v}_C \\ \text{or} & \vec{\omega}\times\vec{r}_{C/D} & = & -\dot{\phi}\hat{k}\times\vec{r}_{C/O} \\ \text{or} & \omega\hat{k}\times r(-\hat{\lambda}_{OC}) & = & -\dot{\phi}\hat{k}\times(R-r)\hat{\lambda}_{OC} \\ \text{or} & -\omega r(\hat{k}\times\hat{\lambda}_{OC}) & = & -\dot{\phi}(R-r)(\hat{k}\times\hat{\lambda}_{OC}) \\ \Rightarrow & \frac{r}{R-r}\omega & = & \dot{\phi}. \end{array}$$

Integrating the last expression with respect to time, we obtain

$$\phi = \frac{r}{R-r}\omega t.$$

Let

$$q = \frac{r}{R-r},$$

then, the position vector of point P may now be written as

$$\vec{r}_P = [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\imath} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\jmath}. \quad (14.37)$$

2. **Velocity of point P:** Differentiating Eqn. (14.37) once with respect to time we get

$$\vec{v}_P = -\omega[(R-r)q\cos(q\omega t) + \ell\cos(\omega t)]\hat{\imath} + \omega[(R-r)q\sin(q\omega t) - \ell\sin(\omega t)]\hat{\jmath}.$$

Substituting $(R-r)q = r$ in $\vec{v}_P$ we get

$$\vec{v}_P = -\omega r[\{\cos(q\omega t) + \frac{\ell}{r}\cos(\omega t)\}\hat{\imath} - \{\sin(q\omega t) - \frac{\ell}{r}\sin(\omega t)\}\hat{\jmath}]. \quad (14.38)$$

3. **Acceleration of point P:** Differentiating Eqn. (14.38) once with respect to time we get

$$\vec{a}_P = -\omega^2 r[-\{q\sin(q\omega t) + \frac{\ell}{r}\sin(\omega t)\}\hat{\imath} - \{q\cos(q\omega t) - \frac{\ell}{r}\cos(\omega t)\}\hat{\jmath}]. \quad (14.39)$$

$$\begin{aligned} \vec{r}_P &= [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\imath} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\jmath} \\ \vec{v}_P &= -\omega r[\{\cos(q\omega t) + \tfrac{\ell}{r}\cos(\omega t)\}\hat{\imath} - \{\sin(q\omega t) - \tfrac{\ell}{r}\sin(\omega t)\}\hat{\jmath}] \\ \vec{a}_P &= -\omega^2 r[-\{q\sin(q\omega t) + \tfrac{\ell}{r}\sin(\omega t)\}\hat{\imath} - \{q\cos(q\omega t) - \tfrac{\ell}{r}\cos(\omega t)\}\hat{\jmath}] \end{aligned}$$

SAMPLE 14.14 The rolling disk: instantaneous kinematics. For the rolling disk in Sample 14.13, let $R = 4\,\text{ft}$, $r = 1\,\text{ft}$ and point P be on the rim of the disk. Assume that at $t = 0$, the center of the disk is vertically below the center of the cylinder and point P is on the vertical line joining the two centers. If the disk is rolling at a constant speed $\omega = \pi$ rad/s, find

1. the position of point P and center C at $t = 1\,\text{s}$, $3\,\text{s}$, and $5.25\,\text{s}$,
2. the velocity of point P and center C at those instants, and
3. the acceleration of point P and center C at the same instants as above.

Draw the position of the disk at the three instants and show the velocities and accelerations found above.

Solution The general expressions for position, velocity, and acceleration of point P obtained in Sample 14.13 can be used to find the position, velocity, and acceleration of any point on the disk by substituting an appropriate value of ℓ in equations (14.37), (14.38), and (14.39). Since $R = 4r$,

$$q = \frac{r}{R-r} = \frac{1}{3}.$$

Now, point P is on the rim of the disk and point C is the center of the disk. Therefore,

$$\begin{array}{lcll} \text{for point P:} & \ell & = & r, \\ \text{for point C:} & \ell & = & 0. \end{array}$$

Substituting these values for ℓ, and $q = 1/3$ in equations (14.37), (14.38), and (14.39) we get the following.

1. **Position:**

$$\begin{aligned} \vec{r}_C &= -3r\left[\sin\left(\frac{\omega t}{3}\right)\hat{\imath} + \cos\left(\frac{\omega t}{3}\right)\hat{\jmath}\right], \\ \vec{r}_P &= \vec{r}_C + r\left[-sin\,(\omega t)\,\hat{\imath} + \cos\,(\omega t)\,\hat{\jmath}\right]. \end{aligned}$$

2. **Velocity:**

$$\begin{aligned} \vec{v}_C &= -\omega r\left[\cos\left(\frac{\omega t}{3}\right)\hat{\imath} - \sin\left(\frac{\omega t}{3}\right)\hat{\jmath}\right], \\ \vec{v}_P &= -\omega r\left[\left\{\cos\left(\frac{\omega t}{3}\right) + \cos\,(\omega t)\right\}\hat{\imath} - \left\{\sin\left(\frac{\omega t}{3}\right) - \sin\,(\omega t)\right\}\hat{\jmath}\right]. \end{aligned}$$

3. **Acceleration:**

$$\begin{aligned} \vec{a}_C &= \frac{\omega^2 r}{3}\left[\sin\left(\frac{\omega t}{3}\right)\hat{\imath} + \cos\left(\frac{\omega t}{3}\right)\hat{\jmath}\right], \\ \vec{a}_P &= \omega^2 r\left[\left\{\frac{1}{3}\sin\left(\frac{\omega t}{3}\right) + \sin\,(\omega t)\right\}\hat{\imath} + \left\{\frac{1}{3}\cos\left(\frac{\omega t}{3}\right) - \cos\,(\omega t)\right\}\hat{\jmath}\right]. \end{aligned}$$

We can now use these expressions to find the position, velocity, and acceleration of the two points at the instants of interest by substituting $r = 1$ ft, $\omega = \pi$ rad/s, and appropriate values of t. These values are shown in Table 14.1.

The velocity and acceleration of the two points are shown in Figures 14.43(a) and (b) respectively.

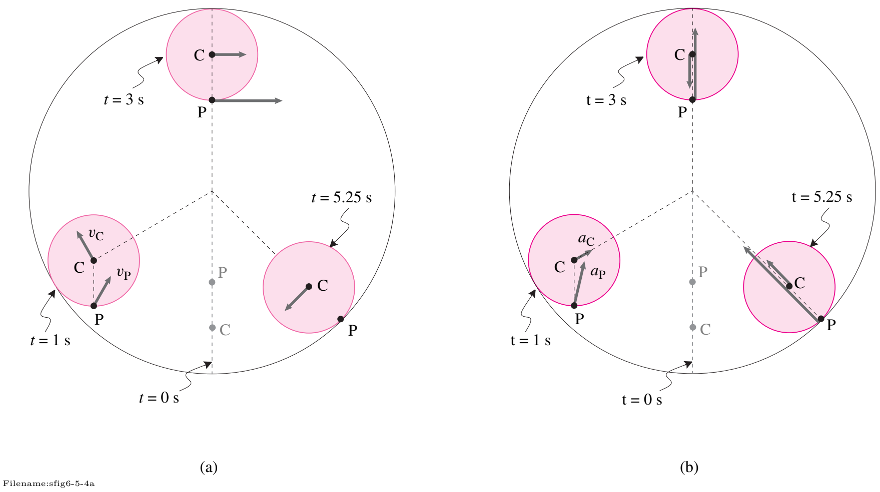

Figure 14.43: (a) Velocity and (b) Acceleration of points P and C at $t = 1$ s, 3 s, and 5.25 s.

It is worthwhile to check the directions of velocities and the accelerations by thinking about the velocity and acceleration of point P as a vector sum of the velocity (same for acceleration) of the center of the disk and the velocity (same for acceleration) of point P with respect to the center of the disk. Since the motions involved are circular motions at constant rate, a visual inspection of the velocities and the accelerations is not very difficult. Try it.

t	1 s	3 s	5.25 s
$\vec{r}_C$ (ft)	$3(-\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath})$	$3\hat{\jmath}$	$3(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{r}_P$ (ft)	$\vec{r}_C - \hat{\jmath}$	$\vec{r}_C - \hat{\jmath}$	$4(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_C$ (ft/s)	$\pi(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$\pi\hat{\imath}$	$\pi(-\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s)	$\pi(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$2\pi\hat{\imath}$	$\vec{0}$
$\vec{a}_C$ (ft/s^2)	$\frac{\pi^2}{3}(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath})$	$-\frac{\pi^2}{3}\hat{\jmath}$	$\frac{\pi^2}{3}(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s^2)	$11.86(.24\hat{\imath} + .97\hat{\jmath})$	$\frac{2\pi^2}{3}\hat{\jmath}$	$13.16(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$

Table 14.1: Position, velocity, and acceleration of point P and point C

SAMPLE 14.15 The rolling disk: path of a point on the disk. For the rolling disk in Sample 14.13, take $\omega = \pi$ rad/s. Draw the path of a point on the rim of the disk for one complete revolution of the center of the disk around the cylinder for the following conditions:

1. $R = 8r$,
2. $R = 4r$, and
3. $R = 2r$.

Solution In Sample 14.13, we obtained a general expression for the position of a point on the disk as a function of time. By computing the position of the point for various values of time t up to the time required to go around the cylinder for one complete cycle, we can draw the path of the point. For the various given conditions, the variable that changes in Eqn. (14.37) is q. We can write a computer program to generate the path of any point on the disk for a given set of R and r. Here is a pseudocode to generate the required path on a computer according to Eqn. (14.37).

A pseudocode to plot the path of a point on the disk:

```
(pseudo-code) program rollingdisk
%-------------------------------------------------------------
% This code plots the path of any point on a disk of radius
% 'r' rolling with speed 'w' inside a cylinder of radius 'R'.
% The point of interest is distance 'l' away from the center of
% the disk. The coordinates x and y of the specified point P are
% calculated according to the relation mentioned above.
%-------------------------------------------------------------
phi = pi/50*[1,2,3,...,100]     % make a vector phi from 0 to 2*pi
x = R*cos(phi)                  % create points on the outer cylinder
y = R*sin(phi)
plot y vs x                     % plot the outer cylinder
hold this plot                  % hold to overlay plots of paths
q = r/(R-r)                     % calculate q.
T = 2*pi/(q*w)                  % calculate time T for going around-
                                % the cylinder once at speed 'w'.
t = T/100*[1,2,3, ..., 100]     % make a time vector t from 0 to T-
                                % taking 101 points.

rcx = -(R-r)*(sin(q*w*t))       % find the x coordinates of pt. C.
rcy = -(R-r)*(cos(q*w*t))       % find the y coordinates of pt. C.
rpx = rcx-l*sin(w*t)            % find the x coordinates of pt. P.
rpy = rcy + l*cos(q*t)          % find the y coordinates of pt. P.
plot rpy vs rpx                 % plot the path of P and the path
plot rcy vs rcx                 % of C. For path of C
```

Once coded, we can use this program to plot the paths of both the center and the point P on the rim of the disk for the three given situations. Note that for any point on the rim of the disk $l = r$ (see Fig 14.42).

1. Let $R = 4$ units. Then $r = 0.5$ for $R = 8r$. To plot the required path, we run our program `rollingdisk` with desired input,

   ```
   R = 4
   r = 0.5
   w = pi
   l = 0.5
   execute rollingdisk
   ```

   The plot generated is shown in Fig.14.44 with a few graphic elements added for illustrative purposes.

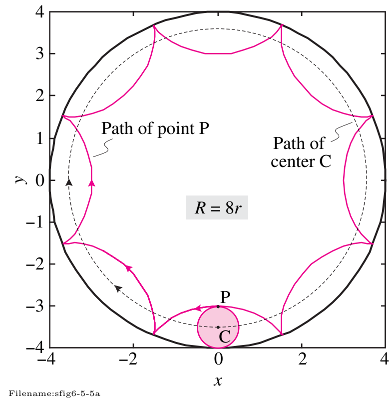

Figure 14.44: Path of point P and the center C of the disk for $R = 8r$.

2. Similarly, for $R = 4r$ we type:

   ```
   R = 4
   r = 1
   w = pi
   l = 1
   execute rollingdisk
   ```

   to plot the desired paths. The plot generated in this case is shown in Fig.14.45

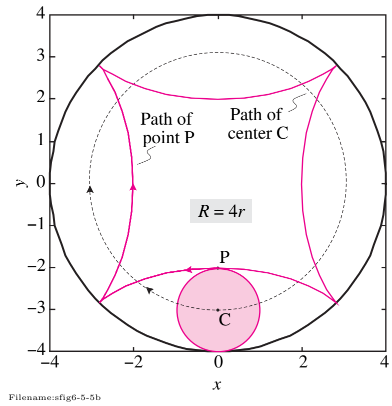

Figure 14.45: Path of point P and the center C of the disk for $R = 4r$.

3. The last one is the most interesting case. The plot obtained in this case by typing:

   ```
   R = 4
   r = 2
   w = pi
   l = 2
   execute rollingdisk
   ```

   is shown in Fig.14.46. Point P just travels on a straight line! In fact, every point on the rim of the disk goes back and forth on a straight line. Most people find this motion odd at first sight. You can roughly verify the result by cutting a whole twice the diameter of a coin (say a US quarter or dime) in a piece of cardboard and rolling the coin around inside while watching a marked point on the perimeter.

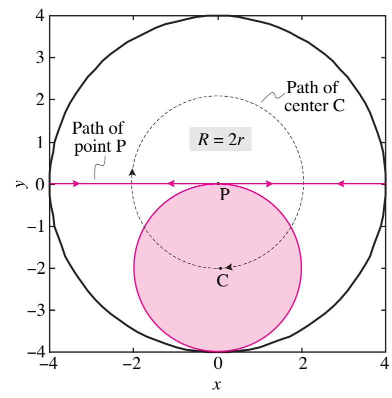

Figure 14.46: Path of point P and the center C of the disk for $R = 2r$.

A curiosity. We just discovered something simple about the path of a point on the edge of a circle rolling in another circle that is twice as big. The edge point moves in a straight line. In contrast one might think about the motion of the center G of a *straight* line segment that *slides* against two *straight* walls as in sample 14.23. A problem couldn't be more different. Naturally the path of point G is a circle (as you can check physically by looking at the middle of a ruler as you hold it as you sliding against a wall-floor corner).

14.4 Mechanics of contacting bodies: rolling and sliding

A typical machine part has forces that come from contact with other parts. In fact, with the major exceptions of

- Gravity,
- Electromagnetic forces inside motors, and
- Magnetic attraction/repulsion

most of the forces that act on bodies of engineering interest come from contact. Many of the forces you have drawn in free body diagrams have been contact forces: The force of the ground on a wheel, of an axle on a bearing, of any part on any other part it touches.

We'd now like to consider some mechanics problems that involve sliding or rolling contact. Once you understand the kinematics from the previous section, there is nothing new in the mechanics. As always, the mechanics is linear momentum balance, angular momentum balance and energy balance. Because we are considering single rigid objects in 2D the expressions for the motion quantities are especially simple (as you can look up in Table I at the back of the book):

$$\dot{\vec{L}} = m_{\text{tot}}\vec{a}_{\text{cm}}, \tag{14.40}$$

$$\dot{\vec{H}}_{/\text{C}} = \vec{r}_{\text{cm/C}} \times (m_{\text{tot}}\vec{a}_{\text{cm}}) + I\dot{\omega}\hat{k} \qquad (\text{where} I = I_{zz}^{\text{cm}}) \tag{14.41}$$

$$E_{\text{K}} = m_{\text{tot}}v_{cm}^2/2 + I\omega^2/2 \tag{14.42}$$

The key to success, as usual, is the drawing of appropriate free body diagrams (see Chapter 3). For rolling and sliding the two cases one needs to consider as possible are, self-evidently, rolling, where the contact point has no relative velocity and the tangential reaction force is unknown but less than μN, and sliding where the relative velocity could be anything and the tangential reaction force is usually assumed to have a magnitude of μN but oppose the relative motion.

Work-energy relations and impulse-momentum relations are useful to solve some problems both with and without slip. In pure rolling contact the contact force does no work because the material point of contact has no velocity. However, when there is sliding mechanical energy is dissipated. The rate of loss of kinetic and potential energy is

$$\text{Rate of frictional dissipation} = P_{diss} = F_{\text{friction}} \cdot v_{\text{slip}} \tag{14.43}$$

where v_{slip} is the relative velocity of the contacting slipping points. If either the friction force (ideal lubrication) or sliding velocity (no slip) is zero there is no dissipation.

As for various problems throughout the text, it is often a savings of calculation to use angular momentum balance (or moment balance in statics) relative to a point where there are unknown reaction forces. For rolling and slipping problems this often means making use of contact points.

Example: Pure rolling on level ground
A ball or wheel rolling on level ground, with no air friction etc, rolls at constant speed (see fig. 14.47). This is most directly deduced from angular momentum balance about the contact point C:

$$\begin{array}{lcccc}
\vec{M}_C = \dot{\vec{H}}_C & \Rightarrow & \vec{r}_{G/C} \times -mg\hat{\jmath} & = & \vec{r}_{G/C} \times m\vec{a}_G + \dot{\omega} I_{zz}^{cm} \hat{k} \\
 & \Rightarrow & \vec{0} & = & R\hat{\jmath} \times (-m\dot{\omega}R\hat{\imath}) + \dot{\omega} I_{zz}^{cm} \hat{k} \\
\text{dotting with } \hat{k} & \Rightarrow & \dot{\omega} = 0 & \Rightarrow & \omega = \text{constant.}
\end{array}$$

Because for rolling $v_G = -\omega R$ we thus have that v_G is a constant. [The result can also be obtained by combining angular momentum balance about the center-of-mass with linear momentum balance.]

Finally, linear momentum balance gives the reaction force at C to be $\vec{F}_{tot} = F\hat{\imath} + n\hat{\jmath} = mg\hat{\jmath}$. So,

assuming point contact, there is no rolling resistance.

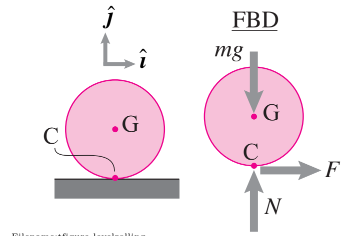

Figure 14.47: A ball rolls or slides on level ground.

Example: Bowling ball with initial sliding
A bowling ball is released with an initial speed of v_0 and no rotation rate. What is its subsequent motion? To start with, the motion is incompatible with rolling, the bottom of the ball is sliding to the right. So there is a frictional force which opposes motion and $F = -\mu N$ (see fig. 14.47). Linear and angular momentum balance give:

$$\begin{array}{llll}
\text{LMB:} & & \Rightarrow & \{-F\hat{\imath} + N\hat{\jmath} - mg\hat{\jmath} = ma\hat{\imath}\} \\
 & \{\}\cdot\hat{\jmath} & \Rightarrow & N = mg \\
 & \{\}\cdot\hat{\imath} & \Rightarrow & a = -\mu g \\
\text{AMB}_{/G}: & & \Rightarrow & -R\mu mg = I_{zz}^{cm}\dot{\omega}
\end{array}$$

$$\Rightarrow \qquad v = v_0 - \mu g t \qquad \text{and} \qquad \omega = -\mu R m g t / I_{zz}^{cm}$$

Thus the forward speed of the ball decreases linearly with time while the counter-clockwise angular velocity decreases linearly with time.

This solution is only appropriate so long as there is rightward slip, $v_G > -\omega R$. Just like for a sliding block, there is no impetus for reversal, and the block switches to pure rolling when

$$v = -\omega R \Rightarrow v_0 - \mu g t = -(-\mu R m g t / I_{zz}^{cm})\, R \Rightarrow t = \frac{v_0}{\mu g \left(1 + \frac{mR^2}{I_{zz}^{cm}}\right)}.$$

Note that the energy lost during sliding is less than μmg times the distance the center of the ball moves during slip.

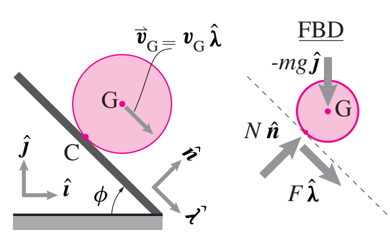

Figure 14.48: A ball rolls or slides down a slope.

Example: Ball rolling down hill.
Assuming rolling we can find the acceleration of a ball as it rolls downhill (see fig. 14.48). We start out with the kinematic observations that $\vec{a}_G = a_G\hat{\lambda}$, that $R\omega = -v_G$ and that $R\dot{\omega} = -a_G$. Angular momentum balance about the stationary

point on the ground instantaneously coinciding with the contact point gives

$$
\begin{aligned}
\text{AMB}_{/C} \quad &\Rightarrow \quad \vec{r}_{G/C} \times (-mg\hat{\jmath}) = \vec{r}_{G/C} \times m\vec{a}_G + I_{zz}^{cm}\dot{\omega}\hat{k} \\
& \qquad \left\{-R\sin\phi mg\hat{k} = (R\hat{n}) \times (ma_G\hat{\lambda}) + I_{zz}^{cm}\dot{\omega}\hat{k}\right\} \\
\{\}\cdot\hat{k} \quad &\Rightarrow \quad -Rmg\sin\phi = -Rma_G - I_{zz}^{cm}a_G/R \\
&\Rightarrow \quad a_G = \frac{g\sin\phi}{1+I_{zz}^{cm}/(mR^2)}.
\end{aligned}
$$

Which is less than the acceleration of a block sliding on a ramp without friction: $a = g\sin\phi$ (unless the mass of the rolling ball is concentrated at the center with $I_{zz}^{cm} = 0$). Note that a very small ball rolls just as slowly. In the limit as the ball radius goes to zero the behavior does not approach that of a point mass that slides; the rolling remains significant.

Example: Ball rolling down hill: energy approach

We can find the acceleration of the rolling ball using power balance or conservation of energy. For example

$$
\begin{aligned}
0 = \tfrac{d}{dt}E_T \Rightarrow \quad 0 &= \dot{E}_K + \dot{E}_P \\
&= \tfrac{d}{dt}\left(mv^2/2 + I_{zz}^{cm}\omega^2/2\right) + \tfrac{d}{dt}(mgy) \\
&= mv\dot{v} + I_{zz}^{cm}\omega\dot{\omega} + mg\dot{y} \\
&= mv\dot{v} + I_{zz}^{cm}(v/R)\dot{v}/R - mg(\sin\phi)v \\
\text{assuming } v \neq 0 \Rightarrow \quad 0 &= (m + I_{zz}^{cm}/R^2)\dot{v} - mg\sin\phi \\
\Rightarrow \quad \dot{v} &= \frac{g\sin\phi}{1+I_{zz}^{cm}/(mR^2)}
\end{aligned}
$$

as before.

Example: Does the ball slide?

How big is the coefficient of friction μ needed to prevent slip for a ball rolling down a hill? Use angular momentum balance to find the normal and frictional components of the contact force, using the rolling example above.

$$
\begin{aligned}
\text{LMB } (\vec{F}_{tot} = m\vec{a}_G) \quad &\Rightarrow \quad \left\{N\hat{n} + F\hat{\lambda} - mg\hat{\jmath} = ma_G\hat{\lambda}\right\} \\
\{\}\cdot\hat{n} \quad &\Rightarrow \quad N = mg\cos\phi \\
\{\}\cdot\hat{\lambda} \quad &\Rightarrow \quad F + mg\sin\phi = m\frac{g\sin\phi}{1+I_{zz}^{cm}/(mR^2)} \\
& \qquad F = \frac{-mg\sin\phi}{1+mR^2/I_{zz}^{cm}}
\end{aligned}
$$

$$
\text{Critical condition:} \quad \Rightarrow \quad \mu = \frac{|F|}{N} = \frac{\tan\phi}{1+mR^2/I_{zz}^{cm}}
$$

If I_{zz}^{cm} is very small (the mass concentrated near the center of the ball) then small friction is needed to prevent rolling. For a uniform rubber ball on pavement (with $\mu \approx 1$ and $I_{zz}^{cm} \approx 2mR^2/5$) the steepest slope for rolling without slip is a steep $\phi = \tan^{-1}(7/2) \approx 74^\circ$. A metal hoop on the other hand (with $\mu \approx .3$ and $I_{zz}^{cm} \approx mR^2$) will only roll without slip for slopes less than about $\phi = \tan^{-1}(.6) \approx 31^\circ$.

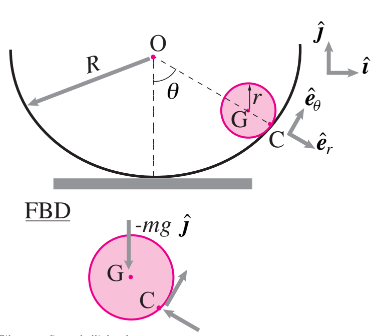

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Figure 14.49: A ball rolls in a round cross-section bowl.

Example: Oscillations of a ball in a bowl.

A round ball can oscillate back and forth in the bottom of a circular cross section bowl or pipe (see fig. 14.49). Similarly, a cylindrical object can roll inside a pipe.

What is the period of oscillation? Start with angular momentum balance about the contact point

$$\begin{aligned}
\vec{r}_{\mathrm{G/C}} \times (-mg\,\hat{\jmath}) &= \vec{r}_{\mathrm{G/C}} \times m\vec{a}_{\mathrm{G}} + I_{zz}^{\mathrm{cm}}\dot{\omega}\hat{k} \\
rmg\sin\theta\hat{k} &= -r\hat{e}_r \times \left(m\left((R-r)\ddot{\theta}\hat{e}_\theta - (R-r)\dot{\theta}^2\hat{e}_r\right)\right) \\
&\quad + I_{zz}^{\mathrm{cm}}\dot{\omega}\hat{k}.
\end{aligned}$$

Evaluating the cross products (using that $\hat{e}_r \times \hat{e}_\theta = \hat{k}$) and using the kinematics from the previous section (that $(R-r)\dot{\theta} = -r\omega$) and dotting the left and right sides with $\hat{k}$ gives

$$(R-r)\ddot{\theta} = \frac{-g\sin\theta}{1 + I_{zz}^{\mathrm{cm}}/mr^2},$$

the tangential acceleration is the same as would have been predicted by putting the ball on a constant slope of $-\theta$. Using the small angle approximation that $\sin\theta = \theta$ the equation can be rearranged as a standard harmonic oscillator equation

$$\ddot{\theta} + \left(\frac{g}{(R-r)(1 + I_{zz}^{\mathrm{cm}}/mr^2)}\right)\theta = 0.$$

If all the ball's mass were concentrated in its middle, keeping r a fixed non-zero size, (so $I_{zz}^{\mathrm{cm}} = 0$, like a lead pellet inside a styrofoam ball) this is the same as for a simple pendulum with length $R-r$. For any parameter values the period of small oscillation is

$$T = 2\pi\sqrt{\frac{(R-r)(1 + I_{zz}^{\mathrm{cm}}/mr^2)}{g}}.$$

For a marble or ball bearing in a sideways glass (with $R - r \approx 2\,\mathrm{cm} = .04\,\mathrm{m}$, $I_{zz}^{\mathrm{cm}}/mr^2 \approx 2/5$ and $g \approx 10\,\mathrm{m/s^2}$) this gives about one oscillation every half second. See page 936 for the energy approach to this problem.

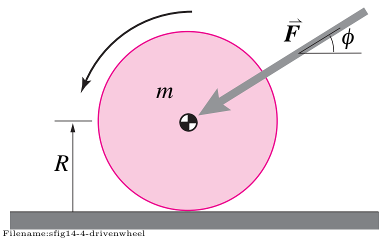

Figure 14.50:

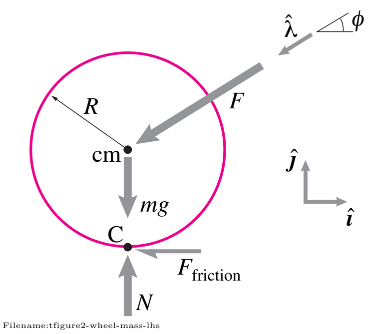

Figure 14.51: FBD of a wheel with mass m. Force F is applied by the axle.

SAMPLE 14.16 Equation of motion of a driven wheel Consider a wheel of mass m and radius R driven by an axle force $\vec{F}$ as shown in the figure. The wheel rolls to the left without slipping. Write the equation of motion of the wheel.

Solution The free body diagram of the wheel is shown in figure 14.51. We can write the equation of motion of the wheel in terms of either the center-of-mass position x or the angular displacement of the wheel θ. Since in pure rolling, these two variables share a simple relationship ($x = R\theta$), we can easily get the equation of motion in terms of x if we have the equation in terms of θ and vice versa. Let $\vec{\omega} = \omega\hat{k}$ and $\dot{\vec{\omega}} = \dot{\omega}\hat{k}$.

Since all the forces are shown in the free body diagram, we can readily write the angular momentum balance for the wheel. We choose the point of contact C as our reference point for the angular momentum balance (because the gravity force, $-mg\,\hat{\jmath}$, the friction force $-F_{\text{friction}}\hat{\imath}$, and the normal reaction of the ground $N\,\hat{\jmath}$, all pass through the contact point C and therefore, produce no moment about this point). We have

$$\sum \vec{M}_C = \dot{\vec{H}}_{/C}$$

where

$$\begin{aligned}
\sum \vec{M}_C &= \overbrace{\vec{r}_{cm/C}}^{R\hat{\jmath}} \times (F\hat{\lambda}) \\
&= R\hat{\jmath} \times F(-\cos\phi\hat{\imath} - \sin\phi\,\hat{\jmath}) \\
&= FR\cos\phi\,\hat{k}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\vec{H}}_{/C} &= \vec{r}_{\text{cm}/C} \times m\vec{a}_{\text{cm}} + I_{zz}^{\text{cm}}\dot{\vec{\omega}} \\
&= R\hat{\jmath} \times m\underbrace{\ddot{x}}_{-\dot{\omega}R}\hat{\imath} + I_{zz}^{\text{cm}}\dot{\omega}\hat{k} \\
&= m\dot{\omega}R^2\hat{k} + I_{zz}^{\text{cm}}\dot{\omega}\hat{k} \\
&= (I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
FR\cos\phi\hat{k} &= (I_{zz}^{\text{cm}} + mR^2)\dot{\omega}\hat{k} \\
\Rightarrow \quad \dot{\omega} \equiv \ddot{\theta} &= \frac{FR\cos\phi}{I_{zz}^{\text{cm}} + mR^2}
\end{aligned}$$

which is the equation of motion we are looking for. Note that we can easily substitute $\ddot{\theta} = -\ddot{x}/R$ in the equation of motion above to get the equation of motion in terms of the center-of-mass displacement x as

$$\ddot{x} = -\frac{FR^2\cos\phi}{I_{zz}^{\text{cm}} + mR^2}.$$

$$\boxed{\ddot{\theta} = \frac{FR\cos\phi}{I_{zz}^{\text{cm}}+mR^2}}$$

Comments: We could have, of course, used linear momentum balance with angular momentum balance about the center-of-mass to derive the equation of motion. Note, however, that the linear momentum balance will essentially give two scalar equations in the x and y directions involving all forces shown in the free-body diagram . The angular momentum balance , on the other hand, gets rid of some of them. Depending on which forces are known, we may or may not need to use all the three scalar equations. In the final equation of motion, we must have only one unknown.

SAMPLE 14.17 Energy and power of a rolling wheel. A wheel of diameter 2 ft and mass 20 lbm rolls without slipping on a horizontal surface. The kinetic energy of the wheel is 1700 ft·lbf. Assume the wheel to be a thin, uniform disk.

1. Find the rate of rotation of the wheel.
2. Find the average power required to bring the wheel to a complete stop in 5 s.

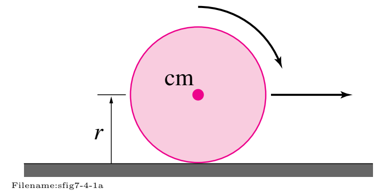

Figure 14.52:

Solution

1. Let ω be the rate of rotation of the wheel. Since the wheel rotates without slip, its center-of-mass moves with speed $v_{cm} = \omega r$. The wheel has both translational and rotational kinetic energy. The total kinetic energy is

$$\begin{aligned}
E_K &= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I^{cm}\omega^2 \\
&= \frac{1}{2}m\omega^2r^2 + \frac{1}{2}I^{cm}\omega^2 \\
&= \frac{1}{2}(mr^2 + \underbrace{I^{cm}}_{\frac{1}{2}mr^2})\omega^2 \\
&= \frac{3}{4}mr^2\omega^2 \\
\Rightarrow \quad \omega^2 &= \frac{4E_K}{3mr^2} \\
&= \frac{4 \times 1700\,\text{ft}\cdot\text{lbf}}{3 \times 20\,\text{lbm}\cdot 1\,\text{ft}^2} \\
&= \frac{4 \times 1700 \times 32.2\,\cancel{\text{lbm}}\cdot\cancel{\text{ft}}\text{t}/\text{s}^2}{3 \times 20\,\cancel{\text{lbm}}\cdot\cancel{\text{ft}}} \\
&= 3649.33\frac{1}{\text{s}^2} \\
\Rightarrow \quad \omega &= 60.4\,\text{rad/s}.
\end{aligned}$$

$$\boxed{\omega = 60.4\,\text{rad/s}}$$

Note: This rotational speed, by the way, is extremely high. At this speed the center-of-mass moves at 60.4 ft/s!

2. Power is the rate of work done on a body or the rate of change of kinetic energy. Here we are given the initial kinetic energy, the final kinetic energy (zero) and the time to achieve the final state. Therefore, the average power is,

$$\begin{aligned}
P &= \frac{E_{K1} - E_{K2}}{\Delta t} \\
&= \frac{1700\,\text{ft}\cdot\text{lbf} - 0}{5\,\text{s}} = 340\,\text{ft}\cdot\text{lbf/s} \\
&= 340\,\text{ft}\cdot\text{lbf/s} \cdot \frac{1\,\text{hp}}{550\,\text{ft}\cdot\text{lbf/s}} \\
&= 0.62\,\text{hp}
\end{aligned}$$

$$\boxed{P = 0.62\,\text{hp}}$$

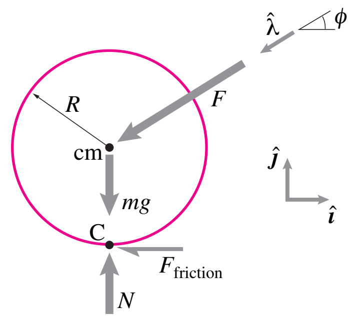

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Figure 14.53: FBD of a rolling wheel.

SAMPLE 14.18 Equation of motion of a rolling wheel from energy balance. Consider the wheel with mass m from figure 14.53. The free-body diagram of the wheel is shown here again. Derive the equation of motion of the wheel using energy balance. Assume CCW (counter clockwise) rotation is positive.

Solution From energy balance, we have

$$P = \dot{E}_K$$

where

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= -F_{\text{friction}}\hat{\imath}\cdot \overbrace{\vec{v}_C}^{\vec{0}} + N\hat{\jmath}\cdot \overbrace{\vec{v}_C}^{\vec{0}} - mg\hat{\jmath}\cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} + F\hat{\lambda}\cdot \overbrace{\vec{v}_{cm}}^{v\hat{\imath}} \\ &= -mgv(\underbrace{\hat{\imath}\cdot\hat{\jmath}}_{0}) + Fv(\underbrace{\hat{\lambda}\cdot\hat{\imath}}_{-\cos\phi}) \\ &= -Fv\cos\phi \end{aligned}$$

and

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_{zz}^{\text{cm}}\overbrace{\omega^2}^{(\dot{x}/R)^2}\right) \\ &= \frac{1}{2}\frac{d}{dt}\left[\left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}^2\right] \\ &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}\ddot{x}. \end{aligned}$$

Thus,

$$\begin{aligned} -Fv\cos\phi &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\dot{x}\ddot{x} \\ \text{or}\quad -F\cancel{\dot{x}}\cos\phi &= \left(m + \frac{I_{zz}^{\text{cm}}}{R^2}\right)\cancel{\dot{x}}\ddot{x} \\ \Rightarrow\quad \ddot{x} &= -\frac{F\cos\phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}. \end{aligned}$$

We can also write the equation of motion in terms of θ by replacing $\ddot{x}$ with $-\ddot{\theta}R$ giving,

$$\ddot{\theta} = \frac{(F/R)\cos\phi}{m + \frac{I_{zz}^{\text{cm}}}{R^2}}.$$

$$\ddot{x} = -\frac{F\cos\phi}{m + I_{zz}^{\text{cm}}/R^2}$$

Comments: In the equations above (for calculating P), we have set $\vec{v}_C = \vec{0}$ because in pure rolling, the instantaneous velocity of the contact point is zero. Note that the force due to gravity is normal to the direction of the velocity of the center-of-mass. So, the only power supplied to the wheel is due to the force $F\hat{\lambda}$ acting at the center-of-mass.

SAMPLE 14.19 Equation of motion of a rolling disk on an incline. A uniform circular disk of mass $m = 1\,\text{kg}$ and radius $R = 0.4\,\text{m}$ rolls down an inclined shown in the figure. Write the equation of motion of the disk assuming pure rolling, and find the distance travelled by the center-of-mass in 2 s.

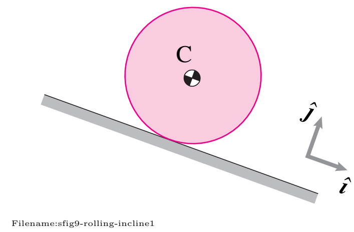

Figure 14.54:

Solution The free-body diagram of the disk is shown in fig. 14.55. In addition to the base unit vectors $\hat{\imath}$ and $\hat{\jmath}$, let us use unit vectors $\hat{\lambda}$ and $\hat{n}$ along the plane and perpendicular to the plane, respectively, to express various vectors. We can write the equation of motion using linear momentum balance or angular momentum balance. However, note that if we use linear momentum balance we have two unknown forces in the equation. On the other hand, if we use angular momentum balance about the contact point C, these forces do not show up in the equation. So, let us use angular momentum balance about point C:

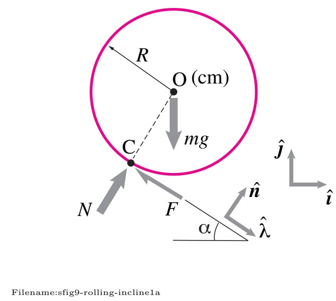

Figure 14.55:

$$\sum \vec{M}_C = \dot{\vec{H}}_{/C}$$

where

$$\begin{aligned}\sum \vec{M}_C &= \vec{r}_{O/C} \times m\vec{g} = R\hat{n} \times (-mg\,\hat{\jmath})\\ &= -Rmg\sin\alpha\hat{k}\end{aligned}$$

and

$$\begin{aligned}\dot{\vec{H}}_{/C} &= -I_{zz}^{cm}\dot{\omega}\hat{k} + \overbrace{\vec{r}_{O/C}}^{R\hat{n}} \times m \overbrace{\vec{a}_{cm}}^{R\dot{\omega}\hat{\lambda}}\\ &= -I_{zz}^{cm}\dot{\omega}\hat{k} + mR^2\dot{\omega}(\hat{n}\times\hat{\lambda})\\ &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k}.\end{aligned}$$

Thus,

$$\begin{aligned}-Rmg\sin\alpha\hat{k} &= -(I_{zz}^{cm} + mR^2)\dot{\omega}\hat{k}\\ \Rightarrow \quad \dot{\omega} &= \frac{g\sin\alpha}{R[1 + I_{zz}^{cm}/(mR^2)]}.\end{aligned}$$

$$\boxed{\dot{\omega} = \frac{g\sin\alpha}{R[1+I_{zz}^{cm}/(mR^2)]}}$$

Note that in the above equation of motion, the right hand side is constant. So, we can solve the equation for ω and θ by simply integrating this equation and substituting the initial conditions $\omega(t=0) = 0$ and $\theta(t=0) = 0$. Let us write the equation of motion as $\dot{\omega} = \beta$ where $\beta = g\sin\alpha/R(1 + I_{zz}^{cm}/mR^2)$. Then,

$$\begin{aligned}\omega &\equiv \dot{\theta} = \beta t + C_1\\ \theta &= \frac{1}{2}\beta t^2 + C_1 t + C_2.\end{aligned}$$

Substituting the given initial conditions $\dot{\theta}(0) = 0$ and $\theta(0) = 0$, we get $C_1 = 0$ and $C_2 = 0$, which implies that $\theta = \frac{1}{2}\beta t^2$. Now, in pure rolling, $x = R\theta$. Therefore,

$$\begin{aligned}x(t) &= R\theta(t) = \frac{1}{2}\beta t^2 = \cancel{R}\cdot\frac{1}{2}\frac{g\sin\alpha}{\cancel{R}(1 + I_{zz}^{cm}/mR^2)}t^2\\ &= \frac{1}{2}\frac{g\sin\alpha}{1+\frac{\frac{1}{2}mR^2}{mR^2}}t^2 = \frac{1}{3}(g\sin\alpha)t^2\\ x(2\,\text{s}) &= \frac{1}{3}\cdot 9.8\,\text{m/s}^2\cdot\sin(30^\circ)\cdot(2\,\text{s})^2 = 6.53\,\text{m}.\end{aligned}$$

$$\boxed{x(2\,\text{s}) = 6.53\,\text{m}}$$

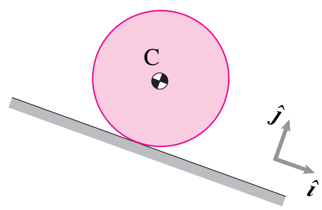

Figure 14.56:

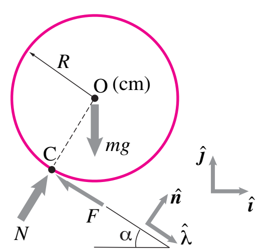

Figure 14.57:

SAMPLE 14.20 Using Work and energy in pure rolling. Consider the disk of Sample 14.19 rolling down the incline again. Suppose the disk starts rolling from rest. Find the speed of the center-of-mass when the disk is 2 m down the inclined plane.

Solution We are given that the disk rolls down, starting with zero initial velocity. We are to find the speed of the center-of-mass after it has travelled 2 malong the incline. We can, of course, solve this problem using equation of motion, by first solving for the time t the disk takes to travel the given distance and then evaluating the expression for speed $\omega(t)$ or $x(t)$ at that t. However, it is usually easier to use work energy principle whenever positions are specified at two instants, speed is specified at one of those instants, and speed is to be found at the other instant. This is because we can, presumably, compute the work done on the system in travelling the specified distance and relate it to the change in kinetic energy of the system between the two instants. In the problem given here, let ω_1 and ω_2 be the initial and final (after rolling down by $d = 2$ m) angular speeds of the disk, respectively. We know that in rolling, the kinetic energy is given by

$$E_K = \frac{1}{2}m \overbrace{v_{cm}^2}^{(\omega R)^2} + \frac{1}{2}I_{zz}^{cm}\omega^2 = \frac{1}{2}(mR^2 + I_{zz}^{cm})\omega^2.$$

Therefore,

$$\Delta E_K = E_{K2} - E_{K1} = \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2). \tag{14.44}$$

Now, let us calculate the work done by all the forces acting on the disk during the displacement of the mass-center by d along the plane. Note that in ideal rolling, the contact forces do no work. Therefore, the work done on the disk is only due to the gravitational force:

$$W = (-mg\hat{\jmath}) \cdot (d\hat{\lambda}) = -mgd(\overbrace{\hat{\jmath} \cdot \hat{\lambda}}^{-\sin\alpha}) = mgd\sin\alpha. \tag{14.45}$$

From work-energy principle (integral form of power balance, $P = \dot{E}_K$), we know that $W = \Delta E_K$. Therefore, from eqn. (14.44) and eqn. (14.45), we get

$$\begin{aligned} mgd\sin\alpha &= \frac{1}{2}(mR^2 + I_{zz}^{cm})(\omega_2^2 - \omega_1^2) \\ \Rightarrow \quad \omega_2^2 &= \omega_1^2 + \frac{2mgd\sin\alpha}{mR^2 + I_{zz}^{cm}} = \omega_1^2 + \frac{2gd\sin\alpha}{R^2\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)} \\ &= \omega_1^2 + \frac{4gd\sin\alpha}{3R^2}. \end{aligned}$$

Substituting the values of g, d, α, R, etc., and setting $\omega_1 = 0$, we get

$$\begin{aligned} \omega_2^2 &= \frac{4 \cdot (9.8\,\text{m/s}^2) \cdot (2\,\text{m}) \cdot (\sin(30^\circ))}{3 \cdot (0.4\,\text{m})^2} = 81.67/s^2 \\ \Rightarrow \quad \omega_2 &= 9.04\,\text{rad/s}. \end{aligned}$$

The corresponding speed of the center-of-mass is

$$v_{cm} = \omega_2 R = 9.04\,\text{rad/s} \cdot 0.4\,\text{m} = 3.61\,\text{m/s}.$$

$v_{cm} = 3.61$ m/s

SAMPLE 14.21 Impulse and momentum calculations in pure rolling. Consider the disk of Sample 14.19 rolling down the incline again. Find an expression for the rolling speed (ω) of the disk after a finite time Δt, given the rolling speed ω_1 at some instant t_1.

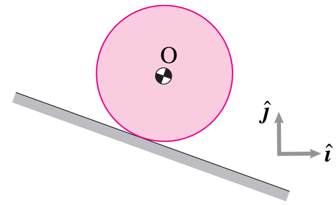

Figure 14.58:

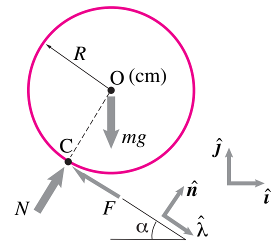

Figure 14.59:

Solution Once again, this problem can be solved by integrating the equation of motion (as done in Sample 14.19). However, we will solve this problem here using impulse-momentum relationship. Let the desired angular speed at $t_2 = t_1 + \Delta t$ be ω_2. We need to find ω_2, given the ω_1 at $t = t_1$. Since the forces acting on the disk do not change during this time (assuming pure rolling), it is easy to calculate impulse and then relate it to the change in the momenta of the disk between the two instants. Now, from the linear impulse momentum relationship, $\sum \vec{F} \cdot \Delta t = \vec{L}_2 - \vec{L}_1$, we have

$$(-F\hat{\lambda} + N\hat{\imath} - mg\hat{\jmath})\Delta t = m(v_2 - v_1)\hat{\lambda}. \quad (14.46)$$

Dotting eqn. (14.46) with $\hat{\lambda}$ gives

$$\begin{aligned}(-F - mg(\underbrace{\hat{\jmath}\cdot\hat{\lambda}}_{-\sin\alpha}))\Delta t &= m(v_2 - v_1)\\ (-F + mg\sin\alpha)\Delta t &= mR(\omega_2 - \omega_1). \quad (14.47)\end{aligned}$$

Similarly, the angular impulse-momentum relationship about the mass-center, $\vec{M}_O \Delta t = (\vec{H}_{/O})_2 - (\vec{H}_{/O})_1$, gives

$$\begin{aligned}(-FR\hat{k})\Delta t &= -I_{zz}^{cm}(\omega_2 - \omega_1)\hat{k}\\ \Rightarrow \quad FR\Delta t &= I_{zz}^{cm}(\omega_2 - \omega_1). \quad (14.48)\end{aligned}$$

Note that the other forces (N and mg) do not produce any moment about the mass-center as they pass through this point. We can now eliminate the unknown force F from eqn. (14.47) and eqn. (14.48) by multiplying eqn. (14.47) with R and adding to eqn. (14.48):

$$\begin{aligned}(-F + mg\sin\alpha)\Delta t \cdot R + FR\Delta t &= mR(\omega_2 - \omega_1)\cdot R + I_{zz}^{cm}(\omega_2 - \omega_1)\\ \text{or} \quad mgR\sin\alpha\Delta t &= (I_{zz}^{cm} + mR^2)(\omega_2 - \omega_1)\\ \text{or} \quad g\sin\alpha\Delta t &= R\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)(\omega_2 - \omega_1)\\ \Rightarrow \quad \omega_2 &= \omega_1 + \frac{g\sin\alpha}{R\left(1 + \frac{I_{zz}^{cm}}{mR^2}\right)}\Delta t.\end{aligned}$$

$$\boxed{\omega_2 = \omega_1 + \frac{g\sin\alpha}{R\left(1+\frac{I_{zz}^{cm}}{mR^2}\right)}\Delta t}$$

Note: From the answer obtained, we clearly see that $\omega_2 = \omega_1 + \dot{\omega}\Delta t$ where $\dot{\omega}$ is the angular acceleration given by the expression $\dot{\omega} = \frac{g\sin\alpha}{R(1+I_{zz}^{cm}/mR^2)}$. This is the same expression we obtained for the angular acceleration in Sample 14.19.

SAMPLE 14.22 Falling ladder. A ladder AB, modeled as a uniform rigid rod of mass m and length ℓ, rests against frictionless horizontal and vertical surfaces. The ladder is released from rest at $\theta = \theta_o$ ($\theta_o < \pi/2$). Assume the motion to be planar (in the vertical plane).

1. As the ladder falls, what is the path of the center-of-mass of the ladder?
2. Find the equation of motion (e.g., a differential equation in terms of θ and its time derivatives) for the ladder.
3. How does the angular speed ω ($= \dot{\theta}$) depend on θ?

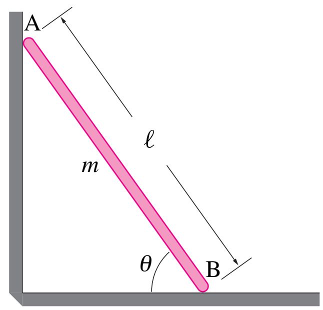

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Figure 14.60: A ladder, modeled as a uniform rod of mass m and length ℓ, falls from a rest position at $\theta = \theta_o$ ($< \pi/2$) such that its ends slide along frictionless vertical and horizontal surfaces.

Solution Since the ladder is modeled by a uniform rod AB, its center-of-mass is at G, half way between the two ends. As the ladder slides down, the end A moves down along the vertical wall and the end B moves out along the floor. Note that it is a single degree of freedom system as angle θ (a single variable) is sufficient to determine the position of every point on the ladder at any instant of time.

1. **Path of the center-of-mass:** Let the origin of our x-y coordinate system be the intersection of the two surfaces on which the ends of the ladder slide (see Fig. 14.61). The position vector of the center-of-mass G may be written as

$$\begin{aligned} \vec{r}_G &= \vec{r}_B + \vec{r}_{G/B} \\ &= \ell\cos\theta\hat{\imath} + \frac{\ell}{2}(-\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\ &= \frac{\ell}{2}(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}). \end{aligned} \tag{14.49}$$

Thus the coordinates of the center-of-mass are

$$x_G = \frac{\ell}{2}\cos\theta \quad \text{and} \quad y_G = \frac{\ell}{2}\sin\theta,$$

from which we get

$$x_G^2 + y_G^2 = \frac{\ell^2}{4}$$

which is the equation of a circle of radius $\frac{\ell}{2}$. Therefore, the center-of-mass of the ladder follows a circular path of radius $\frac{\ell}{2}$ centered at the origin. Of course, the center-of-mass traverses only that part of the circle which lies between its initial position at $\theta = \theta_o$ and the final position at $\theta = 0$.

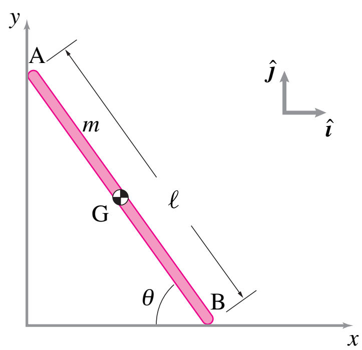

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Figure 14.61:

2. **Equation of motion:** The free-body diagram of the ladder is shown in Fig. 14.62. Since there is no friction, the only forces acting at the end points A and B are the normal reactions from the contacting surfaces. Now, writing the the linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the ladder we get

$$N_1\hat{\imath} + (N_2 - mg)\hat{\jmath} = m\vec{a}_G = m\ddot{\vec{r}}_G.$$

Differentiating eqn. (14.49) twice we get $\ddot{\vec{r}}_G$ as

$$\ddot{\vec{r}}_G = \frac{\ell}{2}[(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta)\hat{\imath} + (\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)\hat{\jmath}].$$

Substituting this expression in the linear momentum balance equation above and dotting both sides of the equation by $\hat{\imath}$ and then by $\hat{\jmath}$ we get

$$\begin{aligned} N_1 &= -\frac{1}{2}m\ell(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta) \\ N_2 &= \frac{1}{2}m\ell(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) + mg. \end{aligned}$$

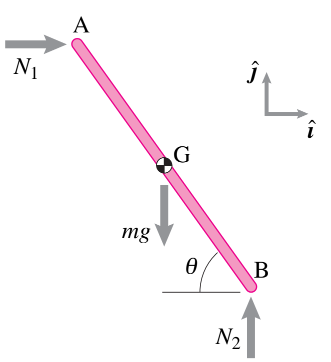

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Figure 14.62: The free-body diagram of the ladder.

Next, we write the angular momentum balance for the ladder about its center-of-mass, $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$, where

$$
\begin{aligned}
\sum \vec{M}_{/G} &= \left(-N_1\frac{\ell}{2}\sin\theta + N_2\frac{\ell}{2}\cos\theta\right)\hat{\boldsymbol{k}} \\
&= \frac{1}{2}m\ell(\ddot{\theta}\,\sin\theta + \dot{\theta}^2\cos\theta)\frac{\ell}{2}\sin\theta\hat{\boldsymbol{k}} \\
&\quad + \left[\frac{1}{2}m\ell(\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta) + mg\right]\frac{\ell}{2}\cos\theta\hat{\boldsymbol{k}} \\
&= \left(\frac{1}{4}m\ell^2\ddot{\theta} + \frac{1}{2}mg\ell\cos\theta\right)\hat{\boldsymbol{k}}
\end{aligned}
$$

and

$$
\dot{\vec{H}}_{/G} = I_{zz/G}\dot{\vec{\omega}} = \frac{1}{12}m\ell^2\ddot{\theta}(-\hat{\boldsymbol{k}}),
$$

where $\dot{\vec{\omega}} = \ddot{\theta}(-\hat{\boldsymbol{k}})$ because θ is measured positive in the clockwise direction $(-\hat{\boldsymbol{k}})$. Now, equating the two quantities $\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$ and dotting both sides with $\hat{\boldsymbol{k}}$ we get

$$
\begin{aligned}
\frac{1}{4}\cancel{m}\ell^2\ddot{\theta} + \frac{1}{2}\cancel{m}g\ell\cos\theta &= -\frac{1}{12}\cancel{m}\ell^2\ddot{\theta} \\
\text{or} \qquad (\frac{1}{12} + \frac{1}{4})\ell^2\ddot{\theta} &= -\frac{1}{2}g\ell\cos\theta \\
\text{or} \qquad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta
\end{aligned}
\tag{14.50}
$$

which is the required equation of motion. Unfortunately, it is a nonlinear equation which does not have a nice closed form solution for $\theta(t)$.

3. **Angular Speed of the ladder:** To solve for the angular speed ω $(= \dot{\theta})$ as a function of θ we need to express eqn. (14.50) in terms of ω, θ, and derivatives of ω with respect to θ. Now,

$$
\ddot{\theta} = \dot{\omega} = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\cdot\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}.
$$

Substituting in eqn. (14.50) and integrating both sides from the initial rest position to an arbitrary position θ we get

$$
\begin{aligned}
\int_0^\omega \omega\, d\omega &= -\int_{\theta_0}^{\theta}\frac{3g}{2\ell}\cos\theta\, d\theta \\
\Rightarrow \qquad \frac{1}{2}\omega^2 &= -\frac{3g}{2\ell}(\sin\theta - \sin\theta_0) \\
\Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}.
\end{aligned}
$$

Since end B is sliding to the right, θ is decreasing; hence it is the negative sign in front of the square root which gives the correct answer, *i.e.,*

$$
\vec{\omega} = \dot{\theta}(-\hat{\boldsymbol{k}}) = -\sqrt{\frac{3g}{\ell}(\sin\theta_0 - \sin\theta)}\;\hat{\boldsymbol{k}}.
$$

Note: To do this problem we have assumed that the upper end of the ladder stays in contact with the wall as it slides down. If a real ladder were sliding against a slippery wall and floor, would it not lose contact? The answer is yes, it would. One way of finding when the contact would be lost is to calculate the normal reaction N_1 and finding out at what value of θ it passes through zero. The answer depends on θ_0. For example, if $\theta_0 = 80^\circ$, then N_1 is zero at about $\theta = 41°$. You can verify this result by substituting the expressions for $\ddot{\theta}$ and $\dot{\theta}$ in the expression for N_1 and solving for θ when $N_1 = 0$.

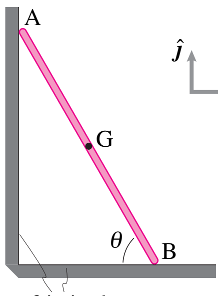

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Figure 14.63:

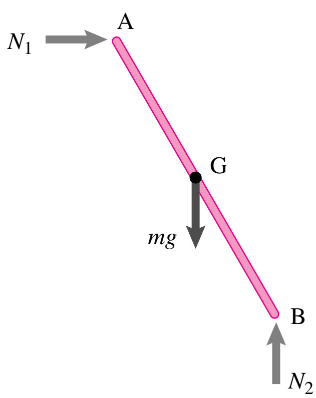

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Figure 14.64:

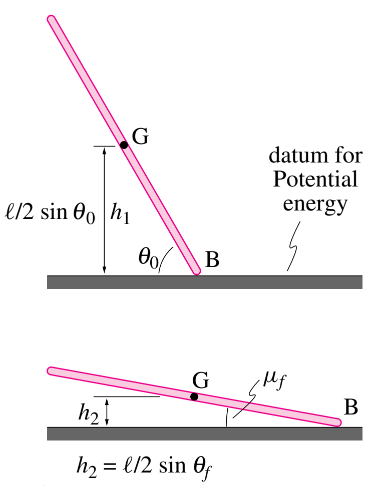

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Figure 14.65:

SAMPLE 14.23 The falling ladder again. Consider the falling ladder of Sample 14.10 again. The mass of the ladder is m and the length is ℓ. The ladder is released from rest at $\theta = 80^{\circ}$.

1. At the instant when $\theta = 45^{\circ}$, find the speed of the center-of-mass of the ladder using energy.
2. Derive the equation of motion of the ladder using work-energy balance.

Solution

1. Since there is no friction, there is no loss of energy between the two states: $\theta_0 = 80^{\circ}$ and $\theta_f = 45^{\circ}$. The only external forces on the ladder are N_1, N_2, and mg as shown in the free body diagram. Since the displacements of points A and B are perpendicular to the normal reactions of the walls, N_1 and N_2, respectively, no work is done by these forces on the ladder. The only force that does work is the force due to gravity. But this force is conservative. Therefore, the conservation of energy holds between any two states of the ladder during its fall.

Let E_1 and E_2 be the total energy of the ladder at θ_0 and θ_f, respectively. Then

$$E_1 = E_2 \quad \text{(conservation of energy).}$$

$$\begin{aligned}
\text{Now} \quad E_1 &= \underbrace{E_{K_1}}_{\text{K.E.}} + \underbrace{E_{P_1}}_{\text{P.E.}} \\
&= 0 + mgh_1 \\
&= mg\frac{\ell}{2}\sin\theta_0 \\
\text{and} \quad E_2 &= E_{K_2} + E_{P_2} = \underbrace{\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^{G}\omega^2}_{E_{K_2}} + mgh_2.
\end{aligned}$$

Equating E_1 and E_2 we get

$$\begin{aligned}
\cancel{m}g\frac{\ell}{\cancel{2}}(\sin\theta_0 - \sin\theta_f) &= \frac{1}{\cancel{2}}(\cancel{m}v_G^2 + \underbrace{\frac{1}{12}\cancel{m}\ell^2}_{I_{zz}^{G}}\omega^2) \\
\text{or} \quad g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\ell^2\omega^2. \qquad (14.51)
\end{aligned}$$

Clearly, we cannot find v_G from this equation alone because the equation contains another unknown, ω. So we need to find another equation which relates v_G and ω. To find this equation we turn to kinematics. Note that

$$\begin{aligned}
\vec{r}_G &= \frac{\ell}{2}(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) \\
\Rightarrow \quad \vec{v}_G &= \dot{\vec{r}}_G = \frac{\ell}{2}(-\sin\theta\cdot\dot{\theta}\,\hat{\imath} + \cos\theta\cdot\dot{\theta}\,\hat{\jmath}) \\
\Rightarrow \quad v_G &= |\vec{v}_G| = \sqrt{\frac{\ell^2}{4}(\cos^2\theta + \sin^2\theta)\,\dot{\theta}^2} \\
&= \frac{\ell}{2}\dot{\theta} = \frac{\ell}{2}\omega \\
\Rightarrow \quad \omega &= \frac{2v_G}{\ell}.
\end{aligned}$$

Substituting the expression for ω in eqn. (14.51) we get

$$\begin{aligned}
g\ell(\sin\theta_0 - \sin\theta_f) &= v_G^2 + \frac{1}{12}\cancel{\ell^2}\cdot\frac{4v_G^2}{\cancel{\ell^2}} \\
&= \frac{4}{3}v_G^2 \\
\Rightarrow \quad v_G &= \sqrt{\frac{3g\ell}{4}(\sin\theta_0 - \sin\theta_f)} \\
&= 0.46\sqrt{g\ell}.
\end{aligned}$$

$$\boxed{v_G = 0.46\sqrt{g\ell}}$$

2. Equation of motion: Since the ladder is a single degree of freedom system, we can use the power equation to derive the equation of motion:

$$P = \dot{E}_K.$$

For the ladder, the only force that does work is mg. This force acts at the center-of-mass G. Therefore,

$$\begin{aligned}
P &= \vec{F}\cdot\vec{v} = -mg\hat{\boldsymbol{\jmath}}\cdot\vec{v}_G \\
&= -mg\hat{\boldsymbol{\jmath}}\cdot\left[\frac{\ell}{2}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}})\,\dot{\theta}\right] \\
&= -mg\frac{\ell}{2}\dot{\theta}\cos\theta.
\end{aligned}$$

Now, the rate of change of kinetic energy is

$$\begin{aligned}
\dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}mv_G^2 + \frac{1}{2}I_{zz}^{G}\omega^2\right) \\
&= \frac{d}{dt}\left(\frac{1}{2}m\frac{\ell^2\omega^2}{4} + \frac{1}{2}\frac{m\ell^2}{12}\omega^2\right) \\
&= \frac{m\ell^2}{4}\omega\dot{\omega} + \frac{m\ell^2}{12}\omega\dot{\omega} \\
&= \frac{m\ell^2}{3}\omega\dot{\omega} \equiv \frac{m\ell^2}{3}\dot{\theta}\ddot{\theta} \quad (\text{since } \omega = \dot{\theta} \text{ and } \dot{\omega} = \ddot{\theta}).
\end{aligned}$$

Now equating P and $\dot{E}_K$ we get

$$\begin{aligned}
\frac{\cancel{m}\ell^2}{3}\cancel{\dot{\theta}}\ddot{\theta} &= -\cancel{m}g\frac{\ell}{2}\cancel{\dot{\theta}}\cos\theta \\
\Rightarrow \quad \ddot{\theta} &= -\frac{3g}{2\ell}\cos\theta
\end{aligned}$$

which is the same expression as obtained in Sample 14.22 (b).

$$\boxed{\ddot{\theta} = -\frac{3g}{2\ell}\cos\theta}$$

Note: Just like in the last problem, we have assumed that the upper end of the ladder stays in contact with the vertical wall. This assumption is used implicitly in the kinematics of the ladder. In energy calculations, the normal reactions of the two walls make no contribution as the work done by them is zero and hence the assumption of wall contact does not figure in the calculations above.

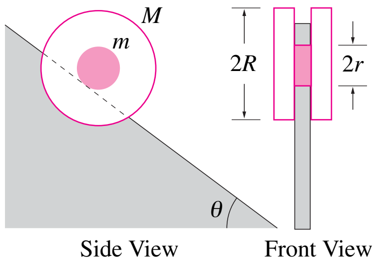

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Figure 14.66: A composite wheel made of three uniform disks rolls down an inclined wedge without slipping.

SAMPLE 14.24 Rolling on an inclined plane. A wheel is made up of three uniform disks— the center disk of mass $m = 1\,\text{kg}$, radius $r = 10\,\text{cm}$ and two identical outer disks of mass $M = 2\,\text{kg}$ each and radius R. The wheel rolls down an inclined wedge without slipping. The angle of inclination of the wedge with horizontal is $\theta = 30^{\circ}$. The radius of the bigger disks is to be selected such that the linear acceleration of the wheel center does not exceed 0.2g. Find the radius R of the bigger disks.

Solution Since a bound is prescribed on the linear acceleration of the wheel and the radius of the bigger disks is to be selected to satisfy this bound, we need to find an expression for the acceleration of the wheel (hopefully) in terms of the radius R.

The free-body diagram of the wheel is shown in Fig. 14.67. In addition to the weight $(m+2M)g$ of the wheel and the normal reaction N of the wedge surface there is an unknown force of friction F_f acting on the wheel at point C. This friction force is necessary for the condition of rolling motion. You must realize, however, that $F_f \neq \mu N$ because there is neither slipping nor a condition of impending slipping. Thus the magnitude of F_f is not known yet.

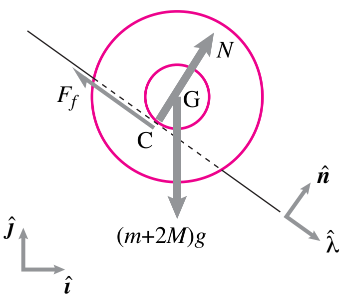

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Figure 14.67: Free body diagram of the wheel.

Let the acceleration of the center-of-mass of the wheel be

$$\vec{\boldsymbol{a}}_G = a_G \hat{\boldsymbol{\lambda}}$$

and the angular acceleration of the wheel be

$$\dot{\vec{\boldsymbol{\omega}}} = -\dot{\omega}\hat{\boldsymbol{k}}.$$

We assumed $\dot{\vec{\boldsymbol{\omega}}}$ to be in the *negative* $\hat{\boldsymbol{k}}$ direction. But, if this assumption is wrong, we will get a negative value for $\dot{\omega}$.

Now we write the equation of linear momentum balance for the wheel:

$$\begin{aligned}\sum \vec{\boldsymbol{F}} &= m_{\text{total}}\vec{\boldsymbol{a}}_{\text{cm}}\\ -(m+2M)g\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{n}} - F_f\hat{\boldsymbol{\lambda}} &= (m+2M)a_G\hat{\boldsymbol{\lambda}}\end{aligned}$$

This 2-D vector equation gives (at the most) two independent scalar equations. But we have three unknowns: N, F_f, and a_G. Thus we do not have enough equations to solve for the unknowns including the quantity of interest a_G. So, we now write the equation of angular momentum balance for the wheel about the point of contact C (using $\vec{\boldsymbol{r}}_{G/C} = r\hat{\boldsymbol{n}}$):

$$\sum \vec{\boldsymbol{M}}_{\text{C}} = \dot{\vec{\boldsymbol{H}}}_{/\text{C}}$$

where

$$\begin{aligned}\vec{\boldsymbol{M}}_{\text{C}} &= \vec{\boldsymbol{r}}_{G/C} \times (m+2M)g(-\hat{\boldsymbol{\jmath}})\\ &= r\hat{\boldsymbol{n}} \times (m+2M)g(-\hat{\boldsymbol{\jmath}})\\ &= -(m+2M)gr\sin\theta\hat{\boldsymbol{k}} \quad \text{(see Fig. 14.68)}\end{aligned}$$

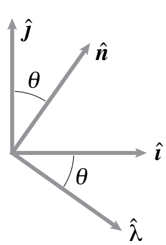

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Figure 14.68: Geometry of unit vectors. This diagram can be used to find various dot and cross products between any two unit vectors. For example, $\hat{\boldsymbol{n}} \times \hat{\boldsymbol{\jmath}} = \sin\theta\hat{\boldsymbol{k}}$.

and

$$\begin{aligned}\dot{\vec{\boldsymbol{H}}}_{/\text{C}} &= I_{zz}^{G}\dot{\vec{\boldsymbol{\omega}}} + \vec{\boldsymbol{r}}_{G/C} \times m_{\text{total}}\vec{\boldsymbol{a}}_G\\ &= I_{zz}^{G}(-\dot{\omega}\hat{\boldsymbol{k}}) - m_{\text{total}}\dot{\omega}r^2\hat{\boldsymbol{k}}\\ &= (I_{zz}^{G} + m_{\text{total}}r^2)(-\dot{\omega}\hat{\boldsymbol{k}})\\ &= \left[(\frac{1}{2}mr^2 + 2\cdot\frac{1}{2}MR^2) + \overbrace{(m+2M)}^{m_{\text{total}}}r^2\right](-\dot{\omega}\hat{\boldsymbol{k}})\\ &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}}.\end{aligned}$$

Thus,

$$\begin{aligned}
-(m+2M)gr\sin\theta\hat{\boldsymbol{k}} &= -\left[\frac{3}{2}mr^2 + M(R^2+2r^2)\right]\dot{\omega}\hat{\boldsymbol{k}} \\
\Rightarrow \quad \dot{\omega} &= \frac{(m+2M)gr\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}. \qquad (14.52)
\end{aligned}$$

Now we need to relate $\dot{\omega}$ to a_G. From the kinematics of rolling,

$$a_G = \dot{\omega}r.$$

Therefore, from Eqn. (14.52) we get

$$a_G = \frac{(m+2M)gr^2\sin\theta}{\frac{3}{2}mr^2 + M(R^2+2r^2)}.$$

Now we can solve for R in terms of a_G:

$$\begin{aligned}
\frac{3}{2}mr^2 + M(R^2+2r^2) &= \frac{(m+2M)gr^2\sin\theta}{a_G} \\
\Rightarrow \quad M(R^2+2r^2) &= \frac{(m+2M)g}{a_G}r^2\sin\theta - \frac{3}{2}mr^2 \\
\Rightarrow \quad R^2 &= \frac{(m+2M)g}{Ma_G}r^2\sin\theta - \frac{3m}{2M}r^2 - 2r^2.
\end{aligned}$$

Since we require $a_G \leq 0.2g$ we get

$$\begin{aligned}
R^2 &\geq \left(\frac{(m+2M)g}{M\cdot 0.2g}\sin\theta - \frac{3m}{2M} - 2\right)r^2 \\
&\geq \left(\frac{5\,\mathrm{kg}}{0.4\,\mathrm{kg}}\cdot\frac{1}{2} - \frac{3\,\mathrm{kg}}{4\,\mathrm{kg}} - 2\right)(0.1\,\mathrm{m})^2 \\
&\geq 0.035\,\mathrm{m}^2 \\
\Rightarrow \quad R &\geq 0.187\,\mathrm{m}.
\end{aligned}$$

Thus the outer disks of radius 20 cm will do the job.

$$\boxed{R \geq 18.7\,\mathrm{cm}}$$

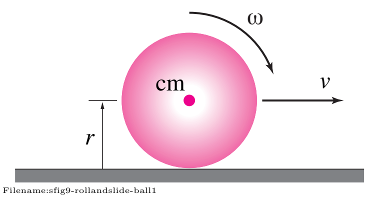


Figure 14.69:

SAMPLE 14.25 Which one starts rolling first — a marble or a bowling ball? A marble and a bowling ball, made of the same material, are launched on a horizontal platform with the same initial velocity, say v_0. The initial velocity is large enough so that both start out sliding. Towards the end of their motion, both have pure rolling motion. If the radius of the bowling ball is 16 times that of the marble, find the instant, for each ball, when the sliding motion changes to rolling motion.

Solution Let us consider one ball, say the bowling ball, first. Let the radius of the ball be r and mass m. The ball starts with center-of-mass velocity $\vec{v}_o = v_0\hat{\imath}$. The ball starts out sliding. During the sliding motion, the force of friction acting on the ball must equal μN (see the FBD). The friction force creates a torque about the mass-center which, in turn, starts the rolling motion of the ball. However, rolling and sliding coexist for a while, till the speed of the mass-center slows down enough to satisfy the pure rolling condition, $v = \omega r$. Let the instant of transition from the mixed motion to pure rolling be t^*. From linear momentum balance , we have

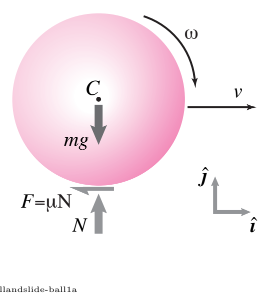


Figure 14.70: Free-body diagram of the ball during sliding.

$$
\begin{aligned}
& & m\dot{v}\hat{\imath} &= -\mu N\hat{\imath} + (N - mg)\hat{\jmath} & (14.53)\\
\text{eqn. (14.53)}\cdot\hat{\jmath} &\Rightarrow & N &= mg &\\
\text{eqn. (14.53)}\cdot\hat{\imath} &\Rightarrow & m\dot{v} &= -\mu N = -\mu mg &\\
&\Rightarrow & \dot{v} &= -\mu g &\\
&\Rightarrow & v &= v_0 - \mu g t. & (14.54)
\end{aligned}
$$

Similarly, from angular momentum balance about the mass-center, we get

$$
\begin{aligned}
-I_{zz}^{\text{cm}}\dot{\omega}\hat{k} &= -\mu N r\hat{k} = -\mu m g r\hat{k}\\
\Rightarrow \quad \dot{\omega} &= \frac{\mu m g r}{I_{zz}^{\text{cm}}}\\
\Rightarrow \quad \omega &= \underbrace{\omega_0}_{0} + \frac{\mu m g r}{I_{zz}^{\text{cm}}} t. \qquad (14.55)
\end{aligned}
$$

At the instant of transition from mixed rolling and sliding to pure rolling, *i.e.,* at $t = t^*$, $v = \omega r$. Therefore, from eqn. (14.54) and eqn. (14.55), we get

$$
\begin{aligned}
v_0 - \mu g t^* &= \frac{\mu m g r^2}{I_{zz}^{\text{cm}}} t^*\\
\Rightarrow \quad v_0 &= \mu g t^* \left(1 + \frac{m r^2}{I_{zz}^{\text{cm}}}\right)\\
\Rightarrow \quad t^* &= \frac{v_0}{\mu g\left(1 + \frac{m r^2}{I_{zz}^{\text{cm}}}\right)}.
\end{aligned}
$$

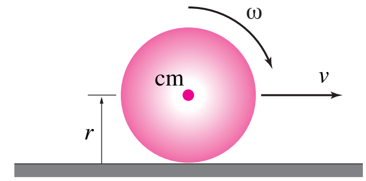
Now, for a sphere, $I_{zz}^{\text{cm}} = \frac{2}{5} m r^2$. Therefore,

$$
t^* = \frac{v_0}{\mu g\left(1 + \frac{m r^2}{\frac{2}{5} m r^2}\right)} = \frac{2 v_0}{7 \mu g}.
$$

Note that the expression for t^* is independent of mass and radius of the ball! Therefore, the bowling ball and the marble are going to change their mixed motion to pure rolling at exactly the same instant. This is not an intuitive result.

$t^* = \frac{2v_0}{7\mu g}$ for both.

SAMPLE 14.26 Transition from a mix of sliding and rolling to pure rolling, using impulse-momentum. Consider the problem in Sample 14.25 again: A ball of radius $r = 10\,\mathrm{cm}$ and mass $m = 1\,\mathrm{kg}$ is launched horizontally with initial velocity $v_0 = 5\,\mathrm{m/s}$ on a surface with coefficient of friction $\mu = 0.12$. The ball starts sliding, rolls and slides simultaneously for a while, and then rolls without sliding. Find the time it takes to start pure rolling.



Filename:sfig9-rollandslide-ball2

Figure 14.71:

Solution Let us denote the time of transition from mixed motion (rolling and sliding) to pure rolling by t^*. At $t = 0$, we know that $v_{\mathrm{cm}} = v_0 = 5\,\mathrm{m/s}$, and $\omega_0 = 0$. We also know that at $t = t^*$, $v_{\mathrm{cm}} = v_{t^*} = \omega_{t^*} r$, where r is the radius of the ball. We do not know t^* and v_{t^*}. However, we are considering a finite time event (during t^*) and the forces acting on the ball during this duration are known. Recall that impulse momentum equations involve the net force on the body, the time of impulse, and momenta of the body at the two instants. Momenta calculations involve velocities. Therefore, we should be able to use impulse-momentum equations here and find the desired unknowns. From linear impulse-momentum, we have

$$\begin{aligned}\left(\sum \vec{F}\right) t^* &= m v_{t^*}\hat{\imath} - m v_0 \hat{\imath}\\ (-\mu N \hat{\imath} + (N - mg)\hat{\jmath})\, t^* &= m(v_{t^*} - v_0)\hat{\imath}.\end{aligned}$$

Dotting the above equation with $\hat{\jmath}$ and $\hat{\imath}$, respectively, we get

$$\begin{aligned} N &= mg\\ -\mu \underbrace{N}_{mg} t^* &= m(v_{t^*} - v_0)\\ \Rightarrow \quad -\mu g t^* &= v_{t^*} - v_0. \end{aligned} \tag{14.56}$$

Similarly, from angular impulse-momentum relation about the mass-center, we get

$$\begin{aligned} \sum \vec{M}_{\mathrm{cm}} t^* &= (\vec{H}_{\mathrm{cm}})_{t^*} - (\vec{H}_{\mathrm{cm}})_0\\ (-\mu N r \hat{k}) t^* &= (I^{\mathrm{cm}}_{zz}\omega_{t^*} - I^{\mathrm{cm}}_{zz}\underbrace{\omega_0}_{0})(-\hat{k})\\ \text{or} \qquad -\mu m g r t^* &= -I^{\mathrm{cm}}_{zz}\omega_{t^*}\\ \Rightarrow \quad \omega_{t^*} &= \mu m g r t^*/I^{\mathrm{cm}}_{zz}\\ \Rightarrow \quad v_{t^*} &\equiv \omega_{t^*} r = \mu m g r^2 t^*/I^{\mathrm{cm}}_{zz}. \end{aligned}$$

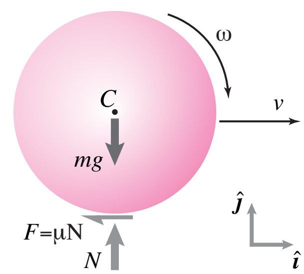
Substituting this expression for v_{t^*} in eqn. (14.56), we get

$$\begin{aligned} -\mu g t^* &= \mu m g r^2 t^*/I^{\mathrm{cm}}_{zz} - v_0\\ \Rightarrow \quad t^* &= \frac{v_0}{\mu g (1 + \frac{mr^2}{I^{\mathrm{cm}}_{zz}})} \end{aligned}$$

which is, of course, the same expression we obtained for t^* in Sample 14.25. Again, noting that $I^{\mathrm{cm}}_{zz} = \frac{2}{5}mr^2$ for a sphere, we calculate the time of transition as

$$t^* = \frac{2v_0}{7\mu g} = \frac{2 \cdot (5\,\mathrm{m/s})}{7 \cdot (0.2) \cdot (9.8\,\mathrm{m/s^2})} = 0.73\,\mathrm{s}.$$

$$t^* = 0.73\,\mathrm{s}$$



Filename:sfig9-rollandslide-ball2a

Figure 14.72: Free-body diagram of the ball during sliding.

14.5 Rigid-object collision mechanics

This section extends the particle collisions discussion in Section 11.2 that starts on page 588.

2D collisions

For collisions between rigid bodies with more general motions before and after the collisions we depend on the three key ideas

I. Collision forces are big,

II. Collisions are quick, and

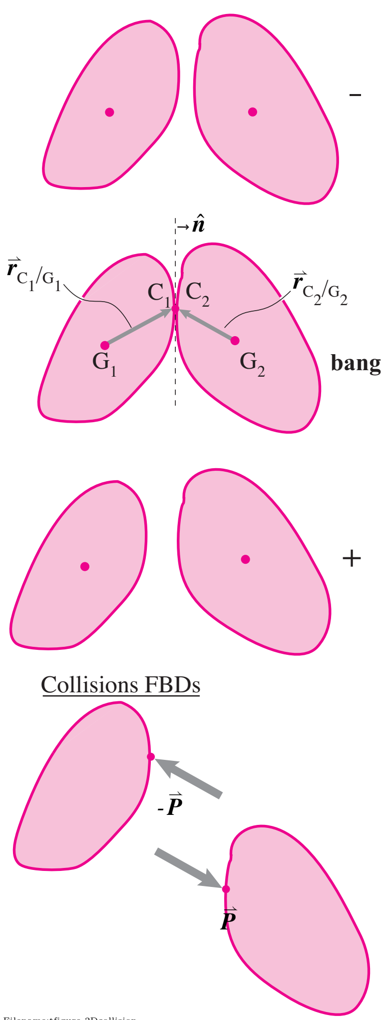
III. The laws of mechanics apply during the collision.

There are two extra assumptions that are needed in simple analysis:

IV. Collision forces are few. For a given rigid body there is one, or at most two non-negligible collision forces. This is the real import of idea (I) above. Because collision forces are big most other forces can be neglected.

V. The collision force(s) act at a well defined point which does not move during the collision.

Based on these assumptions one then uses linear and angular momentum balance in their time-integrated form.



Filename:tfigure-2Dcollision

Figure 14.73: Two bodies collide at point C. The only non-negligible collision impulse is $\vec{\boldsymbol{P}}$ acting on body 2 (and $-\vec{\boldsymbol{P}}$ on body 1) at point C. The material points on the contacting bodies are C_1 and C_2. The outward normal to body 1 at C_1 is $\hat{\boldsymbol{n}}$.

Example: Two bodies in space

Two bodies collide at point C. The impulse acting on body 2 is $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}_{\text{coll}}\, dt$. If the mass and inertia properties of both bodies is known, as are the velocities and rotation rates before the collision we have the following linear and angular momentum balance equations for the two bodies:

$$
\begin{aligned}
-\vec{\boldsymbol{P}} &= m_1\left(\vec{\boldsymbol{v}}^{+}_{G1} - \vec{\boldsymbol{v}}^{-}_{G1}\right)\\
\vec{\boldsymbol{P}} &= m_2\left(\vec{\boldsymbol{v}}^{+}_{G2} - \vec{\boldsymbol{v}}^{-}_{G2}\right)\\
\vec{\boldsymbol{r}}_{C/G1} \times (-\vec{\boldsymbol{P}}) &= I^{cm1}_{zz}\left(\omega^{+}_1 - \omega^{-}_1\right)\hat{\boldsymbol{k}}\\
\vec{\boldsymbol{r}}_{C/G2} \times \vec{\boldsymbol{P}} &= I^{cm2}_{zz}\left(\omega^{+}_2 - \omega^{-}_2\right)\hat{\boldsymbol{k}}.
\end{aligned}
\tag{14.57}
$$

These make up 6 scalar equations (2 for each momentum equation, 1 for each angular momentum equation). There are 8 scalar unknowns: $\vec{\boldsymbol{v}}^{+}_{G1}$ (2), $\vec{\boldsymbol{v}}^{+}_{G2}$ (2), ω^{+}_1 (1), ω^{+}_2 (1), and $\vec{\boldsymbol{P}}$ (2). Thus the motion after the collision cannot be determined.

[Note that linear and angular momentum balance for the system would give equations which could be obtained by adding and subtracting combinations of the equations above. So adding system momentum balance equations does not add information (ie, adds linearly dependent equations).]

So, as for 1-D collisions, momentum balance is not enough to determine the outcome of the collision. Eqns. 14.57 aren't enough. A thousand different models and assumptions could be added to make the system solvable. But there are only two cases that are non-controversial and also relatively simple: 1) sticking collisions, and 2) frictionless collisions.

Sticking collisions

A 'perfectly-plastic' sticking collision is one where the relative velocities of the two contacting points are assumed to go suddenly to zero. That is

$$\vec{\boldsymbol{v}}_{C1}^{+} = \vec{\boldsymbol{v}}_{C2}^{+}$$

Writing $\vec{\boldsymbol{v}}_{\mathrm{C1}}^{+} = \vec{\boldsymbol{v}}_{\mathrm{G1}}^{+} + (\omega_1^{+}\hat{\boldsymbol{k}}) \times \vec{\boldsymbol{r}}_{\mathrm{C/G1}}$ and similarly for $\vec{\boldsymbol{v}}_{\mathrm{C2}}$ thus adds a vector equation (2 scalar equations) to the equation set 14.57. This gives 8 equations in 8 unknowns.

A little cleverness can reduce the problem to one of solving only 4 equations in 4 unknowns. Linear momentum balance for the system, angular momentum balance for the system and angular momentum balance for object 2 make up 4 scalar equations. None of these equations includes the impulse $\vec{\boldsymbol{P}}$. Because the system moves as if hinged at C_1 after the collision, the state of motion after the system is fully characterized by $\vec{\boldsymbol{v}}_{\mathrm{G1}}^{+}$, ω_1^{+}, and ω_2^{+}. Thus we have 4 equations in 4 unknowns.

Example: One body is hugely massive: collision with an immovable object
If body 2, say, is huge compared to body 1 then it can be taken to be immovable and collision problems can be solved by only considering body 1 (see fig. 14.74). In the case of a sticking collision the full state of the system after the collision is determined by ω_1^{+}. This can be found from the single scalar equation obtained from angular momentum balance about the collision point.

$$\begin{aligned} \vec{\boldsymbol{H}}_{\mathrm{A}}^{-} &= \vec{\boldsymbol{H}}_{\mathrm{A}}^{+} \\ \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}_{\mathrm{G}}^{-} + I_{zz}^{\mathrm{cm}}\omega^{-}\hat{\boldsymbol{k}} &= \vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}_{\mathrm{G}}^{+} + I_{zz}^{\mathrm{cm}}\omega^{+}\hat{\boldsymbol{k}} \end{aligned}$$

Because the state of the system before the collision is assumed known (the left "-" side of the equation, and because the post-collision (+) state is a rotation about A, this equation is one scalar equation in the one unknown ω^{+}. Note that $\vec{\boldsymbol{H}}_{\mathrm{A}}^{+}$ could also be evaluated as $\vec{\boldsymbol{H}}_{\mathrm{A}}^{+} = \omega^{+} I_{zz}^{A}\hat{\boldsymbol{k}}$. So one way of expressing the post-collision state is as

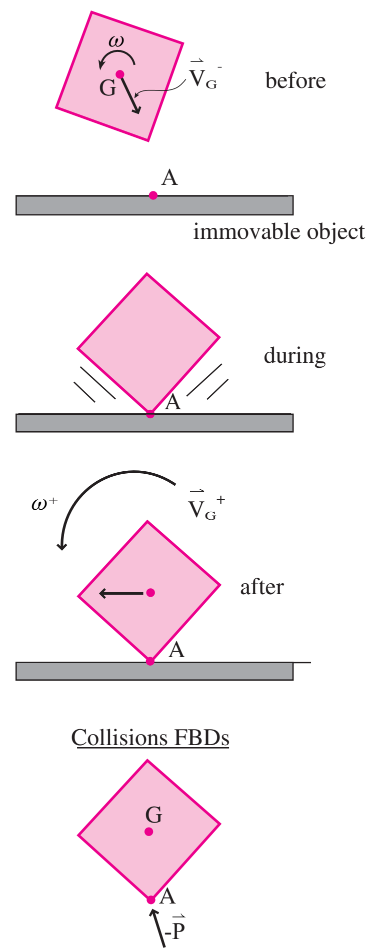
$$\omega^{+} = \frac{\left(\vec{\boldsymbol{r}}_{\mathrm{G/A}} \times m\vec{\boldsymbol{v}}_{\mathrm{G}}^{-} + I_{zz}^{\mathrm{cm}}\omega^{-}\hat{\boldsymbol{k}}\right)\cdot\hat{\boldsymbol{k}}}{I_{zz}^{A}} \quad \text{and} \quad \vec{\boldsymbol{v}}_{\mathrm{G}}^{+} = \omega^{+}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/A}}.$$

Note also that the same $\vec{\boldsymbol{r}}_{\mathrm{G/A}}$ is used on the right and left sides of the equation because only the velocity and not the position is assumed to jump during the collision.

The collision impulse $\vec{\boldsymbol{P}}$ can then be found from linear momentum balance as

$$\vec{\boldsymbol{P}} = m\left(\vec{\boldsymbol{v}}_{\mathrm{G}}^{+} - \vec{\boldsymbol{v}}_{\mathrm{G}}^{-}\right).$$

Sticking collisions are used as models of projectiles hitting targets, of robot and animal limbs making contact with the ground, of monkeys and acrobats grabbing hand holds, and of some particularly dead and frictional collisions between solids (such as when a car trips on a curb).



Filename:tfigure-collimmovable

Figure 14.74: Sticking collision with an immovable object. The box sticks at A and then rotates about A. Angular momentum about point A is conserved in the collision.

Frictionless collisions

The second special case is that of a frictionless collision. Here we add two assumptions:

1. There is no friction so $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$. The number of unknowns is thus reduced from 8 to 7.

2. There is a coefficient of (normal) restitution e.

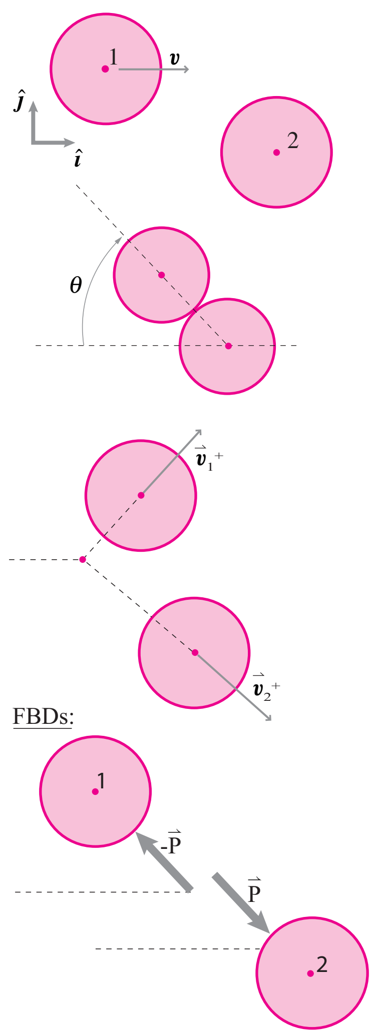
The normal restitution coefficient is taken as a property of the colliding bodies. It is a given number with $0 < e < 1$ with this defining equation:

$$(\vec{v}^{+}_{C2} - \vec{v}^{+}_{C1}) \cdot \hat{\boldsymbol{n}} = -e(\vec{v}^{-}_{C2} - \vec{v}^{-}_{C1}) \cdot \hat{\boldsymbol{n}}.$$

This says that the normal part of the relative velocity of the contacting points reverses sign and its magnitude is attenuated by e. This adds a scalar equation to the set Eqns. 14.57 thus giving 7 scalar equations (4 momentum, 2 angular momentum, 1 restitution) for 7 unknowns (4 velocity components, 2 angular velocities and the normal impulse).

The most popular application of the frictionless collision model is for billiard or pool balls, or carrom pucks. These things have relatively small coefficients of friction.

We state without proof that a frictionless collision with $e = 1$ conserves energy.



Filename:tfigure-poolballs

Figure 14.75: Frictionless collision between two identical round objects. Ball one is initially moving to the right, ball 2 is initially stationary. The impulse of ball 1 *on* ball 2 is $\vec{\boldsymbol{P}}$.

Example: Pool balls

Assume one ball approaches the other with initial velocity $\vec{v}^{+}_{G1} = v\hat{\imath}$ and has an elastic frictionless collision with the other ball at a collision angle of θ as shown in fig. 14.75. Defining $\hat{\boldsymbol{n}} \equiv \cos\theta\hat{\imath} - \sin\theta\,\hat{\jmath}$ we have that $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$. To determine the outcome of the equation we have the angular momentum balance equations (about the center-of-mass) which trivially tell us that

$$\omega_1^{+} = \omega_2^{+} = 0$$

because the balls start with no spin and the frictionless collision impulses $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ and $-\vec{\boldsymbol{P}} = -P\hat{\boldsymbol{n}}$ have no moment about the center-of-mass. Linear momentum balance for each of the balls

$$\begin{aligned} -P\hat{\boldsymbol{n}} &= m\vec{v}^{+}_{G1} - mv\hat{\imath} \\ P\hat{\boldsymbol{n}} &= m\vec{v}^{+}_{G2} - \vec{\boldsymbol{0}} \end{aligned}$$

gives 4 scalar equations which are supplemented by the restitution equation (using $e = 1$)

$$\left(\Delta\vec{v}^{+}\right) \cdot \hat{\boldsymbol{n}} = -e\,(\Delta\vec{v}^{-}) \cdot \hat{\boldsymbol{n}}$$

$$\Rightarrow -v\cos\theta = \vec{v}^{+}_{G2} \cdot \hat{\boldsymbol{n}} - \vec{v}^{+}_{G1} \cdot \hat{\boldsymbol{n}}$$

which together make 5 scalar equations in the 5 scalar unknowns $\vec{v}^{+}_{G1}$, $\vec{v}^{+}_{G2}$, and P (each vector has 2 unknown components). These have the solution

$$\begin{aligned} \vec{v}^{+}_{G1} &= v\sin\theta(\sin\theta\hat{\imath} + \cos\theta\,\hat{\jmath}), \\ \vec{v}^{+}_{G1} &= v\cos\theta(\cos\theta\hat{\imath} - \sin\theta\,\hat{\jmath}), \quad \text{and} \\ P &= mv\cos\theta. \end{aligned}$$

The solution can be checked by plugging back into the momentum and restitution equations. Also, as promised, this $e = 1$ solution conserves kinetic energy. The solution has the interesting property that the outgoing trajectories of the two balls are orthogonal for all θ but $\theta = 0$ in which case ball 1 comes to rest in the collision. [The solution can be found graphically by looking for two outgoing vectors which add to the original velocity of mass 1, where the sum of the squares of the outgoing speeds must add to the square of the incoming speed.]

Frictional collisions

For a collision with friction, but not so much that total sticking is accurate, the modeling is complex and subtle. As of this writing there are no standard acceptable ways of dealing with such situations. Commercial simulation packages should be used for such with skeptical caution. They are generally defective in that they can predict only a limited range of phenomena and/or they can create energy even with innocent input parameters.

Why is it hard to find a good collision law

Ideally one would like a rule to determine how bodies move after a collision from how they move before the collision. Such a rule would be called a collision law or a constitutive relation for collisions. That accurate collision laws are rare at best might be surmised from the basic problem that the phrase *rigid body collisions* is in some sense a contradiction in terms, an oxymoron. The force generated in the contact comes from material deformation, and deformation is just what we generally try to neglect when doing rigid body mechanics.

There is a temptation to say that one wants to continue to neglect deformation during the collision, but for in an infinitesimal contact region. And some collision laws are formulated with this approach. Even then, there are no reliable models for the deformation in that small region, and such laws are doomed to inaccuracy in situations where the deformation is not so limited.

For complex shaped bodies touching at various points that are generally not known *a priori*, no collision law is reliably accurate.

SAMPLE 14.27 For two masses conservation of momentum is expressed by the vector equation $m_1\vec{\boldsymbol{v}}_1 + \mathrm{m}_2\vec{\boldsymbol{v}}_2 = m_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+$. Suppose $\vec{\boldsymbol{v}}_1 = \vec{\boldsymbol{0}}$, $\vec{\boldsymbol{v}}_2 = -v_0\hat{\boldsymbol{\jmath}}$, $\vec{\boldsymbol{v}}_1^+ = v_1^+\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{v}}_2^+ = v_{2_t}^+\hat{\boldsymbol{e}}_t + v_{2_n}^+\hat{\boldsymbol{e}}_n$, where $\hat{\boldsymbol{e}}_t = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_n = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.

1. Obtain two independent scalar equations from the momentum equation corresponding to projections in the $\hat{\boldsymbol{e}}_n$ and $\hat{\boldsymbol{e}}_t$ directions.
2. Assume that you are given another equation $v^+{}_{2_t} = -v_0\sin\theta$. Set up a matrix equation to solve for v_1^+, $v_{2_t}^+$, and $v_{2_n}^+$ from the three equations.

Solution

1. The given equation of conservation of linear momentum is

$$m_1\underbrace{\vec{\boldsymbol{v}}_1}_{\boldsymbol{0}} + \mathrm{m}_2\vec{\boldsymbol{v}}_2 = m_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+$$

$$\text{or} \quad -m_2v_0\hat{\boldsymbol{\jmath}} = m_1v_1^+\hat{\boldsymbol{\imath}} + m_2(v_{2_t}^+\hat{\boldsymbol{e}}_t + v_{2_n}^+\hat{\boldsymbol{e}}_n). \tag{14.58}$$

Dotting both sides of eqn. (14.58) with $\hat{\boldsymbol{e}}_n$ gives

$$-m_2v_0\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\jmath}})}^{\cos\theta} = m_1v_1^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{\imath}})}^{-\sin\theta} + m_2v_{2_t}^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_t)}^{0} + m_2v_{2_n}^+\overbrace{(\hat{\boldsymbol{e}}_n\cdot\hat{\boldsymbol{e}}_n)}^{1}$$

$$\text{or} \quad -m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+. \tag{14.59}$$

Dotting both sides of eqn. (14.58) with $\hat{\boldsymbol{e}}_t$ gives

$$-m_2v_0\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\jmath}})}^{\sin\theta} = m_1v_1^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\imath}})}^{\cos\theta} + m_2v_{2_t}^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_t)}^{1} + m_2v_{2_n}^+\overbrace{(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{e}}_n)}^{0}$$

$$\text{or} \quad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+. \tag{14.60}$$

$$-m_2v_0\cos\theta = -m_1v_1^+\sin\theta + m_2v_{2_n}^+, \quad -m_2v_0\sin\theta = m_1v_1^+\cos\theta + m_2v_{2_t}^+$$

2. Now, we rearrange eqn. (14.59) and 14.60 along with the third given equation, $v^+{}_{2_t} = -v_0\sin\theta$, so that all unknowns are on the left hand side and the known quantities are on the right hand side of the equal sign. These equations, in matrix form, are as follows.

$$\begin{bmatrix} -m_1\sin\theta & 0 & m_2 \\ -m_1\cos\theta & m_2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} v_1^+ \\ v_{2_t}^+ \\ v_{2_n}^+ \end{Bmatrix} = \begin{Bmatrix} -m_2v_0\cos\theta \\ -m_2v_0\sin\theta \\ -v_0\sin\theta \end{Bmatrix}.$$

This equation can be easily solved on a computer for the unknowns.

SAMPLE 14.28 Cueing a billiard ball. A billiard ball is cued by striking it horizontally at a distance $d = 10\,\mathrm{mm}$ above the center of the ball. The ball has mass $m = 0.2\,\mathrm{kg}$ and radius $r = 30\,\mathrm{mm}$. Immediately after the strike, the center-of-mass of the ball moves with linear speed $v = 1\,\mathrm{m/s}$. Find the angular speed of the ball immediately after the strike. Ignore friction between the ball and the table during the strike.

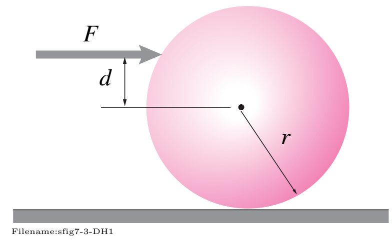


Figure 14.76:

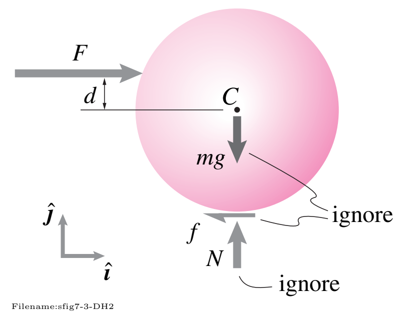


Figure 14.77: FBD of the ball during the strike. The nonimpulsive forces mg, N, and f can be ignored in comparison to the strike force F.

Solution Let the force imparted during the strike be F. Since the ball is cued by giving a blow with the cue, F is an impulsive force. Impulsive forces, such as F, are in general so large that all non-impulsive forces are negligible in comparison during the time such forces act. Therefore, we can ignore all other forces (mg, N, f) acting on the ball from its free body diagram during the strike.

Now, from the linear momentum balance of the ball we get

$$F\hat{\imath} = \dot{\vec{L}} \quad \text{or} \quad (F\hat{\imath})\mathrm{d}t = \mathrm{d}\vec{L} \quad \Rightarrow \quad \int (F\hat{\imath})\mathrm{d}t = \vec{L}_2 - \vec{L}_1$$

where $L_2 - L_1 = \Delta\vec{L}$ is the net change in the linear momentum of the ball during the strike. Since the ball is at rest before the strike, $\vec{L}_1 = m \underbrace{\vec{v}_1}_{0} = \vec{0}$. Immediately after the strike, $\vec{v} = v\hat{\imath} = 1\,\mathrm{m/s}$.

$$\text{Thus} \quad \vec{L}_2 = m\vec{v} = 0.2\,\mathrm{kg}{\cdot}1\,\mathrm{m/s}\hat{\imath} = 0.2\,\mathrm{N{\cdot}s}\hat{\imath}.$$

$$\text{Hence} \quad \int (F\hat{\imath})\mathrm{d}t = 0.2\,\mathrm{N{\cdot}s}\hat{\imath} \quad \text{or} \quad \int F\,\mathrm{d}t = 0.2\,\mathrm{N{\cdot}s}. \tag{14.61}$$

To find the angular speed we apply the angular momentum balance. Let ω be the angular speed immediately after the strike and $\vec{\omega} = \omega\hat{k}$. Now,

$$\sum \vec{M}_{\mathrm{cm}} = \dot{\vec{H}}_{\mathrm{cm}} \quad \Rightarrow \quad \int \sum \vec{M}_{\mathrm{cm}}\,\mathrm{d}t = \int \mathrm{d}\vec{H}_{\mathrm{cm}} = (\vec{H}_{\mathrm{cm}})_2 - (\vec{H}_{\mathrm{cm}})_1.$$

$$\begin{aligned}
\text{Because} \quad \vec{H}_{\mathrm{cm}} &= I_{cm}^{zz}\vec{\omega} \quad \text{and just before the strike, } \vec{\omega} = \vec{0},\\
(\vec{H}_{\mathrm{cm}})_1 &\equiv \text{angular momentum just before the strike} = \vec{0}\\
(\vec{H}_{\mathrm{cm}})_2 &\equiv \text{angular momentum just after the strike} = I_{cm}^{zz}\omega\hat{k},\\
\int \sum \vec{M}_{\mathrm{cm}}\,\mathrm{d}t &= I_{cm}^{zz}\omega\hat{k} = \frac{2}{5}mr^2\omega\hat{k} \ \text{(since for a sphere, } I_{cm}^{zz} = \tfrac{2}{5}mr^2).
\end{aligned}$$

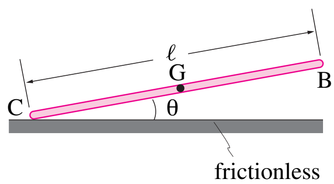
$$\begin{aligned}
\text{But} \quad & \sum \vec{M}_{\mathrm{cm}} = -Fd\hat{k},\\
\text{therefore} \quad & -\int (Fd)\mathrm{d}t\hat{k} = \frac{2}{5}mr^2\omega\hat{k}\\
\text{or} \quad & -\underbrace{d}_{\text{constant}} \int F\,\mathrm{d}t = \frac{2}{5}mr^2\omega \quad \Rightarrow \quad \omega = -\frac{5d}{2mr^2}\int F\,\mathrm{d}t.
\end{aligned}$$

Substituting the given values and $\int F\,\mathrm{d}t = 0.2\,\mathrm{N{\cdot}s}$ from equation 14.61 we get

$$\omega = -\frac{5(0.01\,\mathrm{m})}{2{\cdot}0.2\,\mathrm{kg}{\cdot}(0.03\,\mathrm{m})^2}{\cdot}0.2\,\mathrm{N{\cdot}s} = -27.78\,\mathrm{rad/s}.$$

The negative value makes sense because the ball will spin clockwise after the strike, but we assumed that ω was anticlockwise.

$$\omega = -27.78\,\mathrm{rad/s}.$$



Filename:sfig9-5-fallingbar

Figure 14.78:

SAMPLE 14.29 **Falling stick.** A uniform bar of length ℓ and mass m falls on the ground at an angle θ as shown in the figure. Just before impact at point C, the entire bar has the same velocity v directed vertically downwards. Assume that the collision at C is plastic, *i.e.,* end C of the bar gets stuck to the ground upon impact.

1. Find the angular velocity of the bar just after impact.
2. Assuming θ to be small, find the velocity of end B of the bar just after impact.

Solution We are given that the impact at point C is plastic. That is, end C of the bar has zero velocity after impact. Thus end C gets stuck to the ground. Then we expect the rod to rotate about point C as rest of the bar moves (perhaps faster) to touch the ground. The free-body diagram of the bar is shown in fig. 14.79 during the impact at point C. Note that we can ignore the force of gravity in comparison to the large impulsive force F_c due to impact at C.

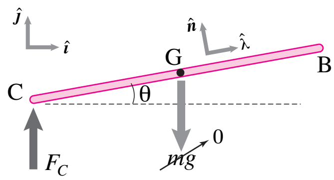


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Figure 14.79: The free-body diagram of the bar during collision. The impulsive force at the point of impact C is so large that the force of gravity can be completely ignored in comparison.

1. Now, if we carry out angular momentum balance about point C, there will be no net moment acting on the bar, and therefore, angular momentum about the impact point C is conserved. Distinguishing the kinematic quantities before and after impact with superscripts '-' and '+', respectively, we get from the conservation of angular momentum about point C,

$$\begin{aligned} \vec{\boldsymbol{H}}_C^- &= \vec{\boldsymbol{H}}_C^+ \\ I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^- + \vec{\boldsymbol{r}}_{\text{G/C}} \times m\vec{\boldsymbol{v}}_G^- &= I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{\boldsymbol{r}}_{\text{G/C}} \times m\vec{\boldsymbol{v}}_G^+. \end{aligned}$$

Now, we know that $\vec{\boldsymbol{\omega}}^- = \vec{\boldsymbol{0}}$ since every point on the bar has the same vertical velocity $\vec{\boldsymbol{v}} = -v\hat{\boldsymbol{\jmath}}$, and that just after impact, $\vec{\boldsymbol{v}}_G^+ = \vec{\boldsymbol{\omega}}^+ \times \vec{\boldsymbol{r}}_{\text{G/C}}$ where we can take $\vec{\boldsymbol{\omega}}^+ = \omega(-\hat{\boldsymbol{k}})$. Thus, ①

$$\begin{aligned} \vec{\boldsymbol{H}}_C^- &= \vec{\boldsymbol{r}}_{\text{G/C}} \times m\vec{\boldsymbol{v}}_G^- = (\ell/2)\hat{\boldsymbol{\lambda}} \times mv(-\hat{\boldsymbol{\jmath}}) \\ &= -\frac{mv\ell}{2}\cos\theta\hat{\boldsymbol{k}} \qquad (\text{since } \hat{\boldsymbol{\lambda}} = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\ \vec{\boldsymbol{H}}_C^+ &= I_{zz}^{\text{cm}}\vec{\boldsymbol{\omega}}^+ + \vec{\boldsymbol{r}}_{\text{G/C}} \times m(\vec{\boldsymbol{\omega}}^+ \times \vec{\boldsymbol{r}}_{\text{G/C}}) \\ &= -I_{zz}^{\text{cm}}\omega\hat{\boldsymbol{k}} + (\ell/2)\hat{\boldsymbol{\lambda}} \times m\,\underbrace{(-\omega\hat{\boldsymbol{k}} \times \ell/2\hat{\boldsymbol{\lambda}})}_{\omega\ell/2(-\hat{\boldsymbol{n}})} \\ &= -\frac{1}{12}m\ell^2\omega\hat{\boldsymbol{k}} - \frac{1}{4}m\ell^2\omega\hat{\boldsymbol{k}} = -\frac{1}{3}m\ell^2\omega\hat{\boldsymbol{k}}. \end{aligned}$$

Now, equating $\vec{\boldsymbol{H}}_C^-$ and $\vec{\boldsymbol{H}}_C^+$ we get

$$\omega = \frac{3v}{2\ell}\cos\theta, \qquad \Rightarrow \qquad \vec{\boldsymbol{\omega}} = -\frac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}.$$

$$\boxed{\vec{\boldsymbol{\omega}} = -\tfrac{3v}{2\ell}\cos\theta\hat{\boldsymbol{k}}}$$

2. The velocity of the end B is now easily found using $\vec{\boldsymbol{v}}_{\text{B}} = \vec{\boldsymbol{v}}_{\text{C}} + \vec{\boldsymbol{v}}_{\text{B/C}} = \vec{\boldsymbol{v}}_{\text{B/C}}$ and $\vec{\boldsymbol{v}}_{\text{B/C}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{B/C}}$. Thus,

$$\begin{aligned} \vec{\boldsymbol{v}}_{\text{B/C}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{B/C}} = -\omega\hat{\boldsymbol{k}} \times \ell\hat{\boldsymbol{\lambda}} \\ &= -\omega\ell\hat{\boldsymbol{n}} = -\frac{3v}{2}\cos\theta(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \end{aligned}$$

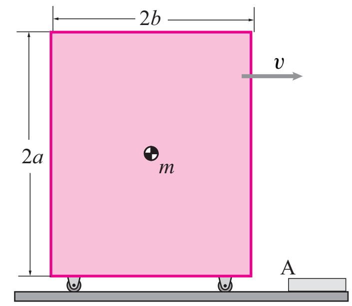
but, for small θ, $\cos\theta \approx 1$, and $\sin\theta \approx 0$. Therefore, $\vec{\boldsymbol{v}}_{\text{B/C}} = -\frac{3v}{2}\hat{\boldsymbol{\jmath}}$. Thus, end B of the bar speeds up by one and a half times its original speed due to the plastic impact at C.

$$\boxed{\vec{\boldsymbol{v}}_{\text{B/C}} = -(3/2)v\hat{\boldsymbol{\jmath}}}$$

①Since C is a fixed point for the motion of the bar after impact, we could calculate $\vec{\boldsymbol{H}}_C^+$ as follows.

$$\vec{\boldsymbol{H}}_C^+ = I_{zz}^C\vec{\boldsymbol{\omega}} = \underbrace{\frac{1}{3}m\ell^2}_{I_{zz}^C}\,\omega(-\hat{\boldsymbol{k}}).$$

SAMPLE 14.30 Tipping box. A box of mass m and dimensions $2a$ and $2b$ moves along a horizontal surface with uniform speed v. Suddenly, it bumps into an obstacle at A. Assume that the impact is plastic and point A is at the lowest level of the box. What is the maximum v the box can have so that it does not tip over after the impact.



Filename:sfig9-5-tipping

Figure 14.80:

Solution Whether the box can tip or not depends on whether it gets sufficient initial angular speed just after collision to overcome the restoring moment due to gravity about the point of rotation A. So, first we need to find the angular velocity of the box immediately following the collision. The free-body diagram of the box during collision is shown in fig. 14.81. There is an impulse $\vec{P}$ acting at the point of impact. If we carry out the angular momentum balance about point A, we see that the impulse at A produces no moment impulse about A, and therefore, the angular momentum about point A has to be conserved. That is, $\vec{H}_A{}^+ = \vec{H}_A{}^-$. Now,

$$\vec{H}_A{}^- = \vec{r}_{G/A} \times m\vec{v}_G{}^- = (-b\hat{\imath} + a\hat{\jmath}) \times mv\hat{\imath} = -mav\hat{k}$$

Let the box have angular velocity $\vec{\omega}^+ = \omega\hat{k}$ just after impact. Then,

$$\begin{aligned}\vec{H}_A{}^+ &= I_{zz}^{cm}\vec{\omega}^+ + \vec{r}_{G/A} \times m\vec{v}_G{}^+ = I_{zz}^{cm}\omega\hat{k} + r\hat{\lambda} \times m(\omega\hat{k} \times r\hat{\lambda})\\ &= I_{zz}^{cm}\omega\hat{k} + mr^2\omega\hat{k} = \frac{1}{12}(4a^2 + 4b^2)m\omega\hat{k} + m(a^2+b^2)\omega\hat{k}\\ &= \frac{4}{3}(a^2+b^2)m\omega\hat{k}.\end{aligned}$$

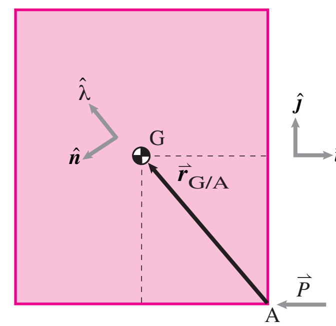


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Figure 14.81: The free-body diagram of the box during collision.

Now equating the two momenta, we get

$$\omega^+ = -\frac{3a}{4(a^2+b^2)}v \quad \Rightarrow \quad \vec{\omega}^+ = -\frac{3a}{4(a^2+b^2)}v\hat{k}. \tag{14.62}$$

Thus we know the angular velocity immediately after impact. Now let us find out if it is enough to get over the hill, so to speak. Once the impact is over (in a few milliseconds), the usual forces show up on the free-body diagram (see fig. 14.82).

In this post-collision part of the motion energy is conserved. Calling just after collision '+' and the top position 't' (where G is directly above A), using that $I_{zz}^A = \frac{4}{3}m(a^2+b^2)$ and solving for the critical condition, that the block has just enough energy to get to the top position, we have,

$$\begin{aligned}E_P{}^+ + E_K{}^+ &= E_K{}^t + E_P{}^t\\ E_P{}^t - E_P{}^+ &= -(E_K{}^t - E_K{}^+)\\ mg(\sqrt{(b^2+a^2)} - b) &= -\left(0 - \frac{1}{2}(\omega^+)^2 I_{zz}^A\right)\\ (\omega^+)^2 &= \frac{3}{2}g\frac{\left(1 - \frac{b}{\sqrt{(b^2+a^2)}}\right)}{\sqrt{(b^2+a^2)}}.\end{aligned}$$

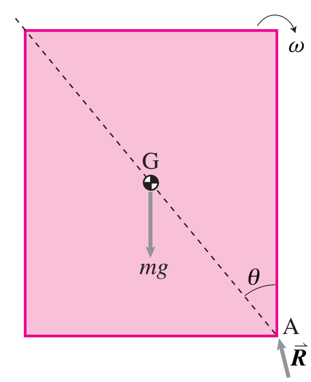

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Figure 14.82: The free-body diagram of the box just after the collision is over.

This tells us how big ω^+ has to be. Substituting this ω^+ into eqn. (14.62) we find how big v has to be. We find that the box does not tip over so long as v is small enough, as given by

$$v \le \frac{2}{a}\sqrt{2g(b^2+a^2)^{3/2}\left(1 - \frac{b}{\sqrt{(b^2+a^2)}}\right)/3}.$$

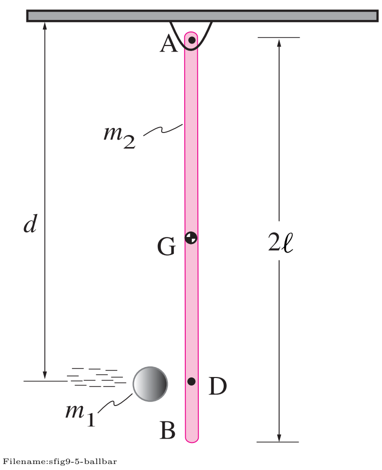


Figure 14.83:

SAMPLE 14.31 **Ball hits the bat.** A uniform bar of mass $m_2 = 1\,\text{kg}$ and length $2\ell = 1\,\text{m}$ hangs vertically from a hinge at A. A ball of mass $m_1 = 0.25\,\text{kg}$ comes and hits the bar horizontally at point D with speed $v = 5\,\text{m/s}$. The point of impact D is located at $d = 0.75\,\text{m}$ from the hinge point A. Assume that the collision between the ball and the bar is plastic.

1. Find the velocity of point D on the bar immediately after impact.
2. Find the impulse on the bar at D due to the impact.
3. Find and plot the impulsive reaction at the hinge point A as a function of d, the distance of the point of impact from the hinge point. What is the value of d which makes the impulse at A to be zero?

Solution The free-body diagram of the ball and the bar as a single system is shown in fig. 14.84 during impact. There is only one external impulsive force $\vec{F}_A$ acting at the hinge point A. We take the ball and the bar together here so that the impulsive force acting between the ball and the bar becomes internal to the system and we are left with only one external force at A. Then, the angular momentum balance about point A yields $\dot{\vec{H}}_A = \vec{0}$ since there is no net moment about A. Thus the angular momentum about A is conserved during the impact.

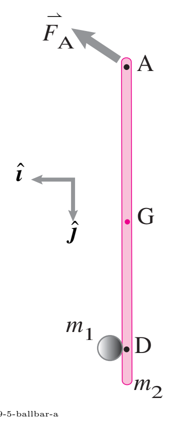


Figure 14.84: The free-body diagram of the ball and the bar together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram.

1. Let us distinguish the kinematic quantities just before impact and immediately after impact with superscripts '-' and '+', respectively. Then, from the conservation of angular momentum about point A, we get $\vec{H}_A^- = \vec{H}_A^+$. Now,

$$\begin{aligned}
\vec{H}_A^- &= (\vec{H}_A^-)_{\text{ball}} + (\vec{H}_A^-)_{\text{bar}} \\
&= \vec{r}_{D/A} \times m_1 \vec{v}^- + I_{zz}^A \vec{\omega}^- \\
&= d\,\hat{\jmath} \times m_1 v(-\hat{\imath}) + \vec{0} = m_1 d v \hat{k}.
\end{aligned}$$

Similarly,

$$\vec{H}_A^+ = \vec{r}_{D/A} \times m_1 \vec{v}^+ + I_{zz}^A \vec{\omega}^+$$

but, $\vec{v}^+ = \vec{\omega}^+ \times \vec{r}_{D/A} = -\omega d\,\hat{\imath}$, where $\vec{\omega}^+ = \omega \hat{k}$ (let). Hence,

$$\begin{aligned}
\vec{H}_A^+ &= d\,\hat{\jmath} \times m_1(-\omega d\,\hat{\imath}) + \frac{1}{3} m_2 (2\ell)^2 \omega \hat{k} \\
&= (m_1 d^2 + \frac{4}{3} m_2 \ell^2)\omega \hat{k}.
\end{aligned}$$

Equating the two momenta, we get

$$\begin{aligned}
\omega &= \frac{m_1 d v}{m_1 d^2 + (4/3) m_2 \ell^2} \\
&= \frac{v}{d\left(1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2\right)} \\
\Rightarrow \quad \vec{v}_D &= \vec{\omega}^+ \times \vec{r}_{D/A} = \omega d(-\hat{\imath}) \\
&= -\frac{v}{1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\imath}.
\end{aligned}$$

Now, substituting the given numerical values, $v = 5\,\text{m/s}$, $m_1 = 0.25\,\text{kg}$, $m_2 = 1\,\text{kg}$, $\ell = 0.5\,\text{m}$, and $d = 0.75\,\text{m}$, we get $\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}$

$$\boxed{\vec{v}_D = -2.08\,\text{m/s}\hat{\imath}}$$

2. To find the impulse at D due to the impact, we can consider either the ball or the bar separately, and find the impulse by evaluating the change in the linear momentum of the body. Let us consider the ball since it has only one impulse acting on it. The

free-body diagram of the ball during impact is shown in fig. 14.85. From the linear impulse-momentum relationship we get,

$$\begin{aligned}\vec{P}_D = \int \vec{F}_D \,dt &= \vec{L}^+ - \vec{L}^- = m_1(\vec{v}^+ - \vec{v}^-) \\ &= m_1 \left(-\frac{v}{1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2}\hat{\imath} + v\hat{\imath} \right) \\ &= m_1 v \left(1 - \frac{1}{1 + \frac{4}{3}\frac{m_2}{m_1}\left(\frac{\ell}{d}\right)^2} \right)\hat{\imath}.\end{aligned}$$

Substituting the given numerical values, we get $\vec{P}_D = 0.73\,\mathrm{kg{\cdot}m/s}\hat{\imath}$. The impulse on the bar is equal and opposite. Therefore, the impulse on the bar is $-\vec{P}_D = -0.73\,\mathrm{kg{\cdot}m/s}\hat{\imath}$.

$$\boxed{\text{Impulse at D} = -0.73\,\mathrm{kg{\cdot}m/s}\hat{\imath}}$$

3. Now that we know the impulse at D, we can easily find the impulse at A by applying impulse-momentum relationship to the bar. Since, the bar is stationary just before impact, its initial momentum is zero. Thus, for the bar,

$$\int (\vec{F}_A - \vec{F}_D)dt = \vec{L}^+ - \vec{L}^- = \vec{L}^+ = m_2 \vec{v}^+_{\mathrm{cm}}.$$

Denoting the impulse at A with $\vec{P}_A$, the mass ratio m_2/m_1 by m_r, and the length ratio ℓ/d byh q, and noting that $\vec{v}^+_{\mathrm{cm}} = \omega\hat{k} \times \ell\hat{\jmath} = -\omega\ell\hat{\imath}$, we get

$$\begin{aligned}\vec{P}_A &\equiv \int \vec{F}_A \,dt = \int \vec{F}_D \,dt + m_2(-\omega\ell\hat{\imath}) \\ &= m_1 v \left(1 - \frac{1}{1 + \frac{4}{3}m_r q^2} \right)\hat{\imath} - m_2 \ell \frac{v}{d\left(1 + \frac{4}{3}m_r q^2\right)}\hat{\imath} \\ &= m_1 v \left(\frac{\frac{4}{3}m_r q^2}{1 + \frac{4}{3}m_r q^2} \right)\hat{\imath} - m_2 v \left(\frac{q}{1 + \frac{4}{3}m_r q^2} \right)\hat{\imath} \\ &= \frac{(4/3)m_2 q^2 - m_2 q}{1 + \frac{4}{3}m_r q^2} v\hat{\imath} = \frac{q(4q-3)}{3\left(1 + \frac{4}{3}m_r q^2\right)} m_2 v\hat{\imath}.\end{aligned}$$

Now, we are ready to graph the impulse at A as a function of $q \equiv \ell/d$. However, note that a better quantity to graph will be $P_A/(m_1 v)$, that is, the nondimensional impulse at A, normalized with respect to the initial linear momentum $m_1 v$ of the ball. The plot is shown in fig. 14.86. It is clear from the plot, as well as from the expression for $\vec{P}_A$, that the impulse at A is zero when $q = 3/4$ or $d = 4\ell/3 = 2/3(2\ell)$, that is, when the ball strikes at two thirds the length of the bar. Note that this location of the impact point is independent of the mass ratio m_r.

$$\boxed{d = 2/3(2\ell) \text{ for } \vec{P}_A = \vec{0}}$$

Comment: This particular point of impact D (when $d = 2/3(2\ell)$) which induces no impulse at the support point A is called the *center of percussion*. If you imagine the bar to be a bat or a racquet and point A to be the location of your grip, then hitting a ball at D gives you an impulse-free shot. In sports, point D is called a *sweet spot*.

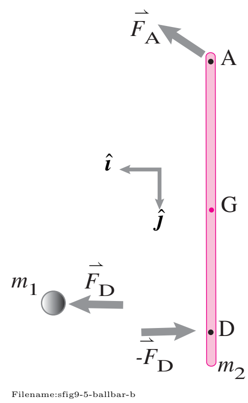


Figure 14.85: Separate free-body diagrams of the ball and the bar during collision.

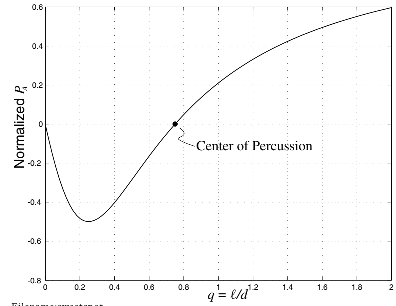
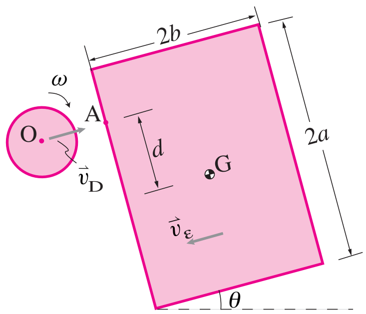


Figure 14.86: Plot of normalized impulse at A as a function of $q = \ell/d$.



Filename:sfig9-5-diskplate

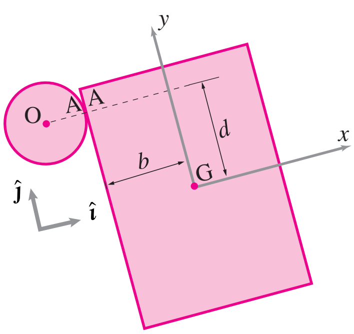
Figure 14.87:

SAMPLE 14.32 Flying dish and the solar panel. A uniform rectangular plate of dimensions $2a = 2\,\mathrm{m}$ and $2b = 1\,\mathrm{m}$ and mass $m_P = 2\,\mathrm{kg}$ drifts in space at a uniform speed $v_p = 10\,\mathrm{m/s}$ (in a local Newtonian reference frame) in the direction shown in the figure. Another circular disk of radius $R = 0.25\,\mathrm{m}$ and mass $m_D = 1\,\mathrm{kg}$ is heading towards the plate at a linear speed $v_D = 1\,\mathrm{m/s}$ directed normal to the facing edge of the plate. In addition, the disk is spinning at $\omega_D = 5\,\mathrm{rad/s}$ in the clockwise direction. The plate and the disk collide at point A of the plate, located at $d = 0.8\,\mathrm{m}$ from the center of the long edge. Assume that the collision is frictionless and purely elastic. Find the linear and angular velocities of the plate and the disk immediately after the collision.

Solution To find the linear as well as the angular velocities of the disk and the plate, we will have to use linear and angular momentum-impulse relations. In total, we have 7 scalar unknowns here — 4 for linear velocities of the disk and the plate (each velocity has two components), 2 for the two angular velocities, and 1 for the collision impulse. Naturally, we need 7 independent equations. We have 6 independent equations from the linear and angular impulse-momentum balance for the two bodies (3 each). We need one more equation. That equation is the relationship between the normal components of the relative velocities of approach and departure with the coefficient of restitution e (=1 for elastic collision). Thus we have enough equations. Let us set up all the required equations. We can then solve the equations using a computer.

The free-body diagrams of the disk and the plate together and the two separately are shown in fig. 14.88 and 14.89, respectively. Using an xy coordinate system oriented as shown in fig. 14.88, we can write

$$
\begin{array}{lrcl}
\text{LMB for disk:} & m_D(\vec{v}_D^{\,+} - \vec{v}_D^{\,-}) & = & -P\hat{\imath} \\
\text{LMB for plate:} & m_P(\vec{v}_P^{\,+} - \vec{v}_P^{\,-}) & = & P\hat{\imath} \\
\text{AMB for disk:} & I_D^{\mathrm{cm}}(\vec{\omega}_D^{\,+} - \vec{\omega}_D^{\,-}) & = & \vec{0} \\
\text{AMB for plate:} & I_P^{\mathrm{cm}}(\vec{\omega}_P^{\,+} - \vec{\omega}_P^{\,-}) & = & \vec{r}_{A/G} \times P\hat{\imath} \\
\text{kinematics:} & \hat{\imath}\cdot\{\vec{v}_{A_D}^{\,+} - \vec{v}_{A_P}^{\,+} & = & e(\vec{v}_{A_P}^{\,-} - \vec{v}_{A_D}^{\,-})\}
\end{array}
$$



Filename:sfig9-5-diskplate-a

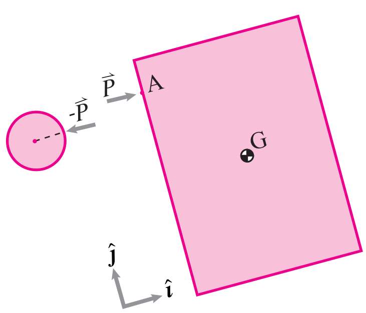
Figure 14.88: The free-body diagram of the disk and the plate together during collision. The impulsive force at the point of impact is internal to the system and hence, does not show on the free-body diagram .

where, in the last equation $\vec{v}_{A_D}$ and $\vec{v}_{A_P}$ refer to the velocities of the material points located at A on the disk and on the plate, respectively. Other linear velocities in the equations above refer to the velocities at the center-of-mass of the corresponding bodies. We are given that $\vec{v}_D^{\,-} = v_D\hat{\imath}$, $\vec{v}_P^{\,-} = -v_P\hat{\imath}$, $\vec{\omega}_D^{\,-} = -\Omega_D\hat{k}$, and $\vec{\omega}_P^{\,-} = \vec{0}$. Let us assume that $\vec{\omega}_D^{\,+} = \omega_D\hat{k}$, $\vec{\omega}_P^{\,+} = \omega_P\hat{k}$, $v v_D^+ = v_{D_x}^+\hat{\imath} + v_{D_y}^+\hat{\jmath}$, and similarly, $v v_P^+ = v_{P_x}^+\hat{\imath} + v_{P_y}^+\hat{\jmath}$. Then,

$$
\begin{aligned}
\vec{v}_{A_D}^{\,-} &= \vec{v}_D^{\,-} + \vec{\omega}_D^{\,-} \times \vec{r}_{A/O} = v_D\hat{\imath} - \omega_D R\hat{\jmath} \\
\vec{v}_{A_D}^{\,+} &= \vec{v}_D^{\,+} + \vec{\omega}_D^{\,+} \times \vec{r}_{A/O} = v_{D_x}^+\hat{\imath} + (v_{D_y}^+ + \omega_D^+ R)\hat{\jmath} \\
\vec{v}_{A_P}^{\,-} &= \vec{v}_P^{\,-} = -v_P\hat{\imath} \\
\vec{v}_{A_P}^{\,+} &= \vec{v}_P^{\,+} + \vec{\omega}_P^{\,+} \times \vec{r}_{A/G} = (v_{P_x}^+ - \omega_P^+ d)\hat{\imath} + (v_{P_y}^+ - \omega_P^+ d)\hat{\jmath}.
\end{aligned}
$$

Substituting these quantities in the kinematics equation above and dotting with the normal direction at A, $\hat{\imath}$, we get

$$
v_{D_x}^+ - v_{P_x}^+ + \omega_P^+ d = \underbrace{e}_{1}(-v_P - v_D) = -v_P - v_D. \tag{14.63}
$$



Filename:sfig9-5-diskplate-b

Figure 14.89: Separate free-body diagrams of the disk and the plate during collision.

Now, let us extract the scalar equations from the impulse-momentum equations for the disk and the plate by dotting with appropriate unit vectors.

Dotting LMB for the disk with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_D(v_{D_x}^+ - v_D) = -P \tag{14.64}$$

$$m_D v_{D_y}^+ = 0. \tag{14.65}$$

Dotting LMB for the plate with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get

$$m_P(v_{P_x}^+ - v_P) = P \tag{14.66}$$

$$m_P v_{P_y}^+ = 0. \tag{14.67}$$

Dotting AMB for the disk and the plate with $\hat{k}$, we get

$$I_D^{cm}(\omega_D^+ - \omega_D) = 0 \tag{14.68}$$

$$I_P^{cm}\omega_P^+ = Pd. \tag{14.69}$$

We have all the equations we need. Let us rearrange these equations in a matrix form, taking the known quantities to the right and putting all unknowns to the left side. We then, write eqns. (14.64)–(14.69), and then eqn. (14.63) as

$$\begin{bmatrix} m_D & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & m_D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_P & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & m_P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_D^{cm} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_P^{cm} & -d \\ 1 & 0 & -1 & 0 & 0 & d & 0 \end{bmatrix} \begin{Bmatrix} v_{D_x}^+ \\ v_{D_y}^+ \\ v_{P_x}^+ \\ v_{P_y}^+ \\ \omega_D^+ \\ \omega_P^+ \\ P \end{Bmatrix} = \begin{Bmatrix} m_D v_D \\ 0 \\ m_P v_P \\ 0 \\ I_D^{cm}\omega_D \\ 0 \\ -v_P - v_D \end{Bmatrix}.$$

Substituting the given numerical values for the masses and the pre-collision velocities, and the moments of inertia, $I_D^{cm} = (1/2)m_D R^2$ and $I_P^{cm} = (1/12)m_P(4a^2 + 4b^2)$, and then solving the matrix equation on a computer, we get,

$$\begin{aligned} \vec{v}_D^+ &= 0.34\,\text{m/s}\hat{\imath}, \quad \vec{v}_P^+ = -9.67\,\text{m/s}\hat{\imath} \\ \vec{\omega}_D^+ &= -5\,\text{rad/s}\hat{k}, \quad \vec{\omega}_P^+ = -1.26\,\text{rad/s}\hat{k} \\ P &= -0.66\,\text{kg·m/s}. \end{aligned}$$

You can easily check that the results obtained satisfy the conservation of linear momentum for the plate and the disk taken together as one system.

$$\vec{v}_D^+ = 0.34\,\text{m/s}\hat{\imath},\ \vec{v}_P^+ = -9.67\,\text{m/s}\hat{\imath},\ \vec{\omega}_D^+ = -5\,\text{rad/s}\hat{k},\ \vec{\omega}_P^+ = -1.26\,\text{rad/s}\hat{k}$$

Comments: In this particular problem, the equations are simple enough to be solved by hand. For example, eqns. (14.65), (14.67), and (14.68) are trivial to solve and immediately give, $v_{D_y}^+ = 0$, $v_{P_y}^+ = 0$, and $\omega_D^+ = \omega^D = 5\,\text{rad/s}$. Rest of the equations can be solved by usual eliminations and substitutions, etc. However, it is important to learn how to set up these equations in matrix form so that no matter how complicated the equations are, they can be easily solved on a computer. What really counts is do you have 7 linear independent equations for the 7 unknowns. If you do, you are home.

Problems for Chapter 14

Planar motion of an object

14.1 Rigid object kinematics

Preparatory Problems

14.1.1 A disk of radius R is hinged at point O at the edge of the disk, approximately as shown. It rotates counterclockwise with angular velocity $\dot{\theta} = \vec{\omega}$. A bolt is fixed on the disk at point P at a distance r from the center of the disk. A frame $x'y'$ is fixed to the disk with its origin at the center C of the disk. The bolt position P makes an angle ϕ with the x'-axis. At the instant of interest, the disk has rotated by an angle θ.

a) Write the position vector of point P relative to C in the $x'y'$ coordinates in terms of given quantities.

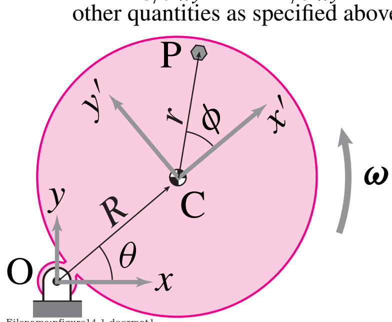
b) Write the position vector of point P relative to O in the xy coordinates in terms of given quantities.

c) Write the expressions for the rotation matrix $R(\theta)$ and the angular velocity matrix $\mathcal{S}(\vec{\omega})$.

d) Find the velocity of point P relative to C using $R(\theta)$ and the angular velocity matrix $\mathcal{S}(\vec{\omega})$.

e) Using $R = 30\,\mathrm{cm}$, $r = 25\,\mathrm{cm}$, $\theta = 60°$, and $\phi = 45°$, find $[\vec{r}_{C/0}]_{xy}$, and $[\vec{r}_{P/0}]_{xy}$ at the instant shown.

f) Assuming that the angular speed is $\omega = 10\,\mathrm{rad/s}$ at the instant shown, find $[\vec{v}_{C/0}]_{xy}$ and $[\vec{v}_{P/0}]_{xy}$ taking other quantities as specified above.



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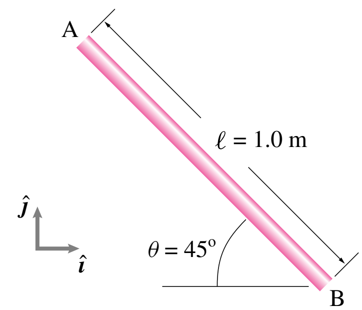
Problem 14.1.1

14.1.2 A uniform rigid rod AB of length $\ell = 1\,\mathrm{m}$ rotates at a constant angular speed ω about an unknown fixed point. At the instant shown, the velocities of the two ends of the rod are $\vec{v}_A = -1\,\mathrm{m/s}\hat{\imath}$ and $\vec{v}_B = 1\,\mathrm{m/s}\hat{\jmath}$.

a) Find the angular velocity of the rod.

b) Find the center of rotation of the rod.



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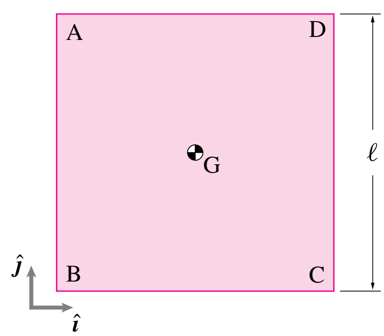
Problem 14.1.2

14.1.3 A square plate ABCD rotates at a constant angular speed about an unknown point in its plane. At the instant shown, the velocities of the two corner points A and D are $\vec{v}_A = -2\,\mathrm{ft/s}(\hat{\imath} + \hat{\jmath})$ and $\vec{v}_D = -(2\,\mathrm{ft/s})\hat{\imath}$, respectively.

a) Find the center of rotation of the plate.

b) Find the acceleration of the center of mass of the plate.



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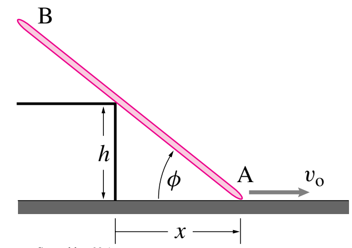
Problem 14.1.3

More-Involved Problems

14.1.4 A ten foot ladder is leaning between a floor and a wall. The top of the ladder is sliding down the wall at one foot per second. (The foot is simultaneously sliding out on the floor). When the ladder makes a 45 degree angle with the vertical what is the speed of the midpoint of the ladder?

14.1.5 The slender rod AB rests against the step of height h, while end "A" is moved along the ground at a constant velocity v_o. Find $\dot{\phi}$ and $\ddot{\phi}$ in terms of x, h, and v_o. Is $\dot{\phi}$ positive or negative? Is $\ddot{\phi}$ positive or negative?



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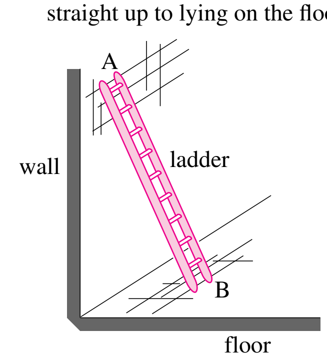
Problem 14.1.5

14.1.6 Consider the motion of a rigid ladder which can slide on a wall and on the floor as shown in the figure. The point A on the ladder moves parallel to the wall. The point B moves parallel to the floor. Yet, at a given instant, both have velocities that are consistent with the ladder rotating about some special point, the center of rotation (COR). Define appropriate dimensions for the problem.

a) Find the COR for the ladder when it is at some given position (and moving, of course). [Hint: if a point A is 'going in circles' about another point C, that other point C must be in the direction perpendicular to the motion of A.

b) As the ladder moves, the COR changes with time. What is the set of points on the plane that are the COR's for the ladder as it falls from straight up to lying on the floor?



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Problem 14.1.6

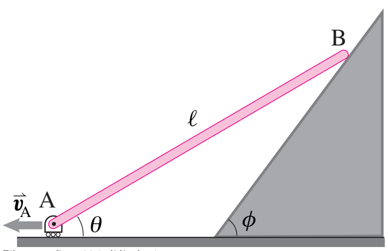
14.1.7 A car driver on a very boring highway is carefully monitoring her speed. Over a one hour period, the car travels on a curve with constant radius of curvature, $\rho = 25\,\mathrm{mi}$, and its speed increases uniformly from 50 mph to 60 mph. What is the acceleration of the of the car half-way through this one hour period, in path coordinates?

14.1.8 The bar AB shown in the figure is 1 m long. The end A of the bar is dragged to the left on the horizontal floor at a constant speed $v_0 = 0.2\,\text{m/s}$. The other end of the bar drags along the inclined plane that makes an angle $\phi = 60°$ with the horizontal.

a) Does the end B of the rod also have a constant speed? Does it imply that the rod has a constant angular velocity? Guess your answer and write down your reasoning. No calculations.

b) Find an expression for the angular velocity of the rod in terms of θ, ϕ, ℓ, and v_0. *

c) Find the velocity of end B when $\theta = 30°$.



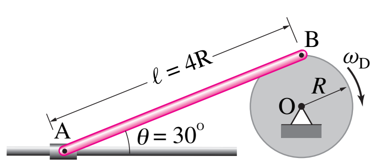
Filename:pfigure14-1-slidingbar1
Problem 14.1.8

14.1.9 The disk shown in the figure drives the slider AB. The disk rotates clockwise at a constant angular speed $\omega_D = 60\,\text{rpm}$. The radius of the disk is $R = 0.25\,\text{m}$.

a) Find the velocity of collar A when $\theta = 30°$.

b) Find the acceleration of collar A when $\theta = 30°$.

c) Find the maximum and minimum velocity of the collar and the corresponding angle θ of the slider.



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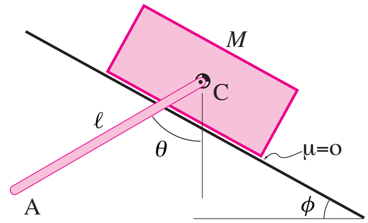
Problem 14.1.9

14.1.10 A rod AC of length $\ell = 2\,\text{m}$ is mounted on a cart at point C. The cart slides down under gravity on a frictionless surface inclined at $\phi = 45°$ from the horizontal. Assume that bar has negligible mass compared to that of the cart. The cart is released from rest and it reaches the position shown in $t = 2\,\text{s}$. At the instant shown, $\theta = 60°$, and rod AC has an angular velocity $\vec{\omega} = 0.5\,\text{rad/s}\hat{\boldsymbol{k}}$ and angular acceleration $\vec{\alpha} = 1\,\text{rad/s}^2\hat{\boldsymbol{k}}$. Find the velocity and acceleration of the center of mass of the rod at the instant shown.

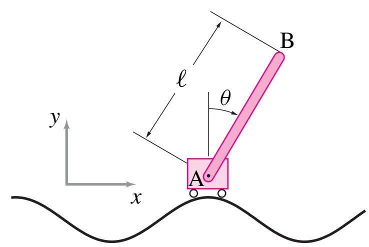


Filename:pfigure14-1-slidingpend
Problem 14.1.10

14.1.11 A vertical bar pendulum AB of length $\ell = 4\,\text{ft}$ is mounted on a cart at point A. The cart has a motor that drives the bar pendulum back and forth about the vertical position such that $\theta(t) = \theta_0 \sin\omega t$ where $\theta_0 = \pi/6$ and $\omega = \pi\,\text{rad/s}$. The cart moves at a constant speed $v_0 = 2\,\text{ft/s}$ on a sinusoidal track represented by $y(x) = y_0 \sin\lambda x$ where $y_0 = 0.5\,\text{ft}$ and $\lambda = \pi/4/\,\text{ft}$.

a) Find the velocity of point B when $\theta = 0$.

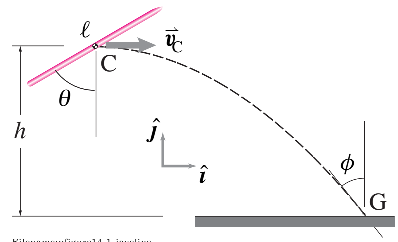
b) Find and plot the position of point B for one full oscillation of the bar, assuming the bar starts from its leftmost angular position (that is, $theta = -\theta_0$) when the cart is on one of the crests of the track.



Filename:pfigure14-1-vertpend1
Problem 14.1.11

14.1.12 The center of mass of a javelin travels on a more or less parabolic path while the javelin rotates during its flight. In a particular throw, the velocity of the center of mass of a javelin is measured to be $\vec{v}_C = 10\,\text{m/s}\hat{\imath}$ when the center of mass is at its highest point $h = 6\,\text{m}$. As the javelin lands on the ground, its nose hits the ground at G such that the javelin is almost tangent to the path of the center of mass at G. Neglect the air drag and lift on the javelin.

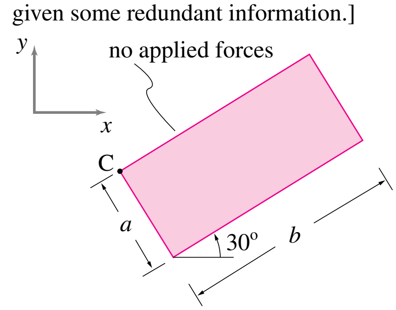
a) Given that the javelin is at an angle $\theta = 45°$ at the highest point, find the angular velocity of the javelin. Assume the angular velocity isconstant during the flight and that the javelin makes less than a full revolution.



Filename:pfigure14-1-javeline
Problem 14.1.12

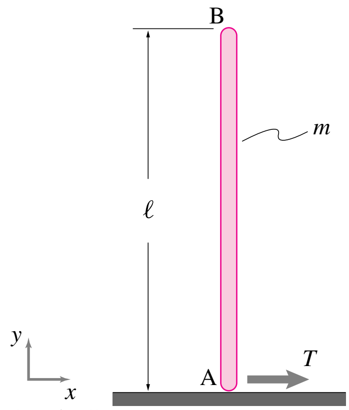
14.2 Dynamics of a rigid object

14.2.1 The uniform rectangle of width $a = 1\,\text{m}$, length $b = 2\,\text{m}$, and mass $m = 1\,\text{kg}$ in the figure is sliding on the xy-plane with no friction. At the moment in question, point C is at $x_C = 3\,\text{m}$ and $y_C = 2\,\text{m}$. The linear momentum is $\vec{\boldsymbol{L}} = 4\hat{\imath} + 3\hat{\jmath}\ (\text{kg·m/s})$ and the angular momentum about the center of mass is $\vec{\boldsymbol{H}}_{cm} = 5\hat{\boldsymbol{k}}(\text{kg·m/s}^2)$. Find the acceleration of any point on the body that you choose. (Mark it.) [Hint: You have been given some redundant information.]



Filename:pfigure-blue-101-1
Problem 14.2.1

14.2.2 The vertical pole AB of mass m and length ℓ is initially, *at rest* on a frictionless surface. A tension T is suddenly applied at A. What is $\ddot{x}_{cm}$? What is $\ddot{\theta}_{AB}$? What is $\ddot{x}_B$? Gravity may be ignored.



Filename:pfigure-blue-95-1
Problem 14.2.2

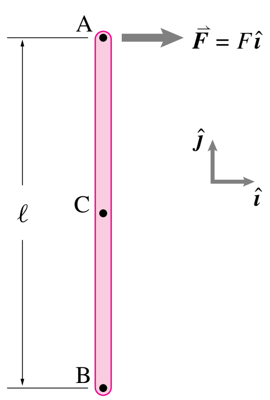
14.2.3 Force on a stick in space. 2-D . No gravity. A uniform thin stick with length ℓ and mass m is, at the instant of interest, parallel to the y axis and has no velocity and no angular velocity. The force $\vec{F} = F\hat{\imath}$ with $F > 0$ is suddenly applied at point A. The questions below concern the instant after the force $\vec{F}$ is applied.

a) What is the acceleration of point C, the center of mass? *

b) What is the angular acceleration of the stick? *

c) What is the acceleration of the point A? *

d) (relatively harder) What additional force would have to be applied to point B to make point B's acceleration zero? *



Filename:pfigure-s94h10p2
Problem 14.2.3

14.2.4 A uniform thin rod of length ℓ and mass m stands vertically, with one end resting on a frictionless surface and the other held by someone's hand. The rod is released from rest, displaced slightly from the vertical. No forces are applied during the release. There is gravity.

a) Find the path of the center of mass.

b) Find the force of the floor on the end of the rod just before the rod is horizontal.

14.2.5 A uniform disk, with mass center labeled as point G, is sitting motionless on the frictionless xy plane. A massless peg is attached to a point on its perimeter. This disk has radius of 1 m and mass of 10 kg. A constant force of $F = 1000\,\mathrm{N}\hat{\imath}$ is applied to the peg for .0001 s (one ten-thousandth of a second).

a) What is the velocity of the center of mass of the center of the disk after the force is applied?

b) Assuming that the idealizations named in the problem statement are exact is your answer to (a) exact or approximate?

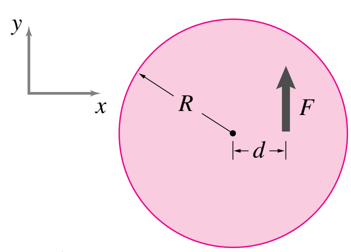
c) What is the angular velocity of the disk after the force is applied?

d) Assuming that the idealizations named in the problem statement are exact is your answer to (c) exact or approximate?

14.2.6 A uniform thin flat disc is floating in space. It has radius R and mass m. A force F is applied to it at a distance d from the center in the y direction. Treat this problem as two-dimensional.

a) What is the acceleration of the center of the disc? *

b) What is the angular acceleration of the disk? *



Filename:pfigure-blue-102-1
Problem 14.2.6

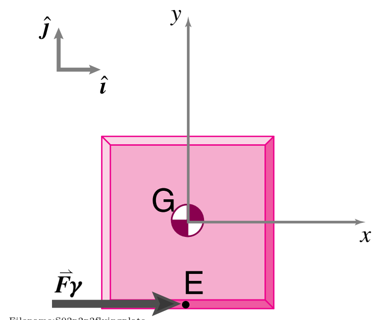
14.2.7 A uniform 1kg plate that is one meter on a side is initially at rest in the position shown. A constant force $\vec{F} = 1\,\mathrm{N}\hat{\imath}$ is applied at $t = 0$ and maintained henceforth. If you need to calculate any quantity that you don't know, but can't do the calculation to find it, assume that the value is given.

a) Find the position of G as a function of time (the answer should have numbers and units).

b) Find a differential equation, and initial conditions, that when solved would give θ as a function of time. θ is the counterclockwise rotation of the plate from the configuration shown.

c) Write computer commands that would generate a drawing of the outline of the plate at $t = 1$ s. You can use hand calculations or the computer for as many of the intermediate commands as you like. Hand work and sketches should be provided as needed to justify or explain the computer work.

d) Run your code and show clear output with labeled plots. Mark output by hand to clarify any points.

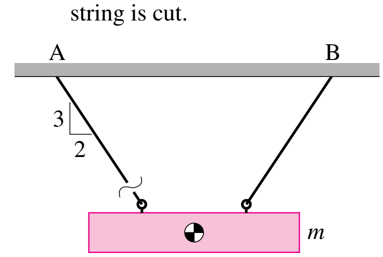


Filename:S02p2p2flyingplate
Problem 14.2.7

14.2.8 A uniform rectangular metal beam of mass m hangs symmetrically by two strings as shown in the figure.

a) Draw a free-body diagram of the beam and evaluate $\sum \vec{F}$.

b) Repeat (a) immediately after the left string is cut.



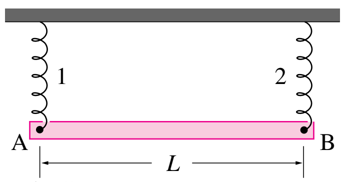
Filename:pfig2-2-rp8
Problem 14.2.8

14.2.9 A uniform slender bar AB of mass m is suspended from two springs (each of spring constant K) as shown. Immediately after spring 2 breaks, determine

a) the angular acceleration of the bar,

b) the acceleration of point A, and

c) the acceleration of point B.



Filename:pfigure-blue-50-2
Problem 14.2.9

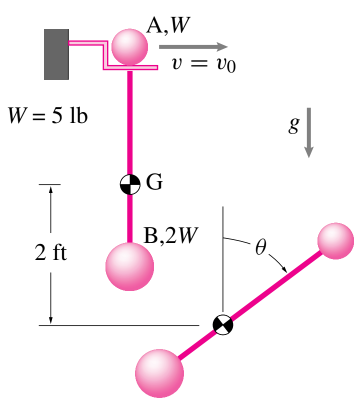
14.2.10 Two small spheres A and B are connected by a rigid rod of length $\ell = 1.0$ ft and negligible mass. The assembly is hung from a hook, as shown. Sphere A is struck, suddenly breaking its contact with the hook and giving it a horizontal velocity $v_0 = 3.0$ ft/s which sends the assembly into free fall. Determine the angular momentum of the assembly about its mass center at point G immediately after A is hit. After the center of mass has fallen two feet, determine:

a) the angle θ through which the rod has rotated,

b) the velocity of sphere A,

c) the total kinetic energy of the assembly of spheres A and B and the rod, and

d) the acceleration of sphere A.



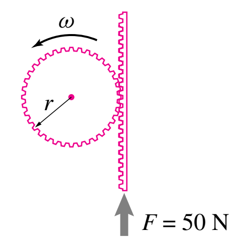
Filename:pfigure-blue-105-1
Problem 14.2.10

The next several problems concern Work, power and energy **14.2.11** Verify that the expressions for work done by a force F, $W = F\Delta S$, and by a moment M, $W = M\Delta\theta$, are dimensionally correct if ΔS and $\Delta\theta$ are linear and angular displacements respectively.

14.2.12 A uniform disc of mass m and radius r rotates with angular velocity $\omega\hat{\boldsymbol{k}}$. Its center of mass translates with velocity $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$ in the xy-plane. What is the total kinetic energy of the disk?

14.2.13 Calculate the energy stored in a spring using the expression $E_P = \frac{1}{2}k\delta^2$ if the spring is compressed by 100 mm and the spring constant is 100 N/m.

14.2.14 In a rack and pinion system, the rack is acted upon by a constant force $F = 50$ N and has speed $v = 2$ m/s in the direction of the force. Find the input power to the system.

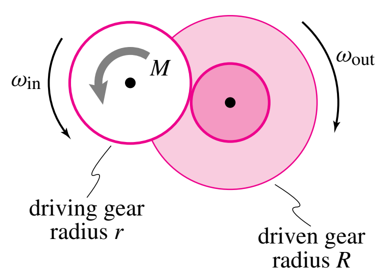


Filename:pfig2-3-rp6
Problem 14.2.14

14.2.15 The driving gear in a compound gear train rotates at constant speed ω_0. The driving torque is M_{in}. If the driven gear rotates at a constant speed ω_{out}, find:

a) the input power to the system, and

b) the output torque of the system assuming there is no power loss in the system; i.e., power in = power out.

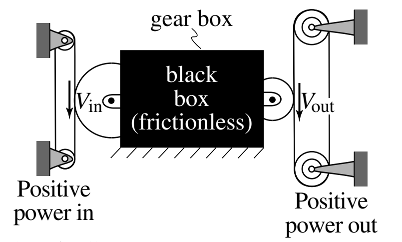


Filename:pfig2-3-rp7
Problem 14.2.15

14.2.16 An elaborate frictionless gear box has an input and output roller with $V_{in} = const$. Assuming that $V_{out} = 7V_{in}$ and the force between the left belt and roller is $F_{in} = 3$ lb:

a) What is F_{out} (draw a picture defining the signs of F_{in} and F_{out})?. *

b) Is F_{out} greater or less than the F_{in}? (Assume $F_{in} > 0$.) Why? *



Filename:pfigure-blue-138-1
Problem 14.2.16

14.3 Kinematics of rolling and sliding

14.3.1 A stone in a wheel. A round wheel rolls to the right. At time $t = 0$ it picks up a stone the road. The stone is stuck in the edge of the wheel. You want to know the direction of the rock's motion just before and after it next hits the ground. Here are some candidate answers:

- When the stone approaches the ground its motion is tangent to the ground.
- The stone approaches the ground at angle x (you name it).
- When the stone approaches the ground its motion is perpendicular to the ground.
- The stone approaches the ground at various angles depending on the following conditions(...you list the conditions.)

Although you could address this question analytically, you are to try to get a clear answer by looking at computer generated plots. In particular, you are to plot the pebble's path for a small interval of time near when the stone next touches the ground. You should pick the parameters that make your case for an answer the strongest. You may make more than one plot.

Here are some steps to follow:

a) Assuming the wheel has radius R_w and the pebble is a distance R_p from the center (not necessarily equal to R_w). The pebble is directly below the center of the wheel at time $t = 0$. The wheel spins at constant clockwise rate ω. The x-axis is on the ground and $x(t = 0) = 0$. The wheel rolls without slipping. Using a clear well labeled drawing (use a compass and ruler or a computer drawing program), show that

$$x(t) = \omega t R_w - R_p \sin(\omega t)$$

$$y(t) = R_w - R_p \cos(\omega t)$$

b) Using this relation, write a program to make a plot of the path of the pebble as the wheel makes a little more than one revolution. Also show the outline of the wheel and the pebble itself at some intermediate time of interest. [Use any software and computer that pleases you.]

c) Change whatever you need to change to make a good plot of the pebble's path for a small amount of time as the pebble approaches and leaves the road. Also show the wheel and the pebble at some time in this interval.

d) In this configuration the pebble moves a *very* small distance in a small time so your axes need to be scaled. But make sure your x- and y- axes have the same scale so that the path of the pebble and the outline of the wheel will not be distorted.

e) How does your computer output buttress your claim that the pebble approaches and leaves the ground at the angles you claim?

f) Think of something about the pebble in the wheel that was not explicitly asked in this problem and explain it using the computer, and/or hand calculation and/or a drawing.

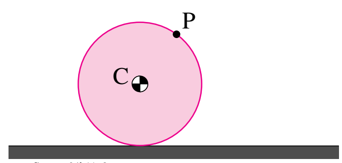
14.3.2 A uniform disk of radius r rolls at a constant rate without slip. A small ball of mass m is attached to the outside edge of the disk. What is the force required to hold the disk in place when the mass is above the center of the disk?

14.3.3 Rolling at constant rate. A round disk rolls on the ground at constant rate. It rolls $1\frac{1}{4}$ revolutions over the time of interest.

a) **Particle paths.** Accurately plot the paths of three points: the center of the disk C, a point on the outer edge that is initially on the ground, and a point that is initially half way between the former two points. [Hint: Write a parametric equation for the position of the points. First find a relation between ω and v_C. Then note that the position of a point is the position of the center plus the position of the point relative to the center.] Draw the paths on the computer, make sure x and y scales are the same.

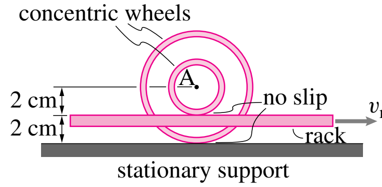
b) **Velocity of points.** Find the velocity of the points at a few instants in the motion: after $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 revolution. Draw the velocity vector (by hand) on your plot. Draw the direction accurately and draw the lengths of the vectors in proportion to their magnitude. You can find the velocity by differentiating the position vector or by using relative motion formulas appropriately. Draw the disk at its position after one quarter revolution. Note that the velocity of the points is perpendicular to the line connecting the points to the ground contact.

c) **Acceleration of points.** Do the same as above but for acceleration. Note that the acceleration of the points is parallel to the line connecting the points to the center of the disk.



Filename:pfigure-s94h11p2
Problem 14.3.3

14.3.4 The concentric wheels are welded to each other and roll without slip on the rack and stationary support. The rack moves to the right at $v_r = 1\,\text{m/s}$. What is the velocity of point A in the middle of the wheels shown?



Filename:pfigure-blue-118-1
Problem 14.3.4

14.3.5 Questions (a) - (e) refer to the cylinders in the configuration shown figure. Question (f) is closely related. Answer the questions in terms of the given quantities (and any other quantities you define if needed).

a) What is the speed (magnitude of velocity) of point c?

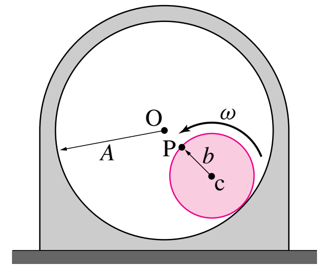
b) What is the speed of point P?

c) What is the magnitude of the acceleration of point c?

d) What is the magnitude of the acceleration of point P?

e) What is the radius of curvature of the path of the particle P at the point of interest?

f) In the special case of $A = 2b$ what is the curve which particle P traces (for all time)? Sketch the path.



Filename:pfigure-blue-50-1
Problem 14.3.5: A little cylinder (with outer radius b and center at point c) rolls without slipping inside a bigger fixed cylinder (with inner radius A and center at point O). The *absolute* angular velocity of the little cylinder ω is *constant*. P is attached to the outside edge of the little cylinder. At the instant of interest, P is on the line between O and c.

14.4 Mechanics of contact

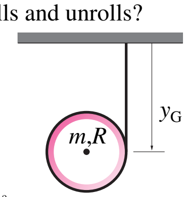
14.4.1 A uniform disc of mass m and radius r rolls without slip at constant rate. What is the total kinetic energy of the disk?

14.4.2 A non-uniform disc of mass m and radius r rolls without slip at constant rate. The center of mass is located at a distance $\frac{r}{2}$ from the center of the disc. What is the total kinetic energy of the disc when the center of mass is directly above the center of the disc?

14.4.3 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R. G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

a) Find y_G at a later time t in terms of any or all of m, R, g, and t.

b) Does G move sideways as the hoop falls and unrolls?

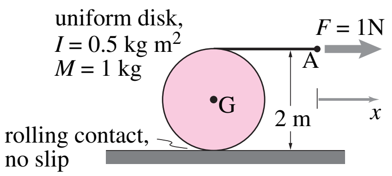


Filename:p-s96-p3-2
Problem 14.4.3

14.4.4 A uniform disk with radius R and mass m has a string wrapped around it. The string is pulled with a force F. The disk rolls without slipping.

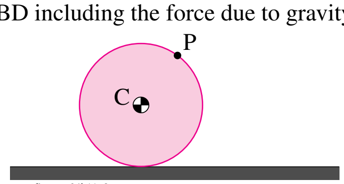
a) What is the angular acceleration of the disk, $\vec{\boldsymbol{\alpha}}_{Disk} = -\ddot{\theta}\hat{\boldsymbol{k}}$? Make any reasonable assumptions you need that are consistent with the figure information and the laws of mechanics. State your assumptions. *

b) Find the acceleration of the point A in the figure. *



Filename:pfigure-blue-43-1
Problem 14.4.4

14.4.5 If a pebble is stuck to the edge of the wheel in problem 14.3.3, what is the maximum speed of the pebble during the motion? When is the force on the pebble from the wheel maximum? Draw a good FBD including the force due to gravity.

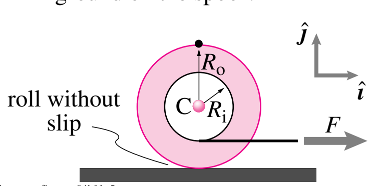


Filename:pfigure-s94h11p2-a
Problem 14.4.5

14.4.6 Spool Rolling without Slip and Pulled by a Cord. The light-weight spool is nearly empty but a lead ball with mass m has been placed at its center. A force F is applied in the horizontal direction to the cord wound around the wheel. Dimensions are as marked. Coordinate directions are as marked.

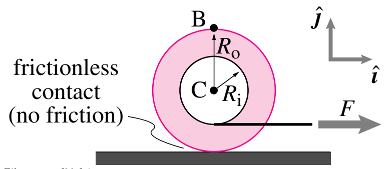
a) What is the acceleration of the center of the spool? *

b) What is the horizontal force of the ground on the spool? *



Filename:pfigure-s94h11p5
Problem 14.4.6

14.4.7 A film spool is placed on a very slippery table. Assume that the film and reel (together) have mass distributed the same as a uniform disk of radius R_i. What, in terms of R_i, R_o, m, g, $\hat{\imath}$, $\hat{\jmath}$, and F are the accelerations of points C and B at the instant shown (the start of motion)?



Filename:p-f96-f-4
Problem 14.4.7

14.4.8 Again, Spool Rolling without Slip and Pulled by a Cord. Reconsider the spool from problem 14.4.6. This time, a force F is applied vertically to the cord wound around the wheel. In this case, what is the acceleration of the center of the spool? Is it possible to pull the cord at some angle between horizontal and vertical so that the angular acceleration of the spool or the acceleration of the center of mass is zero? If so, find the angle in terms of R_i, R_0, m, and F.

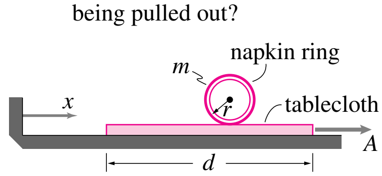
14.4.9 A napkin ring lies on a thick velvet tablecloth. The thin ring (of mass m, radius r) rolls without slip as a mischievous child pulls the tablecloth (mass M) out with acceleration A. The ring starts at the right end ($x = d$). You can make a reasonable physical model of this situation with an empty soda can and a piece of paper on a flat table.

a) What is the ring's acceleration as the tablecloth is being withdrawn?

b) How far has the tablecloth moved to the right from its starting point $x = 0$ when the ring rolls off its left-hand end?

c) Clearly describe the subsequent motion of the ring. Which way does it end up rolling at what speed?

d) Would your answer to the previous question be different if the ring slipped on the cloth as the cloth was being pulled out?



Filename:pfigure-blue-51-1
Problem 14.4.9

14.4.10 A block of mass M is supported by two rollers (uniform cylinders) each of mass m and radius r. They roll without slip on the block and the ground. A force F is applied in the horizontal direction to the right, as shown in the figure. Given F, m, r, and M, find:

a) the acceleration of the block,

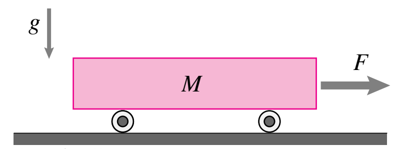
b) the acceleration of the center of mass of this block/roller system,

c) the reaction at the wheel bases,

d) the force of the right wheel on the block,

e) the acceleration of the wheel centers, and

f) the angular acceleration of the wheels.

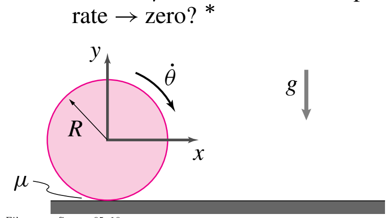


Filename:pfigure-blue-108-2
Problem 14.4.10

14.4.11 Dropped spinning disk. 2-D . A uniform disk of radius R and mass m is gently dropped onto a surface and doesn't bounce. When it is released it is spinning clockwise at the rate $\dot{\theta}_0$. The disk skids for a while and then is eventually rolling.

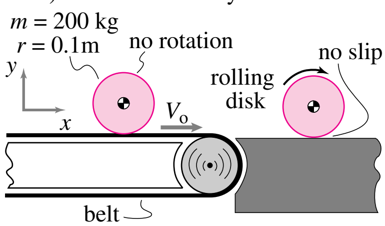
a) What is the speed of the center of the disk when the disk is eventually rolling (answer in terms of g, μ, R, $\dot{\theta}_0$, and m)? *

b) In the transition from slipping to rolling, energy is lost to friction. How does the amount lost depend on the coefficient of friction (and other parameters)? How does this loss make or not make sense in the limit as $\mu \to 0$ and the dissipation rate $\to$ zero? *



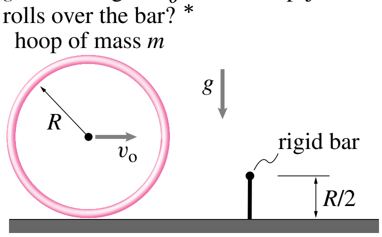
Filename:pfigure-s95q10
Problem 14.4.11

14.4.12 Disk on a conveyor belt. A uniform metal cylinder with mass of 200 kg is carried on a conveyer belt which moves at $V_0 = 3\,\text{m/s}$. The disk is not rotating when on the belt. The disk is delivered to a flat hard platform where it slides for a while and ends up rolling. How fast is it moving (i.e. what is the speed of the center of mass) when it eventually rolls? *



Filename:pfigure-s94b11p3
Problem 14.4.12

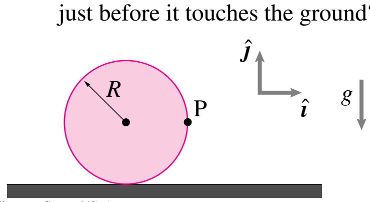
14.4.13 A rigid hoop with radius R and mass m is rolling without slip so that its center has translational speed v_o. It then hits a narrow bar with height $R/2$. When the hoop hits the bar suddenly it sticks and doesn't slide. It does hinge freely about the bar, however. The gravitational constant is g. How big is v_o if the hoop just barely rolls over the bar? *



Filename:pfigure-blue-87-5
Problem 14.4.13

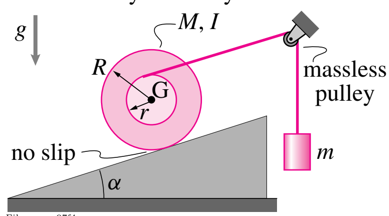
14.4.14 2-D rolling of an unbalanced wheel. A wheel, modeled as massless, has a point mass (mass $= m$) at its perimeter. The wheel is released from rest at the position shown. The wheel makes contact with coefficient of friction μ.

a) What is the acceleration of the point P just after the wheel is released if $\mu = 0$?
b) What is the acceleration of the point P just after the wheel is released if $\mu = 2$?
c) Assuming the wheel rolls without slip (no-slip requires, by the way, that the friction be high: $\mu = \infty$) what is the velocity of the point P just before it touches the ground?



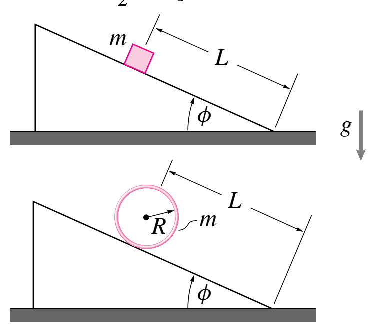
Filename:pfigure-s94f1p1
Problem 14.4.14

14.4.15 Spool and mass. A reel of mass M and moment of inertia $I_{zz}^{cm} = I$ rolls without slipping upwards on an incline with slope-angle α. It is pulled up by a string attached to mass m as shown. Find the acceleration of point G in terms of some or all of M, m, I, R, r, α, g and any base vectors you clearly define.



Filename:s97f4
Problem 14.4.15

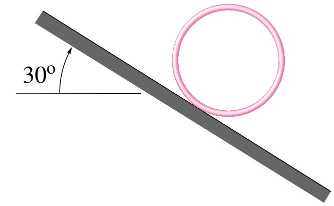
14.4.16 Two objects are released on two identical ramps. One is a sliding block (no friction), the other a rolling hoop (no slip). Both have the same mass, m, are in the same gravity field and have the same distance to travel. It takes the sliding mass 1 s to reach the bottom of the ramp. How long does it take the hoop? [Useful formula: "$s = \frac{1}{2}at^2$"]



Filename:pfigure-blue-51-2
Problem 14.4.16

14.4.17 The hoop is rolled down an incline that is 30° from horizontal. It does not slip. It does not fall over sideways. It is let go from rest at t = 0.

a) At t = 0^+ what is the acceleration of the hoop center of mass?
b) At t = 0^+ what is the acceleration of the point on the hoop that is on the incline?
c) At t = 0^+ what is the acceleration of the point on the hoop that is furthest from the incline?
d) After the hoop has descended 2 vertical meters (and traveled an appropriate distance down the incline) what is the acceleration of the point on the hoop that is (at that instant) furthest from the incline?



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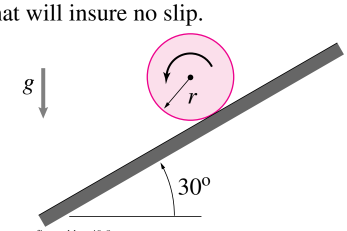
Problem 14.4.17

14.4.18 A uniform cylinder of mass m and radius r rolls down an incline *without slip*, as shown below. Determine: (a) the angular acceleration α of the disk; (b) the minimum value of the coefficient of friction μ that will insure no slip.



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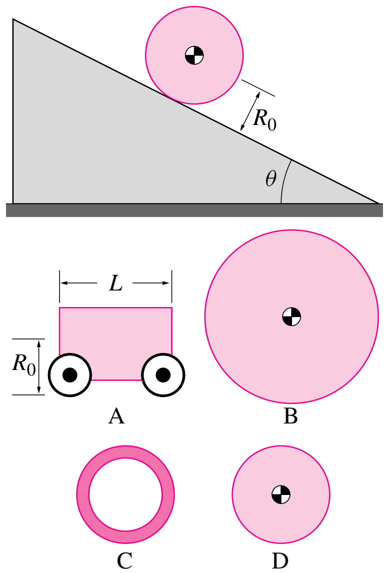
Problem 14.4.18

14.4.19 Race of rollers. A uniform disk with mass M_0 and radius R_0 is allowed to roll down the quite slip-resistant ($\mu = 1$) 30° ramp shown. It is raced against four other objects (A, B, C and D), one at a time. Who wins the races, or are there ties? First try to construct any plausible reasoning. Good answers will be based, at least in part, on careful use of equations of mechanics. *

a) Block A has the same mass and has center of mass a distance R_0 from the ramp. It rolls on massless wheels with frictionless bearings.
b) Uniform disk B has the same mass ($M_B = M_0$) but twice the radius ($R_B = 2R_0$).
c) Hollow pipe C has the same mass ($M_C = M_0$) and the same radius ($R_C = R_0$).
d) Uniform disk D has the same radius ($R_D = R_0$) but twice the mass ($M_D = 2M_0$).

Can you find a round object which will roll as fast as the block slides? How about a massless cylinder with a point mass in its center? Can you find an object which will go slower than the slowest or faster than the fastest of these objects? What would they be and why? (This problem is harder.)



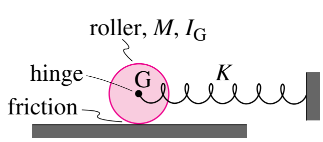
Filename:pfigure-s94h11p4
Problem 14.4.19

14.4.20 A roller of mass M and polar moment of inertia about the center of mass I_G is connected to a spring of stiffness K by a frictionless hinge as shown in the figure. Consider two kinds of friction between the roller and the surface it moves on:

1. Perfect slipping (no friction), and
2. Perfect rolling (infinite friction).

a) What is the period of oscillation in the first case?

b) What is the period of oscillation in the second case?



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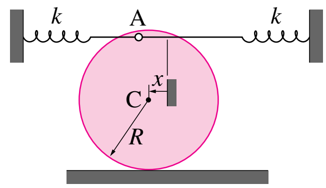
Problem 14.4.20

14.4.21 A uniform cylinder of mass m and radius R rolls back and forth without slipping through small amplitudes (i.e., the springs attached at point A on the rim act linearly and the vertical change in the height of point A is negligible). The springs, which act both in compression and tension, are unextended when A is directly over C.

a) Determine the differential equation of motion for the cylinder's center.

b) Calculate the natural frequency of the system for small oscillations.



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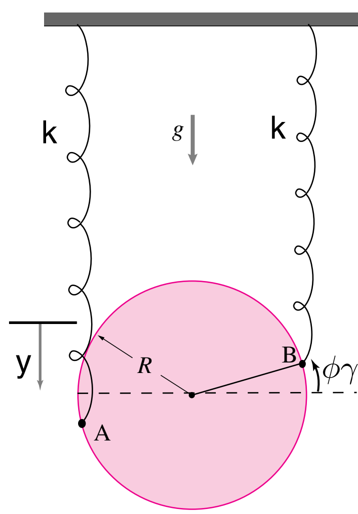
Problem 14.4.21

14.4.22 Hanging disk, 2-D. A uniform thin disk of radius R and mass m hangs in a gravity field g from a pair of massless springs each with constant k. In the static equilibrium configuration the springs are vertical and attached to points A and B on the right and left edges of the disk. In the equilibrium configuration the springs carry the weight, the disk counter-clockwise rotation is $\phi = 0$, and the downwards vertical deflection is $y = 0$. Assume throughout that the center of the disk only moves up and down, and that ϕ is small so that the springs may be regarded as vertical when calculating their stretch ($\sin\phi \approx \phi$ and $\cos\phi \approx 1$).

a) Find $\ddot{\phi}$ and $\ddot{y}$ in terms of some or all of $\phi, \dot{\phi}, y, \dot{y}, k, m, R$, and g.

b) Find the natural frequencies of vibration in terms of some or all of k, m, R, and g.



Filename:pfigure-disktwosprings
Problem 14.4.22

14.4.23 A disk rolls in a cylinder. For all of the problems below, the disk rolls without slip and rocks back and forth due to gravity.

a) **Sketch.** Draw a neat sketch of the disk in the cylinder. The sketch should show all variables, coordinates and dimension used in the problem.

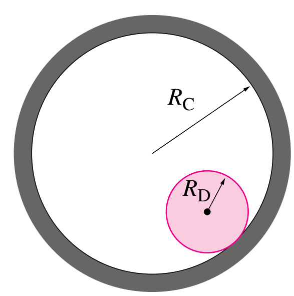
b) **FBD.** Draw a free body diagram of the disk.

c) **Momentum balance.** Write the equations of linear and angular momentum balance for the disk. Use the point on the cylinder which touches the disk for the angular momentum balance equation. Leave as unknown in these equations variables which you do not know.

d) **Kinematics.** The disk rolling in the cylinder is a *one*-degree-of-freedom system. That is, the values of only *one* coordinate and its derivatives are enough to determine the positions, velocities and accelerations of all points. The angle that the line from the center of the cylinder to the center of the disk makes from the vertical can be used as such a variable. Find all of the velocities and accelerations needed in the momentum balance equation in terms of this variable and it's derivative. [Hint: you'll need to think about the rolling contact in order to do this part.]

e) **Equation of motion.** Write the angular momentum balance equation as a single second order differential equation.

f) **Simple pendulum?** Does this equation reduce to the equation for a pendulum with a point mass and length equal to the radius of the cylinder, when the disk radius gets arbitrarily small? Why, or why not?



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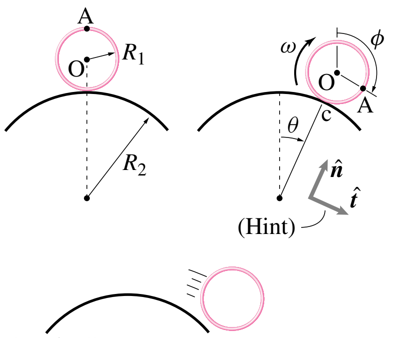
Problem 14.4.23: A disk rolls without slip inside a bigger cylinder.

14.4.24 A uniform hoop of radius R_1 and mass m rolls from rest down a semi-circular track of radius R_2 as shown. Assume that no slipping occurs. At what angle θ does the hoop leave the track and what is its angular velocity ω and the linear velocity $\vec{v}$ of its center of mass at that instant? If the hoop slides down the track without friction, so that it does not rotate, will it leave at a smaller or larger angle θ than if it rolls without slip (as above)? Give a qualitative argument to justify your answer.

HINT: Here is a geometric relationship between angle ϕ hoop turns through and angle θ subtended by its center when no slipping occurs: $\phi = [(R_1 + R_2)/R_2]\theta$. (You may or may not need to use this hint.)



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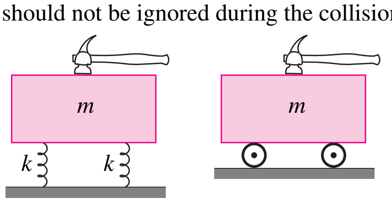
Problem 14.4.24

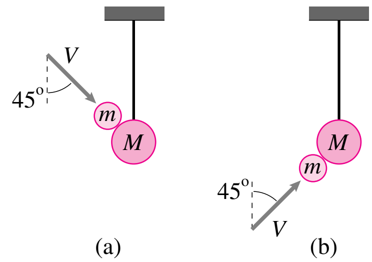
14.5 Collisions

14.5.1 The two blocks shown in the figure are identical except that one rests on two springs while the other one sits on two massless wheels. Draw free-body diagrams of each mass as each is struck by a hammer. Here we are interested in the free-body diagrams only during collision. Therefore, ignore all forces that are much smaller than the *impulsive* forces. State in words why the forces you choose to show should not be ignored during the collision.



Filename:pfig2-1-rp9
Problem 14.5.1

14.5.2 These problems concerns two colliding masses. In the first case In (a) the smaller mass hits the hanging mass from above at an angle 45° with the vertical. In (b) second case the smaller mass hits the hanging mass from below at the same angle. Assuming perfectly elastic impact between the balls, find the velocity of the hanging mass just after the collision. [Note, these problems are *not* well posed and can only be solved if you make additional modeling assumptions.]



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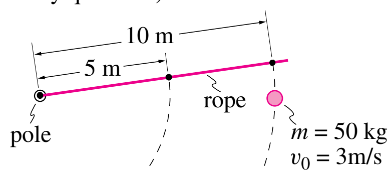
Problem 14.5.2

14.5.3 A narrow pole is in the middle of a pond with a 10 m rope tied to it. A frictionless ice skater of mass 50 kg and speed 3 m/s grabs the rope. The rope slowly wraps around the pole. What is the speed of the skater when the rope is 5 m long? (A tricky question.)



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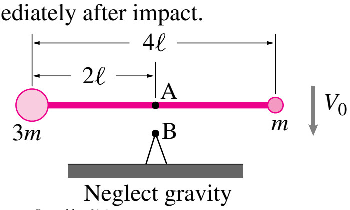
Problem 14.5.3

14.5.4 The masses m and $3m$ are joined by a light-weight bar of length 4ℓ. If point A in the center of the bar strikes fixed point B vertically with velocity V_0, and is not permitted to rebound, find $\dot{\theta}$ of the system immediately after impact.



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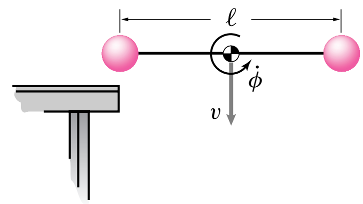
Problem 14.5.4: Neglect gravity!

14.5.5 Two equal masses each of mass m are joined by a massless rigid rod of length ℓ. The assembly strikes the edge of a table as shown in the figure, when the center of mass is moving downward with a linear velocity v and the system is rotating with angular velocity $\dot{\phi}$ in the counter-clockwise sense. The impact is 'elastic'. Find the immediate subsequent motion of the system, assuming that no energy is lost during the impact and that there is no gravity. Show that there is an interchange of translational and rotational kinetic energy.



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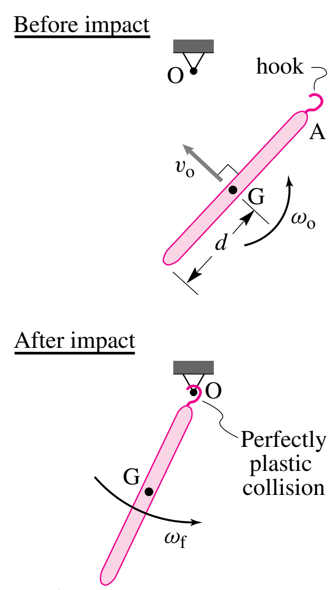
Problem 14.5.5

14.5.6 In the *absence of gravity*, a thin rod of mass m and length ℓ is initially tumbling with constant angular speed ω_o, in the counter-clockwise direction, while its mass center has constant speed v_o, directed as shown below. The end A then makes a perfectly plastic collision with a rigid peg O (via a hook). The velocity of the mass center happens to be perpendicular to the rod just before impact.

a) What is the angular speed ω_f immediately after impact?

b) What is the angular speed 10 seconds after impact? Why?

c) What is the loss in energy in the plastic collision?



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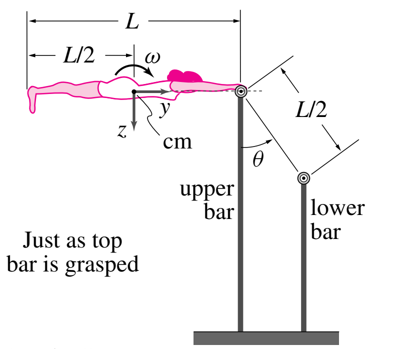
Problem 14.5.6

14.5.7 A gymnast of mass m and extended height L is performing on the uneven parallel bars. About the x, y, z axes which pass through her center of mass, her radii of gyration are k_x, k_y, and k_z, respectively. Just before she grasps the top bar, her fully extended body is horizontal and rotating with angular rate ω; her center of mass is then stationary. Neglect any friction between the bar and her hands and assume that she remains rigid throughout the entire stunt.

a) What is the gymnast's rotation rate just *after* she grasps the bar? State clearly any approximations/assumptions that you make.

b) Calculate the linear speed with which her hips (CM) strike the lower bar. State all assumptions/approximations.

c) Describe (in words, no equations please) her motion immediately after her hips strike the lower bar if she releases her hands just prior to this impact.



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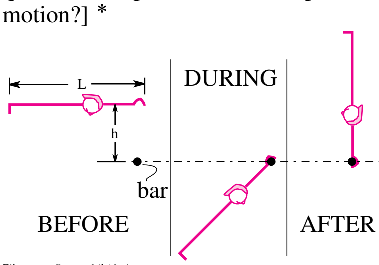
Problem 14.5.7

14.5.8 An acrobat modeled as a rigid body with uniform rigid mass m of length l. She falls without rotation in the position shown from height h where she was stationary. She then grabs a bar with a firm but slippery grip. What is h so that after the subsequent motion the acrobat ends up in a stationary handstand? [Hint: What quantities are preserved in what parts of the motion?] *



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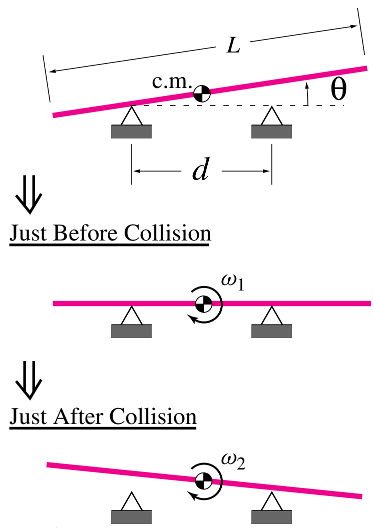
Problem 14.5.8

14.5.9 A crude see-saw is built with two supports separated by distance d about which a rigid plank (mass m, length L) can pivot smoothly. The plank is placed symmetrically, so that its center of mass is midway between the supports when the plan is at rest.

a) While the left end is resting on the left support, the right end of the plank is lifted to an angle θ and released. At what angular velocity ω_1 will the plank strike the right hand support?

b) Following the impact, the left end of the plank can pivot purely about the right end if d/L is properly chosen and the right end does not bounce. Find ω_2 under these circumstances.



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Problem 14.5.9

14.5.10 Baseball bat. In order to convey the ideas without making the calculation to complicated, some of the simplifying assumptions here are highly approximate. Assume that a bat is a uniform rigid stick with length L and mass m_s. The motion of the bat is a pivoting about one end held firmly in place with hands that rotate but do not move. The swinging of the bat occurs by the application of a constant torque M_s at the hands over an angle of $\theta = \pi/2$ until the point of impact with the ball. The ball has mass m_b and arrives perpendicular to the bat at an absolute speed v_b at a point a distance ℓ from the hands. The collision between the bat and the ball is completely elastic.

a) To maximize the speed v_{hit} of the hit ball *How heavy should a baseball bat be? Where should the ball hit the bat?* Here are some hints for one way to approach the problem.

- Find the angular velocity of the bat just before collision by drawing a FBD of the bat etc.
- Find the total energy of the ball and bat system just before the collision.
- Draw a FBD of the ball and of the bat during the collision (with this model there is an impulse at the hands on the bat). Call the magnitude of the impulse of the ball on the bat (and vice versa) $\int F dt$.
- Use various momentum equations to find the angular velocity of the bat and velocity of the ball just after the collision in terms of $\int F dt$ and other quantities above. Use these to find the energy of the system just after collision.
- Solve for the value of $\int F dt$ that conserves energy. As a check you should see if this also predicts that the relative separation speed of the ball and bat (at the impact point) is the same as the relative approach speed (it should be).
- You now know can calculate v_{hit} in terms of m_b, m_s, M_s, L, ℓ, and v_b.
- Find the maximum of the above expression by varying m_s and ℓ. Pick numbers for the fixed quantities if you like.

b) Can you explain in words what is wrong with a bat that is too light or too heavy?

c) Which aspects of the model above do you think lead to the biggest errors in predicting what a real ball player should pick for a bat and place on the bat to hit the ball?

d) Describe as clearly as possible a different model of a baseball swing that you think would give a more accurate prediction. (You need not do the calculation).

CHAPTER 15

Kinematics using time-varying basis vectors

Here is a second approach to the kinematics of particle motion. Now, instead of using constant base vectors, we use time-varying base vectors. The discussion of polar coordinates started in Chapter 13 is completed here. Path coordinates, where one base vector is parallel to the velocity and the others orthogonal to that, are introduced. The challenging kinematics topic of relative motion is introduced in two stages: first using rotating base vectors connected to a moving rigid object and then using the more abstract notation associated with frame-dependent differentiation and the famous "five term acceleration formula."

Contents

Many parts of practical machines and structures move in ways that can be idealized as straight-line motion (Chapter 12) or circular motion (Chapter 13). But often an engineer must analyze parts with more general motions as we began to study in Chapter 14.

In principle one can study all motions of all things using one fixed, say xyz, coordinate system. If one knows the x, y, and z coordinates or all points at all times than one can evaluate the linear momentum, the angular momentum, and their rates of change. In this way one can do all of mechanics. But when a machine has various parts, each moving relative to the other, it turns out it is helpful to make use of additional base vectors besides those fixed to a Newtonian ("fixed") reference frame. That is, the formulas for velocity and acceleration are in some senses simplified (or clarified) by using moving base vectors. Most often these moving base vectors base vectors move with the moving parts.

You have seen the time-varying base vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$, the polar coordinate base vectors used to describe circular motion. These are the ideas on which we build here. Altogether we discuss 4 approaches that use time varying base vectors:

1. Polar coordinates are generalized to include more than just circular motion;
2. Path coordinates and path base vectors;
3. General rotating base vectors and coordinate systems with an origin that moves are introduced; and
4. Formulas are presented for differentiation 'in' moving frames that don't depend on any particular choice of base vectors.

The basic idea is to try to use coordinate systems that most simply describe the motions of interest, even if these coordinate systems are superficially confusing because they rotate and move. Time-varying base vectors are difficult at first. Like many shortcuts, they have a cost. In the end, however, they can aid intuition and simplify calculations.

15.1 Polar coordinates and path coordinates

As you learned in Chapter 13, when a particle moves in a plane while going in circles around the origin its position, velocity, and acceleration can be described like this:

$$\begin{aligned}\vec{r} &= R\hat{e}_R \\ \vec{v} &= R\dot{\theta}\hat{e}_\theta = v_\theta\hat{e}_\theta \\ \vec{a} &= -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta = -\frac{v_\theta^2}{R}\hat{e}_R + \dot{v}_\theta\hat{e}_\theta.\end{aligned}$$

These three equations say that the position is the distance from the origin times a unit vector towards the point; that the velocity is tangent to the circle of motion; and that the acceleration has a centripetal component proportional to the speed squared, and a tangential component, tangent to the circle of motion with magnitude equal to the rate of change of speed.

We will now generalize these results two different ways.

- First we will use polar base vectors for non-constant R.
- Next we will use path base vectors to show that, in a sense to be explained, the 2nd and 3rd formulas above (for $\vec{v}$ and $\vec{a}$) apply to any wild motion in 2D or 3D.

As mentioned, in principle these new methods are not needed. We could just use one fixed coordinate system with base vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ and write the velocity and acceleration of a point at position $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ as

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \quad \text{and} \quad \vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}$$

as in Chapers 9-12. But, as for the circular motion of Chapter 13, rotating base vectors are helpful for simplifying some kinematics and mechanics problems.

Polar coordinates for general (non-circular) motion

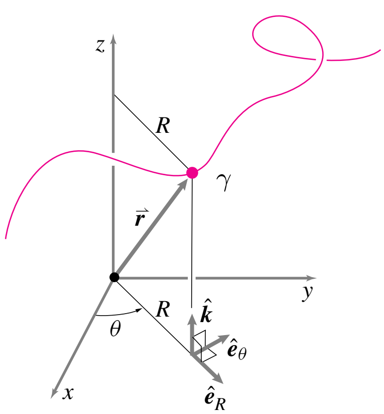

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Figure 15.1: Polar coordinates.

The extension of *polar coordinates* to 3 dimensions as *cylindrical coordinates* is shown in fig. 15.1.

Rather than identifying the location of a point by its x, y and z coordinates, a point is located by its cylindrical coordinates

R, the distance to the point from the z axis,

θ, the angle that the most direct line from the z axis to the point makes with the positive x direction,

z, the conventional z coordinate of the particle,

and base vectors:

$\hat{e}_R$, a unit vector normal to the z axis that points from the z axis to the particle
(in 2-D $\hat{e}_R = \vec{r}/r$,
in 3-D $\hat{e}_R = (\vec{r} - (\vec{r}\cdot\hat{k})\hat{k})/|\vec{r} - (\vec{r}\cdot\hat{k})\hat{k}|$
$=$ a unit vector in the direction of the shadow of $\vec{r}$ in the xy plane)

$\hat{e}_\theta$, a vector in the xy plane normal to $\hat{e}_R$ (Formally $\hat{e}_\theta = \hat{k}\times\hat{e}_R$),

$\hat{\boldsymbol{k}}$, the conventional $\hat{\boldsymbol{k}}$ base vector.

The position vector of a particle is

$$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}}.$$

The z component of position, velocity and acceleration is the same with Cylindrical coordinates as with Cartesian coordinates. If only two-dimensional problems are being considered then

$$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R.$$

As the particle moves, the values of its coordinates R, θ and z change as do the base vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$.

Example: Oblong path

A particle that moves on the oblong path $R = A + B\cos(2\theta)$ with $A > B$ is shown in fig. 15.2. The position vector is

$$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R = (A + B\cos(2\theta))\hat{\boldsymbol{e}}_R.$$

Note that, unlike for circular motion, $\hat{\boldsymbol{e}}_\theta$ is *not* tangent to the particle's path in general. The base vector $\hat{\boldsymbol{e}}_\theta$ is only tangent to the path at those points where the path is closest to, or furthest from, the origin (which, in this example, are also the points where the path crosses the x and y axes).

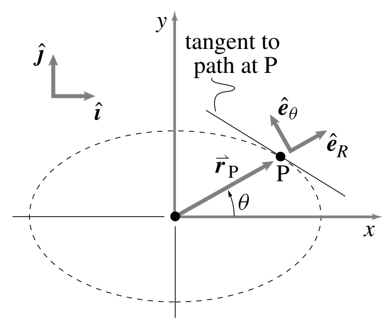

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Figure 15.2: Particle P on an oblong path. Note that the velocity is not parallel to $\hat{\boldsymbol{e}}_\theta$ at most points on the path.

Velocity in polar coordinates

The velocity and acceleration are found by differentiating the position $\vec{\boldsymbol{r}}$, taking account that the base vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ also change with time just as they did for circular motion:

$$\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta \qquad \text{and} \qquad \dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R.$$

We find the velocity by taking the time derivative of the position, using the product rule of differentiation:

$$\begin{aligned}
\vec{\boldsymbol{v}} = \frac{d}{dt}\vec{\boldsymbol{r}} &= \frac{d}{dt}[R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}}] \\
&= \frac{d}{dt}(R\hat{\boldsymbol{e}}_R) + \frac{d}{dt}(z\hat{\boldsymbol{k}}) \\
&= (\dot{R}\hat{\boldsymbol{e}}_R + R\underbrace{\dot{\hat{\boldsymbol{e}}}_R}_{\dot{\theta}\hat{\boldsymbol{e}}_\theta}) + (\dot{z}\hat{\boldsymbol{k}}) \\
&= \underbrace{\dot{R}}_{v_R}\hat{\boldsymbol{e}}_R + \underbrace{R\dot{\theta}}_{v_\theta}\hat{\boldsymbol{e}}_\theta + \underbrace{\dot{z}}_{v_z}\hat{\boldsymbol{k}}. \qquad (15.1)
\end{aligned}$$

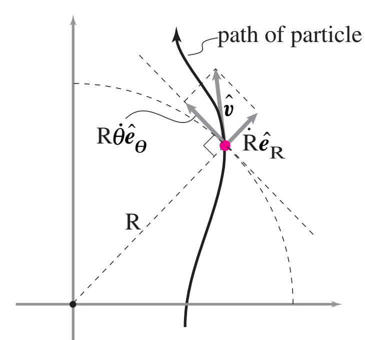

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Figure 15.3: Velocity in polar coordinates for general planar motion. The velocity has a radial $\hat{\boldsymbol{e}}_R$ component $\dot{R}$ and a circumferential $\hat{\boldsymbol{e}}_\theta$ component $R\dot{\theta}$.

This formula is intuitive. The velocity is the sum of three vectors: one, $\dot{R}\hat{\boldsymbol{e}}_R$, due to moving towards or away from the z axis; one, $\dot{\theta}R\hat{\boldsymbol{e}}_\theta$, having to do with the angle being swept; and in 3-D, one, $\dot{z}\hat{\boldsymbol{k}}$, for motion perpendicular to the xy plane. In 2-D this is shown in fig. 15.3.

But, as has been emphasized before, this isn't a new vector $\vec{v}$ but just a new way of representing the same vector:

$$\begin{aligned}\vec{v} &= \vec{v}\\ v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k} &= v_R\hat{e}_R + v_\theta\hat{e}_\theta + v_z\hat{k}.\end{aligned}$$

The vector $\vec{v}$ can be represented in different base vector systems, in this case cartesian and polar.

Note that eqn. (15.1) adds two terms to the circular motion case from Chapter 13: one for variable R and one for variable z.

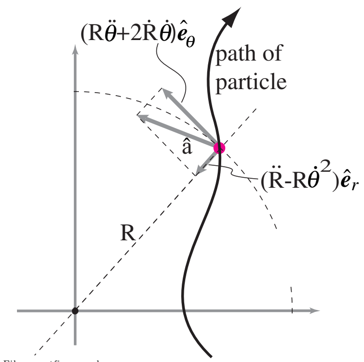

Figure 15.4: Acceleration in polar coordinates for general planar motion. The acceleration has a radial $\hat{e}_R$ component $\ddot{R} - R\dot{\theta}^2$ and a circumferential $\hat{e}_\theta$ component $R\ddot{\theta} + 2\dot{R}\dot{\theta}$.

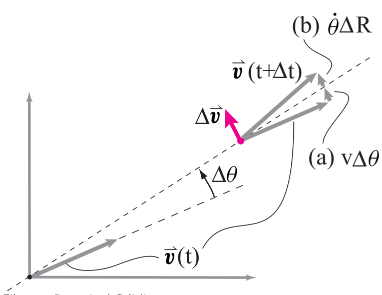

Figure 15.5: The simplest case for understanding of the *Coriolis* term in the acceleration. A particle moves at constant speed $\dot{R}$ out on a line which is rotating at constant rate $\dot{\theta}$. The figure shows a velocity vector as an arrow when the particle is at the origin and a time Δt later. The change in the velocity vector comes from two effects: (a) the radial line has rotated, and (b) a circumferential component is added to the velocity when the particle is away from the origin. For small $\Delta\theta$ these two changes to the velocity are both approximately perpendicular to $\vec{v}$ and parallel to each other.

Acceleration in polar coordinates

To find the acceleration, we differentiate once again. The resulting formula has new terms generated by the product rule of differentiation.

$$\begin{aligned}\vec{a} &= \frac{d}{dt}\vec{v}\\ &= \frac{d}{dt}(\dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k})\\ &= (\ddot{R}\hat{e}_R + \dot{R}\underbrace{\dot{\hat{e}}_R}_{\dot{\theta}\hat{e}_\theta}) + (\dot{R}\dot{\theta}\hat{e}_\theta + R\ddot{\theta}\hat{e}_\theta + R\dot{\theta}\underbrace{\dot{\hat{e}}_\theta}_{-\dot{\theta}\hat{e}_R}) + (\ddot{z}\hat{k})\\ &= (\underbrace{\ddot{R} - \dot{\theta}^2 R}_{a_R})\hat{e}_R + (\underbrace{2\dot{R}\dot{\theta} + R\ddot{\theta}}_{a_\theta})\hat{e}_\theta + \underbrace{\ddot{z}}_{a_z}\hat{k}. \qquad (15.2)\end{aligned}$$

The acceleration for an arbitrary planar path is shown in fig. 15.4. Four of the five terms comprising the polar coordinate formula for acceleration are easy to understand.

$\ddot{R}$ is just the acceleration due to the distance from the origin changing with time.

$\dot{\theta}^2 R$ is the familiar *centripetal* acceleration.

$R\ddot{\theta}$ is the acceleration due to rotation proceeding at a faster and faster rate. And

$\ddot{z}$ is the same as for Cartesian coordinates.

The Coriolis term. The difficult term in the polar coordinate expression for the acceleration is the

$2\dot{\theta}\dot{R}$ term, called the *Coriolis* acceleration, after the civil engineer Gustave-Gaspard Coriolis who first wrote about it in 1835 (in a slightly more general context).

The presence of the '2' in this term is due to the two effects from which it derives: 1 from the change of the $\dot{R}\hat{e}_R$ term in the velocity and 1 from the change of the $R\dot{\theta}\hat{e}_\theta$ term in the velocity $(1 + 1 = 2)$.

The Coriolis acceleration occurs even if both $\dot{\theta}$ and $\dot{R}$ are constant. That is, a particle that moves at constant speed ($\dot{R}$ = constant) on a straight line that is itself rotating at constant rate ($\dot{\theta}$ = constant) does not have a straight-line path, and thus has some acceleration. The Coriolis term catches this acceleration. Here is a situation in which the Coriolis term is the only non-zero term in the general polar-coordinate acceleration formula (eqn. (15.2)).

Example: The simplest Coriolis example
Consider, like above, a particle moving at constant speed $\dot{R}$ along a line which is itself rotating at constant $\dot{\theta}$ about the origin. Let's look at a the particle as it passes through the origin at time t and a small amount of time Δt later (see fig. 15.5). At time $t+\Delta t$ the direction of the scribed line has changed by an angle $\Delta\theta = \dot{\theta}\Delta t$. So that, even if at that later time $\dot{\theta} = 0$ the direction of $\vec{v}$ has changed so $\vec{v}$ has changed by an amount $v\Delta\theta$. But the rotation of the line does continue, so the velocity includes a part in the $\hat{e}_\theta$ direction with magnitude $\dot{\theta}\Delta R$. That is $\vec{v}$ is changed by both $\Delta\theta v$ and by $\dot{\theta}\Delta R$ so

$$\begin{aligned}
\Delta\vec{v} &\approx \left(v\Delta\theta + \dot{\theta}\Delta R\right)\hat{e}_\theta \\
&\approx \left(\dot{R}(\dot{\theta}\Delta t) + \dot{\theta}(\dot{R}\Delta t)\right)\hat{e}_\theta \\
&\approx \left(\dot{R}\dot{\theta} + \dot{\theta}\dot{R}\right)\hat{e}_\theta\Delta t \\
\Rightarrow \frac{\Delta\vec{v}}{\Delta t} &\approx \left(\dot{R}\dot{\theta} + \dot{\theta}\dot{R}\right)\hat{e}_\theta \\
\Rightarrow \vec{a} &\approx \left(\dot{R}\dot{\theta} + \dot{\theta}\dot{R}\right)\hat{e}_\theta = 2\dot{R}\dot{\theta}\hat{e}_\theta
\end{aligned}$$

as predicted by the general polar coordinate acceleration formula.

But one need not get confounded by a desire to understand every term intuitively. Equation (15.2) is a way of describing the same acceleration we have described with cartesian coordinates. Namely,

$$\begin{aligned}
&\vec{a} = \vec{a} \\
\Rightarrow\quad &\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k} = (\underbrace{\ddot{R} - \dot{\theta}^2 R}_{a_R})\hat{e}_R + (\underbrace{2\dot{R}\dot{\theta} + R\ddot{\theta}}_{a_\theta})\hat{e}_\theta + \underbrace{\ddot{z}}_{a_z}\hat{k}.
\end{aligned}$$

Example: $R(t)$ and $\theta(t)$ are given functions.
Say θ and R vary in time according to

$$\theta = at \quad \text{and} \quad R = ct^2$$

with a and c as given constants. Then at any t the position, velocity, and acceleration are (see fig. 15.6)

$$\begin{aligned}
\vec{r} &= R\hat{e}_R = ct^2\hat{e}_R, \\
\vec{v} &= \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta = 2ct\hat{e}_R + act^2\hat{e}_\theta, \quad \text{and} \\
\vec{a} &= (\ddot{R} - \dot{\theta}^2 R)\hat{e}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{e}_\theta \\
&= \left(2c - a^2ct^2\right)\hat{e}_R + (0 + 4act)\,\hat{e}_\theta
\end{aligned}$$

with polar base vectors

$$\begin{aligned}
\hat{e}_R &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} = \cos(at)\hat{\imath} + \sin(at)\hat{\jmath} \quad \text{and} \\
\hat{e}_\theta &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} = -\sin(at)\hat{\imath} + \cos(at)\hat{\jmath}.
\end{aligned}$$

If we substituted these expressions for the polar base vectors into the expressions

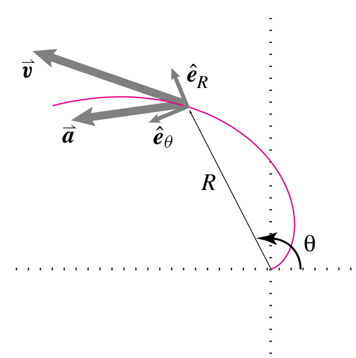

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Figure 15.6: A particle moves with $\theta = at$ and $R = ct^2$. In this drawing $a = .5$ and c is anything since no scale is shown.

for $\vec{r}$, $\vec{v}$, and $\vec{a}$ we would get the same cartesian representation (a giant mess that we don't show) that we would get from using $x = R\cos\theta$ and $y = R\sin\theta$ with $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$, and $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$. That is $\vec{r} = \vec{r}$, $\vec{v} = \vec{v}$, and $\vec{a} = \vec{a}$ even if the representation is different.

Path coordinates

Another way still to describe the velocity and acceleration is to use base vectors which are defined by the motion. In particular, the *path* base vectors used are:

1. the unit tangent to the path $\hat{e}_t$, and
2. the unit normal to the path $\hat{e}_n$.

Somewhat surprisingly at first glance, only two base vectors are needed to define the velocity and acceleration, even in three dimensions.

The base vectors can be described geometrically and analytically. Let's start out with a geometric description.

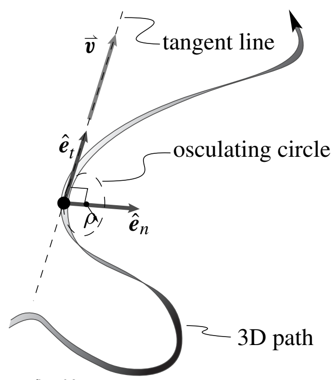

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Figure 15.7: The base vectors for path coordinates are tangent and perpendicular to the path. The osculating circle is a circle that is tangent to the path and has the same curvature as the path. The circle is in the plane containing both the tangent to the path and the rate of change of that tangent. The osculating circle is not defined for a straight-line path.

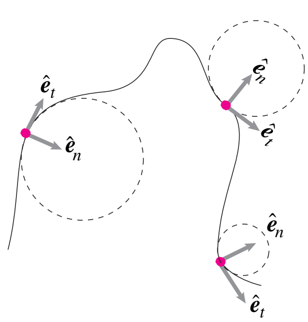

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Figure 15.8: A path in the plane with the path base vectors and osculating circle marked at three points.

The geometry of the path basis vectors.

As a particle moves through space it traces a path $\vec{r}(t)$. At the moment of interest the path has a unique tangent line. The unit tangent $\hat{e}_t$ is along this line in the direction of motion, as shown in fig. 15.7.

Less clear is that the path has a unique 'kissing' *plane*. One line on this plane is the tangent line. The other line needed to define this plane is determined by the position of the particle just before and just after the time of interest. Just before and just after the time of interest the particle is a little off the tangent line (unless the motion happens to be a straight line and the tangent plane is not uniquely determined). Three points, the position of the particle just before, just at, and just after the moment of interest determine the tangent plane.

Another way to picture the tangent plane is to find the circle in space that is tangent to the path and which turns at the same rate and in the same direction as the path turns. This circle, which touches the path so intimately, is called the *osculating* or 'kissing' circle. The tangent plane is the plane of this circle. (See fig. 15.7).

The unit normal $\hat{e}_n$ is the unit vector which is perpendicular to the unit tangent and is in the tangent plane. It is pointed in the direction from the edge of the osculating circle towards the center of the circle as shown in fig. 15.7. For 2-D motion in the xy plane the osculating plane is the xy plane and the osculating circle is in the xy plane. The path base vectors are unit vectors that vary along the path, always tangent and normal to the path (see fig. 15.8).

Formal definition of path basis vectors

The path of a particle $\vec{r}(t)$ can also be parameterized by arc length s along the path, as explained in any introductory calculus text. So the path in space is also $\vec{\boldsymbol{r}}(s)$ ①, where the arc length s is the path "coordinate". The unit tangent is:

$$\hat{\boldsymbol{e}}_t \equiv \frac{d\vec{\boldsymbol{r}}(s)}{ds}.$$

Using the chain rule with $\vec{\boldsymbol{r}}(s(t))$ this is also

$$\hat{\boldsymbol{e}}_t = \frac{d\vec{\boldsymbol{r}}(t)}{dt}\frac{dt}{ds} = \frac{\vec{\boldsymbol{v}}}{v}.$$

To define the unit normal let's first define the *curvature* $\vec{\boldsymbol{\kappa}}$ of the path as the rate of change of the tangent (rate in terms of arc length).

$$\vec{\boldsymbol{\kappa}} \equiv \frac{d\hat{\boldsymbol{e}}_t}{ds}$$

The unit normal $\hat{\boldsymbol{e}}_n$ is the unit vector in the direction of the curvature

$$\hat{\boldsymbol{e}}_n = \frac{\vec{\boldsymbol{\kappa}}}{|\vec{\boldsymbol{\kappa}}|}.$$

Finally, the binormal $\hat{\boldsymbol{e}}_b$ is the unit vector perpendicular to $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$:

$$\hat{\boldsymbol{e}}_b \equiv \hat{\boldsymbol{e}}_t \times \hat{\boldsymbol{e}}_n.$$

For 2-D motion the binormal $\hat{\boldsymbol{e}}_b$ is always in the $\hat{\boldsymbol{k}}$ direction. The radius of the osculating circle ρ is

$$\rho = \frac{1}{|\vec{\boldsymbol{\kappa}}|}.$$

Note that, in general, the polar and path coordinate basis vectors are not parallel; i.e., $\hat{\boldsymbol{e}}_n$ is *not* parallel to $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_t$ is *not* parallel to $\hat{\boldsymbol{e}}_\theta$. For example, consider a particle moving on an elliptical path in the plane shown in fig. 15.9. In this case, the polar coordinate and path coordinate basis vectors are only parallel where the major and minor axes intersect the path.

Velocity and acceleration in path coordinates.

Although it is not necessarily easy to compute the path basis vectors $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$, they lead to simple expressions for the velocity and acceleration:

$$\vec{\boldsymbol{v}} = v\hat{\boldsymbol{e}}_t = \frac{ds}{dt}\hat{\boldsymbol{e}}_t \tag{15.3}$$

$$\begin{aligned}\vec{\boldsymbol{a}} &= \frac{d}{dt}\vec{\boldsymbol{v}} = \frac{d}{dt}(v\hat{\boldsymbol{e}}_t) = \dot{v}\hat{\boldsymbol{e}}_t + v\dot{\hat{\boldsymbol{e}}}_t \\ &= \dot{v}\hat{\boldsymbol{e}}_t + v\frac{d\hat{\boldsymbol{e}}_t}{ds}\frac{ds}{dt} = \dot{v}\hat{\boldsymbol{e}}_t + v\vec{\boldsymbol{\kappa}}v \\ &= \underbrace{\dot{v}\hat{\boldsymbol{e}}_t}_{\vec{\boldsymbol{a}}_t} + \underbrace{\frac{v^2}{\rho}\hat{\boldsymbol{e}}_n}_{\vec{\boldsymbol{a}}_n}.\end{aligned} \tag{15.4}$$

① Mathematicians often do not like this engineering sloppiness of notation. That is, the sequence of points $\vec{\boldsymbol{r}}$ that a particle traverses as both time t and accumulated arc length s increase is not the same mathematical function of both the arguments s and t. For example, when $t = 6\,\mathrm{s}$ the position is not the same as when $s = 6\,\mathrm{cm}$. Here we don't think it is worth the trouble of writing $\vec{\boldsymbol{r}} = \mathbf{f}(t)$ and $\vec{\boldsymbol{r}} = \mathbf{g}(s)$.

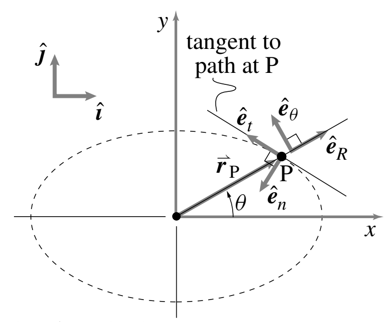

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Figure 15.9: Particle P on an elliptical path.

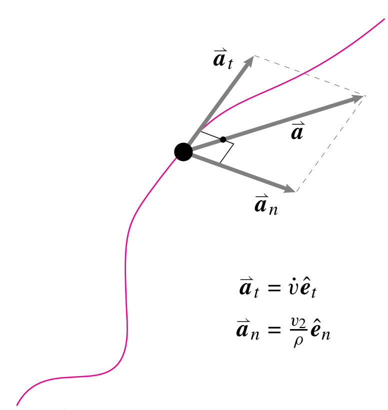

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Figure 15.10: The acceleration of any particle can be broken into a part that is parallel to its path and a part perpendicular to its path.

This formula for velocity is obvious: velocity is speed times a unit vector in the direction of motion. The formula for acceleration is more interesting. It says that the acceleration of any particle at any time is given by the same formula as the formula for acceleration of a particle going around in circles at non-constant rate.

- The first term $\dot{v}\hat{\boldsymbol{e}}_t$ is tangent to the path (also tangent to the osculating circle), as shown in fig. 15.10. If we draw the osculating circle at the point of interest and fix it in space, then this first term is $\rho\ddot{\theta}\hat{\boldsymbol{e}}_\theta$ in a polar coordinate system centered at the center of that fixed circle.
- There is a term $v^2/\rho\hat{\boldsymbol{n}}$ is directed towards the center of the osculating circle. This term is associated with change of direction and does not vanish even if the speed is constant. This normal acceleration is perpendicular to the path.

That is, the two terms in the path-coordinate acceleration formula correspond exactly to the two terms for acceleration for a particle going in circles.

Example: Estimating the direction of acceleration

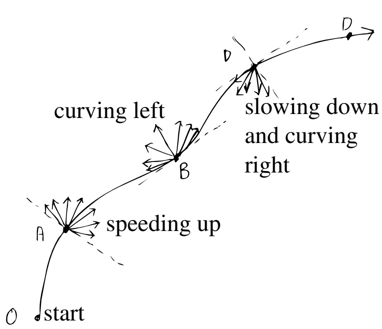
By looking at the path of a particle (*e.g.,* see fig. 15.11) and just knowing whether it is speeding up or slowing down one can estimate the direction of the acceleration.

If the particle is known to be speeding up at A then $\dot{v} > 0$ so the tangential acceleration is in the direction of the velocity. Without thinking about the normal acceleration you know that the acceleration vector is pointed in the half plane of directions shown.

If at point B nothing is known about the rate of change of speed, you still know that the acceleration must be in the half plane shown because that is the direction of $\vec{\boldsymbol{\kappa}}$ and $\hat{\boldsymbol{e}}_n$.

If at C you know that the particle is slowing down then you know that $\dot{v} < 0$. But you also can see the curve is to the right so $\hat{\boldsymbol{e}}_n$ is to the right. So the acceleration must be in the quadrant shown.

One can use the information about the curvature to further restrict the possible accelerations at point A also. At point B there is nothing more to know unless you know how the speed is changing with time.



Filename:tfigure-pathintuition

Figure 15.11: A particle moves from 0 to D on the path shown. At A it is speeding up so $\dot{v} > 0$. At B it is turning to the left so $\hat{\boldsymbol{e}}_n$ points to the left. At C it is slowing down and turning to the right. So the acceleration must be a positive vector pointed in the quadrant shown.

Earlier we found the curvature by assuming the particle's path was parameterized by arc length s. A second way of calculating the curvature $\vec{\boldsymbol{\kappa}}$ (and then the unit normal $\hat{\boldsymbol{e}}_n$) is to calculate the normal part of the acceleration. First calculate the acceleration. Then subtract from the acceleration that part which is parallel to the velocity.

$$\vec{\boldsymbol{a}}_n = \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t = \vec{\boldsymbol{a}} - \frac{(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\vec{\boldsymbol{v}}}{v^2}$$

The normal acceleration is $\vec{\boldsymbol{a}}_n = v^2\vec{\boldsymbol{\kappa}}$ so

$$\vec{\boldsymbol{\kappa}} = \frac{\vec{\boldsymbol{a}}_n}{v^2} = \frac{\vec{\boldsymbol{a}}}{v^2} - \frac{(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{v}})\vec{\boldsymbol{v}}}{v^4}.$$

Recipes for path coordinates.

Assume that you know the position as a function of time in either cartesian or polar coordinates. Then, say, at a particular time of interest when the particle is at $\vec{r}$, you can calculate the velocity of the particle using:

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k} \quad \text{or} \quad \vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k}$$

and the acceleration using

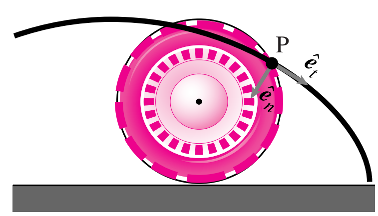
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k} \quad \text{or} \quad \vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}.$$

From these expressions we can calculate all the quantities used in the path coordinate description. So we repeat what we have already said but in an algorithmic form. Here is one set of steps one can follow. This recipe is of little practical use, but does show that the motion explicitly determines the path base vectors as well as the osculating circle.

1. Calculate $\hat{e}_t = \vec{v}/|\vec{v}|$.
2. Calculate $a_t = \vec{a} \cdot \vec{v}/v$.
3. Calculate $\vec{a}_n = \vec{a} - (\vec{a} \cdot \vec{v})\vec{v}/v^2$.
4. Calculate $\hat{e}_n = \frac{\vec{a}_n}{|\vec{a}_n|}$
5. Calculate the radius of curvature as $\rho = \frac{|\vec{v}|^2}{|\vec{a}_n|}$.
6. Write a parametric equation for the osculating circle as

$$\vec{r}_{\text{osculating}} = \underbrace{(\vec{r} + \rho\hat{e}_n)}_{\text{center}} + \underbrace{\rho(-\cos\phi\hat{e}_n + \sin\phi\hat{e}_t)}_{\text{circle}}$$

where ϕ is the parameter used to parameterize the points on the circle $\vec{r}_{\text{osculating}}$ of the point on the curve $\vec{r}$. As ϕ ranges from 0 to 2π the point $\vec{r}_{\text{osculating}}$ goes from $\vec{r}$ around the circle and back. The plane of the osculating circle is determined by $\hat{e}_t$ and $\hat{e}_n$. For planar curves, the osculating circle is in the plane of the curve.



Example: Particle on the rim of a tire

A particle P on the rim of a tire whose center is moving at constant speed v has position r given by

$$\vec{r} = \left(vt - R\sin(vt/R)\right)\hat{\imath} + R\left(1 - \cos(vt/R)\right)\hat{\jmath}$$

where the origin is at the ground contact time $t = 0$. When the particle is at its highest point $vt = \pi R$ and

$$\vec{v} = 2v\hat{\imath} \quad \text{and} \quad \vec{a} = -(v^2/R)\hat{\jmath}.$$

At that midpoint

$$\hat{e}_t = \hat{\imath}, \quad \hat{e}_n = -\hat{\jmath}, \vec{\kappa} = -(1/(4R))\hat{\jmath}$$
$$\text{and} \quad (2v)^2/\rho = v^2/R \Rightarrow \rho = 4R$$

as shown in fig. 15.12. The osculating circle has 4 times the radius of the tire. Note the intimacy of the osculating circle's kiss with the cycloidal path.

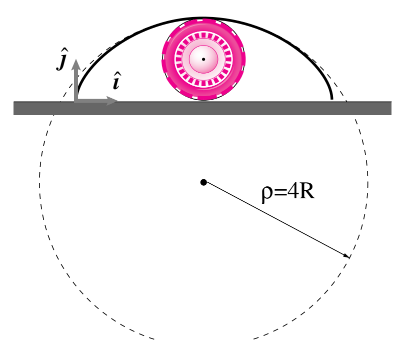


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Figure 15.12: The path of a rock in a tire of radius R is shown as are the unit tangent, the unit normal and the osculating circle at the top position.

Summary of polar(cylindrical) coordinates

See also the inside back cover, table II, row 3 in for future reference.

$$
\begin{aligned}
\vec{\boldsymbol{r}} &= R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}}\\
\vec{\boldsymbol{v}} &= v_R\hat{\boldsymbol{e}}_R + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{k}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}}\\
\vec{\boldsymbol{a}} &= a_R\hat{\boldsymbol{e}}_R + a_\theta\hat{\boldsymbol{e}}_\theta + a_z\hat{\boldsymbol{k}} = (\ddot{R} - \dot{\theta}^2 R)\hat{\boldsymbol{e}}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}\\
\\
\hat{\boldsymbol{e}}_R &= (\vec{\boldsymbol{r}} - z\hat{\boldsymbol{k}})/|\vec{\boldsymbol{r}} - z\hat{\boldsymbol{k}}|\\
\hat{\boldsymbol{e}}_\theta &= \hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_R
\end{aligned}
$$

For 2-D problems just set $z = 0$, $\dot{z} = 0$, and $\ddot{z} = 0$ in these equations.

Summary of path coordinates

See the inside back cover table II, row 4 and the text under the table for future reference:

$$
\begin{aligned}
\vec{\boldsymbol{r}} &= \text{no simple expression in terms of path base vectors}\\
\vec{\boldsymbol{v}} &= \dot{s}\hat{\boldsymbol{e}}_t = v\hat{\boldsymbol{e}}_t\\
\vec{\boldsymbol{a}} &= \vec{\boldsymbol{a}}_t + \vec{\boldsymbol{a}}_n\\
&\quad \vec{\boldsymbol{a}}_t = a_t\hat{\boldsymbol{e}}_t = \dot{v}\hat{\boldsymbol{e}}_t = \ddot{s}\hat{\boldsymbol{e}}_t = (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t\\
&\quad \vec{\boldsymbol{a}}_n = a_n\hat{\boldsymbol{e}}_n = (v^2/\rho)\hat{\boldsymbol{e}}_n = \vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t\\
\\
\hat{\boldsymbol{e}}_t &= d\vec{\boldsymbol{r}}/ds = \vec{\boldsymbol{v}}/v\\
\hat{\boldsymbol{e}}_n &= \vec{\boldsymbol{\kappa}}/|\vec{\boldsymbol{\kappa}}| = \rho\vec{\boldsymbol{\kappa}} = (\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t)/|\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t|\\
\hat{\boldsymbol{e}}_b &= \hat{\boldsymbol{e}}_t \times \hat{\boldsymbol{e}}_n\\
\vec{\boldsymbol{\kappa}} &= d\hat{\boldsymbol{e}}_t/ds = (\vec{\boldsymbol{a}} - (\vec{\boldsymbol{a}} \cdot \hat{\boldsymbol{e}}_t)\hat{\boldsymbol{e}}_t)/v^2 \qquad \rho = 1/|\vec{\boldsymbol{\kappa}}|
\end{aligned}
$$

Both polar coordinates and path coordinates define base vectors in terms of the motion of a particle of interest relative to a fixed coordinate system.

SAMPLE 15.1 Acceleration in polar coordinates. A bug walks along the spiral section of a natural shell. The path of the bug is described by the equation $R = R_0 e^{a\theta}$ where $a = 0.182$ and $R_0 = 5\,\text{mm}$. The bug's radial distance from the center of the spiral is seen to be increasing at a constant rate of $2\,\text{mm/ s}$. Find the x and y components of the acceleration of the bug at $\theta = \pi$.

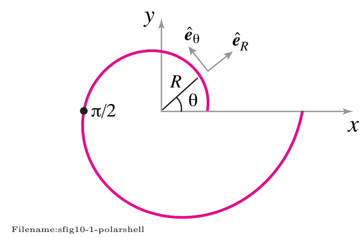

Figure 15.13:

Solution In polar coordinates, the acceleration of a particle in planar motion is

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta.$$

Since we know the position of the bug,

$$\begin{aligned} R &= R_0\, e^{a\theta}, \\ \dot{R} &= R_0\, a e^{a\theta}\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{\dot{R}}{R_0 a\, e^{a\theta}}, \\ \ddot{R} &= R_0\, a e^{a\theta}\ddot{\theta} + R_0\, a^2 e^{a\theta}\dot{\theta}^2. \end{aligned}$$

Since the radial distance R of the bug is increasing at a constant rate $\dot{R} = 2\ \text{mm}/s$, $\ddot{R} = 0$, that is,

$$\begin{aligned} R_0\, a e^{a\theta}(\ddot{\theta} + a\dot{\theta}^2) &= 0 \\ \Rightarrow \qquad \ddot{\theta} &= -a\dot{\theta}^2 \\ &= -\frac{\dot{R}^2}{R_0^2 a\, e^{2a\theta}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \\ &= \left(0 - R_0 e^{a\theta} \cdot \frac{\dot{R}^2}{R_0^2 a^2 e^{2a\theta}}\right)\hat{e}_R + \left(\frac{2\dot{R}^2}{R_0 a\, e^{a\theta}} + R_0 e^{a\theta} \cdot \frac{-\dot{R}^2}{R_0^2 a\, e^{2a\theta}}\right)\hat{e}_\theta \\ &= \frac{\dot{R}^2}{R_0 a\, e^{a\theta}}\left[-\frac{1}{a}\hat{e}_R + (2-1)\hat{e}_\theta\right]. \end{aligned}$$

Now substituting $R_0 = 5\,\text{mm}$, $a = 0.182$, $\dot{R} = 2\,\text{mm}/s$, and $\theta = \pi$ in the above expression, we get

$$\vec{a} = (-13.63\,\text{mm/ s}^2)\,\hat{e}_R + (2.48\,\text{mm/ s}^2)\,\hat{e}_\theta.$$

But, at $\theta = \pi$

$$\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} = -\hat{\imath} \quad \text{and} \quad \hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} = -\hat{\jmath},$$

therefore,

$$\begin{aligned} \vec{a} &= (13.63\,\text{mm/ s}^2)\,\hat{\imath} - (2.48\,\text{mm/ s}^2)\,\hat{\jmath}, \\ \Rightarrow \quad a_x &= 13.63\,\text{mm/ s}^2 \quad \text{and} \quad a_y = -2.48\,\text{mm/ s}^2. \end{aligned}$$

$$a_x = 13.63\,\text{mm/ s}^2,\ a_y = -2.48\,\text{mm/ s}^2$$

SAMPLE 15.2 Going back and forth between (x, y) and (R, θ). Given the position of a particle in polar coordinates (R, θ) and its radial and angular velocity $(\dot{R}, \dot{\theta})$ and radial and angular acceleration $(\ddot{R}, \ddot{\theta})$, find $(\dot{x}, \dot{y})$ and $(\ddot{x}, \ddot{y})$. Also, find the inverse relationship.

Solution

- **Polar to Cartesian:** In polar coordinates, we are given $R, \theta, \dot{R}, \dot{\theta}, \ddot{R}$, and $\ddot{\theta}$. We need to find $\dot{x}, \dot{y}, \ddot{x}$, and $\ddot{y}$. Let us consider the velocity first. The velocity of a point is $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ in cartesian coordinates and $\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$ in polar coordinates. Thus,
$$\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta.$$
where $\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$. Dotting this equation with $\hat{\imath}$ and $\hat{\jmath}$, respectively, we get
$$\begin{aligned}\dot{x} &= \dot{R}\,\overbrace{(\hat{e}_R\cdot\hat{\imath})}^{\cos\theta} + R\dot{\theta}\,\overbrace{(\hat{e}_\theta\cdot\hat{\imath})}^{-\sin\theta}\\ \dot{y} &= \dot{R}\,\underbrace{(\hat{e}_R\cdot\hat{\jmath})}_{\sin\theta} + R\dot{\theta}\,\underbrace{(\hat{e}_\theta\cdot\hat{\jmath})}_{\cos\theta}\end{aligned}$$
or,
$$\begin{aligned}\dot{x} &= \dot{R}\cos\theta + R\dot{\theta}(-\sin\theta)\\ \dot{y} &= \dot{R}\sin\theta + R\dot{\theta}\cos\theta\end{aligned}$$
or,
$$\left\{\begin{array}{c}\dot{x}\\ \dot{y}\end{array}\right\} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\left\{\begin{array}{c}\dot{R}\\ \dot{\theta}\end{array}\right\}. \tag{15.5}$$
Thus given $\dot{R}$ and $\dot{\theta}$ at (R, θ), we can find $\dot{x}$ and $\dot{y}$. Similarly, from the acceleration formula, $\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$, we derive
$$\left\{\begin{array}{c}\ddot{x}\\ \ddot{y}\end{array}\right\} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\left(\begin{bmatrix}1 & 0\\ 0 & R\end{bmatrix}\left\{\begin{array}{c}\ddot{R}\\ \ddot{\theta}\end{array}\right\} + \left\{\begin{array}{c}-R\dot{\theta}^2\\ 2\dot{R}\dot{\theta}\end{array}\right\}\right). \tag{15.6}$$
It is not necessary to split the terms on the right hand side. We could have kept them together as $(\ddot{R} - R\dot{\theta}^2)$ and $(2\dot{R}\dot{\theta} + R\ddot{\theta})$ but we split them to keep the radial acceleration term $\ddot{R}$ and angular acceleration $\ddot{\theta}$ in evidence.

- **Cartesian to Polar:** Given $\dot{x}, \dot{y}, \ddot{x}$, and $\ddot{y}$ at (x, y), we can now find $\dot{R}, \dot{\theta}, \ddot{R}$, and $\ddot{\theta}$ easily by inverting eqn. (15.5) and eqn. (15.6):
$$\left\{\begin{array}{c}\dot{R}\\ \dot{\theta}\end{array}\right\} = \begin{bmatrix}1 & 0\\ 0 & \frac{1}{R}\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\left\{\begin{array}{c}\dot{x}\\ \dot{y}\end{array}\right\}. \tag{15.7}$$
$$\left\{\begin{array}{c}\ddot{R}\\ \ddot{\theta}\end{array}\right\} = \begin{bmatrix}1 & 0\\ 0 & \frac{1}{R}\end{bmatrix}\left(\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\left\{\begin{array}{c}\ddot{x}\\ \ddot{y}\end{array}\right\} - \left\{\begin{array}{c}-R\dot{\theta}^2\\ 2\dot{R}\dot{\theta}\end{array}\right\}\right). \tag{15.8}$$
Note that in eqn. (15.8) we need $\dot{R}$ and $\dot{\theta}$ in order to compute $\ddot{R}$ and $\ddot{\theta}$. This, however, is no problem since we have $\dot{R}$ and $\dot{\theta}$ from eqn. (15.7). Of course, R and θ are required too, which are easily computed as $R = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}(y/x)$.

SAMPLE 15.3 Velocity in path coordinates. The path of a particle, stuck at the edge of a disk rolling on a level ground with constant speed, is called a cycloid. The parametric equations of a cycloid described by a particle is $x = t - \sin t$, $y = 1 - \cos t$ where t is a dimensionless time. Find the velocity of the particle at

1. $t = \frac{\pi}{2}$,
2. $t = \pi$, and
3. $t = 2\pi$

and express the velocity in terms of path basis vectors $(\hat{\boldsymbol{e}}_t, \hat{\boldsymbol{e}}_n)$.

Solution The position of the particle is given:

$$\begin{aligned} \vec{\boldsymbol{r}} &= x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \\ &= (t - \sin t)\hat{\boldsymbol{\imath}} + (1 - \cos t)\hat{\boldsymbol{\jmath}} \\ \Rightarrow \quad \vec{\boldsymbol{v}} &\equiv \frac{d\vec{\boldsymbol{r}}}{dt} \qquad (15.9) \\ &= (1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}, \\ \text{and} \quad v &= |\vec{\boldsymbol{v}}| \qquad (15.10) \\ &= \sqrt{(1 - \cos t)^2 + \sin^2 t} \\ &= \sqrt{2 - 2\cos t}. \qquad (15.11) \end{aligned}$$

In terms of path basis vectors, the velocity is given by

$$\vec{\boldsymbol{v}} = v\hat{\boldsymbol{e}}_t \quad \text{where} \quad \hat{\boldsymbol{e}}_t = \vec{\boldsymbol{v}}/v.$$

Here,

$$\hat{\boldsymbol{e}}_t = \frac{(1 - \cos t)\hat{\boldsymbol{\imath}} + \sin t\,\hat{\boldsymbol{\jmath}}}{\sqrt{2 - 2\cos t}}. \qquad (15.12)$$

Substituting the values of t in equations 15.11 and 15.12 we get

1. at $t = \frac{\pi}{2}$:

$$v = \sqrt{2}, \quad \hat{\boldsymbol{e}}_t = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \quad \vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t.$$

$$\boxed{\vec{\boldsymbol{v}} = \sqrt{2}\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

2. at $t = \pi$:

$$v = 2, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}, \quad \vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t.$$

$$\boxed{\vec{\boldsymbol{v}} = 2\hat{\boldsymbol{e}}_t, \quad \hat{\boldsymbol{e}}_t = \hat{\boldsymbol{\imath}}}$$

3. at $t = 2\pi$:

$$v = 0, \quad \hat{\boldsymbol{e}}_t = \text{undefined}, \quad \vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}.$$

$$\boxed{\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}}$$

SAMPLE 15.4 Path coordinates in 2-D. A particle traverses a limacon $R = (1 + 2\cos\theta)$ ft, with constant angular speed $\dot{\theta} = 3\,\text{rad/s}$.

1. Find the normal and tangential accelerations (a_t and a_n) of the particle at $\theta = \frac{\pi}{2}$.
2. Find the radius of the osculating circle and draw the circle at $\theta = \frac{\pi}{2}$.

Solution

1. The equation of the path is

$$R = (1 + 2\cos\theta)\,\text{ft}.$$

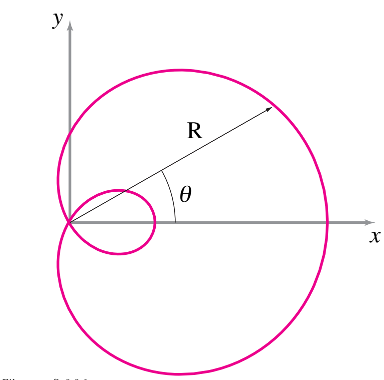

Figure 15.14: The limacon $R = (1 + 2\cos\theta)$ ft.

The path is shown in Fig. 15.14. Since the equation of the path is given in polar coordinates, we can calculate the velocity and acceleration using the polar coordinate formulae:

$$\begin{aligned} \vec{v} &= \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta && (15.13) \\ \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta. && (15.14) \end{aligned}$$

So, we need to find $\dot{R}$, $\ddot{R}$, $\ddot{\theta}$ for computing $\vec{v}$ and $\vec{a}$. From the given equation for R

$$\begin{aligned} R &= (1 + 2\cos\theta)\,\text{ft} \\ \Rightarrow \quad \dot{R} &= -(2\,\text{ft})\sin\theta\,\dot{\theta} \\ \Rightarrow \quad \ddot{R} &= -(2\,\text{ft})\sin\theta\,\ddot{\theta} - (2\,\text{ft})\cos\theta\,\dot{\theta}^2 \\ &= -(2\,\text{ft})\,\dot{\theta}^2\cos\theta \end{aligned}$$

where we set $\ddot{\theta} = 0$ because $\dot{\theta}$ = constant. Substituting these expressions in Eqn. (15.13) and (15.14), we get

$$\begin{aligned} \vec{v} &= -(2\,\text{ft})\dot{\theta}\sin\theta\hat{e}_R + \dot{\theta}(1 + \cos\theta)\,\text{ft}\hat{e}_\theta \\ \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \\ &= [-2\dot{\theta}^2\cos\theta - (1 + 2\cos\theta)\,\dot{\theta}^2]\,\text{ft}\,\hat{e}_R + (-2\dot{\theta}^2\sin\theta)\,\text{ft}\,\hat{e}_\theta \\ &= -\dot{\theta}^2[(1 + 4\cos\theta)\,\hat{e}_R + (2\sin\theta)\,\hat{e}_\theta]\,\text{ft} \end{aligned}$$

which give velocity and acceleration at any θ. Now substituting $\theta = \pi/2$ we get the velocity and acceleration at the desired point:

$$\begin{aligned} \vec{v}|_{\frac{\pi}{2}} &= \dot{\theta}[-2\sin\frac{\pi}{2}\hat{e}_R + (1 + 2\cos\frac{\pi}{2})\hat{e}_\theta]\,\text{ft} \\ &= 3\,\text{ft/s}(-2\hat{e}_R + \hat{e}_\theta) \\ \vec{a}|_{\frac{\pi}{2}} &= -9\,\text{ft/s}^2(\hat{e}_R + 2\hat{e}_\theta). \end{aligned}$$

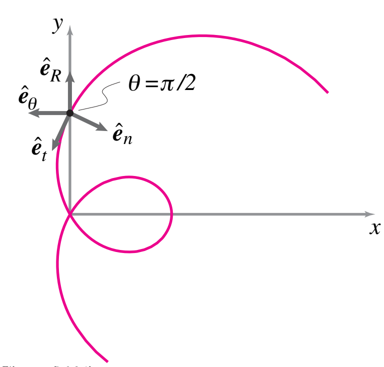

Figure 15.15: The unit vectors $\hat{e}_R$, $\hat{e}_\theta$, and $\hat{e}_t$, $\hat{e}_n$ at $\theta = \pi/2$.

Thus we know the velocity and the acceleration of the particle in polar coordinates. Now we proceed to find the tangential and the normal components of acceleration (acceleration in path coordinates). In path coordinates

$$\begin{aligned} \vec{a} &= \vec{a}_t + \vec{a}_n \\ &\equiv a_t\hat{e}_t + a_n\hat{e}_n \end{aligned}$$

where $\hat{e}_t$ and $\hat{e}_n$ are unit vectors in the directions of the tangent and the principal normal of the path. We compute these unit vectors as follows.

$$
\begin{aligned}
\hat{\boldsymbol{e}}_t &= \frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|} \\
&= (\frac{-6\,\text{ft/s})\hat{\boldsymbol{e}}_R + (3\,\text{ft/s})\hat{\boldsymbol{e}}_\theta}{\sqrt{45}\ \text{ft/s}} \\
&= -\frac{2}{\sqrt{5}}\,\hat{\boldsymbol{e}}_R + \frac{1}{\sqrt{5}}\,\hat{\boldsymbol{e}}_\theta .
\end{aligned}
$$

So,

$$
\begin{aligned}
\vec{\boldsymbol{a}}_t &= (\vec{\boldsymbol{a}}\cdot\hat{\boldsymbol{e}}_t)\,\hat{\boldsymbol{e}}_t \\
&= 9\,\text{ft/s}^2\left(-\frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right)\hat{\boldsymbol{e}}_t \\
&= \vec{\boldsymbol{0}},
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{\boldsymbol{a}}_n &= \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_t \\
&= -9\,\text{ft/s}^2(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\hat{\boldsymbol{e}}_n &= \frac{\vec{\boldsymbol{a}}_n}{|\vec{\boldsymbol{a}}_n|} \\
&= -\frac{1}{\sqrt{5}}(\hat{\boldsymbol{e}}_R + 2\hat{\boldsymbol{e}}_\theta).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\vec{\boldsymbol{a}} &= 9\sqrt{5}\ \text{ft/s}^2\,\hat{\boldsymbol{e}}_n \\
\Rightarrow \quad a_t &= 0 \quad \text{and} \quad a_n = 20.12\,\text{ft/s}^2.
\end{aligned}
$$

$a_t = 0,\ a_n = 20.12\,\text{ft/s}^2$

2. In path coordinates the acceleration is also expressed as

$$
\vec{\boldsymbol{a}} = \dot{v}\hat{\boldsymbol{e}}_t + \frac{v^2}{\rho}\hat{\boldsymbol{e}}_n
$$

where ρ is the radius of the osculating circle. Since we already know the speed v and the normal component of acceleration a_n we can easily compute the radius of the osculating circle.

$$
\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
\Rightarrow \quad \rho &= \frac{v^2}{a_n} = \frac{45(\text{ ft/s})^2}{9\sqrt{5}\,\text{ft/s}^2} = \sqrt{5}\,\text{ft}.
\end{aligned}
$$

$\rho = 2.24\,\text{ft}$

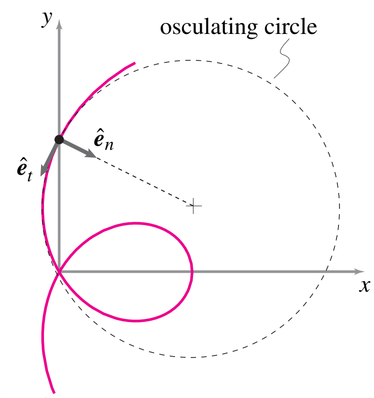

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Figure 15.16: The osculating circle of radius $\rho = 5^{1/2}$ ft at $\theta = \pi/2$. Note that $\hat{\boldsymbol{e}}_n$ points to the center of the osculating circle.

15.2 Rotating reference frames and their time-varying base vectors

In this section you will learn about rotating reference frames, how to take the derivative of a vector 'in' a rotating frame, and how to use that derivative to find the derivative in a Newtonian or fixed frame. We start by showing the alternative, just using one frame with one set of fixed base vectors.

The fixed base-vector method

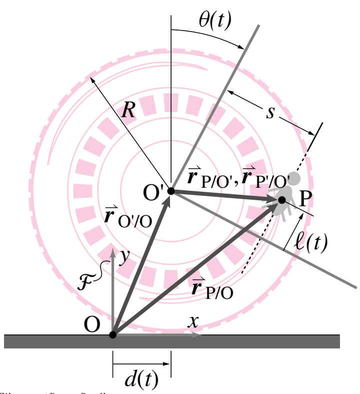

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Figure 15.17: A bug walks at constant speed on a straight line (dotted) marked on a tire that is rotating clockwise (θ relative to a vertical line). The distance of the line from the center of the tire is s the distance the bug has walked from the closest point to the tire center is ℓ. The fixed base vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are aligned with the fixed x and y axis. The line segment connecting the tire center with the closest point on the line is in the direction $\cos\theta\,\hat{\boldsymbol{\imath}} - \sin\theta\,\hat{\boldsymbol{\jmath}}$ the bug is pointing on the line is $\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$.

To motivate the sections that follow we first show the "fixed base vector" method. Consider the task of determining the acceleration of a bug walking at constant speed as it walks on a straight line marked on the surface of a tire rolling at constant rate. Artificial as this problem seems, it is similar to the sort of calculation needed in the kinematics of mechanisms. For now, imagine you really care how strong the bugs legs need to be to hold on (unreasonably neglecting air friction). So knowing the bugs acceleration determines the net force on it by $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Now we try to find $\vec{\boldsymbol{a}}$ by taking two time derivatives of position.

If we want to avoid using rotating base vectors we have to write an expression for the position of the bug in terms of x and y components. Choosing a suitable origin of the coordinate system we have

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{P/0} &= \vec{\boldsymbol{r}}_{0'/0} + \vec{\boldsymbol{r}}_{P/0'} \\
&= \underbrace{(R\theta\hat{\boldsymbol{\imath}} + R\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_{0'/0}} + \underbrace{(s(\cos\theta\,\hat{\boldsymbol{\imath}} - \sin\theta\,\hat{\boldsymbol{\jmath}}) + \ell(\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}))}_{\vec{\boldsymbol{r}}_{P/0'}} \\
&= \big(R\theta + s\cos\theta + \ell\sin\theta\big)\,\hat{\boldsymbol{\imath}} + \big(R - s\sin\theta + \ell\cos\theta\big)\,\hat{\boldsymbol{\jmath}}. \qquad (15.15)
\end{aligned}
$$

To find the velocity we take the time derivative, taking account that both θ and ℓ are functions of t. Thus, for example looking at the term $\ell\cos\theta$ both the product rule and chain rule need be applied. Proceeding we get

$$
\begin{aligned}
\vec{\boldsymbol{v}}_{P/0} &= \Big(R\dot\theta - s\dot\theta\sin\theta + \dot\ell\sin\theta + \ell\dot\theta\cos\theta\Big)\,\hat{\boldsymbol{\imath}} \\
&\quad + \Big(-s\dot\theta\cos\theta + \dot\ell\cos\theta - \ell\dot\theta\sin\theta\Big)\,\hat{\boldsymbol{\jmath}}. \qquad (15.16)
\end{aligned}
$$

To get the acceleration of the bug we differentiate one more time. This time we use the product rule and chain rule again, but get to use the simplification for this problem that the rolling and bug walking are at constant rate so $\ddot\theta = 0$ and $\ddot\ell = 0$:

$$
\begin{aligned}
\vec{\boldsymbol{a}}_{P/0} &= \Big(-s\dot\theta^2\cos\theta + \dot\ell\dot\theta\cos\theta + \dot\ell\dot\theta\cos\theta - \ell\dot\theta^2\sin\theta\Big)\,\hat{\boldsymbol{\imath}} \\
&\quad + \Big(\ s\dot\theta^2\sin\theta - \dot\ell\dot\theta\sin\theta - \dot\ell\dot\theta\sin\theta - \ell\dot\theta^2\cos\theta\Big)\,\hat{\boldsymbol{\jmath}} \\
&= \Big(-\dot\theta^2(s\cos\theta + \ell\sin\theta) + 2\dot\ell\dot\theta\cos\theta\Big)\,\hat{\boldsymbol{\imath}} \\
&\quad + \Big(\ \dot\theta^2(s\sin\theta - \ell\cos\theta) - 2\dot\ell\dot\theta\sin\theta\Big)\,\hat{\boldsymbol{\jmath}} \qquad (15.17)
\end{aligned}
$$

which is a bit of a mess. We could regroup the terms, but there would still be 6 of them.

The moving-reference-frame methods that follow don't change this answer. But they give a somewhat simpler derivation. And they also group the terms in physically meaningful way. One would be hard pressed to make sense of all the terms in eqn. (15.17). With the time-varying base vector methods below we can interpret the terms.

Reference frames

A *reference frame* is a coordinate system①. It has an origin and a set of preferred mutually orthogonal directions represented by base vectors. You can think of a reference frame as a giant piece of graph paper, or in 3-D as a giant jungle gym, that permeates space. It has the look of a wire *frame*. Because we will use various frames, we name them. We always have one frame that we think of as *fixed* for the purposes of Newtonian mechanics. We call this frame $\mathcal{F}$ (or sometimes $\mathcal{N}$). Most often we choose a frame that is 'glued' to the ground with an origin at a convenient point and with at least one base vector lined up with something convenient (e.g., up, sideways, along a slope, along the edge of an important part, *etc.*). $\mathcal{F}$ is a frame in which the mechanics laws we use are accurate. We define it by its origin and the direction of its coordinate axes, thus we would write

$$\mathcal{F} \text{ is } 0xyz \quad \text{or} \quad \mathcal{F} \text{ is } O\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}}.$$

where we would generally have a picture showing the position of the origin and the orientation of the coordinate axes (see fig. 15.18).

When we write casually 'position $\vec{\boldsymbol{r}}$' of a point we mean $\vec{\boldsymbol{r}}_{P/0}$. When we write 'velocity $\vec{\boldsymbol{v}}$' we mean $\frac{d}{dt}\vec{\boldsymbol{r}}$ as calculated in $\mathcal{F}$. That is, if $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$ then we define the derivative of $\vec{\boldsymbol{r}}$ with respect to t in $\mathcal{F}$ as

$$\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}}{dt} = \overset{\mathcal{F}\bullet}{\vec{\boldsymbol{r}}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}.$$

The script $\mathcal{F}$ shows explicitly that when we take the time derivative of the vector we take the time derivative of its components, using the components associated with $\mathcal{F}$ and holding constant the base vectors associated with $\mathcal{F}$. That is

$\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{P/0}}{dt} = \overset{\mathcal{F}\bullet}{\vec{\boldsymbol{r}}}_{P/0}$ are just fancy ways of writing what we have been calling $\vec{\boldsymbol{v}}$.

The elaborate notation just makes explicit how $\vec{\boldsymbol{v}}$ is defined. The only need for this elaborate notation is if there is ambiguity. There is only ambiguity if more than one reference frame is used in a given problem.

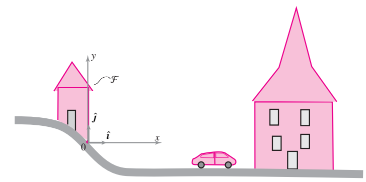
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Figure 15.18: A fixed reference frame $\mathcal{F}$ is defined by an origin 0 and coordinate axes xyz or base vectors $\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}\hat{\boldsymbol{k}}$. Once the xy (or $\hat{\boldsymbol{\imath}}\hat{\boldsymbol{\jmath}}$) directions are chosen the z (or $\hat{\boldsymbol{k}}$) direction is implicitly defined by the right hand rule.

①**A fine point for experts: reference frame *vs* coordinate system.** There is a semantic debate about the degree to which the phrases "coordinate system" and " reference frame" are synonymous. For simplicity we take them to mean the same thing. An alternative definition distinguishes reference frames from coordinate systems. It turns out that coordinate systems which are rotated, but not rotating, with respect to each other both calculate the same time-derivative of a given vector. Because these coordinate systems are equivalent in this regard they are sometimes called the same reference frame. That is, some people consider one reference frame to be the set of all coordinate systems that are glued to each other, no matter what their position or orientation. In this way of thinking, a frame is made manifest by the use of one of its coordinate systems, but no particular coordinate system is unique to the frame.

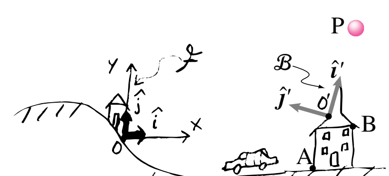

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Figure 15.19: A second reference frame $\mathcal{B}$ is defined by an origin 0' and coordinate axes $x'y'z'$ or base vectors $\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'\hat{\boldsymbol{k}}'$. Once the $x'y'$ (or $\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'$) directions are chosen the z' (or $\hat{\boldsymbol{k}}'$) direction is implicitly defined by the right hand rule.

Using more than one reference frame

Let's add a second reference frame called $\mathcal{B}$ glued to and oriented with the roof of the building. We will always use script capital letters ($\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}$ or $\mathcal{N}$) to name reference frames. We define $\mathcal{B}$ by writing

$$\mathcal{B} \text{ is } 0'x'y'z' \quad \text{or} \quad \mathcal{B} \text{ is } O'\hat{\boldsymbol{\imath}}'\hat{\boldsymbol{\jmath}}'\hat{\boldsymbol{k}}'$$

and by drawing a picture (see fig. 15.19). This new frame, as we have drawn it, is also a good Newtonian or fixed frame. So we could write all positions using the $\mathcal{B}$ coordinates and base vectors and then proceed with all of our mechanics equations with the only confusion being that gravity doesn't point in the $-\hat{\boldsymbol{\jmath}}'$ direction, but in some crooked direction relative to $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ (which we would have to work out from the angle of the roof). Although one hardly notices when using just a single fixed frame, we actually use frames for three somewhat distinct purposes:

I. To define a vector. For example if we were tracking the motion of a canon ball at P we could define its position vector $\vec{\boldsymbol{r}}$ as $\vec{\boldsymbol{r}}_{\mathrm{P}/0}$, using frame $\mathcal{F}$ to define $\vec{\boldsymbol{r}}$. Or we could define $\vec{\boldsymbol{r}}$ as $\vec{\boldsymbol{r}}_{\mathrm{P}/\mathrm{O}'}$ using frame $\mathcal{B}$ to define $\vec{\boldsymbol{r}}$.

II. To assign coordinate values to a given vector. For example, the vector $\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}$ could be written as

$$\begin{aligned} \vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}} &= \vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}, \\ 4\,\mathrm{m}\hat{\boldsymbol{\imath}} + 5\,\mathrm{m}\hat{\boldsymbol{\jmath}} &= 6\,\mathrm{m}\hat{\boldsymbol{\imath}}' - 2.24\,\mathrm{m}\hat{\boldsymbol{\jmath}}'. \end{aligned}$$

Alternatively, if we just want to look at the components of a given vector we use $[]_{\mathcal{F}}$ to indicate the components of the vector in []s using the base vectors of $\mathcal{F}$. Thus

$$[\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}]_{\mathcal{F}} = [4\,\mathrm{m}, \quad 5\,\mathrm{m}]' \quad \text{and} \quad [\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}]_{\mathcal{B}} = [6\,\mathrm{m}, \quad -2.23\,\mathrm{m}]'$$

where we have used $[]'$ to put the components in their standard column form (though this is a picky detail). Note that although $\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}} = \vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}$ that

$$\begin{aligned} [\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}]_{\mathcal{F}} &\neq [\vec{\boldsymbol{r}}_{\mathrm{B}/\mathrm{A}}]_{\mathcal{B}} \\ \text{because} \quad \begin{bmatrix} 4\,\mathrm{m} \\ 5\,\mathrm{m} \end{bmatrix} &\neq \begin{bmatrix} 6\,\mathrm{m} \\ 2.23\,\mathrm{m} \end{bmatrix} \end{aligned}$$

III. To find the rate of change of a given vector. The position of P relative to A changes with time. We can calculate this rate of change two different ways. First using frame $\mathcal{F}$

$$\begin{aligned} \frac{{}^{\mathcal{F}}d\vec{\boldsymbol{v}}_{\mathrm{P}/\mathrm{A}}}{dt} = \overset{\mathcal{F}\bullet}{\vec{\boldsymbol{r}}}_{\mathrm{P}/\mathrm{A}} &= \left(\frac{d}{dt}x_{\mathrm{P}/\mathrm{A}}\right)\hat{\boldsymbol{\imath}} + \left(\frac{d}{dt}y_{\mathrm{P}/\mathrm{A}}\right)\hat{\boldsymbol{\jmath}} \\ \text{or more informally as} \quad \overset{\mathcal{F}\bullet}{\vec{\boldsymbol{r}}} &= \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} \end{aligned}$$

if we are clear in our minds that x and y are the coordinates of P relative to A. But we can also calculate the rate of change of the same vector

$\vec{r}_{P/A}$ using frame $\mathcal{B}$ as

$$\frac{{}^{\mathcal{B}}d\vec{r}_{P/A}}{dt} = \overset{\mathcal{B}\bullet}{\vec{r}}_{P/A} = \left(\frac{d}{dt}x'_{P/A}\right)\hat{\imath}' + \left(\frac{d}{dt}y'_{P/A}\right)\hat{\jmath}'$$

or more informally as $\overset{\mathcal{B}\bullet}{\vec{r}} = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'.$

For the two frames $\mathcal{F}$ and $\mathcal{B}$ $\overset{\mathcal{B}\bullet}{\vec{r}} = \overset{\mathcal{F}\bullet}{\vec{r}}$ because the two frames are not rotating relative to each other. Specifically, for $\mathcal{F}$ and $\mathcal{B}$ the formula for finding x and y from x' and y' does not involve time. Similarly, the formulas for finding $\hat{\imath}'$ and $\hat{\jmath}'$ from $\hat{\imath}$ and $\hat{\jmath}$ do not involve time. For frames that are *rotated* with respect to each other but not *rotating*, the two time derivatives of a given vector are related the same way the vector itself is related to itself in the two frames. The vectors are the same but their coordinates are different. That is, for rotated but not relatively rotating frames

$$\begin{array}{llll} & \vec{r}_{P/A} = \vec{r}_{P/A} & \text{and} & \dfrac{{}^{\mathcal{F}}d\vec{r}_{P/A}}{dt} = \dfrac{{}^{\mathcal{B}}d\vec{r}_{P/A}}{dt} \\ \text{but} & [\vec{r}_{P/A}]_{\mathcal{F}} \neq [\vec{r}_{P/A}]_{\mathcal{B}} & \text{and} & [\dfrac{{}^{\mathcal{F}}d\vec{r}_{P/A}}{dt}]_{\mathcal{F}} \neq [\dfrac{{}^{\mathcal{F}}d\vec{r}_{P/A}}{dt}]_{\mathcal{B}}. \end{array}$$

Going back and forth between these three uses of frames with ease is one of the advanced skills of a person who can analyze the dynamics of complex systems (And being confused about the distinctions is an almost universal part of learning advanced dynamics).

Example: Two fixed frames $\mathcal{F}$ and $\mathcal{B}$
Consider $\mathcal{F}$ and $\mathcal{B}$ both to be fixed to the ground. Let's look at $\vec{r}_{P/A}$ where P is moving at up at constant rate (see fig. 15.20). First look at the position using both frames:

$$\begin{array}{rcll} \vec{r}_{P/A} & = & \vec{r}_{P/A} & \text{and} \\ ct\,\hat{\jmath} & = & \left(\sqrt{2}ct/2\right)\hat{\imath}' + \left(\sqrt{2}ct/2\right)\hat{\jmath}' & \text{but} \\ [\vec{r}_{P/A}]_{\mathcal{F}} = [ct,\quad 0]' & \neq & [\vec{r}_{P/A}]_{\mathcal{B}} = [\sqrt{2}ct/2,\quad \sqrt{2}ct/2]'. & \end{array}$$

Now look at the rate of change of position using both frames. First $\mathcal{F}$:

$$\begin{aligned} \frac{{}^{\mathcal{F}}d\vec{r}_{P/A}}{dt} = \overset{\mathcal{F}\bullet}{\vec{r}}_{P/A} &= \left(\frac{d}{dt}x_{P/A}\right)\hat{\imath} + \left(\frac{d}{dt}y_{P/A}\right)\hat{\jmath} \\ &= c\,\hat{\jmath} \end{aligned}$$

Then the rate of change of $\vec{r}_{B/A}$ as calculated in $\mathcal{B}$:

$$\begin{aligned} \frac{{}^{\mathcal{B}}d\vec{r}_{P/A}}{dt} = \overset{\mathcal{B}\bullet}{\vec{r}}_{P/A} &= \left(\frac{d}{dt}x'_{P/A}\right)\hat{\imath}' + \left(\frac{d}{dt}y'_{P/A}\right)\hat{\jmath}' \\ &= (\sqrt{2}c/2)\hat{\imath}' + (\sqrt{2}c/2)\hat{\jmath}' = c\,\hat{\jmath} \end{aligned}$$

You can quickly verify that $\frac{{}^{\mathcal{F}}d\vec{r}_{P/A}}{dt} = \frac{{}^{\mathcal{B}}d\vec{r}_{P/A}}{dt}$ by noting that $\hat{\imath}' = \sqrt{2}(\hat{\imath} + \hat{\jmath})/2$ and $\hat{\jmath}' = \sqrt{2}(-\hat{\imath} + \hat{\jmath})/2$.

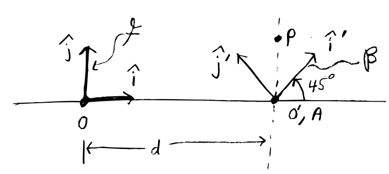

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Figure 15.20: A particle P is moving up at speed c. Two fixed frames $\mathcal{F}$ and $\mathcal{B}$ are used to keep track of it.

So long as $\mathcal{B}$ is not rotating with respect to $\mathcal{F}$ then the rate of change of a given vector is the same in both reference frames.

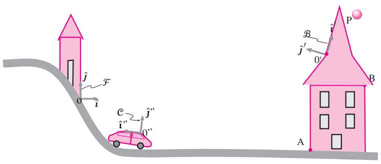

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Figure 15.21: A third reference frame $\mathcal{C}$ is defined by an origin $0''$ and coordinate axes $x''y''z''$ or base vectors $\hat{\imath}''\hat{\jmath}''\hat{k}''$. In this case $\mathcal{C}$ is moving and rotating with respect to $\mathcal{F}$ and $\mathcal{B}$. Once the $x''y''$ (or $\hat{\imath}''\hat{\jmath}''$) directions are chosen the z'' (or $\hat{k}''$) direction is implicitly defined by the right hand rule.

Translating and rotating reference frames

Now look at a third reference frame $\mathcal{C}$ that is glued to the roof of the car as it starts up hill (see fig. 15.21). We define $\mathcal{C}$ by the origin of its coordinate system $0''$ and its time-varying base vectors $\hat{\imath}''$ and $\hat{\jmath}''$. The issues with **defining** a vector with $\mathcal{C}$ and with **writing components** using $\mathcal{C}$ are the same as for $\mathcal{B}$. However taking the **time derivative** of a given vector in $\mathcal{C}$ is different then taking the time derivative in $\mathcal{B}$ or $\mathcal{F}$ because $\mathcal{C}$ is rotating relative to them.

Rate of change of a vector relative to a rotating frame: the $\dot{\vec{Q}}$ formula

Because dynamics involves the time derivatives of so many different vectors (e.g. $\vec{r}$, $\vec{v}$, $\vec{L}$, $\vec{H}_{/\mathrm{C}}$, and $\vec{\omega}$) it is easier to think about the derivative of some arbitrary or general vector, call it $\vec{Q}$, and then apply what we learn to these other vectors.

Recalling our three uses of frames:

I. To define a vector.

II. To express the coordinates of a given vector.

III. To take the time derivative of a vector.

we see that items [III.] and [I.] can be combined. That is, once a vector $\vec{Q}$ is defined clearly by some means then we can define a new vector as the derivative of that vector in, say, moving frame $\mathcal{C}$. Once this new vector ${}^{\mathcal{C}}\dot{\vec{Q}}$ is defined it can be expressed in terms of the coordinates of any convenient frame.

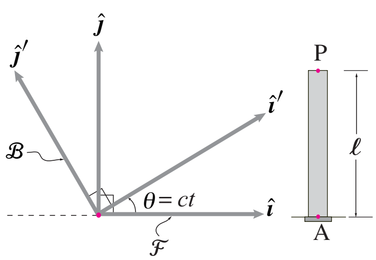

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Figure 15.22: A fixed frame $\mathcal{F}$ and a rotating frame $\mathcal{B}$ both keep track of the position vector $\vec{r}_{\mathrm{P/A}}$.

Example: Derivative in a moving frame of a constant vector
Consider as $\vec{Q}$ the relative position vector $\vec{r}_{\mathrm{P/A}}$ of the points A and P that do not move in the fixed frame $\mathcal{F}$. That is, the points A and B don't move in the ordinary sense of the words (see fig. 15.22). Now also look at the frame $\mathcal{B}$ that is rotating with respect to $\mathcal{F}$ at the rate $\dot{\theta}$. We have

$$\begin{aligned} \vec{r}_{\mathrm{P/A}} &= \vec{r}_{\mathrm{P/A}} \\ \ell\hat{\jmath} &= \ell\sin\theta\hat{\imath}' + \ell\cos\theta\hat{\jmath}' \end{aligned}$$

So we can now calculate the derivative in each frame by holding the corresponding base vectors as constant. So

$$
\begin{aligned}
\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} &= \dot{\ell}\hat{\boldsymbol{j}} = \vec{\boldsymbol{0}} \\
\text{and} \quad \frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} &= \ell\dot{\theta}\cos\theta\,\hat{\boldsymbol{\imath}}' - \ell\dot{\theta}\sin\theta\,\hat{\boldsymbol{\jmath}}' \\
&= \ell\dot{\theta}\,\underbrace{\left(\cos\theta\,\hat{\boldsymbol{\imath}}' - \sin\theta\,\hat{\boldsymbol{\jmath}}'\right)}_{\hat{\boldsymbol{\imath}}} \\
&= \ell\dot{\theta}\hat{\boldsymbol{\imath}}
\end{aligned}
$$

That is, the stationary vector $\vec{\boldsymbol{r}}_{\mathrm{P/A}}$ and the rotating frame $\mathcal{B}$ define a new vector, the derivative of $\vec{\boldsymbol{r}}_{\mathrm{P/A}}$ in $\mathcal{B}$. This is also $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}} - \vec{\boldsymbol{v}}_{\mathrm{A}/\mathcal{B}}$, the difference between the velocity of P and the velocity of A in the frame $\mathcal{B}$. This new vector can be expressed in any coordinate system of choice for example the $\hat{\boldsymbol{\imath}}\,\hat{\boldsymbol{\jmath}}$ system. So we wrote above

$$\frac{{}^{\mathcal{B}}d\vec{\boldsymbol{r}}_{\mathrm{P/A}}}{dt} = \ell\dot{\theta}\hat{\boldsymbol{\imath}}$$

which looks mixed up but isn't. The frame $\mathcal{B}$ is used to help define a vector which is then expressed in the coordinates of $\mathcal{F}$.

Using the moving-frame derivative to calculate the fixed-frame derivative

Given a new vector $\overset{\mathcal{C}\,\bullet}{\vec{\boldsymbol{Q}}}$, the derivative of $\vec{\boldsymbol{Q}}$ as calculated in a rotating frame $\mathcal{C}$, one calculation of common use is the determination of the derivative $\overset{\mathcal{F}\,\bullet}{\vec{\boldsymbol{Q}}}$ of the same vector in the fixed frame $\mathcal{F}$.

First think of a line segment that is marked between two points that are glued to a moving frame $\mathcal{C}$. We know (at least in 2-D and for fixed axis rotation) that

$$\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}.$$

Likewise for any vector which is fixed in $\mathcal{C}$. It is especially useful to apply this formula to unit base vectors, so

$$
\begin{aligned}
\dot{\hat{\boldsymbol{\imath}}}'' &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}'', \\
\dot{\hat{\boldsymbol{\jmath}}}'' &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}'', \quad \text{and} \\
\dot{\hat{\boldsymbol{k}}}'' &= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{k}}''.
\end{aligned}
\tag{15.18}
$$

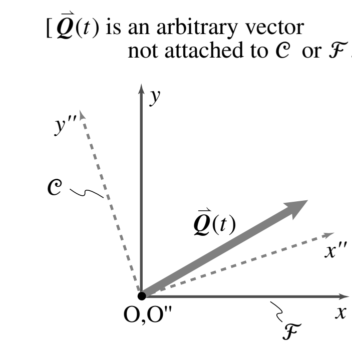

Figure 15.23: A fixed frame $\mathcal{F}$ defined by $Oxyz$ and a moving frame $\mathcal{C}$ defined by $O'x'y'z'$. Also shown is an arbitrary vector $\vec{\boldsymbol{Q}}$ which changes relative to both frames.

In some minds, Eqns. 15.18 are the core of rigid body kinematics. Box 15.1 on 863 shows how these relations give 'the Q dot' formula: For any time dependent vector $\vec{\boldsymbol{Q}}$

$$\overset{\mathcal{F}\,\bullet}{\vec{\boldsymbol{Q}}} = \overset{\mathcal{C}\,\bullet}{\vec{\boldsymbol{Q}}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/F} \times \vec{\boldsymbol{Q}}. \tag{15.19}$$

or more simply, but less explicitly,

$$\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}.$$

where $\vec{Q}_{rel}$ is the time derivative of $\vec{Q}$ relative to the moving frame of interest (in this case $\mathcal{C}$). The 'Q dot' formula says that

The derivative of a vector with respect to a Newtonian frame $\mathcal{F}$ (or 'absolute derivative') can be calculated as the derivative $\overset{\mathcal{C}\bullet}{\vec{Q}}$ of the vector with respect to a moving frame $\mathcal{C}$, plus a term $\vec{\omega}_{\mathcal{C}} \times \vec{Q}$ that corrects for the rotation of frame $\mathcal{C}$ relative to frame $\mathcal{F}$.

Note that if $\vec{Q}$ is a constant in the frame $\mathcal{C}$, like the relative position vector of two points glued to $\mathcal{C}$, then $\overset{\mathcal{C}\bullet}{\vec{Q}} = \vec{Q}_{rel} = \vec{0}$ and the $\dot{\vec{Q}}$ formula reduces to

$$\overset{\mathcal{F}\bullet}{\vec{Q}} = \vec{\omega} \times \vec{Q}.$$

The $\dot{\vec{Q}}$ formula 15.19 is useful for the derivation of a variety of formulas and is also useful in the solution of problems.

While we have shown how to use this formula to calculate the rate of change of a vector with respect to a Newtonian frame, the formula can be used to calculate its rate of change with respect to a non-Newtonian frame. Letting $\mathcal{A}$ and $\mathcal{B}$ be two possibly non-Newtonian frames, the $\dot{\vec{Q}}$ formula for the rate of change of $\vec{Q}$ with respect to frame $\mathcal{A}$ is

$$\overset{\mathcal{A}\bullet}{\vec{Q}} = \overset{\mathcal{B}\bullet}{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{A}} \times \vec{Q}. \tag{15.20}$$

Both $\mathcal{A}$ and $\mathcal{B}$ could be non-Fixed (non-Newtonian).

Summary of the $\dot{\vec{Q}}$ formula

For a vector $\vec{Q}$ fixed in $\mathcal{B}$,

$$\dot{\vec{Q}} = \vec{\omega}_{\mathcal{B}} \times \vec{Q} \qquad \text{or} \qquad \overset{\mathcal{F}\bullet}{\vec{Q}} = \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

For any time dependent vector $\vec{Q}$,

$$\dot{\vec{Q}} = \overset{\mathcal{B}\bullet}{\vec{Q}} + \vec{\omega}_{\mathcal{B}} \times \vec{Q} \qquad \text{or} \qquad \overset{\mathcal{F}\bullet}{\vec{Q}} = \overset{\mathcal{B}\bullet}{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}.$$

Some examples of applying the $\dot{\vec{Q}}$ formula are:

- Absolute velocity of a point P relative to O':

$$\dot{\vec{r}}_{P/O'} = \overset{\mathcal{B}\bullet}{\vec{r}}_{P/O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}$$

- Rate of change of a rotating unit vector which is fixed in $\mathcal{B}$:

$$\dot{\hat{\imath}}' = \underbrace{\dot{\hat{\imath}}'_{rel}}_{\vec{\mathbf{0}}} + \vec{\omega}_{\mathcal{B}} \times \hat{\imath}'$$

The varying base-vectors method of computing velocity and acceleration

One way to calculate velocity, acceleration is to express the position of a particle in terms of a combination of based vectors, some of which change in time. Velocity and acceleration are then determined by directly differentiating the expression for position, taking account that the base vectors themselves are changing. This method is sometimes convenient for bodies connected in series, one body to the next, etc. The overall approach is as follows:

1) Glue a coordinate system to every moving body. If needed, also create moving frames that move independently of any particular body.
2) Call the basis vectors associated with these frames $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ for the fixed frame $\mathcal{F}$; $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ for the moving frame $\mathcal{B}$; and $\hat{\imath}''$, $\hat{\jmath}''$, $\hat{k}''$ for the moving frame $\mathcal{C}$, etc.
3) Evaluate all of the relative angular velocities; $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$, $\vec{\omega}_{\mathcal{C}/\mathcal{B}}$, etc. in terms of the scalar angular rates $\dot{\theta}$, $\dot{\phi}$, etc. and the base vectors glued to the frames.
4) Express all of the absolute angular velocities in terms of the relative angular velocities.
5) Differentiate to get the angular accelerations using, for example,

$$\begin{aligned} \dot{\hat{\imath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\imath}' \text{ or} \\ \dot{\hat{\jmath}}'' &= \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}'' \end{aligned}$$

6) Write the position of all points of interest in terms of the various base vectors.
7) Differentiate the position to get the velocities (again using $\dot{\hat{\jmath}}'' = \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}''$, etc.)
8) Differentiate again to get acceleration.

First, reconsider the bug crawling on the tire in fig. 15.24.

Example: Absolute velocity of a point moving relative to a moving frame: Bug crawling on a tire

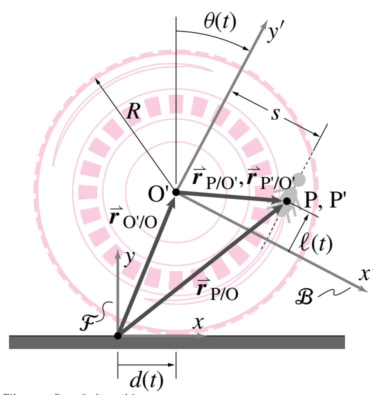

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Figure 15.24: The xyz coordinate system is attached to the fixed frame with basis vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ and the $x'y'z'$ coordinate system is attached to the rolling tire with basis vectors $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$ .The absolute angular velocity of the tire is $\vec{\omega}_{\mathcal{B}/\mathcal{F}} = \vec{\omega}_{\mathcal{B}} = -\dot{\theta}\hat{k}' = -\dot{\theta}\hat{k}$.

We write the position of the bug in terms of the various basis vectors as

$$\begin{aligned}
\vec{r}_{P/O} &= \vec{r}_{O'/O} + \vec{r}_{P/O'} \\
&= \overbrace{d\hat{\imath} + R\hat{\jmath}}^{\vec{r}_{O'/O}} + \overbrace{s\hat{\imath}' + \ell\hat{\jmath}'}^{\vec{r}_{P/O'}}.
\end{aligned}$$

To get the absolute velocity $\frac{{}^{\mathcal{F}}d}{dt}(\vec{r}_{P/O})$ of the bug at the instant shown, we differentiate the position of the bug once, using the product rule and the rates of change of the rotating basis vectors with respect to the fixed frame, to get

$$\begin{aligned}
\dot{\vec{r}}_{P/O} = \vec{v}_P &= \dot{d}\hat{\imath} + \underbrace{\dot{R}}_{0}\hat{\jmath} + \underbrace{\dot{s}}_{0}\hat{\imath}' + s\dot{\hat{\imath}}' + \dot{\ell}\hat{\jmath}' + \ell\dot{\hat{\jmath}}' \\
&= \dot{d}\hat{\imath} + s(\vec{\omega}_{\mathcal{B}} \times \hat{\imath}') + \dot{\ell}\hat{\jmath}' + \ell(\vec{\omega}_{\mathcal{B}} \times \hat{\jmath}') \\
&= \dot{d}\hat{\imath} + s(-\dot{\theta}\hat{k}' \times \hat{\imath}') + \dot{\ell}\hat{\jmath}' + \ell(-\dot{\theta}\hat{k}' \times \hat{\jmath}') \\
&= \dot{d}\hat{\imath} + \ell\dot{\theta}\hat{\imath}' + (\dot{\ell} - s\dot{\theta})\hat{\jmath}'. \qquad (15.21)
\end{aligned}$$

Example: Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again

Differentiating equation 15.21 from the example above again, we get the absolute acceleration $\frac{{}^{\mathcal{F}}d^2}{dt^2}(\vec{r}_{P/O}) = \frac{{}^{\mathcal{F}}d}{dt}(\vec{v}_P)$ of the bug at the instant shown,

$$\begin{aligned}
\ddot{\vec{r}}_{P/O} = \dot{\vec{v}}_P = \vec{a}_P &= \ddot{d}\hat{\imath} + (\ell\ddot{\theta} + \dot{\ell}\dot{\theta})\hat{\imath}' + \ell\dot{\theta}(-\dot{\theta}\hat{k}' \times \hat{\imath}') \\
&\quad + (\ddot{\ell} - s\ddot{\theta} - \underbrace{\dot{s}}_{0}\dot{\theta})\hat{\jmath}' + (\dot{\ell} - s\dot{\theta})(-\dot{\theta}\hat{k}' \times \hat{\jmath}') \\
&= \ddot{d}\hat{\imath} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\ell}\dot{\theta})\hat{\imath}' + (\ddot{\ell} - \ell\dot{\theta}^2 - s\ddot{\theta})\hat{\jmath}'.
\end{aligned}$$

Summary of the varying base-vector method

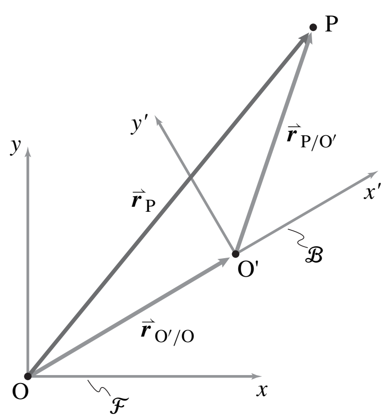

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Figure 15.25: The position of a point P relative to the origin of a fixed frame $\mathcal{F}$ is represented as the sum of two vectors: the position of the new origin relative to the old, and the position of P relative to the new origin. (Here we use "new" as an informal synonym for "moving frame".)

In the varying base vector method, we calculate the velocity of a point by looking at the position as the sum of two position vector, one of which is expressed in the moving base vectors. We then differentiate the position, taking account that the base vectors of the moving frame change with time. In general

$$\begin{aligned}
\vec{v}_P &= \frac{d}{dt}\vec{r}_P \\
&= \frac{d}{dt}\left[\vec{r}_{O'/O} + \vec{r}_{P/O'}\right] \\
&= \frac{d}{dt}\left[(x\hat{\imath} + y\hat{\jmath} + z\hat{k}) + (x'\hat{\imath}' + y'\hat{\jmath}' + z'\hat{k}')\right] \\
&= (\dot{x}\hat{\imath} + \dot{y}\hat{\jmath} + \dot{z}\hat{k}) + (\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' + \dot{z}'\hat{k}') + \\
&\quad \left[x'(\vec{\omega}_{\mathcal{B}} \times \hat{\imath}') + y'(\vec{\omega}_{\mathcal{B}} \times \hat{\jmath}') + z'(\vec{\omega}_{\mathcal{B}} \times \hat{k}')\right]
\end{aligned}$$

We could calculate $\vec{a}_P$ similarly using a combination of the product rule of differentiation and the facts that $\dot{\hat{\imath}}' = \vec{\omega}_{\mathcal{B}} \times \hat{\imath}'$, $\dot{\hat{\jmath}}' = \vec{\omega}_{\mathcal{B}} \times \hat{\jmath}'$, and $\dot{\hat{k}}' = \vec{\omega}_{\mathcal{B}} \times \hat{k}'$, and would get a formula with 15 non-zero terms.

15.1 The $\dot{\vec{Q}}$ formula

We think about some vector $\vec{Q}$ as a quantity that could be represented by an arrow. We can write $\vec{Q}$ using the coordinates of the fixed Newtonian frame with base vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$: $\vec{Q} = Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k}$. Similarly we could write $\vec{Q}$ in terms of the coordinates of some moving and rotating frame $\mathcal{B}$ with base vectors $\hat{\imath}', \hat{\jmath}', \hat{k}'$: $\vec{Q} = Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'$. Now of course

$$\begin{aligned} \vec{Q} &= \vec{Q} \\ \text{so}\quad Q_x\hat{\imath} + Q_y\hat{\jmath} + Q_z\hat{k} &= Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'. \end{aligned}$$

Similarly, $\dot{\vec{Q}} = \dot{\vec{Q}}$ so long as what we mean by $\dot{\vec{Q}}$ is its derivative in a fixed frame. That is, we use $\dot{\vec{Q}}$ as an informal notation for $\overset{\mathcal{F}\bullet}{\vec{Q}} = \frac{{}^{\mathcal{F}}d\vec{Q}}{dt}$. We can calculate $\dot{\vec{Q}}$ the same way we have from the start of the book, namely,

$$\dot{\vec{Q}} = \dot{Q}_x\hat{\imath} + \dot{Q}_y\hat{\jmath} + \dot{Q}_z\hat{k}.$$

We didn't have to use the product rule of differentiation because the unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, associated with a fixed frame, are constant in time.

What if we wanted to use the coordinate information that was given to us by a person who was moving and rotating with the moving frame $\mathcal{B}$? Now we calculate $\dot{\vec{Q}}$ taking account that the $\mathcal{B}$ base vectors change in time.

$$\begin{aligned} \dot{\vec{Q}} &= \frac{d}{dt}[Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'] \\ &= \underbrace{[\dot{Q}_{x'}\hat{\imath}' + \dot{Q}_{y'}\hat{\jmath}' + \dot{Q}_{z'}\hat{k}']}_{\overset{\mathcal{B}\bullet}{\vec{Q}}} \qquad (15.22) \\ &\quad + \underbrace{[Q_{x'}\dot{\hat{\imath}}' + Q_{y'}\dot{\hat{\jmath}}' + Q_{z'}\dot{\hat{k}}']}_{?}. \end{aligned}$$

The first term in the product rule is just the derivative of $\vec{Q}$ in the moving frame $\overset{\mathcal{B}\bullet}{\vec{Q}}$. That is, $\overset{\mathcal{B}\bullet}{\vec{Q}}$ is calculated by differentiating the components in $\mathcal{B}$ holding the base vectors in $\mathcal{B}$ fixed. The second term depends on evaluating $\dot{\hat{\imath}}'$, $\dot{\hat{\jmath}}'$, and $\dot{\hat{k}}'$. We know (at least for 2-D and for fixed-axis rotation in 3-D) that

$$\begin{aligned} \dot{\hat{\imath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\imath}', \\ \dot{\hat{\jmath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\jmath}', \quad \text{and} \\ \dot{\hat{k}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{k}'. \end{aligned} \qquad (15.23)$$

Eqns. 15.23 are the core of rigid body kinematics.

Now we can go back to the second group of terms in Eqn. 15.22.

$$\begin{aligned} ? &= [Q_{x'}\dot{\hat{\imath}}' + Q_{y'}\dot{\hat{\jmath}}' + Q_{z'}\dot{\hat{k}}'] \\ &= [Q_{x'}\vec{\omega}_{\mathcal{B}} \times \hat{\imath}' + Q_{y'}\vec{\omega}_{\mathcal{B}} \times \hat{\jmath}' + Q_{z'}\vec{\omega}_{\mathcal{B}} \times \hat{k}'] \\ &= \vec{\omega}_{\mathcal{B}} \times [Q_{x'}\hat{\imath}' + Q_{y'}\hat{\jmath}' + Q_{z'}\hat{k}'] \\ &= \vec{\omega}_{\mathcal{B}} \times \vec{Q} \end{aligned}$$

Going back to Eqn. 15.22 we get the desired result:

$$\overset{\mathcal{F}\bullet}{\vec{Q}} = \overset{\mathcal{B}\bullet}{\vec{Q}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{Q}. \qquad (15.24)$$

or more simply, but less explicitly,

$$\dot{\vec{Q}} = \dot{\vec{Q}}_{rel} + \vec{\omega} \times \vec{Q}. \qquad (15.25)$$

Geometric 'derivation' of the $\dot{\vec{Q}}$ formula

Here is a geometrical 'derivation' of the $\dot{\vec{Q}}$ formula in two dimensions. Referring to the figure at right, we look at a vector $\vec{Q}$ at two successive times. We then look at how $\vec{Q}$ seems to change in a frame that rotates slightly as $\vec{Q}$ changes. The picture shows how to account for the difference between the change of $\vec{Q}$ as perceived by the two different frames.

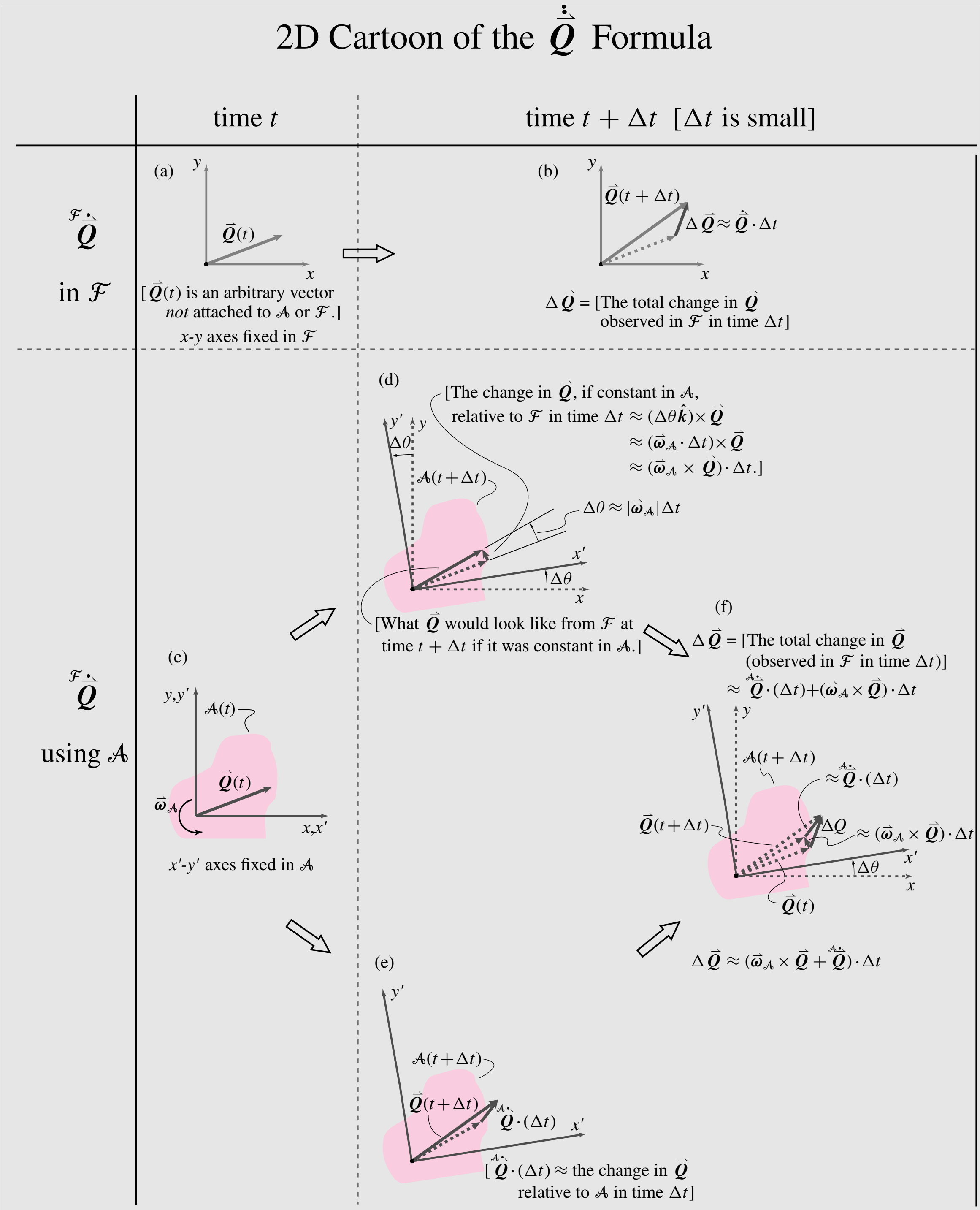
In detail the parts (a) to (e) of the picture show the following.

- Part (a) shows a vector $\vec{Q}$ at time t.
- Part (b) shows $\vec{Q}$ at time $t + \Delta t$ *and* the change in $\vec{Q}$, $\Delta\vec{Q} \approx \dot{\vec{Q}} \cdot \Delta t$.
- Part (c) is like (a) but shows a moving body or frame $\mathcal{A}$.
- Part (d) shows the change in $\vec{Q}$, $(\vec{\omega}_{\mathcal{A}} \times \vec{Q}) \cdot \Delta t$, that would occur if $\vec{Q}$ were fixed (constant) in $\mathcal{A}$.
- Part (e) shows the change in $\vec{Q}$ that would be observed in the moving frame $\mathcal{A}$.
- Part (f) shows the net change in $\vec{Q}$, $\Delta\vec{Q}$, that is the same as that in (b) above; here, it is shown as the sum of the two contributions from (d) and (e).

Thus, using $\mathcal{A}$, $\Delta\vec{Q}$ for small Δt is composed of two parts: (1) the $\Delta\vec{Q}$ observed in $\mathcal{A}(t)$, and (2) the change in $\vec{Q}$ which would occur if $\vec{Q}$ were constant in $\mathcal{A}(t)$ and thus rotating with it. Dividing $\Delta\vec{Q}$ by Δt gives the '$\dot{\vec{Q}}$ formula', $\dot{\vec{Q}} = \overset{\mathcal{A}\bullet}{\vec{Q}} + \vec{\omega}_{\mathcal{A}} \times \vec{Q}$.

(continued...)

15.1 The $\overset{\bullet}{\vec{Q}}$ formula (continued)



Two different looks at the change in the vector $\vec{Q}$, $\Delta\vec{Q}$, over a time interval Δt.

SAMPLE 15.5 Acceleration of a point moving in a rotating frame. Consider the rotating tube of Sample 15.9 again. It is given that the arm OAB rotates with counterclockwise angular acceleration $\dot{\omega} = 3\,\mathrm{rad/s^2}$ and at the instant shown the angular speed $\omega = 5\,\mathrm{rad/s}$. Also, at the same instant, the particle P is falling down with speed $v_{/\mathrm{tube}} = 4\,\mathrm{ft/s}$ and acceleration $a_{/\mathrm{tube}} = 2\,\mathrm{ft/s^2}$. Find the absolute acceleration of the particle at the given instant. Take $\ell = 2\,\mathrm{ft}$ in the figure.

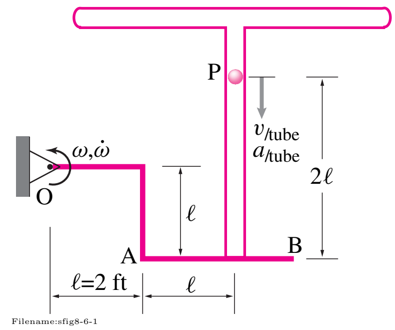


Figure 15.26:

Solution Let us attach a body frame $\mathcal{B}$ to the rigid arm OAB. For calculations we fix a coordinate system $x'y'z'$ in this frame such that the origin O' of the coordinate system coincides with O, and at the given instant, the axes are aligned with the inertial coordinate axes xyz. Since $x'y'z'$ is fixed in the frame $\mathcal{B}$ and $\mathcal{B}$ rotates with the rigid arm with $\dot{\vec{\omega}}_{\mathcal{B}} = 3\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}$ and $\vec{\omega}_{\mathcal{B}} = 5\,\mathrm{rad/s}\hat{\boldsymbol{k}}$, the basis vectors $\hat{\imath}'$, $\hat{\jmath}'$ and $\hat{\boldsymbol{k}}'$ rotate with the same $\dot{\vec{\omega}}_{\mathcal{B}}$ and $\vec{\omega}_{\mathcal{B}}$.

In the rotating (primed) coordinate system,

$$
\begin{aligned}
\vec{r}_P &= x'\hat{\imath}' + y'\hat{\jmath}' \\
\vec{v}_P &= \frac{d}{dt}(\vec{r}_P) = \frac{d}{dt}(x'\hat{\imath}' + y'\hat{\jmath}') \\
&= \dot{x}'\hat{\imath}' + x'\dot{\hat{\imath}}' + \dot{y}'\hat{\jmath}' + y'\dot{\hat{\jmath}}'.
\end{aligned}
$$

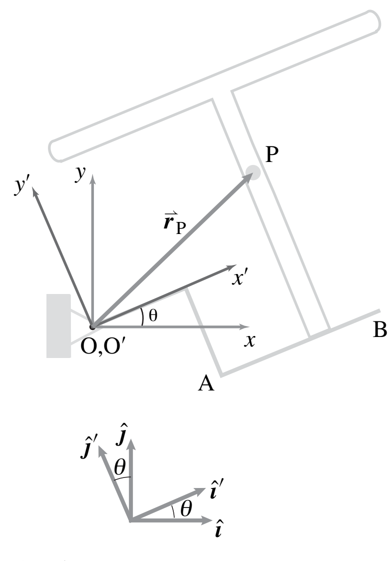


Figure 15.27:

Now, we use the $\dot{\vec{Q}}$ formula to evaluate $\dot{\hat{\imath}}'$ and $\dot{\hat{\jmath}}'$, *i.e.,*

$$
\begin{aligned}
\dot{\hat{\imath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\imath}' = \omega\hat{\boldsymbol{k}}' \times \hat{\imath}' = \omega\hat{\jmath}' \\
\dot{\hat{\jmath}}' &= \vec{\omega}_{\mathcal{B}} \times \hat{\jmath}' = \omega\hat{\boldsymbol{k}}' \times \hat{\jmath}' = -\omega\hat{\imath}'.
\end{aligned}
$$

Also, note that x' is constant since in frame $\mathcal{B}$, the motion of the particle is always along the tube, *i.e.,* along the negative y' axis (see Fig. 15.27). Thus, $x' = 2\ell$, $\dot{x}' = 0$, $y' = \ell$, and $\dot{y}' = -v_{/\mathrm{tube}}$. Substituting these quantities in $\vec{v}_P$, we get:

$$
\begin{aligned}
\vec{v}_P &= x'\omega\hat{\jmath}' - v_{/\mathrm{tube}}\hat{\jmath}' + y'(-\omega\hat{\imath}') \\
&= (x'\omega - v_{/\mathrm{tube}})\hat{\jmath}' - \omega y'\hat{\imath}'. \qquad (15.26)
\end{aligned}
$$

Now substituting $x' = 2\ell = 4\,\mathrm{ft}$, $\omega = 5\,\mathrm{rad/s}$, $y' = \ell = 2\,\mathrm{ft}$, $v_{/\mathrm{tube}} = 4\,\mathrm{ft/s}$ and noting that $\hat{\imath}' = \hat{\imath}$, $\hat{\jmath}' = \hat{\jmath}$ at the given instant, we get:

$$\vec{v}_P = [(20-4)\hat{\jmath} - 10\hat{\imath}]\,\mathrm{ft/s} = (-10\hat{\imath} + 16\hat{\jmath})\,\mathrm{ft/s}.$$

We can find $\vec{a}_P$ by differentiating Eq. (15.26) and noting again that $\hat{\imath}' = \hat{\imath}$, $\hat{\jmath}' = \hat{\jmath}$ at the given instant②:

② We will revisit this problem in Sample 15.9. Here we use the $\dot{\vec{Q}}$ formula on various base vectors. In Sample 15.9 we will use the relative velocity and acceleration formulae.

$$
\begin{aligned}
\vec{a}_P &= \frac{d}{dt}(\vec{v}_P) = \frac{d}{dt}[(x'\omega - v_{/\mathrm{tube}})\hat{\jmath}' - \omega y'\hat{\imath}'] \\
&= (\underbrace{\dot{x}'}_{0}\omega + x'\dot{\omega} - \overbrace{\dot{v}_{/\mathrm{tube}}}^{a_{/\mathrm{tube}}})\hat{\jmath}' + (x'\omega - v_{/\mathrm{tube}})\dot{\hat{\jmath}}' - (\dot{\omega}y' + \omega\overbrace{\dot{y}'}^{-v_{/\mathrm{tube}}})\hat{\imath}' - \omega y'\dot{\hat{\imath}}' \\
&= (x'\dot{\omega} - a_{/\mathrm{tube}})\hat{\jmath}' + (x'\omega - v_{/\mathrm{tube}})(-\omega\hat{\imath}') - (\dot{\omega}y' - \omega v_{/\mathrm{tube}})\hat{\imath}' - \omega y'(\omega\hat{\jmath}') \\
&= -a_{/\mathrm{tube}}\hat{\jmath}' + 2\omega v_{/\mathrm{tube}}\hat{\imath}' - \omega^2(x'\hat{\imath}' + y'\hat{\jmath}') + \dot{\omega}(x'\hat{\jmath}' - y'\hat{\imath}') \\
&= -2\,\mathrm{ft/s^2}\hat{\jmath} + 40\,\mathrm{ft/s^2}\hat{\imath} - 25(4\hat{\imath} + 2\hat{\jmath})\,\mathrm{ft/s^2} + 3(4\hat{\jmath} - 2\hat{\imath})\,\mathrm{ft/s^2} \\
&= -(66\hat{\imath} + 40\hat{\jmath})\,\mathrm{ft/s^2}.
\end{aligned}
$$

$$\boxed{\vec{a}_P = -(66\hat{\imath} + 40\hat{\jmath})\,\mathrm{ft/s^2}}$$

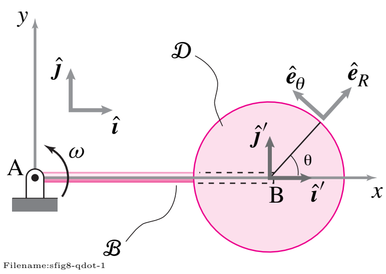


Figure 15.28:

SAMPLE 15.6 **Rate of change of unit vectors.** A circular disk $\mathcal{D}$ is welded to a rigid rod AB. The rod rotates about point A with angular velocity $\vec{\omega} = \omega\hat{k}$. A frame $\mathcal{B}$ is attached to the disk and therefore rotates with the same $\vec{\omega}$. Two coordinate systems, $(\hat{\imath}', \hat{\jmath}')$ and $(\hat{e}_R, \hat{e}_\theta)$ are fixed in frame $\mathcal{B}$ as shown in the figure.

1. Find the rate of change of unit vectors $\hat{e}_R$, $\hat{e}_\theta$, $\hat{\imath}'$ and $\hat{\jmath}'$ using the $\dot{\vec{Q}}$ formula.
2. Express the $\hat{e}_R$ and $\hat{e}_\theta$ vectors in terms of $\hat{\imath}'$ and $\hat{\jmath}'$ and verify the results obtained above for $\dot{\hat{e}}_R$ and $\dot{\hat{e}}_\theta$ by direct differentiation.

Solution Since the disk is welded to the rod and frame $\mathcal{B}$ is fixed in the disk, the frame rotates with $\vec{\omega}_\mathcal{B} = \omega\hat{k}$.

1. To find the rate of change of the unit vectors using the $\dot{\vec{Q}}$ formula, we substitute the desired unit vector in place of $\vec{Q}$ in the formula (eqn. (15.19)). For example,

$$\dot{\hat{e}}_R = {}^{\mathcal{B}}\dot{\hat{e}}_R + \vec{\omega}_\mathcal{B} \times \hat{e}_R.$$

It should be clear that ${}^{\mathcal{B}}\dot{\hat{e}}_R = 0$, since $\hat{e}_R$ does not change with respect to an observer sitting in frame $\mathcal{B}$. Therefore,

$$\dot{\hat{e}}_R = \vec{\omega}_\mathcal{B} \times \hat{e}_R = \omega\hat{k} \times \hat{e}_R = \omega\hat{e}_\theta.$$

Similarly,

$$\begin{aligned}
\dot{\hat{e}}_\theta &= \underbrace{{}^{\mathcal{B}}\dot{\hat{e}}_\theta}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{e}_\theta = \omega\,\overbrace{\hat{k} \times \hat{e}_\theta}^{-\hat{e}_R} = -\omega\hat{e}_R. \\
\dot{\hat{\imath}}' &= \underbrace{{}^{\mathcal{B}}\dot{\hat{\imath}}'}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{\imath}' = \omega\hat{k} \times \hat{\imath}' = \omega\hat{\jmath}'. \\
\dot{\hat{\jmath}}' &= \underbrace{{}^{\mathcal{B}}\dot{\hat{\jmath}}'}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \hat{\jmath}' = \omega\hat{k} \times \hat{\jmath}' = -\omega\hat{\imath}'.
\end{aligned}$$

$$\dot{\hat{e}}_R = \omega\hat{e}_\theta, \quad \dot{\hat{e}}_\theta = -\omega\hat{e}_R, \quad \dot{\hat{\imath}}' = \omega\hat{\jmath}', \text{ and } \dot{\hat{\jmath}}' = -\omega\hat{\imath}'$$

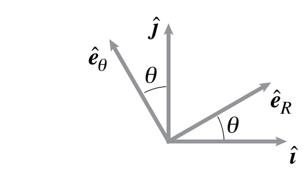


Figure 15.29:

2. Since $\hat{e}_R = \cos\theta\hat{\imath}' + \sin\theta\hat{\jmath}'$ and $\hat{e}_\theta = -\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}'$, we get their rates of change by direct differentiation as

$$\begin{aligned}
\dot{\hat{e}}_R &= \cos\theta\dot{\hat{\imath}}' + \sin\theta\dot{\hat{\jmath}}' \\
&= \cos\theta(\omega\hat{\jmath}') + \sin\theta(-\omega\hat{\imath}') \\
&= \omega(-\sin\theta\hat{\imath}' + \cos\theta\hat{\jmath}') = \omega\hat{e}_\theta, \\
\dot{\hat{e}}_\theta &= -\sin\theta\dot{\hat{\imath}}' + \cos\theta\dot{\hat{\jmath}}' \\
&= -\sin\theta(\omega\hat{\jmath}') + \cos\theta(-\omega\hat{\imath}') \\
&= -\omega(\cos\theta\hat{\imath}' + \sin\theta\hat{\jmath}') = -\omega\hat{e}_R.
\end{aligned}$$

Here we have used the fact that θ, the angle between the unit vectors $\hat{e}_R$ and $\hat{\imath}'$, remains constant during the motion. The results obtained are the same as in part (a).

SAMPLE 15.7 **Rate of change of a position vector.** A rigid rod OAB rotates counterclockwise about point O with constant angular speed $\omega = 5\,\mathrm{rad/s}$. A collar C slides out on the bent arm AB with constant speed $v = 0.5\,\mathrm{m/s}$ with respect to the arm. Find the velocity of the collar using the $\dot{\vec{Q}}$ formula.

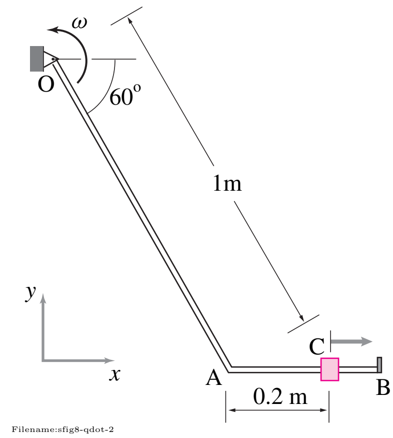


Figure 15.30:

Solution Let $\vec{r}_C$ be the position vector of the collar. Then the velocity of the collar is $\dot{\vec{r}}_C$. Let the rod OAB be the rotating frame $\mathcal{B}$. Now we can find $\dot{\vec{r}}_C$ using the $\dot{\vec{Q}}$ formula:

$$\dot{\vec{r}}_C = {}^{\mathcal{B}}\dot{\vec{r}}_C + \vec{\omega}_{\mathcal{B}} \times \vec{r}_C$$

To compute $\dot{\vec{r}}_C$, let us first find ${}^{\mathcal{B}}\dot{\vec{r}}_C$, the rate of change of $\vec{r}_C$ as seem in frame $\mathcal{B}$ (this term represents the velocity of the collar you see if you sit on the rod and watch the collar; also called $\vec{v}_{rel}$).

$$\begin{aligned} \vec{r}_C &= \vec{r}_A + \vec{r}_{C/A} \\ {}^{\mathcal{B}}\dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_A + {}^{\mathcal{B}}\dot{\vec{r}}_{C/A} \end{aligned}$$

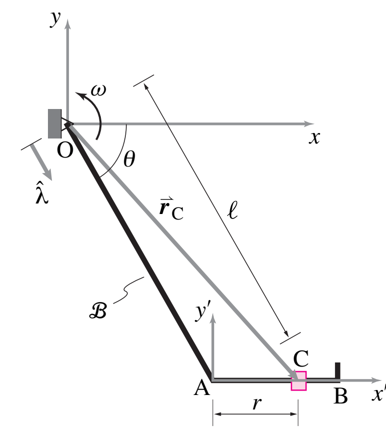


Figure 15.31:

Note that the vector $\vec{r}_A = \ell\hat{\boldsymbol{\lambda}}$ does not change in frame $\mathcal{B}$ since both its magnitude, ℓ, and direction, $\hat{\boldsymbol{\lambda}}$, remain fixed in $\mathcal{B}$. Therefore,

$${}^{\mathcal{B}}\dot{\vec{r}}_A = 0$$

$$\text{Now} \quad \vec{r}_{C/A} = r\hat{\imath}'. \quad \Rightarrow \quad {}^{\mathcal{B}}\dot{\vec{r}}_{C/A} = \dot{r}\hat{\imath}' = 0.5\,\mathrm{m/s}\hat{\imath}'$$

because $\hat{\imath}'$ does not change in $\mathcal{B}$ and $\dot{r}$ = speed of the collar with respect to the arm. (see Figure 15.31) Thus,

$${}^{\mathcal{B}}\dot{\vec{r}}_C = 0.5\,\mathrm{m/s}\hat{\imath}'.$$

Hence,

$$\begin{aligned} \dot{\vec{r}}_C &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega\hat{\boldsymbol{k}} \times (\ell\hat{\boldsymbol{\lambda}} + r\hat{\imath}') \\ &= {}^{\mathcal{B}}\dot{\vec{r}}_C + \omega\ell(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}) + \omega r(\hat{\boldsymbol{k}} \times \hat{\imath}') \\ &= \dot{r}\hat{\imath}' + \omega\ell(\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) + \omega r\,\hat{\jmath}' \\ &= 0.5\,\mathrm{m/s}\hat{\imath}' + 5\,\mathrm{m/s}(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath}) + 1\,\mathrm{m/s}\hat{\jmath}' \\ &= 4.83\,\mathrm{m/s}\hat{\imath} + 3.5\,\mathrm{m/s}\hat{\jmath} \end{aligned}$$

where we have used the fact that at the given instant, $\hat{\imath}' = \hat{\imath}$ and $\hat{\jmath}' = \hat{\jmath}$.

$$\boxed{\vec{v}_C = 4.83\,\mathrm{m/s}\hat{\imath} + 3.5\,\mathrm{m/s}\hat{\jmath}}$$

15.3 General expressions for velocity $\vec{v}$ and acceleration $\vec{a}$ of a moving point

Now that we have some comfort with moving frames we can develope formulas that are not so strongly attached to base vectors. That is, we take account that the base vectors rotate with the frame, but develop formulas that don't use the base vectors explicitly. Thus the formulas we develop here work equally for any frame that is glued to the rotating frame of choice, independent of its orientation.

Absolute velocity of a point moving relative to a moving frame

Imagine that you know the absolute velocity $\vec{\boldsymbol{v}}_{0'/\mathcal{F}}$ of some point O' on an object $\mathcal{B}$, say the center of a car tire and the angular velocity of the tire, $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}}$. Finally, imagine you also know the relative velocity of point P, $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}}$, say of a bug crawling on the tire.

If the frame $\mathcal{B}$ is translating or rotating, the velocity of particle P relative to the frame $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}}$ is not the absolute velocity (the velocity relative to a Newtonian frame). The absolute velocity in this case is $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{F}}$, or more simply $\vec{\boldsymbol{v}}_{\mathrm{P}}$, or more simply still, just $\vec{\boldsymbol{v}}$. The relationship between the absolute velocity $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{B}}$ and the relative velocity $\vec{\boldsymbol{v}}_{\mathrm{P}/\mathcal{F}} \equiv \vec{\boldsymbol{v}}$ is of interest.

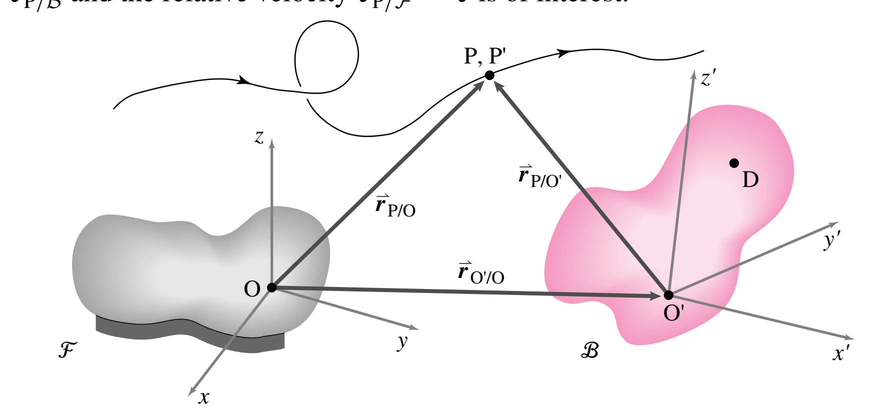


Figure 15.32: The position of point P relative to the origin O' of moving frame $\mathcal{B}$ is $\vec{\boldsymbol{r}}_{\mathrm{P}/\mathrm{O}'}$. The position of the origin of the frame $\mathcal{B}$ relative to the origin O of the fixed frame $\mathcal{F}$ is $\vec{\boldsymbol{r}}_{\mathrm{O}'/\mathrm{O}}$. The position of point P relative to O is the sum of $\vec{\boldsymbol{r}}_{\mathrm{P}/\mathrm{O}'}$ and $\vec{\boldsymbol{r}}_{\mathrm{O}'/\mathrm{O}}$. The motion of point P relative to the fixed frame may be complicated.

Let's start by looking at the position. The position of a point P that is moving is:

$$\vec{\boldsymbol{r}}_{\mathrm{P}/\mathrm{O}} = \vec{\boldsymbol{r}}_{\mathrm{O}'/\mathrm{O}} + \vec{\boldsymbol{r}}_{\mathrm{P}/\mathrm{O}'}$$

where O' is the origin of a coordinate system which is glued to the rigid object, as shown in fig. 15.32.

To find the absolute velocity of point P we will use the $\dot{\vec{\boldsymbol{Q}}}$ formula, equation 15.19, for computing the rate of change of a vector. The velocity of P is the rate of change of its position. Here, we use $\vec{Q} = \vec{r}_{\mathrm{P/O'}}$

$$\begin{aligned}
\vec{v}_{\mathrm{P}/\mathcal{F}} &= {}^{\mathcal{F}}\dot{\vec{r}}_{\mathrm{P/O}} \\
&= {}^{\mathcal{F}}\dot{\vec{r}}_{O'/O} + {}^{\mathcal{F}}\dot{\vec{r}}_{\mathrm{P/O'}} \\
&= \vec{v}_{\mathrm{O'}/\mathcal{F}} + \overbrace{\underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{\mathrm{P/O'}}}_{\vec{v}_{\mathrm{P}/\mathcal{B}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{\mathrm{P'/O'}}}^{{}^{\mathcal{F}}\dot{\vec{r}}_{\mathrm{P/O'}}}
\end{aligned}$$

The $\dot{\vec{Q}}$ formula 15.19 was used in the calculation to compute ${}^{\mathcal{F}}\dot{\vec{r}}_{P/O'}$

$$ {}^{\mathcal{F}}\dot{\vec{r}}_{P/O'} = {}^{\mathcal{B}}\dot{\vec{r}}_{P/O'} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{\mathrm{P'/O'}}. \tag{15.27}$$

Thus, the 'three term velocity formula'.

$$\overbrace{\vec{v}_{\mathrm{P}}}^{\text{Absolute velocity of point P}} = \underbrace{\vec{v}_{\mathrm{O'}}}_{\text{1. Velocity of reference point } O' \text{ on object } \mathcal{B}} + \overbrace{\vec{v}_{\mathrm{P}/\mathcal{B}}}^{\text{2. Velocity of point P relative to moving frame } \mathcal{B}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P'/O'}}}_{\text{3. Velocity relative to point } O' \text{ of the point } P' \text{ glued to object } \mathcal{B} \text{ which is instantaneously coincident with point P}} \tag{15.28}$$

Another way to write the formula for absolute velocity is as

$$\vec{v}_{\mathrm{P}} = \vec{v}_{\mathrm{P'}} + \vec{v}_{\mathrm{P}/\mathcal{B}}$$

where P' is a point glued to $\mathcal{B}$ which is instantaneously coincident with P, so the absolute velocity of P' is

$$\vec{v}_{\mathrm{P'}} = \vec{v}_{\mathrm{O'}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P'/O'}}. \tag{15.29}$$

Reconsider the bug crawling on the tire, object $\mathcal{B}$, in fig. 15.33. To find the absolute velocity of the bug, we need be concerned with how the bug moves relative to the tire *and* how the tire moves relative to the ground.

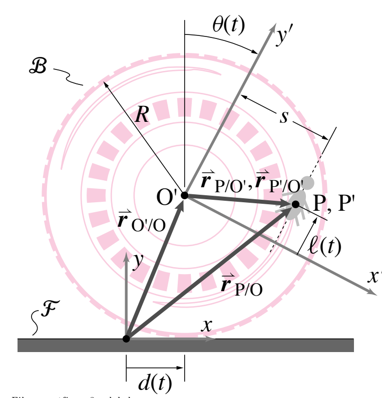


Figure 15.33:

Example: **Absolute velocity of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again**

Referring to equation 15.27 on page 869, the absolute velocity of the bug is

$$\begin{aligned}\vec{v}_{P/\mathcal{F}} &= \vec{v}_{O'/\mathcal{F}} + \underbrace{{}^{\mathcal{B}}\dot{\vec{r}}_{P/O'}}_{\vec{v}_{P/\mathcal{B}}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{P/O'} \\ &= R\dot{\theta}\hat{\imath} \quad + \quad \dot{\ell}\hat{\jmath}' \quad + \quad (-\dot{\theta}\hat{k}') \times (s\hat{\imath}' + \ell\hat{\jmath}') \\ &= R\dot{\theta}\hat{\imath} \quad + \quad \ell\dot{\theta}\hat{\imath}' \quad + \quad (\dot{\ell} - s\dot{\theta})\hat{\jmath}'.\end{aligned}$$

At the instant of interest, the direction of the bug's absolute velocity depends upon the relative magnitudes of $\dot{\ell}$ and $\dot{\theta}$ as well as the orientation of $\hat{\imath}'$ and $\hat{\jmath}'$.

As we noted earlier, another way to write the formula for absolute velocity is

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{P/\mathcal{B}}$$

where, in the example above, $\vec{v}_{P'} = R\dot{\theta}\hat{\imath} + \dot{\theta}(\ell\hat{\imath}' - s\hat{\jmath}')$ and $\vec{v}_{P/\mathcal{B}} = \dot{\ell}\hat{\jmath}'$. At the instant of concern, we can think of the absolute velocity of the bug as the velocity of the mark labeled P' under the bug *plus* the velocity of the bug relative to the tire.

Acceleration

We would like to find acceleration of a point using information about its motion relative to a moving frame. The result, the 'five term acceleration formula' is the most complicated formula in this book. (For reference, it is in Table II, 5c).

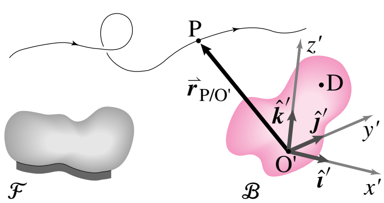


Figure 15.34: An object or reference frame $\mathcal{B}$ and a point P move around. We are interested in the acceleration of point P.

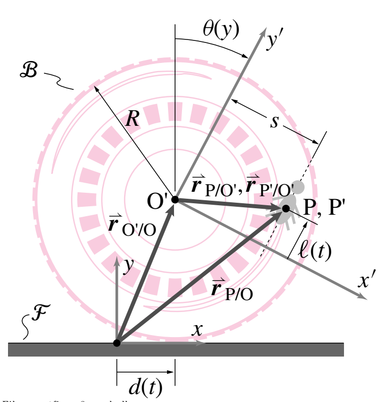


Figure 15.35: A bug crawling on a tire at point P. Point P' is on the tire and instantaneously corresponds to point P.

Acceleration relative to a object or frame

The acceleration of a point relative to a object or frame is the acceleration you would calculate if you were looking at the particle while you translated and rotated with the frame and took no account of the outside world. That is, if the position of a particle P relative to the origin O' of a coordinate system in a moving frame $\mathcal{B}$ is given by:

$$\vec{r}_{P/O'} = \underbrace{r_{Px'/O'}}_{x'}\hat{\imath}' + \underbrace{r_{Py'/O'}}_{y'}\hat{\jmath}' + \underbrace{r_{Pz'/O'}}_{z'}\hat{k}',$$

then the acceleration of the particle P relative to the frame is:

$$\vec{a}_{P/\mathcal{B}} = \underbrace{\ddot{r}_{Px'/O'}}_{\ddot{x}'}\hat{\imath}' + \underbrace{\ddot{r}_{Py'/O'}}_{\ddot{y}'}\hat{\jmath}' + \underbrace{\ddot{r}_{Pz'/O'}}_{\ddot{z}'}\hat{k}'.$$

That is, the acceleration relative to the frame takes no account of (a) the motion of the frame or of (b) the rotation of the base vectors with the frame to which they are fixed.

Reconsider the bug labeled point P crawling on the tire, object $\mathcal{B}$, in fig. 15.35.

Example: Acceleration relative to a frame (2-D): Bug crawling on a tire, again
If we are sitting on the tire, all that we see is the bug crawling in a straight line at non-constant rate relative to us. Thus, its acceleration relative to the tire is

$$\vec{a}_{P/\mathcal{B}} = \ddot{\ell}\hat{\jmath}'.$$

So, at the instant of interest, the bug has an acceleration relative to the tire frame parallel to the y'-axis.

Absolute acceleration of a point P′ glued to a moving frame

Imagine that you know the absolute acceleration of some point O' at the center of a frame $\mathcal{B}$, say the center of a car tire. Imagine you also know the angular velocity of the tire, $\vec{\omega}_{\mathcal{B}/\mathcal{F}}$, and the angular acceleration, $\vec{\alpha}_{\mathcal{B}/\mathcal{F}}$. Then, you can find the absolute acceleration of a piece of gum labeled point D stuck to the sidewall (see fig. 15.37). If we start with the equation 15.29 for the absolute velocity of a point glued to a moving frame on page 869 and differentiate with respect to time, we get the absolute acceleration of a point D fixed in a moving frame $\mathcal{B}$ as follows:

$$\begin{aligned}\vec{a}_{\mathrm{D}} &= \frac{d}{dt}[\vec{v}_{\mathrm{O'}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{D/O'}}] \\ &= \vec{a}_{\mathrm{O'/O}} + [\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{\mathrm{D/O'}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{D/O'}})] \\ &= \vec{a}_{\mathrm{O'/O}} + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{\mathrm{D/O'}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{D/O'}}) \qquad (15.30)\end{aligned}$$

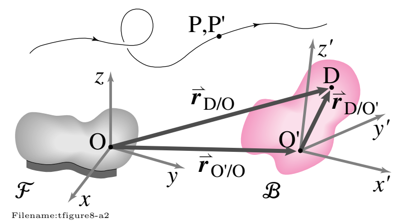


Figure 15.36: Tracking a moving particle from two different reference frames.

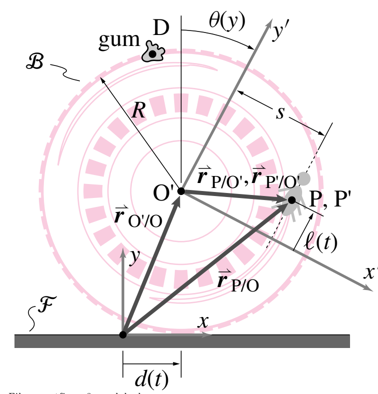


Figure 15.37: Gum is glued to a rolling tire at D. The point P′ glued to the tire is just where the point P happens to be passing at the moment of interest.

Example: Absolute acceleration of a point glued to a moving frame (2-D): Bug crawling on a tire, again

Here, the acceleration of point P′ glued to the tire, relative to the tire is zero, $\vec{a}_{\mathrm{P'}/\mathcal{B}} = 0$ (see fig. 15.37). The angular velocity of the wheel with respect to the ground is $\vec{\omega}_{\mathcal{B}/\mathcal{F}} = -\dot{\theta}\hat{k} = -\dot{\theta}\hat{k}'$. The angular speed is increasing at a rate $\ddot{\theta}$. Thus, $\vec{\alpha}_{\mathcal{B}/\mathcal{F}} = -\ddot{\theta}\hat{k} = -\ddot{\theta}\hat{k}'$. The position of P' relative to O' is $\vec{r}_{\mathrm{P'/O'}} = s\hat{\imath}' + \ell\hat{\jmath}'$.

Using equation 15.30 on page 871, we get the absolute acceleration of point P' to be

$$\vec{a}_{\mathrm{P'}/\mathcal{F}} = \vec{a}_{\mathrm{O'}/\mathcal{F}} + \vec{\omega}_{\mathcal{B}/\mathcal{F}} \times (\vec{\omega}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{\mathrm{P'/O'}}) + \vec{\alpha}_{\mathcal{B}/\mathcal{F}} \times \vec{r}_{\mathrm{P'/O'}}$$

$$= \underbrace{R\ddot{\theta}\hat{\imath}}_{\text{acceleration of origin of moving frame}} \overbrace{-\dot{\theta}^2(s\hat{\imath}' + \ell\hat{\jmath}')}^{\text{centripetal term}} + \overbrace{R\ddot{\theta}(s\hat{\jmath}' - \ell\hat{\imath}')}^{\text{tangential term}}$$

In this example, the absolute acceleration P' is due to:

1. the increase in the translational speed of the tire relative to the ground (acceleration of origin of moving frame),
2. its going in circles at non-constant rate about point O' relative to the ground ('tangential term'), and
3. 'centripetal term' towards the origin of the moving frame. (In three-dimensional problems, this term is directed towards an axis through $\vec{\omega}$ that goes through O').

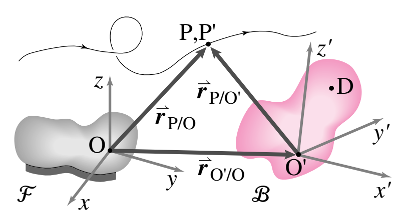


Figure 15.38: Keeping track of P which instantaneously coincides with point P' which is glued to frame $\mathcal{B}$.

Absolute acceleration of a point moving relative to a moving frame

If we start with the equation for absolute velocity 15.27 on page 869 and differentiate with respect to time we get the absolute acceleration of a point P using a moving frame $\mathcal{B}$. To do this calculation we need to use the product rule of differentiation. Refer to the $\dot{\vec{Q}}$ formula, eqn. (15.24) on page 863. Here is the calculation:

$$
\begin{aligned}
\vec{a}_{\mathrm{P}} &= \frac{d}{dt}[\vec{v}_{\mathrm{O'/O}} + \vec{v}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}] \\
&= \vec{a}_{\mathrm{O'/O}} + (\vec{a}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}) \\
&\quad + \ [\dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}})] \\
&= \underbrace{\vec{a}_{\mathrm{O'/O}} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}) + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}}_{\vec{a}_{\mathrm{P'}}} + \vec{a}_{\mathrm{P}/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}.
\end{aligned}
$$

The collection of terms $\vec{a}_{\mathrm{P'}}$ is the acceleration of a point P' which is glued to body $\mathcal{B}$ and is instantaneously coincident with P. It is the same as $\vec{a}_{\mathrm{D}}$ using $D = P'$ in equation 15.30. To repeat, the result is

The 'five term' acceleration formula

$$
\underbrace{\vec{a}_{\mathrm{P}}}_{\text{The absolute acceleration}} = \overbrace{\vec{a}_{\mathrm{O'/O}}}^{\text{1. acceleration of reference point } O' \text{ on object } \mathcal{B}} + \underbrace{\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}})}_{\text{2. centripetal acceleration: acceleration relative to } O' \text{ of point glued to object, coincident with P, and going in circles around } O' \text{ at constant rate}} + \overbrace{\vec{\alpha}_{\mathcal{B}} \times \vec{r}_{\mathrm{P/O'}}}^{\text{3. tangential acceleration: acceleration relative to } O' \text{ of point glued to object, coincident with P, due to non-constant } \vec{\omega}} + \underbrace{\vec{a}_{\mathrm{P}/\mathcal{B}}}_{\text{4. acceleration of P relative to moving frame } \mathcal{B}} + 2\,\underbrace{\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}}}_{\text{5. Coriolis accel.}} \tag{15.31}
$$

$$
= \underbrace{\vec{a}_{\mathrm{P'}}}_{\text{The first 3 terms above}} + \vec{a}_{\mathrm{P}/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\mathrm{P}/\mathcal{B}} \tag{15.32}
$$

This 'five-term-acceleration' formula is both famous and infamous. It's famous because it is given a lot of emphasis by some instructors[1]. and infamous because it takes some getting used to. Eqn. 15.32 is the 'three term acceleration formula'. It combines the first three terms in the 5-term formula and interprets them as the acceleration of the point P′ on $\mathcal{B}$ that instantaneously coincides with P. The best way to get used to the five term acceleration formula is to find situations where some of the terms drop out.

Reconsider the bug labeled point P crawling on the tire, object $\mathcal{B}$, in fig. 15.37. To find the absolute acceleration of the bug we need to think about how the bug moves relative to the tire *and* how the tire moves relative to the ground.

Example: **Absolute acceleration of a point moving relative to a moving frame (2-D): Bug crawling on a tire, again**

From the previous bug examples on page 869 and page 870 we know that

$$\vec{v}_{P/\mathcal{B}} = \dot{\ell}\hat{\jmath}', \quad \text{and } \vec{a}_{P/\mathcal{B}} = \ddot{\ell}\hat{\jmath}'.$$

Referring to the five term acceleration formula, equation 15.31 on page 872, the absolute acceleration of the bug is

$$\begin{aligned}
\vec{a}_{P/\mathcal{F}} &= \vec{a}_{O'/O} \\
&\quad +\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'}) \\
&\quad +\vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P/O'} \\
&\quad +\vec{a}_{P/\mathcal{B}} \\
&\quad +2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} \\
&= \underbrace{R\ddot{\theta}\hat{\imath} - \dot{\theta}^2(s\hat{\imath}' + \ell\hat{\jmath}') + \ddot{\theta}(\ell\hat{\imath}' - s\hat{\jmath}')}_{\vec{a}_{P'/\mathcal{F}}} + \ddot{\ell}\hat{\jmath}' + 2\dot{\theta}\dot{\ell}\hat{\imath}' \\
&= R\ddot{\theta}\hat{\imath} + (\ell\ddot{\theta} - s\dot{\theta}^2 + 2\dot{\theta}\dot{\ell})\hat{\imath}' + (-s\ddot{\theta} + \ddot{\ell} - \ell\dot{\theta}^2)\hat{\jmath}'.
\end{aligned}$$

So, at the instant of interest, the bug's absolute acceleration is due to:

1. the translational acceleration of the tire, $\vec{a}_{O'/O} = R\ddot{\theta}\hat{\imath}$
2. the centripetal acceleration of going in circles of radius $\sqrt{s^2 + \ell^2}$ about the center of the tire as it rolls, $-\dot{\theta}^2(s\hat{\imath}' + \ell\hat{\jmath}')$, pointing at the center of the tire,
3. the tangential acceleration of going in circles about the center of the tire as the tire rolls at non-constant rate, $\ddot{\theta}(\ell\hat{\imath}' - s\hat{\jmath}')$,
4. the acceleration of the bug relative to the tire as it crawls on the line, $\vec{a}_{P/\mathcal{B}} = \ddot{\ell}\hat{\jmath}'$, and
5. the Coriolis acceleration caused, in part, by the change in direction, relative to the ground, of the velocity of the bug relative to the tire, $2\dot{\theta}\dot{\ell}\hat{\imath}'$.

Items 1, 2 and 3 sum to be the acceleration of point P' *on* the tire but instantaneously coinciding with moving point P.

① **True story.** Once a teacher wrote eqn. (15.31) in big letters on a fresh clean black board, including the explanations in boxes. Then he talked it through, telling the class that this was what they had been building up to. That after all the vector notation, the more involved kinematic development and the messy notation that this was finally the climax. There was still silence in the room as, so the teacher was hoping, the students could feel satisfied with the accomplishment. After time for a slow breath a student asked "Was it as good for you as it was for us?"

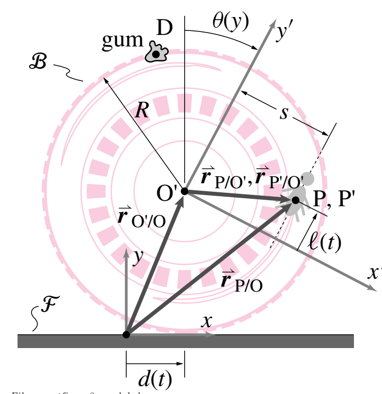


Figure 15.39:

Motion relative to a point versus motion relative to a frame

We can now give a different interpretation of the expressions we have been using $\vec{v}_{B/A}$ and $\vec{a}_{B/A}$. Rather than thinking of $\vec{v}_{B/A}$ as the difference between $\vec{v}_B$ and $\vec{v}_A$ we can think of $\vec{v}_{B/A}$ as the $\vec{v}_{B/\mathcal{A}}$ where $\mathcal{A}$ is a frame with origin that moves with point A and which has no rotation rate relative to $\mathcal{F}$. That is

$$\vec{v}_{B/A} \quad \text{means} \quad \vec{v}_{B/\mathcal{A}}$$

Similarly,

$$\vec{a}_{B/A} \quad \text{means} \quad \vec{a}_{B/\mathcal{A}}.$$

15.2 Relation between moving frame formulae and polar coordinate formulae

A similarity exists between the polar coordinate velocity formula

$$\vec{v} = \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$$

and the second two terms in the 'three-term' velocity formula

$$\vec{v}_P = \vec{v}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}}.$$

In fact, we have tried to build your understanding of moving frames by means of that connection.

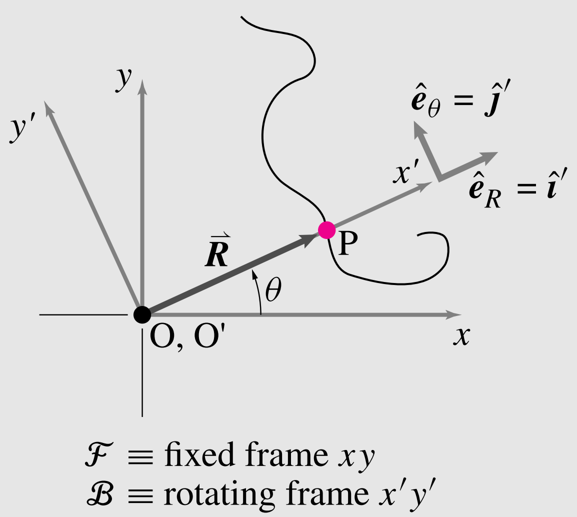
Similarly, the polar coordinate formula for acceleration

$$\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$$

is somehow closely linked to the last four terms of the '5-term' acceleration formula

$$\begin{aligned} \vec{a}_P &= \vec{a}_{O'/O} + \vec{a}_{P/\mathcal{B}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) \\ &\quad + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}}. \end{aligned}$$

Let's make these connections explicit. Imagine a particle P moving around on the xy-plane.



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Let's create a moving frame $\mathcal{B}$ with rotating coordinate system $x'y'$ attached to it whose origin O' is coincident with origin O of a coordinate system xy attached to a fixed frame $\mathcal{F}$. Let this frame rotate in exactly such a way so that the particle is always on the x'-axis. So, in this frame, $\vec{r}_{P/O'} = R\hat{\imath}'$, $\vec{v}_{P/\mathcal{B}} = \dot{R}\hat{\imath}'$, and $\vec{a}_{P/\mathcal{B}} = \ddot{R}\hat{\imath}'$. Also, the frame motion is characterized by $\vec{v}_{O'/O} = \vec{0}$, $\vec{a}_{O'/O} = \vec{0}$, $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{k}$, and $\dot{\vec{\omega}}_{\mathcal{B}} = \ddot{\theta}\hat{k}$. So, if we plug in the three-term velocity formula, we get

$$\begin{aligned} \vec{v}_P &= \vec{v}_{O'/O} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P/O'} + \vec{v}_{P/\mathcal{B}} \\ &= \vec{0} + (\dot{\theta}\hat{k}) \times (R\hat{\imath}') + \dot{R}\hat{\imath}' \\ &= R\dot{\theta}\hat{\jmath}' + \dot{R}\hat{\imath}' \\ &= R\dot{\theta}\hat{e}_\theta + \dot{R}\hat{e}_r \qquad \text{(because } \hat{\imath}' \parallel \hat{e}_r \text{ and } \hat{\jmath}' \parallel \hat{e}_\theta\text{)} \end{aligned}$$

which is the polar coordinate velocity formula.

Similarly, if we plug into the five-term acceleration formula, we get

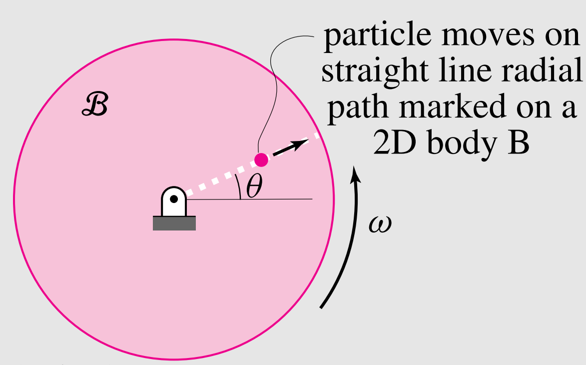
$$\begin{aligned} \vec{a}_P &= \vec{a}_{O'/O} + \vec{a}_{P/\mathcal{B}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/O'}) \\ &\quad + \dot{\vec{\omega}} \times \vec{r}_{P/O'} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} \\ &= \vec{0} + \ddot{R}\hat{\imath}' + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times R\hat{\imath}') \\ &\quad + (\ddot{\theta}\hat{k}) \times (R\hat{\imath}') + 2(\dot{\theta}\hat{k}) \times \dot{R}\hat{\imath}' \\ &= (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta \end{aligned}$$

Again, we recover the appropriate polar coordinate formula.

We have just shown how the polar coordinate formulae are special cases of the relative motion formulae.

Warning!

In problems where we want the rotating frame to be a rotating object on which a particle moves, the polar coordinate formulae only correspond term by term with the relative motion formulae if the particle path is a straight radial line fixed on a $2D$ object, as in the example of a bug walking on a straight line scribed on the surface of a rotating CD or a bead sliding in a tube rotating about an axis perpendicular to the tube.



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Because we are sometimes interested in more general relative motions, the polar coordinate formulae do not always apply and we must make use of the more general relative motion formulae.

SAMPLE 15.8 A 'T' shaped tube is welded to a massless rigid arm OAB which rotates about O at a constant rate $\omega = 5\,\mathrm{rad/s}$. At the instant shown a particle P is falling down in the vertical section of the tube with speed $v_{/\text{tube}} = 4\,\mathrm{ft/s}$. Find the absolute velocity of the particle. Take $\ell = 2\,\mathrm{ft}$ in the figure.

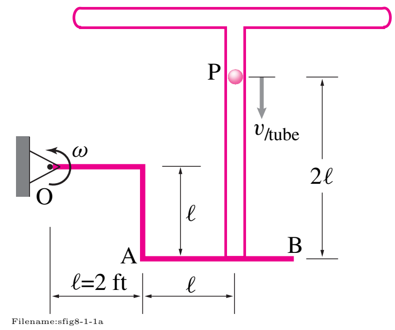


Figure 15.40:

Solution Let us attach a frame $\mathcal{B}$ to arm OAB. Thus $\mathcal{B}$ rotates with OAB with angular velocity $\vec{\omega}_{\mathcal{B}} = \omega\hat{k}$ where $\omega = 5\,\mathrm{rad/s}$. To do calculations in $\mathcal{B}$ we attach a coordinate system $x'y'z'$ to $\mathcal{B}$ at point O. At the instant of interest the rotating coordinate system $x'y'z'$ coincides with the fixed coordinate system xyz. (Since the entire motion is in the xy-plane, the z-axis is not shown in the figure). Let P' be a point coincident with P but fixed in $\mathcal{B}$. Now,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}}$$

where

$$\begin{aligned}
\vec{v}_{P'} &= \underbrace{\vec{v}_{O'}}_{\vec{0}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'} \\
&= \omega\hat{k} \times (2\ell\hat{\imath} + \ell\hat{\jmath}) \\
&= 2\omega\ell\hat{\jmath} - \omega\ell\hat{\imath},
\end{aligned}$$

and

$$\begin{aligned}
\vec{v}_{\text{rel}} &= \text{Velocity relative to the frame } \mathcal{B} \\
&= -v_{/\text{tube}}\hat{\jmath}.
\end{aligned}$$

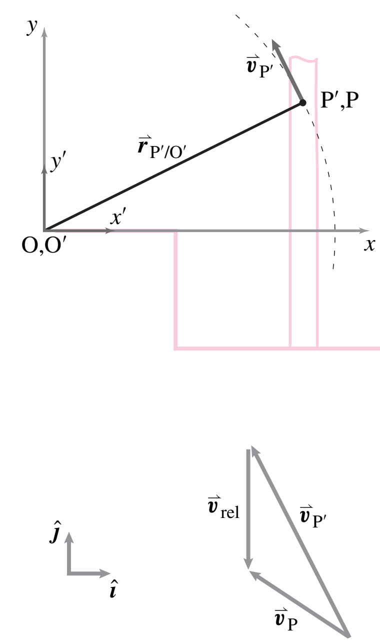


Figure 15.41:

Thus,

$$\begin{aligned}
\vec{v}_P &= \vec{v}_{P'} + \vec{v}_{rel} \\
&= -\omega\ell\hat{\imath} + (2\omega\ell - v_{/\text{tube}})\hat{\jmath} \\
&= -5\,\mathrm{rad/s}\cdot 2\,\mathrm{ft}\hat{\imath} + (2\cdot 5\,\mathrm{rad/s}\cdot 2\,\mathrm{ft} - 4\,\mathrm{ft/s})\hat{\jmath} \\
&= -10\,\mathrm{ft/s}\hat{\imath} + 16\,\mathrm{ft/s}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{v}_P = -10\,\mathrm{ft/s}\hat{\imath} + 16\,\mathrm{ft/s}\hat{\jmath}}$$

Comments: The kinematics calculation is equivalent to the vector addition shown in Figure 15.41. The velocity of P is the sum of $\vec{v}_{P'}$ and $\vec{v}_{\text{rel}} = \vec{v}_{P/\mathcal{B}}$.

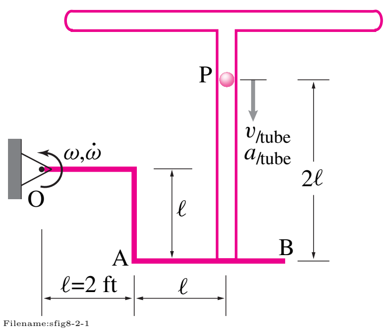
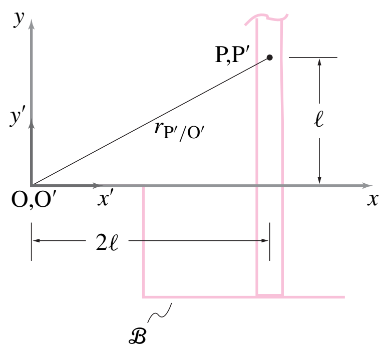


Figure 15.42:



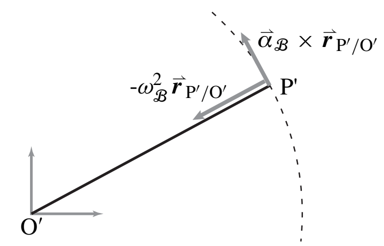


Figure 15.43: Acceleration of point P′. P′ is fixed to the frame $\mathcal{B}$ and at the moment coincides with point P. Therefore, $\vec{a}_{P'} = \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P'/O'} - \omega_{\mathcal{B}}^2 \vec{r}_{P'/O'}$.

SAMPLE 15.9 Acceleration of a point in a rotating frame. Consider the rotating tube of Sample 15.8 again. The arm OAB rotates with counterclockwise angular acceleration $\dot{\omega} = 3\,\text{rad/s}^2$ and, at the instant shown, its angular speed $\omega = 5\,\text{rad/s}$. Also, at the same instant, the particle P falls down with speed $v_{/\text{tube}} = 4\,\text{ft/s}$ and acceleration $a_{/\text{tube}} = 2\,\text{ft/s}^2$. Find the absolute acceleration of the particle at the given instant. Take $\ell = 2\,\text{ft}$ in the figure.

Solution We consider a frame $\mathcal{B}$, with coordinate axes $x'y'z'$, fixed to the arm OAB and thus rotating with $\vec{\omega}_{\mathcal{B}} = \omega\hat{k} = 5\,\text{rad/s}\hat{k}$ and $\vec{\alpha}_{\mathcal{B}} = \dot{\omega}\hat{k} = 3\,\text{rad/s}^2\hat{k}$. The acceleration of point P is given by

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}}$$

where

$$\begin{aligned}
\vec{a}_{P'} &= \text{acceleration of a point P}' \text{ that is fixed in } \mathcal{B} \\
& \text{and at the moment coincides with P,} \\
\vec{a}_{\text{cor}} &= \text{Coriolis acceleration, and} \\
\vec{a}_{\text{rel}} &= \text{acceleration of P relative to frame } \mathcal{B}.
\end{aligned}$$

Now we calculate each of these terms separately. For calculating $\vec{a}_{P'}$, imagine a rigid rod from point O to point P, rotating with the frame $\mathcal{B}$. Mark the far end of the rod as P′ (same as point P). The acceleration of this end of the rod is $\vec{a}_{P'}$. To find the relative terms $\vec{v}_{\text{rel}}$ and $\vec{a}_{\text{rel}}$, freeze the motion of the frame $\mathcal{B}$ at the given moment and watch the motion of point P. The non-intuitive term $\vec{a}_{\text{cor}}$ has no such simple physical interpretation but has a simple formula. Thus,

$$\begin{aligned}
\vec{a}_{P'} &= \overbrace{\vec{a}_{O'}}^{\vec{0}} + \vec{\alpha}_{\mathcal{B}} \times \vec{r}_{P'/O'} + \overbrace{\vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'})}^{-\omega_{\mathcal{B}}^2 \vec{r}_{P'/O'}} \\
&= \dot{\omega}\hat{k} \times (2\ell\hat{\imath} + \ell\hat{\jmath}) - \omega^2(2\ell\hat{\imath} + \ell\hat{\jmath}) \\
&= 2\dot{\omega}\ell\hat{\jmath} - \dot{\omega}\ell\hat{\imath} - 2\omega^2\ell\hat{\imath} - \omega^2\ell\hat{\jmath} \\
&= -\ell(\dot{\omega} + 2\omega^2)\hat{\imath} + \ell(2\dot{\omega} - \omega^2)\hat{\jmath} \\
&= -(106\hat{\imath} + 38\hat{\jmath})\,\text{ft/s}^2,
\end{aligned}$$

$$\begin{aligned}
\vec{a}_{\text{cor}} &= 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\text{rel}} \\
&= 2\omega\hat{k} \times v_{/\text{tube}}(-\hat{\jmath}) \\
&= 2\omega v_{/\text{tube}}\hat{\imath} = 2 \cdot 5\,\text{rad/s} \cdot 4\,\text{ft/s} \\
&= 40\,\text{ft/s}^2\hat{\imath},
\end{aligned}$$

$$\vec{a}_{\text{rel}} = a_{/\text{tube}}(-\hat{\jmath}) = -2\,\text{ft/s}^2\hat{\jmath}.$$

Adding the three terms together, we get

$$\begin{aligned}
\vec{a}_P &= -106\,\text{ft/s}^2\hat{\imath} - 38\,\text{ft/s}^2\hat{\jmath} + 40\,\text{ft/s}^2\hat{\imath} - 2\,\text{ft/s}^2\hat{\jmath} \\
&= -66\,\text{ft/s}^2\hat{\imath} - 40\,\text{ft/s}^2\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{a}_P = -(66\hat{\imath} + 40\hat{\jmath})\,\text{ft/s}^2}$$

Note that the single term $\vec{a}_{P'}$ encompasses three terms of the five term acceleration formula.

SAMPLE 15.10 A small collar P is pinned to a rigid rod AB at length $\ell = 1\,\mathrm{m}$ along the rod. The collar is free to slide in a straight track on a disk of radius $r = 400\,\mathrm{mm}$. The disk rotates about its center O at a constant $\omega = 2\,\mathrm{rad/s}$. At the instant shown, when $\theta = 45^\circ$ and the collar is at a distance $\frac{3}{4}r$ in the track from the center O, find

1. the angular velocity of the rod AB and
2. the velocity of point P relative to the disk.

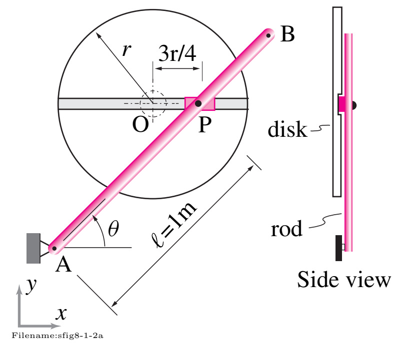
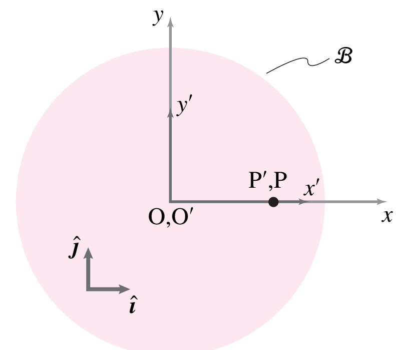


Figure 15.44:

Solution We will think of P in two ways: one as attached to the rod and the other as sliding in the slot. First, let us attach a frame $\mathcal{B}$ to the disk. Thus $\mathcal{B}$ rotates with the disk with angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega\hat{\boldsymbol{k}} = 2\,\mathrm{rad/s}\hat{\boldsymbol{k}}$. We attach a coordinate system $x'y'z'$ to $\mathcal{B}$ at point O. At the instant of interest, the rotating coordinate system $x'y'z'$ coincides with the fixed coordinate system xyz. Now let us consider point P' which is fixed on the disk (and hence in $\mathcal{B}$) and coincides with point P at the moment of interest. We can write the velocity of P as:



xy is the fixed frame. $x'y'$ rotates with the disk with ω_{B}. P′ is fixed in the rotating frame.

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{P'} + \vec{\boldsymbol{v}}_{\mathrm{rel}}$$

where

$$\begin{aligned}
\vec{\boldsymbol{v}}_{P'} &= \overbrace{\vec{\boldsymbol{v}}_{O'}}^{\vec{\boldsymbol{0}}} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} = \omega\hat{\boldsymbol{k}} \times \left(\frac{3}{4}r\hat{\boldsymbol{\imath}}'\right) = \frac{3}{4}\omega r\hat{\boldsymbol{\jmath}}' = \frac{3}{4}\omega r\hat{\boldsymbol{\jmath}}, \\
\vec{\boldsymbol{v}}_{\mathrm{rel}} &\equiv \vec{\boldsymbol{v}}_{P/\mathcal{B}} = v_{\mathrm{rel}}\hat{\boldsymbol{\imath}}' = v_{\mathrm{rel}}\hat{\boldsymbol{\imath}}.
\end{aligned}$$

In the last expression, $\vec{\boldsymbol{v}}_{\mathrm{rel}} = v_{\mathrm{rel}}\hat{\boldsymbol{\imath}}$, we do not know the magnitude of $\vec{\boldsymbol{v}}_{\mathrm{rel}}$ and hence have left it as an unknown v_{rel}, but its direction is known because $\vec{\boldsymbol{v}}_{\mathrm{rel}}$ has to be along the track and the track at the given instant is along the x-axis. Thus,

$$\vec{\boldsymbol{v}}_P = \frac{3}{4}\omega r\hat{\boldsymbol{\jmath}} + v_{\mathrm{rel}}\hat{\boldsymbol{\imath}}. \tag{15.33}$$

Now let us consider the motion of rod AB. Let $\vec{\boldsymbol{\Omega}} = \Omega\hat{\boldsymbol{k}}$ be the angular velocity of AB at the instant of interest where Ω is unknown. Since P is pinned to the rod, it executes circular motion about A with radius $AP = \ell$. Therefore,

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{\Omega}} \times \vec{\boldsymbol{r}}_{P/A} = \Omega\hat{\boldsymbol{k}} \times \ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) = \Omega\ell(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}). \tag{15.34}$$

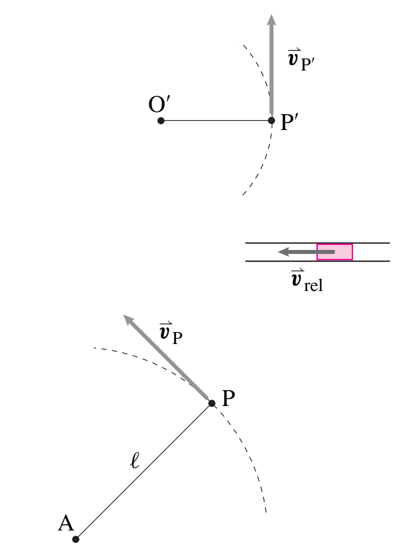


Figure 15.45:

But, and this trivial formula is the key, $\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_P$. Therefore, from Eqn. (15.33) and (15.34),

$$\frac{3}{4}\omega r\hat{\boldsymbol{\jmath}} + v_{\mathrm{rel}}\hat{\boldsymbol{\imath}} = \Omega\ell(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}). \tag{15.35}$$

Taking dot product of both sides of the above equation with $\hat{\boldsymbol{\jmath}}$ we get

$$\begin{aligned}
\frac{3}{4}\omega r &= \Omega\ell\cos\theta \\
\Rightarrow \quad \Omega &= \frac{3\omega r}{4\ell\cos\theta} = \frac{3\cdot 2\,\mathrm{rad/s}\cdot 0.4\,\mathrm{m}}{4\cdot 1\,\mathrm{m}\cdot\frac{1}{\sqrt{2}}} = 0.85\,\mathrm{rad/s}.
\end{aligned}$$

Again taking the dot product of both sides of Eqn. (15.35) with $\hat{\boldsymbol{\imath}}$ we get

$$v_{\mathrm{rel}} = -\Omega\ell\sin\theta = -0.85\,\mathrm{rad/s}\cdot 1\,\mathrm{m}\cdot\frac{1}{\sqrt{2}} = -0.6\,\mathrm{m/s}.$$

$$(i)\ \vec{\boldsymbol{\Omega}} = 0.85\,\mathrm{rad/s}\hat{\boldsymbol{k}}, \quad (ii)\ \vec{\boldsymbol{v}}_{\mathrm{rel}} = -0.6\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$$

SAMPLE 15.11 Spinning wheel on a rotating rod in 2-D . A rigid body OA is attached to a wheel that is massless except for three point masses P, Q, and R, placed symmetrically on the wheel. Each of the three masses is $m = 0.5\,\text{kg}$. The rod OA rotates about point O in the counterclockwise direction at a constant rate $\omega_1 = 3\,\text{rad/s}$. The wheel rotates with respect to the arm about point A with angular acceleration $\dot{\omega}_2 = 1\,\text{rad/s}^2$ and at the instant shown it has angular speed $\omega_2 = 5\,\text{rad/s}$. Note that both ω_2 and $\dot{\omega}_2$ are given with respect to the arm.

Using a rotating frame $\mathcal{B}$ attached to the rod and a coordinate system attached to the frame with origin at O, find

1. the velocity of the mass P and
2. the acceleration of the mass P.

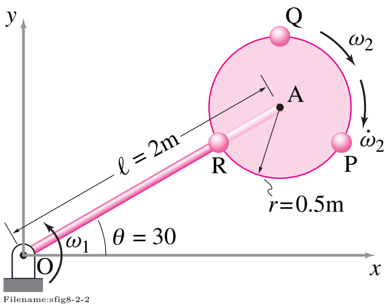


Figure 15.46:

Solution Frame $\mathcal{B}$ is attached to the rod. We choose a coordinate system $x'y'z'$ in frame $\mathcal{B}$ with its origin at O and, at the instant, aligned with the fixed coordinate system xyz. We consider a point P$'$ momentarily coincident with point P but fixed in frame $\mathcal{B}$. Since P$'$ is fixed in $\mathcal{B}$, it rotates with $\mathcal{B}$ with $\vec{\omega}_\mathcal{B} = \omega_1\hat{k}$. To visualize the motion of P$'$ imagine a rigid rod from O to P$'$ (see Fig. 15.47). Now we can calculate the velocity and acceleration of point P$'$ as follows.

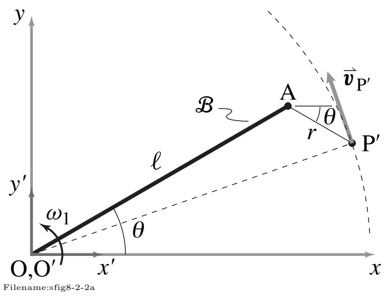


Figure 15.47: P$'$ goes in circles around point O with radius OP$'$. Therefore, $\vec{v}_{P'}$ is tangential to the circular path at P$'$.

1. **Velocity of point P:**

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}}.$$

Now we calculate the two terms separately:

$$\begin{aligned}
\vec{v}_{P'} &= \underbrace{\vec{v}_{O'}}_{\vec{0}} + \vec{\omega}_\mathcal{B} \times \vec{r}_{P'/O'} \\
&= \omega_1\hat{k} \times (\vec{r}_{A/O'} + \vec{r}_{P'/A}) \\
&= \omega_1\hat{k} \times [\underbrace{\ell(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})}_{\vec{r}_{A/O'}} + \underbrace{r(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath})}_{\vec{r}_{P'/A}}] \\
&= \omega_1(\ell + r)\cos\theta\hat{\jmath} - \omega_1(\ell - r)\sin\theta\hat{\imath} \\
&= 3\,\text{rad/s} \cdot 2.5\,\text{m} \cdot \cos 30^\circ\,\hat{\jmath} - 3\,\text{rad/s} \cdot 1.5\,\text{m} \cdot \sin 30^\circ\,\hat{\imath} \\
&= (6.50\hat{\jmath} - 2.25\hat{\imath})\,\text{m/s}.
\end{aligned}$$

Since the wheel rotates with angular speed ω_2 with respect to the rod, an observer sitting in frame $\mathcal{B}$ would see a circular motion of point P about point A. Therefore,

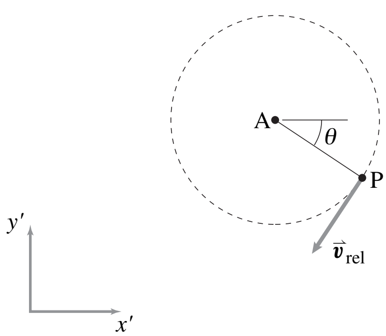


Figure 15.48: Velocity of point P with respect to the frame $\mathcal{B}$.

$$\begin{aligned}
\vec{v}_{\text{rel}} &= \vec{\omega}_{\text{wheel}/\mathcal{B}} \times \vec{r}_{P/A} \\
&= -\omega_2\hat{k}' \times r(\cos\theta\hat{\imath}' - \sin\theta\hat{\jmath}') \\
&= -\omega_2 r(\cos\theta\hat{\jmath}' + \sin\theta\hat{\imath}') \\
&= -(2.16\hat{\jmath}' + 1.25\hat{\imath}')\,\text{m/s}.
\end{aligned}$$

But at the instant of interest, $\hat{\imath}' = \hat{\imath}$, $\hat{\jmath}' = \hat{\jmath}$, and $\hat{k}' = \hat{k}$. So,

$$\vec{v}_{\text{rel}} = -(2.16\hat{\jmath} + 1.25\hat{\imath})\ \text{m/s}$$

Therefore,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}} = 4.33\,\text{m/s}\,\hat{\jmath} - 3.50\,\text{m/s}\,\hat{\imath}.$$

$$\boxed{\vec{v}_P = (-3.50\hat{\imath} + 4.33\hat{\jmath})\,\text{m/s}}$$

2. **Acceleration of point P:** We can similarly find the acceleration of point P:

$$\vec{a}_P = \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}}$$

where

$$\begin{aligned}
\vec{a}_{P'} &= \text{acceleration of point P}' \\
&= \underbrace{\vec{a}_{O'}}_{\vec{0}} + \underbrace{\vec{\alpha}_{\mathcal{B}}}_{\vec{0}} \times \vec{r}_{P'/O'} + \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\
&= -\omega_{\mathcal{B}}^2 \vec{r}_{P'/O'} \\
&= -\omega_1^2[(\ell + r)\cos\theta\,\hat{\imath} + (\ell - r)\sin\theta\,\hat{\jmath}] \\
&= -9(\,\text{rad/s})^2[2.5\,\text{m}\cdot\cos 30^\circ\,\hat{\imath} + 1.5\,\text{m}\cdot\sin 30^\circ\,\hat{\jmath}] \\
&= -(19.48\hat{\imath} + 6.75\hat{\jmath})\,\text{m/s}^2,
\end{aligned}$$

$$\begin{aligned}
\vec{a}_{\text{cor}} &= \text{Coriolis acceleration} \\
&= 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{\text{rel}} \\
&= 2\omega_1 \hat{k} \times \vec{v}_{\text{rel}} \qquad \text{(see part (a) above for } \vec{v}_{\text{rel}}). \\
&= (6\,\text{rad/s})\hat{k} \times (-2.16\hat{\jmath} - 1.25\hat{\imath})\,\text{m/s} \\
&= (12.99\hat{\imath} - 7.50\hat{\jmath})\,\text{m/s}^2,
\end{aligned}$$

$$\begin{aligned}
\vec{a}_{\text{rel}} &= \text{acceleration of P relative to frame } \mathcal{B} \\
&= \vec{a}_{P/\mathcal{B}} = \dot{\vec{\omega}}_2 \times \vec{r}_{P/A} - \omega_2^2 \vec{r}_{P/A} \\
&= -\dot{\omega}_2 \hat{k}' \times r(\cos\theta\,\hat{\imath}' - \sin\theta\,\hat{\jmath}') - \omega_2^2 r(\cos\theta\,\hat{\imath}' - \sin\theta\,\hat{\jmath}') \\
&= -r[(\dot{\omega}_2 \sin\theta + \omega_2^2\cos\theta)\hat{\imath}' + (\dot{\omega}_2\cos\theta - \omega_2^2\sin\theta)\hat{\jmath}'] \\
&= -0.5\,\text{m}[(1\,\text{rad/s}^2\cdot\sin 30^\circ + 25(\,\text{rad/s})^2\cdot\cos 30^\circ)\hat{\imath}' \\
&\qquad + (1\,\text{rad/s}^2\cdot\cos 30^\circ - 25(\,\text{rad/s})^2\cdot\sin 30^\circ)\hat{\jmath}'] \\
&= (-11.08\hat{\imath}' + 5.82\hat{\jmath}')\,\text{m/s}^2 \\
&= (-11.08\hat{\imath} + 5.82\hat{\jmath})\,\text{m/s}^2.
\end{aligned}$$

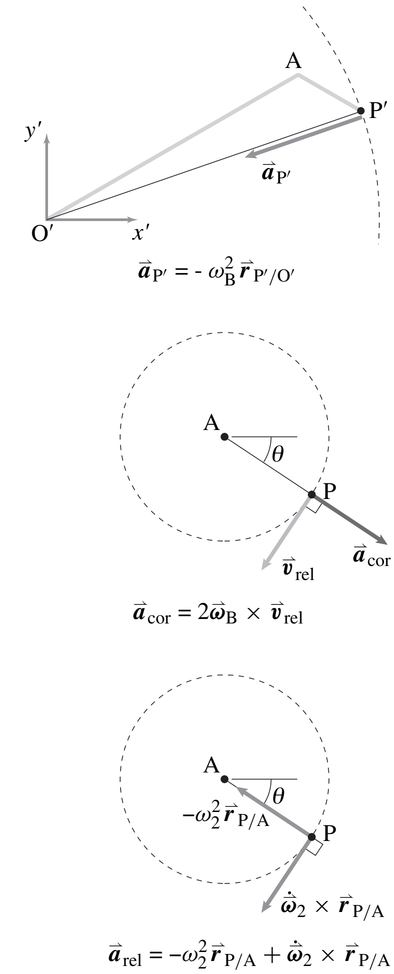


Figure 15.49:

The term $\vec{a}_{P'}$ encompasses three terms of the five term acceleration formula. The last line in the calculation of $\vec{a}_{\text{rel}}$ follows from the fact that at the instant of interest $\hat{\imath}' = \hat{\imath}$ and $\hat{\jmath}' = \hat{\jmath}$.

Now adding the three parts of $\vec{a}_P$ we get

$$\begin{aligned}
\vec{a}_P &= \vec{a}_{P'} + \vec{a}_{\text{cor}} + \vec{a}_{\text{rel}} \\
&= -17.57\,\text{m/s}^2\hat{\imath} - 8.43\,\text{m/s}^2\,\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{a}_P = -(17.57\hat{\imath} + 8.43\hat{\jmath})\,\text{m/s}^2}$$

15.4 Kinematics of 2-D mechanisms

An *ideal mechanism* or *linkage* is a collection of rigid objects constrained to move relative to the ground and each other by hinges but which still has some possible motion(s). People also use the word mechanism or linkage more loosely to include any collection of machine parts connected by any means.

The analysis of the kinematics of mechanisms is an important part of machine design. Mechanisms *synthesis*, coming up with a mechanism design which has desired motions, is obviously key in creative design, and now days in computer aided design.

Finally, the determination of the dynamics of a mechanism, how it will move and with what forces, is completely dependent on understanding the kinematics of the mechanism. The whole subject of mechanism kinematic analysis, although in some sense a subset of dynamics, is actually a huge and infinitely complex subject in itself① and also a useful subject in itself. Often kinematics is the central interest in machine design , and mechanics (force and acceleration) analysis is only carried out if something that shouldn't do so, breaks or shakes. This section presents some of the basic ideas in kinematic analysis. The overall question in mechanism kinematic analysis is this:

①**Very advanced aside.** The general kinematics of planar linkages is as complicated as the classification of closed surfaces in any number of dimensions: spheres, donuts, spheres with two holes, etc. For every surface there is a corresponding 2D mechanism. In math-speak: Any orientable manifold is a connected component of the configuration space of some planar linkage. This idea was proposed by Bill Thurston in the 1970's and later proved by Kapovich and Millson. We report this just so you don't feel bad that you don't now, nor ever will, know all there is to know about planar mechanisms.

> Given a collection of parts and a description of how they are connected, in what ways can they move?

Without getting into the details of the motions yet, the first question to answer is simpler than finding the motions, but just counting them: In how many ways can the mechanism move?

Degrees of freedom (DOF)

The number of degrees of freedom (DOF) n_{DOF} of a mechanism is the number of different ways it can move. More precisely

> The number of degrees of freedom n_{DOF} of a mechanism is the minimum number of configuration variables needed to describe all possible configurations of the mechanism.

The minimum number of configuration variables n_{DOF} is a property of the mechanism. The choice of what these variables are, however is not unique to a given mechanism.

Example: A particle in a plane has 2 degrees of freedom.
The set of 'configurations' of a particle in a plane is the set of positions of the particle. This is fully described by its x and y coordinates. Thus $n_{\text{DOF}} = 2$. But the configuration is also determined by the particles polar coordinates R and θ. And there are an infinite number of other pairs of numbers that could be used to describe the configurations (*e.g.*, the x' and y' coordinates, the w and z coordinates

with $w = e^x$ and $z = e^y$, etc). The minimum number of configuration variables, 2, is unique, but the choice of variables is not.

For planar mechanisms one can often determine the number of degrees of freedom by the following formula

$$3 \cdot \underbrace{(\text{\# of rigid objects})}_{n_{\text{bod}}} + 2 \cdot \underbrace{(\text{\# of particles})}_{n_{\text{part}}} - \underbrace{(\text{\# of constraints})}_{n_{\text{con}}} = n_{\text{DOF}} \tag{15.36}$$

The formula starts with the number of ways one rigid object can move (2 translations and a rotation makes 3) and one particle can move (just 2 translations) and then subtracts the restrictions to the motion. In eqn. (15.36) the number n_{con} of constraints is counted as the number of degrees of freedom restricted by the connections.

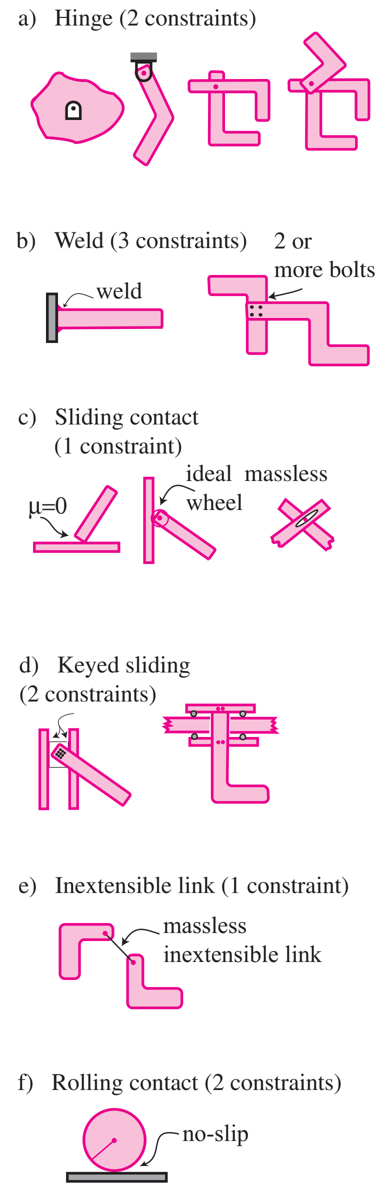


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Figure 15.50: The number of degrees of freedom that are eliminated by various constraints. See text for explanation.

Examples of connections and their effect on n_{DOF}

See fig. 15.50 for some standard idealized connections and their number of constraints (assuming they are already constrained to a plane).

a) **2** for a **pin joint**: a pin joint restricts relative motion in 2 directions but still allows relative rotation. If three objects are connected at one pin then it counts as two pin joints and thus $2 \times 2 = 4$ reductions in the number of degrees of freedom. There are 6 reductions for 4 objects connected at one pin, etc.

b) **3** for a **welded connection**: a weld restricts relative translation in two directions as well as relative rotation ($2 + 1 = 3$). So two parts that are welded together have $2 \cdot 3 - 3 = 3$ degrees of freedom. That is, any collection of rigid objects welded together is the same as one rigid object. The word ‘weld’ is meant to include any collection of bolts, glue, string, rivets or bailing wire that prevents any relative motion.

c) **1** for a **sliding contact**: the sliding contact restricts relative translation normal to the contact surfaces and allows translation tangent to the surfaces. Relative rotation is also allowed.

d) **2** for a **keyed sliding contact**: allows relative translation in one direction but disallows translation in one direction as well as rotation.

e) **1** for a **massless link hinged at its ends to two objects**: this keeps the distance between two points fixed which is one restriction (alternatively the bar adds 3 degrees of freedom and each hinge subtracts 2 ($+3 - 2 \times 2 = -1$ degree of freedom).

f) **2** for a **rolling contact**: relative slip is not allowed nor is interpenetration.

Be warned

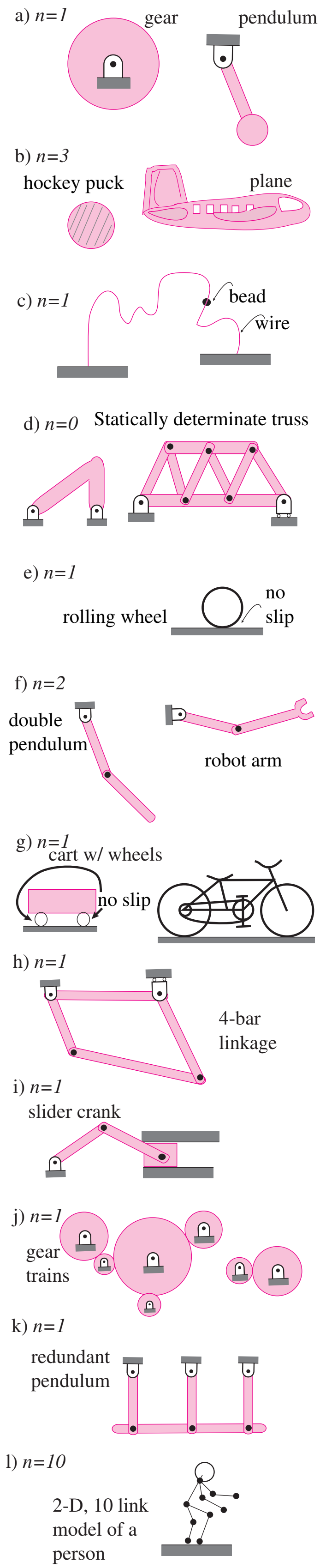

Figure 15.51: Some simple mechanisms and their number of degrees of freedom, n (called n_{DOF} in the text).

If some of the constraints are redundant then a system can have more degrees of freedom than eqn. (15.36) indicates. But a mechanism never has fewer degrees of freedom than eqn. (15.36) indicates.

Some simple mechanisms

Figure 15.51 shows some examples of simple mechanisms and the number of degrees of freedom. In each case we look at eqn. (15.36): $3 \cdot n_{\text{bod}} + 2 \cdot n_{\text{part}} - n_{\text{con}} = n_{\text{DOF}}$.

a) A **object connected to ground by a hinge** has **1** degree of freedom; the set of all possible configurations can be described by the angle of the object: $n_{\text{bod}} = 1, n_{\text{con}} = 2 \quad \Rightarrow \quad n_{\text{DOF}} = 1$.

b) An **unconstrained object** has **3** degrees of freedom; the set of all configurations can be described by the x and y coordinates of a reference point and by the rotation θ: $n_{\text{bod}} = 1, n_{\text{con}} = 0 \quad \Rightarrow \quad n_{\text{DOF}} = 3$.

c) A **bead on a wire** has **1** degree of freedom; its configuration is fully determined by the distance the bead has advanced along the wire relative to a reference mark: $n_{\text{part}} = 1, n_{\text{con}} = 1 \quad \Rightarrow \quad n_{\text{DOF}} = 1$.

d) A **statically determinate truss** has **0** degrees of freedom; a statically determinate truss has no ways to move: $n_{\text{bod}} = n_{\text{bars}}, n_{\text{part}} = 0, n_{\text{con}} = 2 \cdot n_{\text{pins}} + n_{\text{ground const}} \quad \Rightarrow n_{\text{DOF}} = 0$. Note that the number of pin restrictions in our count here is more than twice the number of joints in the study of trusses in statics. In statics we focussed on joints as restricted by bars. Here we look at bars as restricted by joints and a given joint counts 2, 4 or 6 pin restrictions depending on whether it connects 2, 3 or 4 bars. Thus the 11 bar truss shown has 7 joints (by truss-analysis counting) but, by the counting here it has $2\times2+2\times4+3\times6 = 30$ pin restrictions plus 3 ground restrictions for 33 restrictions in all. Without restrictions the 11 bars had 33 degrees of freedom. So, because 33-33 = 0, the truss has zero degrees of freedom.

e) A **rolling wheel** has **1** degree of freedom; its configuration is fully determined either by the net angle θ it has rolled or by the x coordinate of its center: $n_{\text{bod}} = 1, n_{\text{con}} = 2 \quad \Rightarrow \quad n_{\text{DOF}} = 1$.

f) A **double pendulum** or two-link robot arm has **2** degrees of freedom; its configuration is determined by the net rotations of its two links (or by the rotation of the first link and the relative rotation of the second link): $n_{\text{bod}} = 2, n_{\text{con}} = 4 \quad \Rightarrow \quad n_{\text{DOF}} = 2$.

g) A **cart with two rolling wheels** or a planar rolling bicycle has **1** degree of freedom; both the rotation of the wheels and the position of the bicycle are determined by the 1 variable, say, the x coordinate of a reference point on the vehicle(*e.g.,* the bicycle seat). $n_{\text{bod}} = 3, n_{\text{con}} = 4\times2 = 8$ (2 hinges and 2 rolling contacts, the hinge for bicycle steering isn't relevant for a planar analysis) $\quad \Rightarrow \quad n_{\text{DOF}} = 1$.

h) A **"four" bar linkage** has **1** degree of freedom; the angle of any one of the bars determines the angles of the others: $n_{\text{bod}} = 4, n_{\text{con}} = 2 \times 4 + 2 + 1 = 11$ (there are 4 pin joints between the bars, one pin joint to ground and one roller connection to the ground) $\Rightarrow \quad n_{\text{DOF}} = 1$.

i) A **slider crank** has **1** degree of freedom; the rotation of the crank determines the configuration of the system $n_{\text{bod}} = 3, n_{\text{con}} = 3 \cdot 2 + 2$ (there are 3 pins and one keyed connection) $\Rightarrow \quad n_{\text{DOF}} = 1$.

j) An ideal **gear train** (with all gears pinned to ground) has **1** degree of freedom; the amount of rotation of any one gear determines the rotation of all of the gears: In this case the counting formula is *wrong*. Say there are 2 gears, then $n_{\text{bod}} = 2, n_{\text{con}} = 3 \cdot 2 = 6$ (two pins and one rolling contact) $\Rightarrow \quad n_{\text{DOF}} = 0 \neq 1$. The rolling constraint prevents interpenetration, but this was already prevented by the hinges at the center of the gears. The constraints are redundant and the system has more degrees of freedom than eqn. (15.36) indicates.

k) A **redundant swing** with one horizontal bar suspended by 3 parallel struts has **1** degree of freedom; the angle of one upright links determines the full configuration of the mechanism. The counting formula is again wrong: $n_{\text{bod}} = 4, n_{\text{con}} = 6 \cdot 2 = 12 \quad \Rightarrow \quad n_{\text{DOF}} = 0$ underestimates the number of degrees of freedom because the constraints are redundant.

l) A **2-D 10-link model of a person** with one foot on the ground has **10** degrees of freedom; the angles of the 10 links determine the full configuration of the mechanism. There are no redundant constraints and the counting formula works: $n_{\text{bod}} = 10, n_{\text{con}} = 10 \cdot 2 = 20$ (counting a hinge at the ground contact and two hinges at both the hips and the shoulders) $\Rightarrow \quad n_{\text{DOF}} = 10$.

Configuration variables

Once we know the number of degrees of freedom n_{DOF} of a system it is often useful to settle on one set of n_{DOF} configuration variables. In this book n_{DOF} will be 1, 2 or at most 3. Thus we pick 1,2 or 3 variables.

Example: Straight line motion
Chapter 6 on straight line motion was mostly about one-degree-of-freedom systems ($n_{\text{DOF}} = 1$). These systems could all be characterized by the single configuration variable x, the displacement along the line of a reference point on the object relative to a reference point on the ground. All the positions,velocities and accelerations of all points in the system could be found in terms of x, $\dot{x}$ and $\ddot{x}$ (in fact all points had ($\vec{v} = \dot{x}\hat{\imath}$ and $\vec{a} = \ddot{x}\hat{\imath}$.

Example: Circular motion about a fixed axis
In chapters 7 and 8 we were almost entirely focussed on systems with one degree of freedom well characterized by the one configuration variable, the rotation angle θ. For such motions positions, velocities, and accelerations of all points were determined by the initial positions of the points θ, $\dot{\theta}$ and $\ddot{\theta}$ by equations which you know well by now.

For more general motions we almost always take inspiration from the two examples above. We use the translation of a conspicuous point, or we use the rotation of a conspicuous object for a configuration variable. And more of the same if the system has more than 1 degree of freedom. The natural choice of configuration variables for some simple mechanisms is given in the text discussing fig. 15.51.

Often our main kinematic task is to express the full configuration of the system as well as all the velocities and accelerations of all its parts in terms of the positions of the parts, the configuration variables, and their first and second time derivatives.

Adding relative angular velocities

One last simple kinematic fact is needed before we can plug and chug with the theory we have so far and apply it to general kinematic mechanisms. It concerns the addition of rotations and rotation rates. The following example basically tells the whole story

Example: Double pendulum and the addition of rotation rates
The commonly used configuration variables for the double pendulum shown in fig. 15.52 are θ_1 and θ_2. To actually know the configuration of the system obviously we need to know ϕ which is given by

$$\phi = \theta_1 + \theta_2 \quad \text{so} \quad \dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2 \quad \text{and so} \quad \ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2.$$

[Aside: One reason for choosing θ_2 instead of ϕ as a configuration variable is that if one was measuring or controlling the second link, say as a robotic arm, the angle θ_2 can be measured more easily than ϕ. Also, it turns out (in hindsight) that the differential equations of motion are slightly simpler using θ_2 instead of ϕ.]

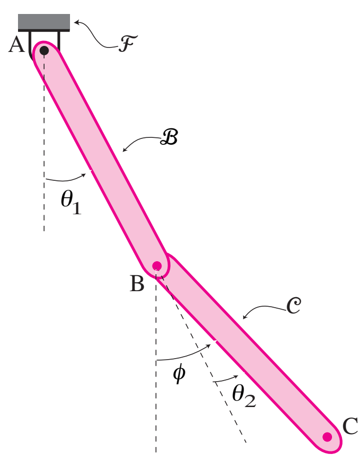

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Figure 15.52: A double pendulums configuration variables are most often taken to be θ_1 and θ_2. But one then needs to know that $\phi = \theta_1 + \theta_2$ and that $\dot{\phi} = \dot{\theta}_1 + \dot{\theta}_2$ and $\ddot{\phi} = \ddot{\theta}_1 + \ddot{\theta}_2$.

Looking at the bars as being glued to reference frames ($\mathcal{F}$ for the fixed frame, $\mathcal{B}$ for bar AB, and $\mathcal{C}$ for bar BC), the above example shows that

$$\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} + \vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} \tag{15.37}$$

which is often written with the simple notation

$$\vec{\boldsymbol{\omega}} = \vec{\boldsymbol{\omega}}_1 + \vec{\boldsymbol{\omega}}_2$$

Which can only be given strict meaning by the more elaborate eqn. (15.37) above it.

Example: Double pendulum (see previous example)
Take $\vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} = \dot{\theta}_1 \hat{\boldsymbol{k}}$, $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{B}} = \dot{\theta}_2 \hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{\omega}}_{\mathcal{C}/\mathcal{F}} = \dot{\phi} \hat{\boldsymbol{k}}$ and eqn. (15.37) is self evident from the addition of angles.

Similarly we have

$$\vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{F}} = \vec{\boldsymbol{\alpha}}_{\mathcal{B}/\mathcal{F}} + \vec{\boldsymbol{\alpha}}_{\mathcal{C}/\mathcal{B}} \tag{15.38}$$

which is often written with the simple notation

$$\vec{\boldsymbol{\alpha}} = \vec{\boldsymbol{\alpha}}_1 + \vec{\boldsymbol{\alpha}}_2.$$

For those going on to study 3-D mechanics (Chapter 12), one should note that, unlike eqn. (15.37), eqn. (15.38) does *not* hold in 3-D.

Kinematics of mechanisms

One approach to mechanisms is to do what one can with high-school geometry and trigonometry, the laws of sines (see page 95), and so on.

Example: Rod on step using geometry and trigonometry
One end of a rod slides on the ground. The other end slides on a corner at A (see fig. 15.53). Given that $\vec{v}_{\mathrm{B}} = v_B\hat{\imath}$ we can find $\dot{\phi}$ as follows:

$$\frac{h}{\ell_{AB}} = \tan\phi \Rightarrow \left\{\frac{\ell_{AB}}{h} = \frac{\cos\phi}{\sin\phi}\right\}$$
$$\frac{d}{dt}\left\{\quad\right\} \Rightarrow \frac{\dot{\ell}_{AB}}{h} = \frac{-\dot{\phi}}{\sin^2\phi} \Rightarrow \dot{\phi} = -\frac{v_B\sin^2\phi}{h}$$

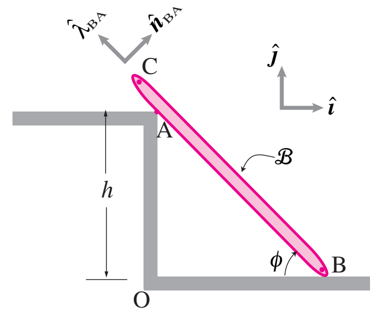

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Figure 15.53: A rigid rod slides so that it always touches the ground and the corner at A. One would like to know the relation between $\dot{\phi}$ and $\vec{v}_{\mathrm{B}}$.

As the above calculation shows, this problem doesn't need the heavy machinery of our moving-frame vector methods. But it provides an instructive example.

Example: Rod on step using moving-frame methods (see previous example)
We look at point A and note that we can think of it as a fixed point in the fixed frame $\mathcal{F}$ and also as a point that is moving relative to the translating and rotating frame $\mathcal{B}$. We evaluate its velocity both ways.

$$\begin{array}{rcll}
\vec{v}_{\mathrm{A}} &=& \vec{v}_{\mathrm{A}} & \\
\vec{0} &=& \vec{v}_{\mathrm{B}} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{\mathrm{A/B}} + \vec{v}_{\mathrm{A}/\mathcal{B}} & \text{(eqn. (15.28))} \\
\{\vec{0} &=& v_B\hat{\imath} - \dot{\phi}\hat{k} \times \left(\ell_{BA}\hat{\lambda}_{BA}\right) + v_{A/\mathcal{B}}\hat{\lambda}_{BA}\} & \\
\Rightarrow 0 &=& v_B\underbrace{\hat{\imath}\cdot\hat{n}_{BA}}_{\sin\phi} + \dot{\phi}\ell_{BA} & (\{\}\cdot\hat{n}_{BA}) \\
\Rightarrow \dot{\phi} &=& -\dfrac{v_B\sin\phi}{\ell_{BA}} = -\dfrac{v_B\sin^2\phi}{h} & (\sin\phi = \dfrac{h}{\ell_{AB}})
\end{array}$$

as we had before. The key equation was the 'three term velocity formula' eqn. (15.28) on page 869 and the observation that relative to frame $\mathcal{B}$ point A slides along the rod. Note that we never had to explicitly use the rotating coordinates associated with frame $\mathcal{B}$ to do this calculation.

You should understand the examples above, and the needed background material, before going on to the following examples.

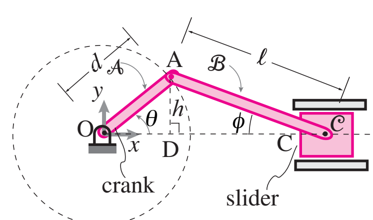

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Figure 15.54: The slider crank mechanism.

Example: Slider Crank using geometry and trigonometry
The slider crank mechanism (fig. 15.54) was briefly introduced in the context of statics where its forces could be analyzed assuming inertial terms were negligible (see 358). But it is a commonly used mechanism (*e.g.,* in every car) and its motions are of central interest. The angle θ is the most natural configuration variable for this $n_{\mathrm{DOF}} = 1$ system. One would like to know the position, velocity and acceleration of the slider (x_C, $\dot{x}_C$ and $\ddot{x}_C$) in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$.

$$\begin{aligned}
x_C &= x_D + \ell_{DC} \\
&= d\cos\theta + \sqrt{\ell^2 - h^2} \\
&= d\cos\theta + \sqrt{\ell^2 - (d\sin\theta)^2} \\
&= d\left(\cos\theta + \sqrt{(\ell/d)^2 - \sin^2\theta}\right) \\
&= d\left(\cos\theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2}\right) \qquad (15.39)
\end{aligned}$$

The positive $\sqrt{\ }$ corresponds to C being to the right of 0. The negative $\sqrt{\ }$ corresponds to point C being to the left of 0. The mechanism just doesn't work for a full revolution of the link 0A if $\ell < d$ as you can see from the picture or from that the $\sqrt{\ }$ above giving imaginary values for $\cos 2\theta$ near -1, $\sin\theta$ near 1, and θ near $\pi/2$.

To get the velocity of point C we just take the derivative of eqn. (15.39) above.

$$\begin{aligned}
v_C &= \dot{x}_C \qquad (15.40) \\
&= \frac{d}{dt}\left\{d\left(\cos\theta + \sqrt{(\ell/d)^2 + (\cos 2\theta - 1)/2}\right)\right\} \\
&= \dot{\theta} d\left\{-\sin(\theta) - \sin(2\,\theta)\frac{1}{\sqrt{4\,\frac{l^2}{d^2} + 2\,\cos(2\,\theta) - 2}}\right\}
\end{aligned}$$

To get the acceleration we differentiate once again. For simplicity lets assume the crank rotates at constant rate, so $\dot{\theta}$ is a constant and $\ddot{\theta} = 0$. Cranking out the derivative of eqn. (15.40), so to speak, we get

$$\begin{aligned}
a_c = \dot{v}_C &= \frac{d}{dt}\{\text{the mess on the right of eqn. (15.40)}\} \qquad (15.41) \\
&= -d\dot{\theta}^2\left\{\cos(\theta)\right. \\
&\qquad +2\,\sin^2(2\,\theta)\left(4\,\frac{l^2}{d^2} + 2\,\cos(2\,\theta) - 2\right)^{-3/2} \\
&\qquad \left. +2\,\cos(2\,\theta)\frac{1}{\sqrt{4\,\frac{l^2}{d^2} + 2\,\cos(2\,\theta) - 2}}\right\}
\end{aligned}$$

So we now know the position, velocity and acceleration of point C in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$. You should commit the solution eqn. (15.41) to memory. Just kidding.

Plots of x_C, v_C, and a_C from these equations are shown in fig. 15.55ab for two different extremes of slider crank design: one with a very long connecting rod that gives sinusoidal motion, and one with a connecting rod just barely long enough to prevent locking that gives intermittent motion.

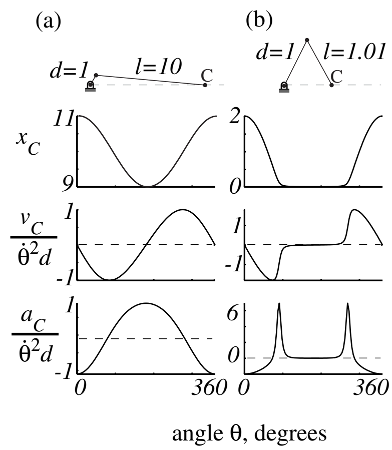


Figure 15.55: Position x_C of the slider, its velocity v_C and its acceleration a_c (Strictly, *e.g.,* $\vec{a}_C = a_C\hat{\imath}$). Two sets of curves are shown. **(a)** a long connecting rod, and **(b)** a connecting rod a hair longer than the crank. In case (a) the connecting rod is nearly horizontal at all times and the displacement of point C is almost entirely due to the horizontal displacement of the end of the crank arm. Thus point C moves with almost exactly a cosine wave with amplitude equal to the length of the crank. The velocity and acceleration curves are thus also sine and cosine waves. In case (b) the motion is close to a cosine curve approximately when $-60^\circ < \theta < 60^\circ$. That is, the displacement of C is about twice the horizontal displacement of the end of the crank arm when the end is to the right of its base bearing. When the end of the crank arm is to the left of the bearing point C is nearly stationary and is just to the right of the crank base bearing. The transition between these two cases involves a sudden large acceleration.

Unlike some more complex mechanisms, the slider crank is solvable in that one can write a formula for the position of any point of interest in terms of the single configuration variable θ. For more complex mechanisms this may not be possible. Further, even if possible the above example shows that the differentiation required to find velocity and acceleration can lead to a bit of a mess.

A different approach is to assume that at some value of the configuration variable (θ for the slider crank) that the full configuration of the system is known. That is, that the locations of all points are known. Then we can use our vector methods to find velocities and accelerations of all points of interest.

Example: Slider crank using vector methods (see previous example)
Take the slider crank of fig. 15.54 to be in some known configuration. We now try to find the velocity and acceleration of point C in terms of the positions of the points 0, B, and C as well as θ and $\dot\theta$.

The basic approach is to write true things, and then solve for unknowns. First work on velocities. The basic idea is to look at the *closure* condition. That is, the velocity of point C as calculated by working down the linkage from 0 to A to C has to be consistent with the velocity of C as calculated in the fixed frame.

$$\begin{aligned}
\vec{v}_{\mathrm{C}} &= \vec{v}_{\mathrm{C}}\\
v_C\hat{\imath} &= \vec{v}_{\mathrm{A/0}} + \vec{v}_{\mathrm{C/A}}\\
v_C\hat{\imath} &= (\dot\theta\hat{k})\times\vec{r}_{\mathrm{A/0}} + (-\dot\phi\hat{k})\times\vec{r}_{\mathrm{C/A}} \qquad (15.42)
\end{aligned}$$

eqn. (15.42) is a 2-D vector equation in the 2 unknown scalars v_C and $\dot\phi$. It could be solved as a pair of equations, or solved directly by first dotting both sides with $\hat{\jmath}$ to find $\dot\phi$ and dotting both sides with $\vec{r}_{\mathrm{C/A}}$ to find v_C. These yield

$$\begin{aligned}
\dot\phi &= \frac{\left((\dot\theta\hat{k})\times\vec{r}_{\mathrm{A/0}}\right)\cdot\hat{\jmath}}{\left(\hat{k}\times\vec{r}_{\mathrm{C/A}}\right)\cdot\hat{\jmath}} \quad \text{and}\\
v_C &= \frac{\dot\theta\left(\hat{k}\times\vec{r}_{\mathrm{A/0}}\right)\cdot\vec{r}_{\mathrm{C/A}}}{\hat{\imath}\cdot\vec{r}_{\mathrm{C/A}}}
\end{aligned}$$

where everything on the right of these equations is assumed known. Without grinding out the vector products in terms, say, of components, we can just know that we can at this point know $\dot\phi$ and v_C.

We proceed to find the accelerations by similar means, assuming $\dot\theta$ is a constant so $\vec{\alpha}_{\mathcal{A}} = \vec{0}$:

$$\begin{aligned}
\vec{a}_{\mathrm{C}} &= \vec{a}_{\mathrm{C}}\\
a_C\hat{\imath} &= \vec{a}_{\mathrm{A/0}} + \vec{a}_{\mathrm{C/A}}\\
a_C\hat{\imath} &= -\dot\theta^2\vec{r}_{\mathrm{A/0}} + (-\ddot\phi\hat{k})\times\vec{r}_{\mathrm{C/A}} - \dot\phi^2\vec{r}_{\mathrm{C/A}} \qquad (15.43)
\end{aligned}$$

eqn. (15.43) is a 2-D vector equation in the two scalar unknowns a_c and $\ddot\phi$. We can set this up as two equations in two unknowns. Or we can solve for $\ddot\phi$ directly by dotting both sides with $\hat{\jmath}$ and we can solve for a_C directly by crossing both sides with $\vec{r}_{\mathrm{C/A}}$ or by dotting with a vector perpendicular to $\vec{r}_{\mathrm{C/A}}$ (e.g., $\vec{r}^{\perp}_{C/A} = \hat{k}\times\vec{r}_{\mathrm{C/A}}$

Although we have presented an algorithm rather than a formula, we have found the velocity and acceleration of C without writing any large equations of the type needed in the previous example. The shortcoming is that this method depends on knowing the full configuration at the time of interest.

Example: Four bar linkage using geometry and trigonometry
fig. 15.56 shows a "four-bar linkage". Please see 357 for an introduction to 4-bar linkages in the context of statics. Four bar linkages are solvable in the sense that one can write equations for the positions of any point of interest in terms of the single configuration variable θ marked in fig. 15.56. But the formulas are really a mess. And the first and second time derivatives are an unbelievable mess.

The four-bar linkage is about as complex a system as can be solved in this sense, and it is probably too-complex for this solution to be useful in the kinematic analysis of accelerations.

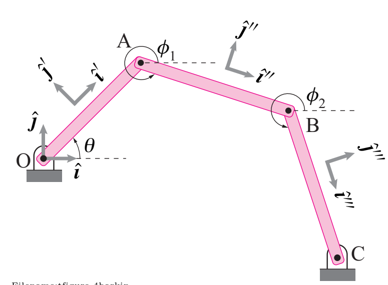


Figure 15.56: A four bar linkage.

For complex mechanisms one is often stuck using vector methods, like we are for practical purposes stuck with the 4-bar linkage. But the vector methods based on the current configuration are not crippled by complexity.

Example: Four-bar linkage using relative velocities and accelerations
Assuming the configuration is known (*i.e.,* that θ, ϕ_1, and ϕ_2 are known), we can proceed with the 4-bar linkage just as we did for the slider crank. We enforce closure by picking a point and thinking about its velocity two different ways②. We could pick any point, say C. From the fixed frame we know that the velocity of C is zero. Working around the linkage, link by link, we know it is the sum of relative velocities as

②**Relative *vs* absolute angles.** In this example, and the examples above, we use the absolute rotations of the links. However, we could also have done the calculations using the relative rotations and then used the formulas for velocity and acceleration relative to a rotating frame. For these examples this would make the calculations slightly. but not intractably, longer.

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\mathrm{C}} &= \vec{\boldsymbol{v}}_{\mathrm{C}} \\
\vec{\mathbf{0}} &= \vec{\boldsymbol{v}}_{\mathrm{A/0}} + \vec{\boldsymbol{v}}_{\mathrm{B/A}} + \vec{\boldsymbol{v}}_{\mathrm{C/B}} \\
&= \left(\dot{\theta}\hat{\boldsymbol{k}}\right) \times \vec{\boldsymbol{r}}_{\mathrm{A/0}} + \left(\dot{\phi}_1\hat{\boldsymbol{k}}\right) \times \vec{\boldsymbol{r}}_{\mathrm{B/A}} + \left(\dot{\phi}_2\hat{\boldsymbol{k}}\right) \times \vec{\boldsymbol{r}}_{\mathrm{C/B}}
\end{aligned}$$

which is equivalent to two scalar equations in the two unknowns $\dot{\phi}_1$ and $\dot{\phi}_2$. This equation can be solved directly for $\dot{\phi}_2$ by taking the dot product of both sides with a vector perpendicular to $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ (such as $\hat{\boldsymbol{\jmath}}''$ or $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}$) and for $\dot{\phi}_1$ by taking the dot product of both sides with for a vector perpendicular to $\vec{\boldsymbol{r}}_{\mathrm{C/B}}$ (such as $\hat{\boldsymbol{\jmath}}'''$ or $\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{C/B}}$) to get

$$\begin{aligned}
\dot{\phi}_1 &= -\dot{\theta}\frac{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{A/0}}\right) \cdot \hat{\boldsymbol{\jmath}}'''}{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}\right) \cdot \hat{\boldsymbol{\jmath}}'''} \qquad \text{and} \\
\dot{\phi}_2 &= -\dot{\theta}\frac{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{A/0}}\right) \cdot \hat{\boldsymbol{\jmath}}''}{\left(\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{C/B}}\right) \cdot \hat{\boldsymbol{\jmath}}''}.
\end{aligned}$$

The dot product with $\hat{\boldsymbol{k}}$ is used to get a scalar on the top and bottom of the fraction, both vectors are already only in the $\hat{\boldsymbol{k}}$ direction. Now that $\dot{\phi}_1$ and $\dot{\phi}_2$ are known the velocity of any point on the mechanism is known. For example

$$\vec{\boldsymbol{v}}_{\mathrm{B}} = \left(\dot{\phi}_2\hat{\boldsymbol{k}}\right) \times \vec{\boldsymbol{r}}_{\mathrm{B/C}}.$$

The angular accelerations of the two links are found by the same method. For simplicity lets assume that the driving crank 0A spins at constant rate so $\ddot{\theta} = 0$. Looking at the acceleration of point C two ways we have

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{C}} &= \vec{\boldsymbol{a}}_{\mathrm{C}} \\
\vec{\mathbf{0}} &= \vec{\boldsymbol{a}}_{\mathrm{A/0}} + \vec{\boldsymbol{a}}_{\mathrm{B/A}} + \vec{\boldsymbol{a}}_{\mathrm{C/B}} \\
&= \underbrace{\left(\ddot{\theta}\hat{\boldsymbol{k}}\right)}_{0} \times \vec{\boldsymbol{r}}_{\mathrm{A/0}} - \dot{\theta}^2\vec{\boldsymbol{r}}_{\mathrm{A/0}} \\
&\quad + \left(\ddot{\phi}_1\hat{\boldsymbol{k}}\right) \times \vec{\boldsymbol{r}}_{\mathrm{B/A}} - \dot{\phi}_1^2\vec{\boldsymbol{r}}_{\mathrm{B/A}} \\
&\quad + \left(\ddot{\phi}_2\hat{\boldsymbol{k}}\right) \times \vec{\boldsymbol{r}}_{\mathrm{C/B}} - \dot{\phi}_2^2\vec{\boldsymbol{r}}_{\mathrm{C/B}}
\end{aligned}$$

Because $\dot{\phi}_1$ and $\dot{\phi}_2$ are already known, this is one equation in the two unknowns $\ddot{\phi}_1$ and $\ddot{\phi}_2$. They can be solved for $\ddot{\phi}_1$ by taking the dot product of both sides with $\hat{\boldsymbol{\jmath}}'''$ and for $\ddot{\phi}_2$ by taking the dot product of both sides with $\hat{\boldsymbol{\jmath}}''$.

At this point you know θ, $\dot{\theta}$, ϕ_1, $\dot{\phi}_1$, $\ddot{\phi}_1$, ϕ_2, $\dot{\phi}_2$, and $\ddot{\phi}_2$ and can thus calculate the position, velocity and acceleration of any point in the mechanism.

SAMPLE 15.12 Two rods connected in a one-DOF mechanism. A mechanism consists of two rods AB and CD connected together at P with a collar pinned to AB but free to slide on CD. Rod AB is driven with $\vec{\omega}_{AB} = 10\,\text{rad/s}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}}_{AB} = 4\,\text{rad/s}^2\hat{\boldsymbol{k}}$. At the instant shown, $\theta = 45^\circ$ and $\beta = 30^\circ$. The length of rod AB is $R = 0.4\,\text{m}$. At the instant shown,

1. Find the angular velocity and angular acceleration of rod CD.
2. Find the velocity and acceleration of the collar with respect to rod CD.

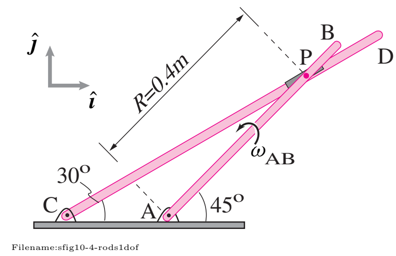


Figure 15.57:

Solution Here, we are interested in instantaneous kinematics of this mechanism. Since point P is on rod AB as well as on rod CD, its velocity and acceleration can be expressed in terms of the angular motion of rod AB or that of rod CD. Let us consider rod AB first. Let $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ be basis vectors attached to rod AB that rotate with the rod. SInce P is fixed on rod AB, it executes simple circular motion about A with $\vec{\omega}_{AB} = \dot{\theta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}}_{AB} = \ddot{\theta}\hat{\boldsymbol{k}}$ where $\dot{\theta} = 10\,\text{rad/s}$ and $\ddot{\theta} = 4\,\text{rad/s}^2$, respectively. Then

$$\vec{v}_B = R\dot{\theta}\hat{\boldsymbol{e}}_\theta \tag{15.44}$$
$$\vec{a}_B = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta. \tag{15.45}$$

Now let us consider rod CD and express the velocity and acceleration of point P in terms of motion of rod CD. Let the angular velocity and angular acceleration of rod CD be $\vec{\omega}_{CD} = \dot{\beta}\hat{\boldsymbol{k}}$ and $\dot{\vec{\omega}}_{CD} = \ddot{\beta}\hat{\boldsymbol{k}}$, respectively. Let $\hat{\boldsymbol{\lambda}}$ and $\hat{\boldsymbol{n}}$ be basis vectors attached to rod CD. Let the instantaneous position of point P on CD be $\vec{r}_P = r\hat{\boldsymbol{\lambda}}$. Since the collar can slide along CD, we can write the velocity and acceleration of point P as

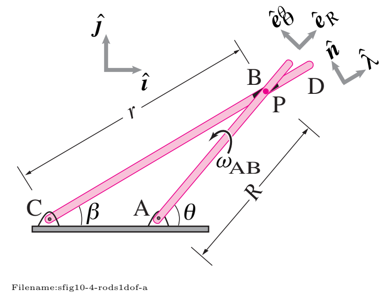


Figure 15.58: The position of point P can be written as $R\hat{\boldsymbol{e}}_R$ or $r\hat{\boldsymbol{\lambda}}$ where $r = R\sin\theta/\sin\beta$ (since $r\sin\beta = R\sin\theta$).

$$\vec{v}_B = \dot{r}\hat{\boldsymbol{\lambda}} + r\dot{\beta}\hat{\boldsymbol{n}} \tag{15.46}$$
$$\vec{a}_B = (\ddot{r} - r\dot{\beta}^2)\hat{\boldsymbol{\lambda}} + (2\dot{r}\dot{\beta} + r\ddot{\beta})\hat{\boldsymbol{n}}. \tag{15.47}$$

From eqn. (15.44) and 15.46 we get,

$$\begin{aligned} \dot{r}\hat{\boldsymbol{\lambda}} + r\dot{\beta}\hat{\boldsymbol{n}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\ \Rightarrow \quad \dot{r} &= R\dot{\theta}(\hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{\lambda}}) \\ \dot{\beta} &= (1/r)R\dot{\theta}(\hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{n}}) \end{aligned}$$

Similarly, from eqn. (15.45) and 15.47 we get,

$$\begin{aligned} \ddot{r} - r\dot{\beta}^2 &= -R\dot{\theta}^2(\hat{\boldsymbol{e}}_R \cdot \hat{\boldsymbol{\lambda}}) + R\ddot{\theta}(\hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{\lambda}}) \\ r\ddot{\beta} + 2\dot{r}\dot{\beta} &= -R\dot{\theta}^2(\hat{\boldsymbol{e}}_R \cdot \hat{\boldsymbol{n}}) + R\ddot{\theta}(\hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{n}}). \end{aligned}$$

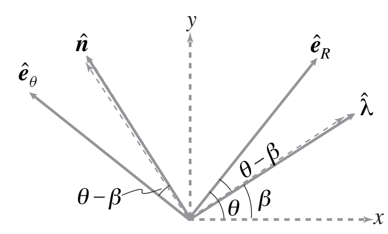


Figure 15.59: Geometry of the unit vectors.

Thus, to find all kinematic quantities of interest, all we need now is to figure out a few dot products between the two sets of basis vectors. This is easily done by writing out $\hat{\boldsymbol{e}}_R$, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{\lambda}}$, and $\hat{\boldsymbol{n}}$. ③ Substituting the dot products in the expressions for $\dot{r}$, $\dot{\beta}$, $\ddot{r}$, and $\ddot{\beta}$ we get

$$\begin{aligned} \dot{r} &= R\dot{\theta}\sin(\beta - \theta), \\ \dot{\beta} &= r^{-1}R\dot{\theta}\cos(\beta - \theta) \\ \ddot{r} &= r\dot{\beta}^2 - R\dot{\theta}^2\cos(\beta - \theta) + R\ddot{\theta}\sin(\beta - \theta), \\ \ddot{\beta} &= r^{-1}[-R\dot{\theta}^2\sin(\theta - \beta) + R\ddot{\theta}\cos(\theta - \beta) - 2\dot{r}\dot{\beta}]. \end{aligned}$$

Substituting the given values of $\dot{R}, \dot{\theta}, \ddot{\theta}, R, \theta, \beta$, and $r = R\sin\theta/\sin\beta$, we get

$$\dot{r} = -1.04\,\text{m/s}, \ \dot{\beta} = 6.83\,\text{rad/s}, \quad \ddot{r} = -12.66\,\text{m/s}^2, \ \ddot{\beta} = 9.43\,\text{rad/s}^2.$$

(a)	$\vec{\omega}_{CD} = 6.83\,\text{rad/s}\hat{\boldsymbol{k}}$,	$\dot{\vec{\omega}}_{CD} = 9.43\,\text{rad/s}^2\hat{\boldsymbol{k}}$
(b)	$\vec{v}_{/CD} = -1.04\,\text{m/s}\hat{\boldsymbol{\lambda}}$,	$\vec{a}_{/CD} = -12.66\,\text{m/s}^2\hat{\boldsymbol{\lambda}}$

③ We have,
$\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$,
$\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$,
$\hat{\boldsymbol{\lambda}} = \cos\beta\hat{\boldsymbol{\imath}} + \sin\beta\hat{\boldsymbol{\jmath}}$,
$\hat{\boldsymbol{n}} = -\sin\beta\hat{\boldsymbol{\imath}} + \cos\beta\hat{\boldsymbol{\jmath}}$.
Therefore,
$\hat{\boldsymbol{e}}_R \cdot \hat{\boldsymbol{\lambda}} = \cos(\beta - \theta)$,
$\hat{\boldsymbol{e}}_R \cdot \hat{\boldsymbol{n}} = \sin(\beta - \theta)$,
$\hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{\lambda}} = \sin(\theta - \beta)$,
$\hat{\boldsymbol{e}}_\theta \cdot \hat{\boldsymbol{n}} = \cos(\theta - \beta)$.

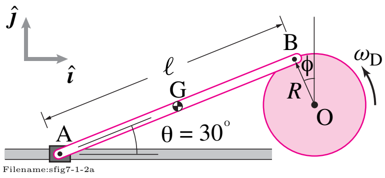


Figure 15.60:

SAMPLE 15.13 Kinematics of a Link rod in a one-DOF mechanism. In machines we often encounter mechanisms and links in which the ends of a link or a rod are constrained to move on a specified geometric path. A simplified typical link AB is shown in Fig. 15.60.

Link AB is a uniform rigid rod of length $\ell = 2\,\mathrm{m}$. End A of the rod is attached to a collar which slides on a horizontal track. End B of the rod is attached to a uniform disk of radius $R = 0.5\,\mathrm{m}$ which rotates about its center O. At the instant shown, when $\theta = 30^{\circ}$, end A is observed to move at $2\,\mathrm{m/s}$ to the left.

1. Find the angular velocity of the rod.
2. Find the angular velocity of the disk.
3. Find the velocity of the center-of-mass of the rod.

Solution Let the angular velocities of the rod and the disk be $\vec{\omega}_{AB} = \dot{\theta}\hat{k}$ and $\vec{\omega}_D = \dot{\phi}\hat{k}$ respectively, where $\dot{\theta}$ and $\dot{\phi}$ are unknowns. We are given $\vec{v}_A = -v_A\hat{\imath}$ where $v_A = 2\,\mathrm{m/s}$.

1. Point B is on the rod as well as the disk. Hence, the velocity of point B can be found by considering either the motion of the rod or the disk. Considering the motion of the rod we write,

$$\begin{aligned}
\vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\
&= \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A} \\
&= -v_A\hat{\imath} + \dot{\theta}\hat{k} \times \ell(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) \\
&= -v_A\hat{\imath} + \dot{\theta}\ell\cos\theta\hat{\jmath} - \dot{\theta}\ell\sin\theta\hat{\imath} \\
&= -(v_A + \dot{\theta}\ell\sin\theta)\hat{\imath} + \dot{\theta}\ell\cos\theta\hat{\jmath}. \qquad (15.48)
\end{aligned}$$

Now considering the motion of the disk we write,

$$\begin{aligned}
\vec{v}_B &= \vec{\omega}_D \times \vec{r}_{B/O} \\
&= \dot{\phi}\hat{k} \times R(-\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}) \\
&= -\dot{\phi}R\sin\theta\hat{\jmath} - \dot{\phi}R\cos\theta\hat{\imath}. \qquad (15.49)
\end{aligned}$$

But $\vec{v}_B = \vec{v}_B$, therefore, from equations (15.48) and (15.49) we get

$$-(v_A + \dot{\theta}\ell\sin\theta)\hat{\imath} + \dot{\theta}\ell\cos\theta\hat{\jmath} = -\dot{\phi}R\sin\theta\hat{\jmath} - \dot{\phi}R\cos\theta\hat{\imath}$$

By equating the $\hat{\imath}$ and $\hat{\jmath}$ components of the above equation we get

$$\begin{aligned}
-(v_A + \dot{\theta}\ell\sin\theta) &= -\dot{\phi}R\cos\theta, \qquad (15.50) \\
\text{and} \quad \dot{\theta}\ell\cos\theta &= -\dot{\phi}R\sin\theta \\
\Rightarrow \quad \dot{\theta} &= -\dot{\phi}\frac{R}{\ell}\tan\theta. \qquad (15.51)
\end{aligned}$$

Dividing Eqn. (15.50) by (15.51) we get

$$\begin{aligned}
-\left(\frac{v_A}{\dot{\theta}} + \ell\sin\theta\right) &= \frac{\ell\cos\theta}{\tan\theta} = \ell\frac{\cos^2\theta}{\sin\theta} \\
\Rightarrow \quad -\frac{v_A}{\dot{\theta}} &= \ell\left(\frac{\cos^2\theta}{\sin\theta} + \sin\theta\right) = \ell\left(\frac{\cos^2\theta + \sin^2\theta}{\sin\theta}\right) = \frac{\ell}{\sin\theta} \\
\Rightarrow \quad \dot{\theta} &= -\frac{v_A}{\ell}\sin\theta = -\frac{v_A}{\ell}\sin 30^{\circ} \\
&= -\frac{2\,\mathrm{m/s}}{2\,\mathrm{m}} \cdot \frac{1}{2} = -0.5\,\mathrm{rad/s}.
\end{aligned}$$

$$\boxed{\vec{\omega}_{AB} = -0.5\,\mathrm{rad/s}\hat{k}}$$

2. From eqn. (15.51)

$$\dot\phi = -\frac{\dot\theta\ell}{R\tan\theta} = -\frac{\overbrace{-\frac{v_A}{\ell}}^{\dot\theta}\sin\theta\,\ell}{R\frac{\sin\theta}{\cos\theta}} = \frac{v_A}{R}\cos\theta$$
$$= -\frac{2\,\mathrm{m/s}}{0.5\,\mathrm{m}}\frac{\sqrt{3}}{2} = 2\sqrt{3}\,\mathrm{rad/s}.$$

Thus $\vec{\boldsymbol{\omega}}_D = \dot\phi\hat{\boldsymbol{k}} = 3.46\,\mathrm{rad/s}\hat{\boldsymbol{k}}$.

$$\boxed{\vec{\boldsymbol{\omega}}_D = 3.46\,\mathrm{rad/s}\hat{\boldsymbol{k}}}$$

[At this point, it is a good idea to check our algebra by substituting the values of $\dot\theta$ and $\dot\phi$ in equations (15.48) and (15.49) to calculate $\vec{\boldsymbol{v}}_B$.] ④

3. Now we can calculate the velocity of the center-of-mass of the rod by considering either point A or point B as a reference:

$$\begin{aligned}
\vec{\boldsymbol{v}}_G &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{G/A}\\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{AB}\times\vec{\boldsymbol{r}}_{G/A}\\
&= -v_A\hat{\boldsymbol{\imath}} + \dot\theta\hat{\boldsymbol{k}}\times\frac{\ell}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})\\
&= -(v_A + \dot\theta\frac{\ell}{2}\sin\theta)\hat{\boldsymbol{\imath}} + \dot\theta\frac{\ell}{2}\cos\theta\hat{\boldsymbol{\jmath}}\\
&= -(2\,\mathrm{m/s} - 0.5\,\mathrm{rad/s}\cdot\frac{2\,\mathrm{m}}{2}\cdot\frac{1}{2})\hat{\boldsymbol{\imath}} + (-0.5\,\mathrm{rad/s}\cdot\frac{2\,\mathrm{m}}{2}\cdot\frac{\sqrt{3}}{2})\hat{\boldsymbol{\jmath}}\\
&= -\frac{7}{4}\,\mathrm{m/s}\hat{\boldsymbol{\imath}} - \frac{\sqrt{3}}{4}\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_G = -(1.75\hat{\boldsymbol{\imath}} + 0.43\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}}$$

We could easily check our calculation by taking point B as a reference and writing

$$\begin{aligned}
\vec{\boldsymbol{v}}_G &= \vec{\boldsymbol{v}}_B + \vec{\boldsymbol{v}}_{G/B}\\
&= \vec{\boldsymbol{v}}_B + \vec{\boldsymbol{\omega}}_{AB}\times\vec{\boldsymbol{r}}_{G/B}
\end{aligned}$$

By plugging in appropriate values we get, of course, the same value as above.

Comment: We used the standard basis vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ for all our vector calculations in this sample. We can shorten these calculations by choosing other appropriate basis vectors as we show in the following samples.

④ Substituting $\dot\theta = -0.5\,\mathrm{rad/s}$ in Eqn. (15.48) and plugging in the given values of other variables we get

$$\vec{\boldsymbol{v}}_B = -(\frac{3}{2}\hat{\boldsymbol{\imath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}.$$

Similarly, substituting $\dot\phi = 2\sqrt{3}\,\mathrm{rad/s}$ in Eqn. (15.48) and plugging in the other given values we get

$$\vec{\boldsymbol{v}}_B = -(\frac{3}{2}\hat{\boldsymbol{\imath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}})\,\mathrm{m/s},$$

which checks with the $\vec{\boldsymbol{v}}_B$ found above.

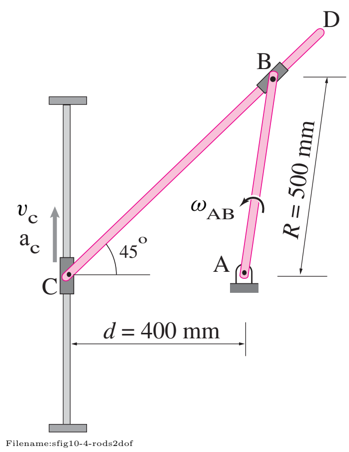


Figure 15.61:

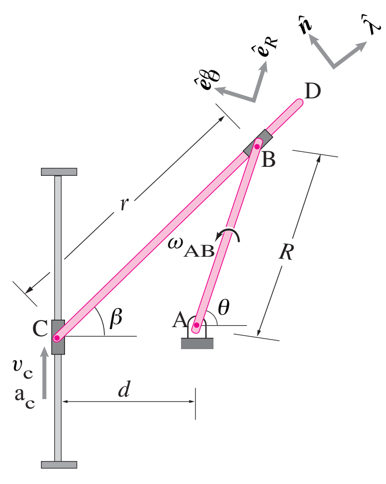


Figure 15.62:

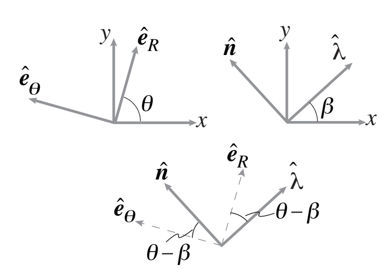


Figure 15.63:

SAMPLE 15.14 A two-DOF mechanism. A two degree-of-freedom mechanism made of three rods and two sliders is shown in the figure. At the instant shown, the crank AB is rotating with angular velocity $\vec{\omega}_{AB} = 12\,\text{rad/s}\hat{k}$ and angular acceleration $\dot{\vec{\omega}}_{AB} = 10\,\text{rad/s}^2\hat{k}$. At the same instant, the collar at end C of the link rod CD is sliding on the vertical rod with velocity $\vec{v}_C = 0.5\,\text{m/s}\,\hat{\jmath}$ and acceleration $\vec{a}_C = 10\,\text{m/s}^2\,\hat{\jmath}$. Find the angular velocity and angular acceleration of the link rod CD.

Solution Once again, we are interested in instantaneous kinematics — we wish to find the angular velocity and acceleration of rod CD at the given instant. This problem is just like the previous sample problem except that end C of the link rod CD is not fixed but free to slide on the vertical bar. But the velocity and acceleration of point C is given; so it is exactly like the previous sample (there, the velocity and acceleration of point C was identically zero). So, we adopt the same line of attack. We figure out the velocity and acceleration of point B using the kinematics of rod AB. We then write the velocity and acceleration of the same point using the kinematics of rod CD (this will involve the unknown angular velocity and acceleration of CD that we are interested in). Equate the two and solve for the unknowns we are interested in.

Let the angular velocity and acceleration of rod CD be $\vec{\omega}_{CD} = \dot{\beta}\hat{k}$ and $\dot{\vec{\omega}}_{CD} = \ddot{\beta}\hat{k}$, respectively. Let $\hat{e}_R$ and $\hat{e}_\theta$ be base vectors rotating with rod AB, and $\hat{\lambda}$ and $\hat{n}$ be the base vectors rotating with rod CD (see fig. 15.62). Considering rod AB, we have

$$\begin{aligned}\vec{v}_B &= R\dot{\theta}\hat{e}_\theta && (15.52)\\ \vec{a}_B &= -R\dot{\theta}^2\hat{e}_R + R\ddot{\theta}\hat{e}_\theta. && (15.53)\end{aligned}$$

Considering rod CD, we have

$$\begin{aligned}\vec{v}_B = \vec{v}_C + \vec{v}_{B/C} &= v_C\,\hat{\jmath} + \dot{r}\hat{\lambda} + r\dot{\beta}\hat{n} && (15.54)\\ \vec{a}_B = \vec{a}_C + \vec{a}_{B/C} &= a_C\,\hat{\jmath} + (\ddot{r} - r\dot{\beta}^2)\hat{\lambda} + (2\dot{r}\dot{\beta} + r\ddot{\beta})\hat{n}. && (15.55)\end{aligned}$$

Now equating eqn. (15.52) and (15.54), and dotting both sides with $\hat{\lambda}$ and $\hat{n}$, we get

$$\begin{aligned}\dot{r} &= R\dot{\theta}\,\overbrace{(\hat{e}_\theta\cdot\hat{\lambda})}^{-\sin(\theta-\beta)} - v_C\,\overbrace{(\hat{\jmath}\cdot\hat{\lambda})}^{\sin\beta} = -R\dot{\theta}\sin(\theta-\beta) - v_C\sin\beta && (15.56)\\ r\dot{\beta} &= R\dot{\theta}\,\underbrace{(\hat{e}_\theta\cdot\hat{n})}_{\cos(\theta-\beta)} - v_C\,\underbrace{(\hat{\jmath}\cdot\hat{n})}_{\cos\beta} = R\dot{\theta}\cos(\theta-\beta) - v_C\cos\beta && (15.57)\end{aligned}$$

where the dot products among the basis vectors are easily found from either their geometry (see fig. 15.63) or from their component representation (see previous sample). Following exactly the same procedure, we get, from eqn. (15.53) and 15.55,

$$\begin{aligned}\ddot{r} &= -R\dot{\theta}^2\cos(\theta-\beta) + R\ddot{\theta}(-\sin(\theta-\beta)) - a_C\sin\beta + r\dot{\beta}^2 && (15.58)\\ r\ddot{\beta} &= -R\dot{\theta}^2\sin(\theta-\beta) + R\ddot{\theta}\cos(\theta-\beta) - a_C\cos\beta - 2\dot{r}\dot{\beta}. && (15.59)\end{aligned}$$

Now, note that although we are only interested in finding $\dot{\beta}$ and $\ddot{\beta}$. So, we only need eqn. (15.57) and eqn. (15.59). But, eqn. (15.59) requires $\dot{r}$ on the right hand side and, therefore, we do need eqn. (15.56). We can, however, happily ignore eqn. (15.58).

Now, to find the numerical values of $\dot{\beta}$ and $\ddot{\beta}$, we need to find r and θ in addition to all other given values. Consider triangle ABC in fig. 15.62. Using the law of sines ($\frac{R}{\sin\beta} = \frac{r}{\sin\theta} = \frac{d}{\sin(\theta-\beta)}$), we get $r = 0.7\,\text{m}$ and $\theta = 79.45^\circ$. Now, substituting all known numerical values in eqns. (15.56), (15.57), and (15.59), we get

$$\dot{r} = -3.75\,\text{m/s}, \quad \dot{\beta} = 6.61\,\text{rad/s}, \quad \ddot{\beta} = 8.43\,\text{rad/s}^2.$$

$$\vec{\omega}_{CD} = (6.61\,\text{rad/s})\hat{k}, \quad \dot{\vec{\omega}}_{CD} = (8.43\,\text{rad/s}^2)\hat{k}$$

15.5 Advanced kinematics of planar motion

In this section we consider three types of problems where the kinematics involves solution of differential equations. In most cases this means computer solution is involved for this type of problem. Here are the three problem types:

- **I. Closed kinematic chains.** The main simple example is a 4-bar linkage with one bar grounded. This system has one degree of freedom, but it is difficult to directly calculate the positions, velocities and accelerations of all points in terms of one variable. Instead, the constraint that the linkage is closed is sometimes most-easily expressed as a differential equation.
- **II. Rolling contact with not-round objects.** For non-round rollers and cams solving for configuration, velocity and acceleration can sometimes be best done with integration. A side benefit from studying this topic is the observation that all non-translational motions are equivalent to rolling of some kind.
- **III. Contact with ideal wheels and skates, looking down.** Cars, tricycles, trailers, grocery carts and sleighs have wheels and have dynamics that is sometimes well characterized by planar analysis, where the plane is the horizontal plane. In this view the simple model of a wheel is as something that prevents sideways motion but allows motion in the direction of travel (like for some of the trike and car problems in 1-D constrained motion). Such problems are called *non-holonomic* (see box 15.3 on page 895).

Closed kinematic chains

When a series of mechanical links is *open* you can not go from one link to the next successively and get back to your starting point. Such chains include a pendulum (1 link), a double pendulum (2 links), a 100 link pendulum, and a model of the human body (so long as only one foot is on the ground). A *closed* chain has at least one loop in it. You can go from link to next and get back to where you started. A slider-crank, a 4-bar linkage, and a person with two feet on the ground are closed chains.

Closed chains are kinematically difficult because they have fewer degrees of freedom than do they have joints. So some of the joint angles depend on the others. The values of any minimal set of configuration variables, say some of the joint angles, determines all of the joint angles, but by geometry that is difficult or impossible to express with formulas.

Example: Four bar linkage: configuration variables
It is impractically difficult to write the positions velocities and accelerations of a 4-bar linkage in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$ of any one of its joints.

However, given a configuration, the constraint on the rates and accelerations is relatively easy to express, always yielding linear equations.

Example: Four bar linkage: configuration rates
If you write the relative velocities of the ends of the bars in terms of configuration rates $\dot{\theta}_1$, $\dot{\theta}_2$, and $\dot{\theta}_3$ and then write the chain closure equation you get a linear equation in the rates. Likewise if you write the closure condition in terms of acceleration. The coefficients in these equations are likely to be complex functions of the configuration, so integrating these equations requires numerics. But the constraint is linear.

Thus, as shown in the last sample of Sect. 10.4, one way to calculate the evolving configurations of a closed chain is to integrate the velocity relations numerically.

Rolling of not round objects

When two objects roll on each other they maintain contact and do not slip relative to each other. That is to say rolling of one rigid curve $\mathcal{B}$ on another $\mathcal{A}$ means:

- The instantaneous relative motion of $\mathcal{B}$ with respect to $\mathcal{A}$ is a rotation about the contact point at the common tangent C, and
- The sequence of points C moves the same distance on both curves.

For simplicity let's take $\mathcal{A}$ to be a curve fixed in space on which rigid curve $\mathcal{B}$ rolls. Take a reference point of interest fixed on body $\mathcal{B}$ to be O'. So,

$$\vec{\boldsymbol{v}}_{O'} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{O'/C} \qquad \text{where } \vec{\boldsymbol{\omega}} = \dot{\theta}_{\mathcal{B}} \hat{\boldsymbol{k}} \tag{15.64}$$

and $\theta_{\mathcal{B}}$ is, say, the rotation of a $\hat{\boldsymbol{\imath}}'$ axis fixed in $\mathcal{B}$ relative to a $\hat{\boldsymbol{\imath}}$ axis fixed in $\mathcal{A}$. If we use the rotation $\theta_{\mathcal{B}}$ of body $\mathcal{B}$ as our configuration variable, we now know how to find the velocity of all points in terms of their positions and the rotation rate. Thus we can find the rate of change of the configuration. To proceed as time progresses we also need to know how the position of point C evolves. Not the material point C on either body, but the location of mutual contact.

If we assume that both curves are parameterized by arc-length going counter-clock wise, if we take curvature as positive if directed towards the interior of each curve's body, then the condition of maintaining contact requires that

$$\vec{\boldsymbol{v}}_C = \dot{s}\hat{\boldsymbol{e}}_t \qquad \text{where } \hat{\boldsymbol{e}}_t \text{ is the tangent to fixed curve } \mathcal{A}.$$

and s is the advance along curve $\mathcal{A}$. To maintain tangency, the angles must be maintained so

$$\dot{s} = \frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\dot{\theta}.$$

Altogether this gives

$$\vec{\boldsymbol{v}}_C = \frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\dot{\theta}\hat{\boldsymbol{e}}_t \quad \text{(not the velocity of any material point).}$$

15.3 Skates, wheels and non-holonomic constraints

Of the words in this book "non-holonomic" is probably the most obscure. This is because the subject of mechanics was mostly stolen from engineers by physicists about 100 years ago. And physicists, the authors of most introductions to mechanics, had no use for non-holonomic mechanics as it wasn't useful for the development of quantum mechanics. So many people are unaware of the word, the subject or its utility.

In two dimensions the word non-holonomic in effect means the mechanics of objects constrained by ideal skates or massless ideal wheels. Often these decades non-holonomic constraints are described as "non integrable". Literally, the word non-holonomic means "not whole". But in what sense is a rolling ideal wheel "non-integrable" or less "whole" than anything else?

A constrained rigid body. Consider a rigid body that is free to slide on a plane. It has three degrees of freedom described by $x_{O'}, y_{O'}$ and θ, all measured relative to a fixed reference frame $O\hat{\imath}\hat{\jmath}$. Point C on the body has relative position $\vec{r}_{C/O'} = x'_C\hat{\imath}' + y'_C\hat{\jmath}'$ where x'_C and y'_C are constants. Now lets constrain the body at point C one of these two different ways:

- **a)** Pin the body to the ground with an ideal hinge at point C. This keeps point C from moving but allows the body to rotate (holonomic).
- **b)** Put an ideal wheel or skate under the body at C that prevents sliding sideways to the skate but allows point C to move parallel to the skate and also allows rotation about the skate (non-holonomic).

Pin Constraint. In the first case, for the pin, we could describe the constraint with the phrase 'point C on the body can have no velocity' and the write and calculate:

$$
\begin{aligned}
\vec{0} &= \vec{v}_C \\
\vec{0} &= \vec{v}_{O'} + \vec{\omega} \times \vec{r}_{C/O'} \qquad (15.60)\\
\vec{0} &= \dot{x}_{O'}\hat{\imath} + \dot{y}_{O'}\hat{\jmath} + \dot{\theta}\hat{k} \times (x'_C\hat{\imath}' + y'_C\hat{\jmath}') \\
\{\vec{0} &= \dot{x}_{O'}\hat{\imath} + \dot{y}_{O'}\hat{\jmath} + \left(\dot{\theta}x'_C\hat{\jmath}' - \dot{\theta}y'_C\hat{\imath}'\right)\}
\end{aligned}
$$

$$\{\}\cdot\hat{\imath} \quad \Rightarrow \quad \dot{x}_{O'} - \dot{\theta}\sin\theta x'_C - \dot{\theta}\cos\theta y'_C = 0$$

$$\{\}\cdot\hat{\jmath} \quad \Rightarrow \quad \dot{y}_{O'} + \dot{\theta}\cos\theta x'_C - \dot{\theta}\sin\theta y'_C = 0.$$

$$\Rightarrow \quad \frac{d}{dt}\left(x_{O'} + \cos\theta x'_C - \sin\theta y'_C\right) = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(y_{O'} + \sin\theta x'_C + \cos\theta y'_C\right) = 0$$

The last two equations are two differential equations in the three variables $x_{O'}, y_{O'}$ and θ. They are "integrable" in the sense that they are equivalent to

$$
\begin{aligned}
x_{O'} + \cos\theta x'_C - \sin\theta y'_C &= C_1 \\
\text{and} \quad y_{O'} + \sin\theta x'_C + \cos\theta y'_C &= C_2 \qquad (15.61)
\end{aligned}
$$

where C_1 and C_2 are integration constants that need to be set by the starting configuration. Solving for $x_{O'}$ and $y_{O'}$ in terms of θ:

$$
\begin{aligned}
x_{O'} &= C_1 - \cos\theta x'_C + \sin\theta y'_C \\
y_{O'} &= C_2 - \sin\theta x'_C - \cos\theta y'_C.
\end{aligned}
$$

Here we have derived the obvious, that a pinned body has one independent configuration variable θ, but we did so starting with a vector expression of constraint in terms of velocities (eqn. (15.60)). Then we wrote the constraint as two scalar constraints on the derivatives of configuration variables and then "integrated" them to write constraints on the configuration variables, finally eliminating two of the configuration variables.

Skate constraint. Now consider the same body constrained by a skate or ideal wheel at C instead of a pin. The skate is aligned with the $\hat{\imath}'$ so point C can only move in the $\hat{\imath}'$ direction. The body is still free to rotate about the point C (to steer). Thus,

$$
\begin{aligned}
0 &= \vec{v}_C \cdot \hat{\jmath}' \\
0 &= (\vec{v}_{O'} + \vec{\omega} \times \vec{r}_{C/O'}) \cdot \hat{\jmath}' \qquad (15.62)\\
0 &= \left(\dot{x}_{O'}\hat{\imath} + \dot{y}_{O'}\hat{\jmath} + \dot{\theta}\hat{k} \times (x'_C\hat{\imath}' + y'_C\hat{\jmath}')\right) \cdot \hat{\jmath}' \\
0 &= -\dot{x}_{O'}\sin\theta + \dot{y}_{O'}\cos\theta + \dot{\theta}x'_C \\
\Rightarrow \quad 0 &= \frac{d}{dt}F(x_{O'}, y_{O'}, \theta)\ ?
\end{aligned}
$$

As for the hinge where we found 2 constant functions, we might want to find the function $F(x_{O'}, y_{O'}, \theta)$ that satisfies the differential equation above, namely

$$\frac{d}{dt}F(x_{O'}, y_{O'}, \theta) = -\dot{x}_{O'}\sin\theta + \dot{y}_{O'}\cos\theta + \dot{\theta}x'_C. \qquad (15.63)$$

Another math nightmare. How do we find this F? You can't find one. This is the crux of the matter. Neither your calculus professor nor Ramanujan could find one either. No computer can find one, or even a numerical approximation of a solution. There is no function $F(x_{O'}, y_{O'}, \theta)$ that solves eqn. (15.63). The solution fundamentally does not exist. That is why we say the skate/wheel constraint eqn. (15.62) is "non-integrable".

Parallel parking We can use physical reasoning to show that no function F can solve eqn. (15.63). If such an F did exist it would mean that only the set of configurations with position $x_{O'}, y_{O'}$ and angles θ consistent with $F = constant$ would be allowed by the skate constraint (assuming F depends nontrivially on at least one of the variables). This means there would be some angles and positions that the body couldn't get to. Remember, we are not doing mechanics, just kinematics. So we can see what configurations are geometrically allowed while still respecting the constraint. The simple observation that motivates the answer is this:

> Even though the skate constrains $\vec{v}_C$ to not have a sideways component, point C can get to a point that is straight sideways.

How? Like a car can move sideways into a parking space without skidding sideways; by parallel parking. More generally, the body can get to any position and any orientation by the following moves. First rotate the body so the skate aims to its new goal. Then slide the skate to its new goal. And finally rotate the body to its new desired orientation.

Thus, the skate constraint does not disallow *any* configurations! Yet the constraint does disallow some velocities (the skate can't go sideways). In this way, the skate constraint is not "whole". It constrains velocities without constraining configurations.

Counting degrees of freedom. How many degrees of freedom does a body with a skate constraint have? There are two different answers. By counting possible configurations there are three degrees of freedom (it takes three variables to describe all possible configurations). But at any configuration the velocity can be described by 2 numbers ($\dot{\theta}$ and $v_{\hat{\imath}'}$). Whenever the number of configuration degrees of freedom is greater than the number of velocity degrees of freedom (for example, 3¿2) there are non-holonomic constraints.

One might like more examples. But besides artificial mathematical ones, there are none. The only smooth non-holonomic constraint in 2D mechanics is the ideal skate or wheel.

To find the acceleration of material point O' on $\mathcal{B}$ we differentiate eqn. (15.64) with respect to time:

$$\begin{aligned}
\vec{a}_{O'} &= \frac{d}{dt}\vec{v}_{O'} \\
&= \frac{d}{dt}\left(\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/C}\right) \\
&= \frac{d}{dt}\left(\vec{\omega}_{\mathcal{B}} \times (\vec{r}_{O'} - \vec{r}_{C})\right) \\
&= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{O'/C} + \vec{\omega}_{\mathcal{B}} \times \vec{v}_{O'} - \vec{\omega} \times \vec{v}_{C} \\
&= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{O'/C} - \omega_{\mathcal{B}}^2 \vec{r}_{O'/C} - \vec{\omega} \times \left(\frac{1}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\dot{\theta}\hat{e}_t\right) \\
&= \dot{\vec{\omega}}_{\mathcal{B}} \times \vec{r}_{O'/C} - \omega_{\mathcal{B}}^2 \vec{r}_{O'/C} + \left(\frac{\dot{\theta}_{\mathcal{B}}^2}{\kappa_{\mathcal{B}} + \kappa_{\mathcal{A}}}\hat{e}_n\right)
\end{aligned}$$

where $\hat{e}_n$ is normal to the curves and directed towards the interior of $\mathcal{B}$. Thus the acceleration of all points on $\mathcal{B}$ is the same as if the body were pinned at C *plus* an acceleration due to rolling. This rolling acceleration is small if either of the bodies is sharp (has very large κ) and large if the bodies are nearly conformal.

Body curve and space curve

As one rigid body moves arbitrarily on a plane with some non-zero rotation rate we can find a point at relative position $\vec{r}_{C/O'}$ where $\vec{r}_C = \vec{0}$. That is a place where the velocity due to rotation about O' exactly cancels the velocity *of* O'.

$$\vec{0} = \vec{v}_{O'} + \vec{\omega}_{\mathcal{B}} \times \vec{r}_{C/O'}$$

Crossing both sides with $\hat{k}$ and using that $\vec{\omega} = \omega\hat{k}$ we get

$$\vec{r}_{C/O'} = \frac{\hat{k} \times \vec{v}_{O'}}{\omega_{\mathcal{B}}}$$

as the point "on" the body that has no velocity. This point does not literally have to be on the body, rather it is fixed to the reference frame defined by the body.

As motion progresses a sequence of such points C is traced on the ground. Similarly a sequence of points is traced on the body. These two sequences are called the space curve and the body curve (or "polohodie" and "herpolhodie" in older books). The motion of body $\mathcal{B}$ is thus a rolling of the body curve on the space curve.

As a machine designer this means you can generate any desired motion by rolling of appropriate shapes. Move the object in the desired manner, draw the space curve and body curve, make parts with those shapes, and the desired motion occurs by a rolling of those shapes.

Ideal wheels and skates, looking down

If we look down on an ideal skate or wheel at point C on a rigid body and assume that the skate is oriented with the positive $\hat{\boldsymbol{\imath}}'$ axis at point C on the body then we know that

$$\vec{\boldsymbol{v}}_{\mathrm{C}} = v_C \hat{\boldsymbol{\imath}}'$$

and hence the velocity of any point G on the body of interest is

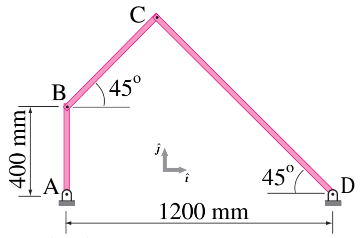
$$\begin{aligned}
\vec{\boldsymbol{v}}_{\mathrm{G}} &= \vec{\boldsymbol{v}}_{\mathrm{C}} + \vec{\boldsymbol{v}}_{\mathrm{G/C}} \\
&= v_c \hat{\boldsymbol{\imath}}' + \dot{\theta} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}}.
\end{aligned}$$

The acceleration is found by differentiating this expression as

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{G}} = \frac{d}{dt} \vec{\boldsymbol{v}}_{\mathrm{G}} &= \vec{\boldsymbol{v}}_{\mathrm{C}} + \vec{\boldsymbol{v}}_{\mathrm{G/C}} \\
&= \frac{d}{dt} \left(v_c \hat{\boldsymbol{\imath}}' + \dot{\theta} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}} \right) \\
&= \dot{v}_C \hat{\boldsymbol{\imath}}' + v_C \dot{\hat{\boldsymbol{\imath}}} + \ddot{\theta} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}} - \dot{\theta}^2 \vec{\boldsymbol{r}}_{\mathrm{G/C}} \\
&= \underbrace{\dot{v}_C \hat{\boldsymbol{\imath}}' + v_C \dot{\theta} \hat{\boldsymbol{\jmath}}'}_{\vec{\boldsymbol{a}}_{\mathrm{C}}} + + \underbrace{\ddot{\theta} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{G/C}} - \dot{\theta}^2 \vec{\boldsymbol{r}}_{\mathrm{G/C}}}_{\vec{\boldsymbol{a}}_{\mathrm{G/C}}}.
\end{aligned}$$

It is interesting to note that the Coriolis-like term $v_C \dot{\theta} \hat{\boldsymbol{\jmath}}'$ does not have the usual factor of 2 one encounters in holonomic problems. To find the trajectory of the point C, say, one needs to integrate the velocity like this:

$$\begin{aligned}
\dot{x} &= \vec{\boldsymbol{v}}_{\mathrm{C}} \cdot \hat{\boldsymbol{\imath}} \\
&= v_C \hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\imath}} \\
&= v_C \cos\theta \\
\dot{y} &= \vec{\boldsymbol{v}}_{\mathrm{C}} \cdot \hat{\boldsymbol{\jmath}} \\
&= v_C \hat{\boldsymbol{\imath}}' \cdot \hat{\boldsymbol{\jmath}} \\
&= v_C \sin\theta.
\end{aligned}$$



Filename:sfig10-4-fourbar

Figure 15.64:

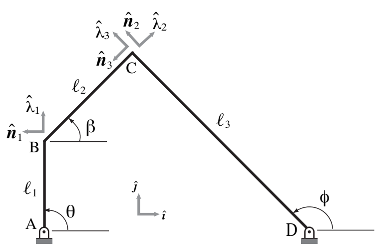
SAMPLE 15.15 Kinematics of a four bar linkage. A four bar linkage ABCD is shown in the figure (fourth bar is the ground AD) at some instant t_0. The driving bar AB rotates with angular velocity $\vec{\omega}_{AB} = \dot{\theta}(t)\hat{\boldsymbol{k}}$. Find the angular velocities of rods BC and CD as a function of $\dot{\theta}$. How can you solve for the positions of the bars at any t if the initial configuration is as shown in the figure?

Solution Let the angles that rods AB, BC, and CD make with the horizontal (x-axis) be θ, β, and ϕ, respectively. Then, we can write $\vec{\omega}_{BC} = \dot{\beta}\hat{\boldsymbol{k}}$ and $\vec{\omega}_{CD} = \dot{\phi}\hat{\boldsymbol{k}}$. We have to find $\dot{\beta}$ and $\dot{\phi}$.

Note that the motion of point C is a simple circular motion about point A with given angular velocity $\vec{\omega}_{AB}$. Thus, the velocity of point B is known. Now, we can find the velocity of point C two ways: (i) by considering rod CB: $\vec{v}_{\rm C} = \vec{v}_{\rm B} + \vec{v}_{\rm C/B} = \vec{v}_{\rm B} + \vec{\omega}_{BC} \times \vec{r}_{\rm C/B}$, and (ii) by considering rod CD: $\vec{v}_{\rm C} = \vec{\omega}_{CD} \times \vec{r}_{\rm C/D}$. Either way the velocity must be the same. Thus, we have a 2-D vector equation with two unknowns $\dot{\beta}$ and $\dot{\phi}$. We can get two independent scalar equations from the vector equation and thus we can solve for the desired unknowns.

Let us use the rotating base vectors $(\hat{\boldsymbol{\lambda}}_1, \hat{\boldsymbol{n}}_1)$, $(\hat{\boldsymbol{\lambda}}_2, \hat{\boldsymbol{n}}_2)$, and $(\hat{\boldsymbol{\lambda}}_3, \hat{\boldsymbol{n}}_3)$ with rods AB, BC, and CD, respectively. Note that these base vectors are basically the $(\hat{e}_R, \hat{e}_\theta)$ pairs; we use $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}})$ just for the sake of easy subscripting. Now,

$$\begin{aligned} \vec{v}_{\rm B} &= \vec{\omega}_{AB} \times \vec{r}_{\rm B/A} = \ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 \\ \vec{v}_{\rm C} &= \vec{v}_{\rm B} + \vec{\omega}_{BC} \times \vec{r}_{\rm C/B} = \ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 + \ell_2 \dot{\beta} \hat{\boldsymbol{n}}_2 \qquad (15.65) \\ \text{also,} \quad \vec{v}_{\rm C} &= \vec{\omega}_{CD} \times \vec{r}_{\rm C/D} = \ell_3 \dot{\phi} \hat{\boldsymbol{n}}_3 \qquad (15.66) \end{aligned}$$



Filename:sfig10-4-fourbar-a

Figure 15.65:

Thus, from eqn. (15.65) and (15.66), we have,

$$\ell_1 \dot{\theta} \hat{\boldsymbol{n}}_1 + \ell_2 \dot{\beta} \hat{\boldsymbol{n}}_2 = \ell_3 \dot{\phi} \hat{\boldsymbol{n}}_3 \qquad (15.67)$$

Dotting eqn. (15.67) with $\hat{\boldsymbol{\lambda}}_2$ (to eliminate $\dot{\beta}$ term), we get

$$\begin{aligned} \ell_1 \dot{\theta} (\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2) &= \ell_3 \dot{\phi} (\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2) \\ \Rightarrow \quad \dot{\phi} &= \frac{\ell_1}{\ell_3} \frac{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2)}{(\hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2)} \dot{\theta}. \qquad (15.68) \end{aligned}$$

Similarly, dotting eqn. (15.67) with $\hat{\boldsymbol{\lambda}}_3$ (to eliminate $\dot{\phi}$ term), we get

$$\dot{\beta} = -\frac{\ell_1}{\ell_2} \frac{(\hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_3)}{(\hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{\lambda}}_3)} \dot{\theta}. \qquad (15.69)$$

We are practically done at this point with the kinematics — we have found $\dot{\beta}$ and $\dot{\phi}$ as functions of $\dot{\theta}$. The various dot products are just geometry and vector algebra. To write them explicitly, we note that $\hat{\boldsymbol{\lambda}}_1 = \cos\theta \hat{\imath} + \sin\theta \hat{\jmath}$, $\hat{\boldsymbol{n}}_1 = -\sin\theta \hat{\imath} + \cos\theta \hat{\jmath}$, $\hat{\boldsymbol{\lambda}}_2 = \cos\beta \hat{\imath} + \sin\beta \hat{\jmath}$, etc. Thus,

$$\begin{aligned} \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_2 &= -\sin\theta\cos\beta + \cos\theta\sin\beta = \sin(\beta - \theta) \\ \hat{\boldsymbol{n}}_3 \cdot \hat{\boldsymbol{\lambda}}_2 &= \sin(\beta - \phi) \\ \hat{\boldsymbol{n}}_1 \cdot \hat{\boldsymbol{\lambda}}_3 &= \sin(\phi - \theta), \quad \hat{\boldsymbol{n}}_2 \cdot \hat{\boldsymbol{\lambda}}_3 = \sin(\phi - \beta). \end{aligned}$$

Substituting the appropriate expressions in eqn. (15.69) and (15.68), we get

$$\dot{\beta} = -\frac{\ell_1}{\ell_2} \frac{\sin(\phi - \theta)}{\sin(\phi - \beta)} \dot{\theta}, \qquad \dot{\phi} = \frac{\ell_1}{\ell_3} \frac{\sin(\beta - \theta)}{\sin(\beta - \phi)} \dot{\theta}.$$

$$\vec{\omega}_{BC} = -\frac{\ell_1}{\ell_2} \frac{\sin(\phi-\theta)}{\sin(\phi-\beta)} \dot{\theta}\, \hat{\boldsymbol{k}}, \qquad \vec{\omega}_{CD} = \frac{\ell_1}{\ell_3} \frac{\sin(\beta-\theta)}{\sin(\beta-\phi)} \dot{\theta}\, \hat{\boldsymbol{k}}$$

Note that the expressions for $\dot{\beta}$ and $\dot{\phi}$ are coupled, nonlinear, first order ordinary differential equations. To be able to find $\dot{\theta}$, $\beta(t)$ and $\phi(t)$, we need to integrate $\dot{\theta}$, $\dot{\beta}$, and $\dot{\phi}$. Here, we set up these differential equations for numerical integration. Although, we can use any given $\dot{\theta}(t)$ (*e.g.,* αt or $\dot{\theta}_0 \sin(\Omega t)$ or whatever), for definiteness in our numerical integration, let us take a constant $\dot{\theta}$, that is, let $\dot{\theta} = C = 10\,\mathrm{rad/s}$ (say). So, our equations are,

$$\dot{\theta} = C, \qquad \dot{\beta} = \frac{\ell_1}{\ell_2}\frac{\sin(\phi-\theta)}{\sin(\beta-\phi)}C, \qquad \dot{\phi} = \frac{\ell_1}{\ell_3}\frac{\sin(\beta-\theta)}{\sin(\beta-\phi)}C,$$

and the initial conditions are $\theta(0) = \pi/2, \quad \beta(0) = \pi/4, \quad \phi(0) = 3\pi/4.$

Here is a pseudocode that we use to integrate these equations numerically for a period of $2\pi/C = \pi/5$ seconds (one complete revolution of AB).

```
ODEs = {thetadot = C,
        betadot  = (l1/l2)*sin(phi-theta)/sin(beta-phi)*C,
        phidot   = (l1/l3)*sin(beta-theta)/sin(beta-phi)*C}
IC   = {theta(0) = pi/2, beta(0) = pi/4, phi(0)  = 3*pi/4}
Set  C=10, l1=.4, l2=.4*sqrt(2), l3=.8*sqrt(2)
Solve ODEs with IC for t=0 to t=pi/5
```

After we get the angles, we can compute the xy coordinates of points B and C at each instant and plot the mechanism at those instants. Plots thus obtained from our numerical solution are shown in fig. 15.66 where the configuration of the mechanism is shown at 9 equally spaced times between $t = 0$ to $t = \pi/5$.

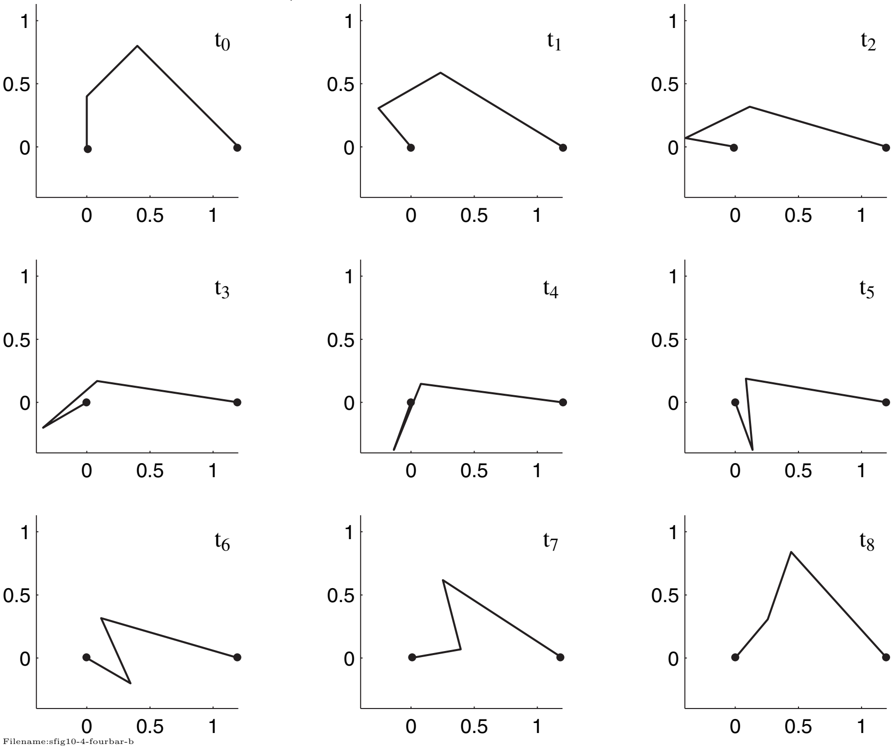


Figure 15.66: Several configurations of the mechanism at equal intervals of time during one complete revolution of the driving link. After t_8 the mechanism returns to the initial configuration.

Problems for Chapter 15

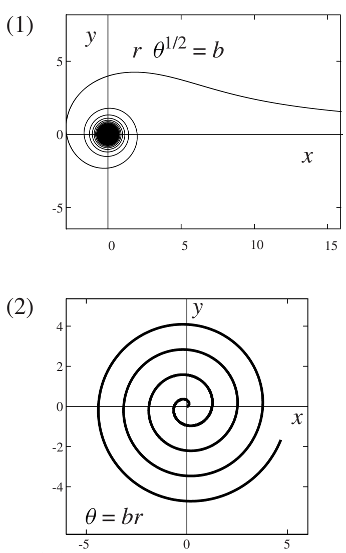
Kinematics using time-varying basis vectors

15.1 Polar coordinates and path coordinates

15.1.1 A particle moves along the two paths (1) and (2) as shown.

a) In each case, determine the velocity of the particle in terms of b, θ, and $\dot{\theta}$. *

b) Find the x and y coordinates of the path as functions of b and r or b and θ. *



Filename:polarplot0
Problem 15.1.1

15.1.2 For the particle path (1) in problem 15.1.1, find the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

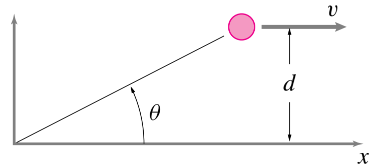
15.1.3 For the particle path (2) in problem 15.1.1, find the acceleration of the particle in terms of b, θ, and $\dot{\theta}$.

15.1.4 A body moves with constant velocity V in a straight line parallel to and at a distance d from the x-axis.

a) Calculate $\dot{\theta}$ in terms of V, d, and θ.

b) Calculate the $\hat{e}_\theta$ component of acceleration.

Carefully define, using a sketch and/or words, any variables, coordinate systems, and reference frames you use. Express your answers using any convenient coordinate system (just make sure its orientation has been clearly defined).



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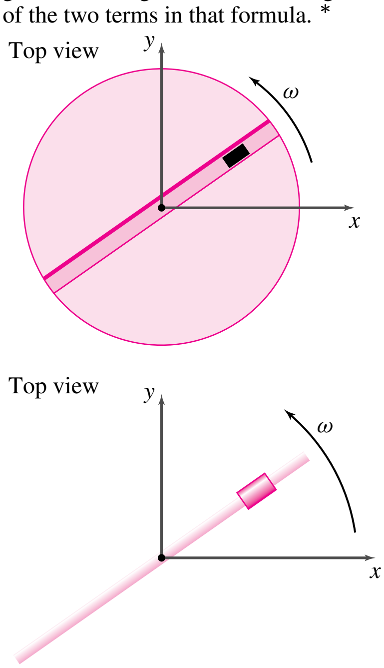
Problem 15.1.4

15.1.5 Picking apart the polar coordinate formula for velocity. This problem concerns a small mass m that sits in a slot in a turntable. Alternatively you can think of a small bead that slides on a rod. The mass always stays in the slot (or on the rod). Assume the mass is a little bug that can walk as it pleases on the rod (or in the slot) and you control how the turntable/rod rotates. Name two situations in which one of the terms is zero but the other is not in the two term polar coordinate formula for velocity, $\dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta$. You should thus gain some insight into the meaning of each of the two terms in that formula. *



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Problem 15.1.5

15.1.6 Picking apart the polar coordinate formula for acceleration. Reconsider the configurations in problem 15.1.5. This time, name four situations in which all of the terms, but one, in the four term polar coordinate formula for acceleration, $\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$, are zero. Each situation should pick out a different term. You should thus gain some insight into the meaning of each of the four terms in that formula. *

15.1.7 The two differential equations:

$$
\begin{aligned}
\ddot{R} - R\dot{\theta}^2 &= 0 \\
2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0
\end{aligned}
$$

have the general solution

$$
\begin{aligned}
R &= \sqrt{d^2 + (v(t - t_0))^2} \\
\theta &= \theta_0 + \tan^{-1}(v(t - t_0)/d)
\end{aligned}
$$

where θ_0, d, t_0, and v are arbitrary constants. This solution could be checked by plugging back into the differential equations — you need not do this (tedious) substitution. The solution describes a curve in the plane. That is, if for a range of values of t, the values of R and θ were calculated and then plotted using polar coordinates, a curve would be drawn. *What can you say about the shape of this curve?*

[Hints:

a) Actually make a plot using some random values of the constants and see what the plot looks like.

b) Write the equation $\mathbf{F} = m\mathbf{a}$ for a particle in polar coordinates and think of a force that would be relevant to this problem.

c) The answer is something simple.]

15.1.8 A car driver on a very boring highway is carefully monitoring her speed. Over a one hour period, the car travels on a curve with constant radius of curvature, $\rho = 25\,\text{mi}$, and its speed increases uniformly from 50 mph to 60 mph. What is the acceleration of the of the car half-way through this one hour period, in path coordinates?

15.1.9 Find expressions for $\hat{e}_t$, a_t, a_n, $\hat{e}_n$, and the radius of curvature ρ, at any position (or time) on the given particle paths for

a) problem 10.1.11, *

b) problem 10.1.12,

c) problem 15.2.4,

d) problem 10.1.14,

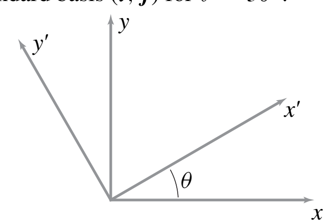
e) problem 10.1.13, and

f) problem 10.1.10.

15.1.10 A particle travels at non-constant speed on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$. Carefully sketch the ellipse for particular values of a and b. For various positions of the particle on the path, sketch the position vector $\vec{r}(t)$; the polar coordinate basis vectors $\hat{e}_r$ and $\hat{e}_\theta$; and the path coordinate basis vectors $\hat{e}_n$ and $\hat{e}_t$. At what points on the path are $\hat{e}_r$ and $\hat{e}_n$ parallel(or $\hat{e}_\theta$ and $\hat{e}_t$ parallel)?

15.2 Rotating reference frames

15.2.1 Express the basis vectors $(\hat{\imath}', \hat{\jmath}')$ associated with axes x' and y' in terms of the standard basis $(\hat{\imath}, \hat{\jmath})$ for $\theta = 30^\circ$.



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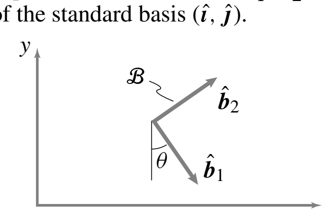
Problem 15.2.1

15.2.2 Body frames are frames of reference attached to a body in motion. The orientation of a coordinate system attached to a body frame $\mathcal{B}$ is shown in the figure at some moment of interest. For $\theta = 60^\circ$, express the basis vectors $(\hat{b}_1, \hat{b}_2)$ in terms of the standard basis $(\hat{\imath}, \hat{\jmath})$.



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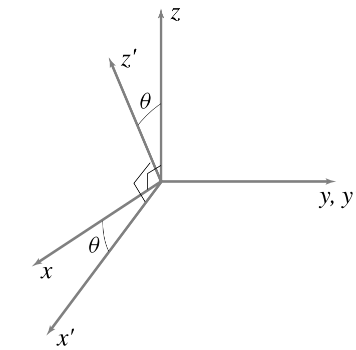
Problem 15.2.2

15.2.3 Find the components of (a) $\vec{v} = 3\,\text{m/s}\hat{\imath} + 2\,\text{m/s}\hat{\jmath} + 4\,\text{m/s}\hat{k}$ in the rotated basis $(\hat{\imath}', \hat{\jmath}', \hat{k}')$ and (b) $\vec{a} = -0.5\,\text{m/s}^2\hat{\jmath}' + 3.0\,\text{m/s}^2\hat{k}'$ in the standard basis $(\hat{\imath}, \hat{\jmath}, \hat{k})$. ($y$ and y' are in the same direction and x' and z' are in the xz plane.)



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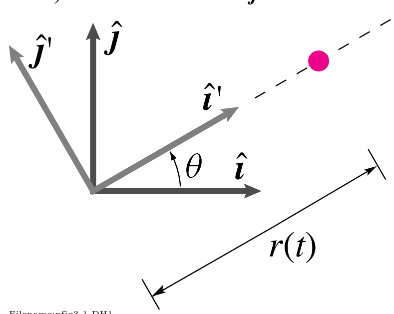
Problem 15.2.3

15.2.4 A particle travels in a straight line in the xy-plane parallel to the x-axis at a distance $y = \ell$ in the positive x direction. The position of the particle is denoted by $\vec{r}(t)$. The angle of $\vec{r}$ measured positive counter-clockwise from the x axis is decreasing at a constant rate with magnitude ω. If the particle starts on the y axis at $\theta = \pi/2$, what is $\vec{r}(t)$ in cartesian coordinates?

15.2.5 Given that $\vec{r}(t) = ct^2\hat{\imath}'$ and that $\theta(t) = d\,\sin(\lambda t)$, find $\vec{v}(t)$

a) in terms of $\hat{\imath}$ and $\hat{\jmath}$,

b) in terms of $\hat{\imath}'$ and $\hat{\jmath}'$.



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Problem 15.2.5

15.2.6 A bug walks on a turntable. In polar coordinates, the position of the bug is given by $\vec{R} = R\hat{e}_r$, where the origin of this coordinate system is at the center of the turntable. The $(\hat{e}_r, \hat{e}_\theta)$ coordinate system is attached to the turntable and, hence, rotates with the turntable. The kinematical quantities describing the bugs motion are $\dot{R} = -\dot{R}_0$, $\ddot{R} = 0$, $\dot{\theta} = \omega_0$, and $\ddot{\theta} = 0$. A fixed coordinate system Oxy has origin O at the center of the turntable. As the bug walks through the center of the turntable:

a) What is its speed?

b) What is its acceleration?

c) What is the radius of the osculating circle (i.e., what is the radius of curvature of the bug's path?).

15.3 General expressions for velocity and acceleration

15.3.1 A bug walks on a straight line engraved on a rotating turntable (the bug's path in the room is not a straight line). The line passes through the center of the turntable. The bug's speed on this line is 1 in/s (the bug's absolute speed is *not* 1 in/s). The turntable rotates at a constant rate of 2 revolutions every π s in a positive sense about the z-axis. It's surface is always in the xy-plane. At time $t = 0$, the engraved line is aligned with the x-axis, the bug is at the origin and headed towards the positive x direction.

a) What is the x component of the bug's velocity at $t = \pi$ s?

b) What is the y component of the bug's velocity at $t = \pi$ seconds?

c) What is the y component of the bug's acceleration at $t = \pi$?

d) What is the radius of curvature of the bug's path at $t = 0$?

15.3.2 Actual path of bug trying to walk a straight line. A straight line is inscribed on a horizontal turntable. The line goes through the center. Let ϕ be angle of rotation of the turntable which spins at constant rate $\dot{\phi}_0$. A bug starts on the outside edge of the turntable of radius R and walks towards the center, passes through it, and continues to the opposite edge of the turntable. The bug walks at a constant speed v_A, as measured by how far her feet move per step, on the line inscribed on the table. Ignore gravity.

a) **Picture.** Make an accurate drawing of the bug's path as seen in the room (which is not rotating with the turntable). In order to make this plot, you will need to assume values of v_A and $\dot{\phi}_0$ and initial values of R and ϕ. You will need to write a parametric equation for the path in terms of variables that you can plot (probably x and y coordinates).

You will also need to pick a range of times. Your plot should include the instant at which the bug walks through the origin. Make sure your x and y- axes are drawn to the same scale. A computer plot would be nice.

b) Calculate the radius of curvature of the bug's path as it goes through the origin.

c) Accurately draw (say, on the computer) the osculating circle when the bug is at the origin on the picture you drew for (a) above.

d) **Force.** What is the force on the bugs feet from the turntable when she starts her trip? Draw this force as an arrow on your picture of the bug's path.

e) **Force.** What is the force on the bugs feet when she is in the middle of the turntable? Draw this force as an arrow on your picture of the bug's path.

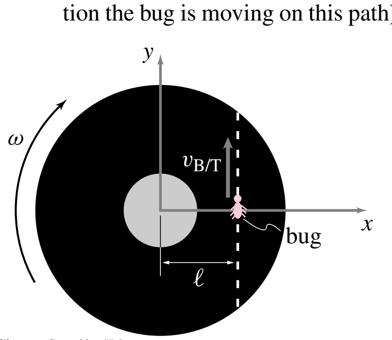
15.3.3 A small bug is crawling on a straight line scratched on an old record. The scratch is a distance $\ell = 6\,\text{cm}$ from the center of the turntable. The turntable is turning clockwise at a constant angular rate $\omega = 2\,\text{rad/s}$. The bug is walking, relative to the turntable, at a constant rate $v_{B/T} = 12\,\text{cm/s}$, straight along the scratch in the y-direction. At the instant of interest, everything is aligned as shown in figure. The bug has a mass $m_B = 1$ gram.

a) What is the bug's velocity?

b) What is the bug's acceleration?

c) What is the sum of all forces acting on the bug?

d) Sketch the path of the bug in the neighborhood of its location at the time of interest (indicate the direction the bug is moving on this path).



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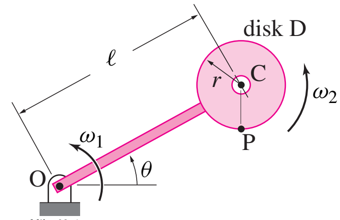
Problem 15.3.3

15.3.4 Arm OC rotates with constant rate ω_1. Disc D of radius r rotates about point C at constant rate ω_2 measured with respect to the arm OC. What are the absolute velocity and acceleration of point P on the disc, $\vec{v}_P$ and $\vec{a}_P$? (To do this problem will require defining a moving frame of reference. More than one choice is possible.) *

a) Pick a suitable moving frame and do the problem.

b) Pick another suitable moving frame and redo the problem. Make sure the answers are the same.

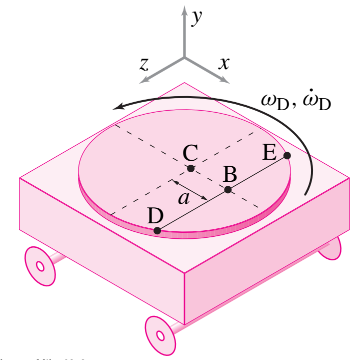


Filename:Mikes93p1
Problem 15.3.4

15.3.5 For the configuration in problem 15.3.4 what is the absolute angular velocity of the disk, D? *

15.3.6 For the configuration in problem 15.3.4, taking ω_2 to be the angular velocity of disk $\mathcal{D}$ relative to the rod, what is the absolute angular acceleration of the disk, D? What is the absolute angular acceleration of the disk if ω_1 and ω_2 are not constant?

15.3.7 A turntable oscillates with displacement $\vec{x}_C(t) = A\sin(st)\hat{\imath}$. The disc of the turntable rotates with angular speed and acceleration ω_D and $\dot{\omega}_D$. A small bug walks along line DE with velocity v_b relative to the turntable. At the instant shown, the turntable is at its maximum amplitude $x = A$, the line DE is currently aligned with the z-axis, and the bug is passing through point B on line DE. Point B is a distance a from the center of the turntable, point C. Find the absolute acceleration of the bug, $\vec{a}_b$. *



Filename:Mikes93p2
Problem 15.3.7

15.3.8 A small 0.1 kg toy train engine is going *clockwise* at a constant rate (relative to the track) of $2\,\text{m/s}$ on a circular track of radius 1 m. The track itself is on a turntable $\mathcal{B}$ that is rotating *counter- clockwise* at a constant rate of $1\,\text{rad/s}$. The dimensions are as shown. At the instant of interest the train is pointing due south ($-\mathbf{j}$) and is *at* the center O of the turntable.

a) What is the velocity of the train relative to the turntable $\vec{v}_{/\mathcal{B}}$?

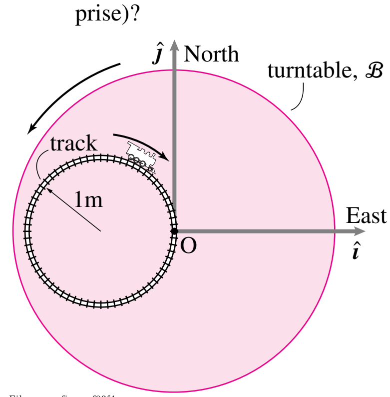
b) What is the absolute velocity of the train $\vec{v}$?

c) What is the acceleration of the train relative to the turntable $\vec{a}_{/\mathcal{B}}$?

d) What is the absolute acceleration of the train $\vec{a}$?

e) What is the total force acting on the train?

f) Sketch the path of the train for one revolution of the turntable (surprise)?



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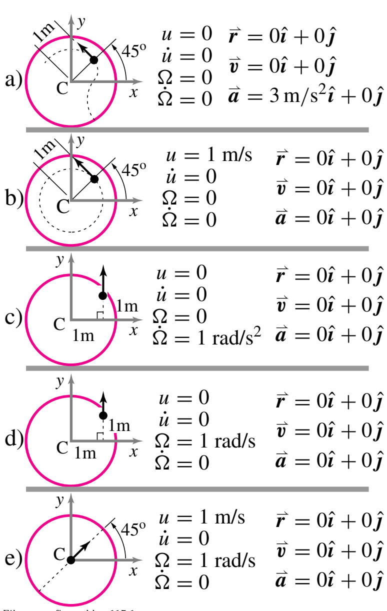
Problem 15.3.8

15.3.9 A giant bug walks on a horizontal disk. An xyz frame is attached to the disk. The disk is rotating about the z axis (out of the paper) and simultaneously translating with respect to an inertial frame XYZ

plane. In each of the cases shown in the figure, determine the total force acting on the bug. In each case, the dotted line is scratched on the disk and is the path the bug follows walking in the direction of the arrow. The location of the bug is marked with a dot. At the instants shown, the xyz coordinate system shown is aligned with the inertial XYZ frame (which is not shown in the pictures). In this problem:

$\vec{r}$ = the position of the center of the disk,
$\vec{v}$ = the velocity of the center of the disk,
$\vec{a}$ = the acceleration of the center of the disk,
Ω the angular velocity of the disk (i.e., $\vec{\boldsymbol{\Omega}} = \Omega \boldsymbol{k}$),
$\dot{\Omega}$ = the angular acceleration of the disk,
u = the speed of the bug traversing the dotted line (arc length on the disk per unit time),
$\dot{u} = du/dt$, and
m = 1 kg = mass of the bug.



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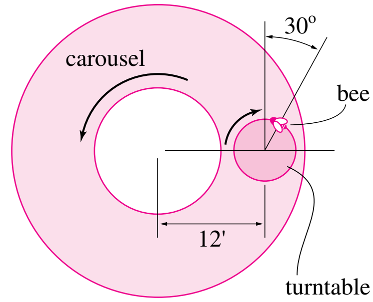
Problem 15.3.9

15.3.10 Repeat questions (a)-(f) for the toy train in problem 15.3.8 going *counter-clockwise* at constant rate (relative to the track) of 2 m/s on the circular track.

15.3.11 A honeybee, sensing that it can get a cheap thrill, alights on a phonograph turntable that is being carried by a carnival goer who is riding on a carousel. The situation is sketched below. The carousel has angular velocity of 5 rpm, which is increasing (accelerating) at 10 rev/min^2; the phonograph rotates at a constant 33 1/3 rpm. The honeybee is at the outer edge of the phonograph record in the position shown in the figure; the radius of the record is 7 inches. Calculate the magnitude of the acceleration of the honeybee.



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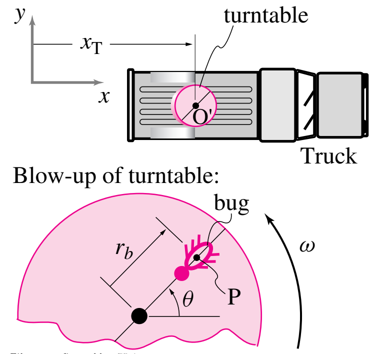
Problem 15.3.11

15.3.12 Consider a turntable on the back of a pick-up truck. A bug walks on a line on the turntable. The line may or may not be drawn through the center of the turntable. The truck may or may not be going at constant rate. The turntable may or may not be spinning, and, if it spins, it may or may not go at a constant rate. The bug may be anywhere on the line and may or may not be walking at a constant rate.

a) **Draw a picture of the situation.** Clearly define all variables you are going to use. What is the moving frame, and what is the reference point on that frame?

b) **One term at a time.** For each term in the five term acceleration formula, find a situation where all but one term is zero. Use the turntable as the moving frame, the bug as the particle of interest. *

c) **Two terms at a time.** (harder) How many situations can you find where a pair of terms is not zero but all other terms are zero? (Don't try to do all 10 cases unless you really think this infamous formula is fun. Try at least one or two.) *



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Problem 15.3.12: **Picking apart the five term acceleration formula.**

15.4 Kinematics of 2-D mechanisms

15.4.1 Slider crank kinematics (No FBD required!). 2-D . Assume $R, \ell, \theta, \dot{\theta}, \ddot{\theta}$ are given. The crank mechanism parts move on the xy plane with the x direction being along the piston. Vectors should be expressed in terms of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ components.

a) What is the angular velocity of the crank OA? *

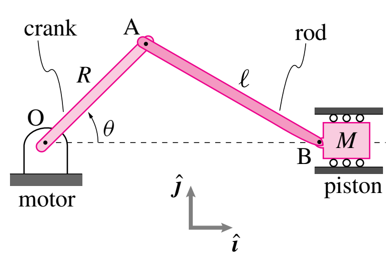
b) What is the angular acceleration of the crank OA? *

c) What is the velocity of point A? *

d) What is the acceleration of point A? *

e) What is the angular velocity of the connecting rod AB? [Geometry fact: $\vec{r}_{AB} = \sqrt{\ell^2 - R^2 \sin^2\theta}\ \hat{\imath} - R\sin\theta\,\hat{\jmath}$] *

f) For what values of θ is the angular velocity of the connecting rod AB equal to zero (assume $\dot{\theta} \neq 0$)? (you need not answer part (e) correctly to answer this question correctly.) *



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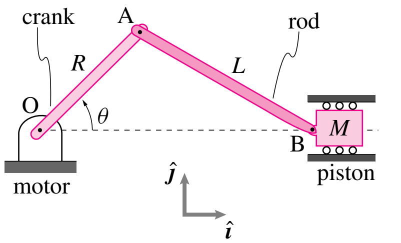
Problem 15.4.1

15.4.2 Slider-Crank. Consider a slider-crank mechanism. Given θ, $\dot{\theta}$, $\ddot{\theta}$, L, and

R, can you find the velocity and acceleration of B? There are many ways to do this problem.



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Problem 15.4.2

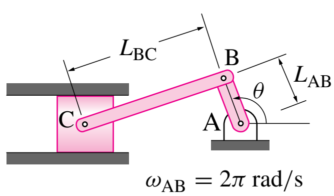
15.4.3 The crank AB with length $L_{AB} = 1$ inch in the crank mechanism shown rotates at a constant rate $\dot{\theta}_{AB} = \omega_{AB} = 2\pi$ rad/s counter-clockwise. The initial angle of rotation is $\theta = 0$ at $t = 0$. The connecting rod BC has a length of $L_{BC} = 2$ in.

a) What is the velocity of point B at the end of the crank when $\theta = \pi/2$ rad?

b) What is the velocity of point C at the end of the connecting rod when $\theta = \pi/2$ rad?

c) What is the angular velocity of the connecting rod BC when $\theta = \pi/2$ rad?

d) What is the angular velocity of the connecting rod BC as a function of time?



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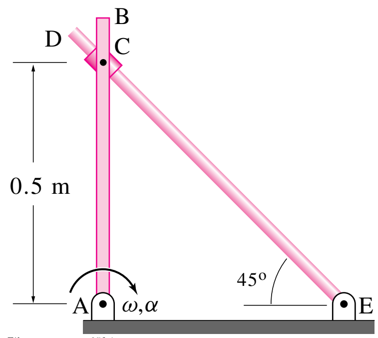
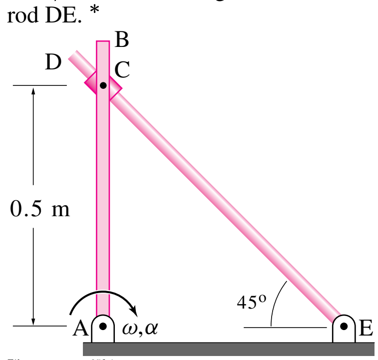
Problem 15.4.3

15.4.4 The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3$ rad/s and angular acceleration $\alpha = 2$ rad/s^2. Find the angular velocity of rod DE. *



Filename:summer95f-4
Problem 15.4.4

15.4.5 Reconsider problem 15.4.4. The two rods AB and DE, connected together through a collar C, rotate in the vertical plane. The collar C is pinned to the rod AB but is free to slide on the frictionless rod DE. At the instant shown, rod AB is rotating clockwise with angular speed $\omega = 3$ rad/s and angular acceleration $\alpha = 2$ rad/s^2. Find the angular acceleration of rod DE. *



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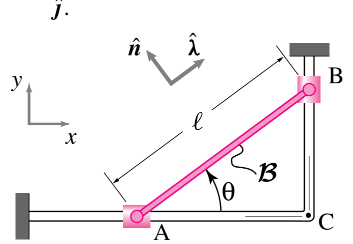
Problem 15.4.5

15.4.6 Collar A is constrained to slide on a horizontal rod to the right at constant speed v_A. It is connected by a pin joint to one end of a rigid bar AB with length ℓ which makes an angle θ with the horizontal at the instant of interest. A second collar B connected by a pin joint to the other end of the rigid bar AB slides on a vertical rod.

a) Find the velocity of point B. Answer in terms of v_A, θ, ℓ, $\hat{\imath}$ and $\hat{\jmath}$.

b) Find the angular speed $\dot{\theta}$ of the rod? Answer in terms of v_A, θ, ℓ, $\hat{\imath}$ and $\hat{\jmath}$.



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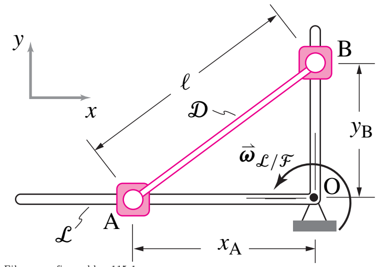
Problem 15.4.6

15.4.7 A bar of length $\ell = 5$ ft, body $\mathcal{D}$, connects sliders A and B on an L-shaped frame, body $\mathcal{L}$, which itself is rotating at constant speed about an axle perpendicular to the plane of the figure through the point O and relative to a fixed frame $\mathcal{F}$, $\vec{\omega}_{\mathcal{L}/\mathcal{F}} = 0.5\,\text{rad/s}\,\boldsymbol{k}$. At the instant shown, body $\mathcal{L}$ is aligned with the $x-y$ axes, slider A is $x_A = 4$ ft from point O, slider B is $y_B = 3$ ft from O. The speed and acceleration of slider B relative to the frame $\mathcal{L}$ are $\vec{v}_{B/\mathcal{L}} = -2\,\text{ft/s}\,\hat{\jmath}$ and $\vec{a}_{B/\mathcal{L}} = -2\,\text{ft/s}^2\,\hat{\jmath}$, respectively.

Determine:

a) The absolute velocity of the slider A, $\vec{v}_{A/\mathcal{F}}$, and

b) The absolute angular velocity of bar AB, body $\mathcal{D}$, $\vec{\omega}_{\mathcal{D}/\mathcal{F}}$.



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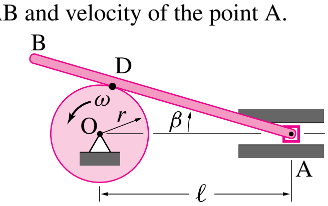
Problem 15.4.7

15.4.8 The link AB is supported by a wheel at D and its end A is constrained so that it only has horizontal velocity. No slipping occurs between the wheel and the link. The wheel has an angular velocity ω and radius r. The distance $OA = \ell$.

Given: ω, r, and ℓ. ($\beta = \sin^{-1}\frac{r}{\ell}$). Determine the angular velocity of the link AB and velocity of the point A.



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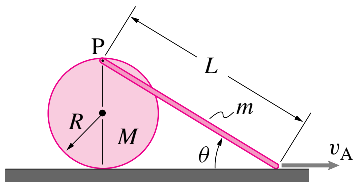
Problem 15.4.8

15.4.9 A solid cylinder of radius R and mass M rolls without slip along the ground. A thin rod of mass m and length L is attached by a frictionless pin P to the cylinder's rim and its right end is dragged at a constant speed $\vec{v}_A$ along the (frictionless) horizontal ground.

a) For the position shown (where P is directly above the contact point), find P's velocity $\vec{v}_P$ and the rod's angular velocity $\vec{\omega}$.

b) Find P's acceleration $\vec{a}_P$ and the angular acceleration $\vec{\alpha}$ of the rod.



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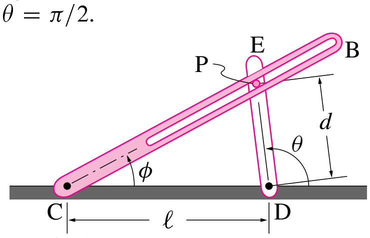
Problem 15.4.9

15.4.10 The slotted link CB is driven in an oscillatory motion by the link ED which rotates about D with constant angular velocity $\dot{\theta} = \omega_D$. The pin P is attached to ED at fixed radius d and engages the slot on CB as shown. Find the angular velocity and acceleration $\dot{\phi}$ and $\ddot{\phi}$ of CB when $\theta = \pi/2$.



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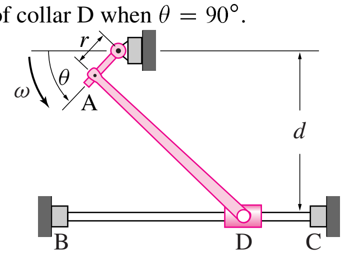
Problem 15.4.10

15.4.11 The rod of radius $r = 50\,\mathrm{mm}$ shown has a constant angular velocity of $\omega = 30\,\mathrm{rad/s}$ counterclockwise. Knowing that rod AD is 250 mm long and distance $d = 150\,\mathrm{mm}$, determine the acceleration of collar D when $\theta = 90°$.



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Problem 15.4.11

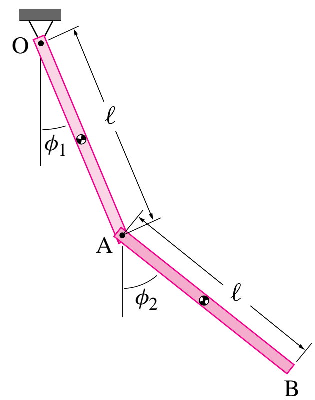
15.5 Advance kinematics of planar motion

15.5.1 Double pendulum. The double pendulum shown is made up of two uniform bars, each of length ℓ and mass m. At the instant shown, ϕ_1, $\dot{\phi}_1$, ϕ_2, and $\dot{\phi}_2$ are known. For the instant shown, answer the following questions in terms of ℓ, ϕ_1, $\dot{\phi}_1$, ϕ_2, and $\dot{\phi}_2$.

a) What is the absolute velocity of point A?

b) What is the velocity of point B relative to point A?

c) What is the absolute velocity of point B?



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Problem 15.5.1

15.5.2 Double pendulum, Again. For the double pendulum in problem 15.5.1, $\ddot{\phi}_1$ and $\ddot{\phi}_2$ are also known at the instant shown. For the instant shown, answer the following questions in terms of ℓ, ϕ_1, $\dot{\phi}_1$, $\ddot{\phi}_1$, ϕ_2, $\dot{\phi}_2$, and $\ddot{\phi}_2$.

a) What is the absolute acceleration of point A?

b) What is the acceleration of point B relative to point A?

c) What is the absolute acceleration of point B?

CHAPTER 16

Mechanics of constrained particles and rigid objects

The dynamics of particles and rigid objects is studied using the relative-motion kinematics ideas from chapter 15. This is the capstone chapter for a two-dimensional dynamics course. After this chapter a good student should be able to navigate through and use most of the skills in the concept map inside the back cover.

Contents

We have studied the mechanics of particles and rigid bodies with constraints that require progressively more involved kinematics. We now proceed to study the mechanics of more complex systems: particles with constrained paths, particles moving relative to moving frames, and mechanisms with several parts.

The basic strategy throughout is to use, in combination, the following skills:

1. Basic modeling. Describe a system in an appropriate way using the language of particle and rigid body mechanics. As described in Chapter 2, the force modeling and kinematic modeling are coupled. Where relative motion is freely allowed there is no force. And where motion is caused or prevented there is a force. Here is where you decide the constitutive (force) laws you are using for springs, contact, gravity, etc.
2. Draw free body diagrams of the system of interest and of its parts. These diagrams should show what you do and do not know about the constraint forces (*e.g.,* at a pin connection cut free in a free body diagram the FBD should show an arbitrary force and no moment). These are exactly the same free body diagrams that one would draw for statics.
3. Kinematics calculations. Pick appropriate configuration variables, as many as there are degrees of freedom. Then write the velocities, accelerations, angular velocities and angular accelerations of interest in terms of the configuration variables and their first and second time derivatives, possibly using methods from Chapter 10. Often this is the hardest part of the analysis.
4. Use appropriate balance equations: linear momentum, angular momentum, or power balance equations.
5. Solve the balance equations for unknown forces or accelerations of interest. Sometimes this can be done by hand by writing out components and solving simultaneous equations or by using appropriate dot products. And sometimes it is best done on by setting up a matrix equation and solving on the computer.
6. Solve the differential equations to find how the basic configuration variables change with time. For some special problems this can be done by hand, but most often involves computer solution.
7. Plug the ODE solution from (6) above into the equations from kinematics (3 above) and the balance laws (4 above). This is not a different skill from (3) or (4), it is just applied at a different time in the work.

8. Make plots of how forces, positions and velocities change with time, or of trajectories. Animations are also often nice.

These skills are used to solve dynamics problems which often fall into one of these 4 categories.

a. Kinematics. These are problems where *only* geometry is used, where the kinematics constraints determine what you are interested in, independent of the forces or time history. A classic example is determining the path of a point on a given four-bar linkage. More basic examples include finding position or acceleration from a given velocity history.

b. Instantaneous dynamics. These are problems where the positions and velocities of all points are given and you need to find forces or accelerations. Often these are "first-motion" problems: what are accelerations and forces immediately after something is released from rest?

c. "Inverse dynamics." These problems are called "inverse" because they are backwards of the original hard dynamics problems ((d) below). In these problems the motion is given as a function of time, and you have to calculate the forces. These problems are easier than non "inverse" problems because the differential equations from the balance laws don't need to be solved. A classic example is a slider-crank where the motion of the crank is known *a priori* to be at constant rate and you need to find the torque required to keep that motion. Usually in science "inverse" problems are harder. In dynamics this kind of "inverse" problem is easier than the non "inverse" problems.

d. Dynamics analysis. You are given some information about forces and constraints and you have to find the motion and more about the forces. These are the capstone problems that require use of all the skills.

A flow chart showing how these problem types are solved using the basic skill components ideas is shown in the chart in the inside back cover. As you solve a problem, at any instant in time you should be able to place your work on this chart.

In the sense of putting all the basic ideas together, this chapter completes the book. But in this chapter we only consider two dimensional models and motions. Three dimensional models and motions involve the same assembly of basic ideas, but more difficult kinematics, so are postponed.

16.1 Mechanics of a constrained particle

The kinematics of time-varying base vectors help us deal with some more difficult particle motion mechanics problems. For one point mass it is easy to write balance of linear momentum. It is:

$$\vec{F} = m\vec{a}.$$

The mass of the particle m times its vector acceleration $\vec{a}$ is equal to the total force on the particle $\vec{F}$. No problem.

Now, however, we can write this equation in five somewhat distinct ways.

1. In general abstract vector form: $\vec{F} = m\vec{a}$.
2. In cartesian coordinates: $F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} = m[\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}]$.
3. In polar coordinates:
$$F_R\hat{e}_R + F_\theta\hat{e}_\theta + F_z\hat{k} = m[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta + \ddot{z}\hat{k}].$$
4. In path coordinates: $F_t\hat{e}_t + F_n\hat{e}_n = m[\dot{v}\hat{e}_t + (v^2/\rho)\hat{e}_n]$

All of these equations are always right. Additionally, for a given particle moving under the action of a given force there are many more correct equations that can be found by shifting the origin and orientation of the coordinate systems. For example for a moving frame $\mathcal{B}$ with origin $0'$, rotation rate $\vec{\omega}_\mathcal{B}$ and angular acceleration $\vec{\alpha}_\mathcal{B}$:

5. $\vec{F} = m\left\{\vec{a}_{0'} \quad - \omega_\mathcal{B}^2\vec{r} \quad + \vec{\alpha}_\mathcal{B} \times \vec{r} \quad + \vec{a}_{/\mathcal{B}} \quad + 2\vec{\omega}_\mathcal{B} \times \vec{v}_{/\mathcal{B}}\right\}$

where, to simplify the notation, all motions are relative to $\mathcal{F}$ and positions relative to 0 unless explicitly indicated by a $/\mathcal{B}$ or $/0'$. This is quite a collection of kinematic tools. In general we want to choose the best tools for the job. But to get a sense lets first look at a simple problem using each of these kinematic approaches, some of which are rather inappropriate.

A particle that moves with no net force

In the special case that a particle has no force on it we know intuitively, or from the verbal statement of Newton's First Law, that the particle travels in a straight line at constant speed. As a first example, let's try to find this result using the vector equations of motion five different ways: in the general abstract form, in cartesian coordinates, in polar coordinates, in path coordinates, and relative to a moving frame (see fig. 16.1).

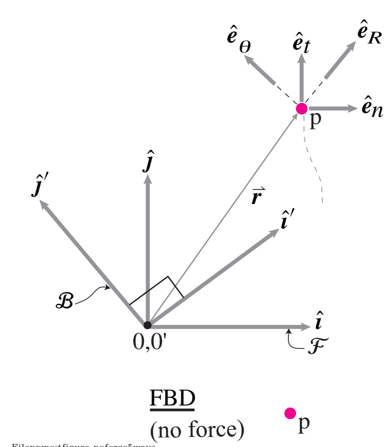

Figure 16.1: A particle P moves. One can track its motion using the general vector form $\vec{r}$, Cartesian coordinates in the fixed frame $\mathcal{F} = 0\hat{\imath}\hat{\jmath}$, Polar coordinates using $\hat{e}_R \& \hat{e}_\theta$, path coordinates $\hat{e}_t \& \hat{e}_n$ and cartesian coordinates in a rotating frame $\mathcal{B} = 0'\hat{\imath}'\hat{\jmath}'$.

General abstract form. The equation of linear momentum balance is $\vec{F} = m\vec{a}$ or, if there is no force, $\vec{a} = \vec{0}$, which means that $d\vec{v}/dt = \vec{0}$. So $\vec{v}$ is a constant. We can call this constant $\vec{v}_0$. So after some time the particle is where it was at $t = 0$, say, $\vec{r}_0$, plus its velocity $\vec{v}_0$ times time. That is:

$$\vec{r} = \vec{r}_0 + \vec{v}_0 t. \tag{16.1}$$

This vector relation is a parametric equation for a straight line. The particle moves in a straight line, as expected.

Cartesian coordinates. If instead we break the linear momentum balance equation into cartesian coordinates we get

$$F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} + \ddot{z}\hat{k}).$$

Because the net force is zero and the net mass is not negligible,

$$\ddot{x} = 0, \qquad \ddot{y} = 0, \text{ and } \qquad \ddot{z} = 0.$$

These equations imply that $\dot{x}$, $\dot{y}$, and $\dot{z}$ are all constants, lets call them v_{x0}, v_{y0}, v_{z0}. So x, y, and z are given by

$$x = x_0 + v_{x0}t, \quad y = y_0 + v_{y0}t, \quad \& \; z = z_0 + v_{z0}t.$$

We can put these components into their place in vector form to get:

$$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}} = (x_0 + v_{x0}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{y0}t)\hat{\boldsymbol{\jmath}} + (z_0 + v_{z0}t)\hat{\boldsymbol{k}}. \quad (16.2)$$

Note that there are six free constants in this equation representing the initial position and velocity. Equation 16.2 is a cartesian representation of equation 16.1; it describes a straight line being traversed at constant rate.

Polar/cylindrical coordinates. When there is no force, in polar coordinates we have:

$$\underbrace{F_R}_{0}\,\hat{\boldsymbol{e}}_R + \underbrace{F_\theta}_{0}\,\hat{\boldsymbol{e}}_\theta + \underbrace{F_z}_{0}\,\hat{\boldsymbol{k}} = m[(\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}].$$

This vector equation leads to the following three scalar differential equations, the first two of which are coupled non-linear equations (neither can be solved without the other).

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \\ \ddot{z} &= 0 \end{aligned}$$

A tedious calculation will show that these equations are solved by the following functions of time:

$$\begin{aligned} R &= \sqrt{d^2 + [v_0(t - t_0)]^2} \quad (16.3) \\ \theta &= \theta_0 + \tan^{-1}[v_0(t - t_0)/d] \\ z &= z_0 + v_{z0}t, \end{aligned}$$

where θ_0, d, t_0, v_0, z_0, and v_{z0} are constants. Note that, though eqn. (16.4) looks different than eqn. (16.2), there are still 6 free constants. From the physical interpretation you know that eqn. (16.4) must be the parametric equation of a straight line. And, indeed, you can verify that picking arbitrary constants and using a computer to make a polar plot of eqn. (16.4) does in fact show a straight line. From eqn. (16.4) it seems that polar coordinates' main function is to obfuscate rather than clarify. For the simple case that a particle moves with no force at all, we have to solve non-linear differential equations whereas using cartesian coordinates we get linear equations which are easy to solve and where the solution is easy to interpret.

But, if we add a central force, a force like earth's gravity acting on an orbiting satellite (the force on the satellite is directed towards the center of the earth), the equations become almost intolerable in cartesian coordinates. But, in polar coordinates, the solution is almost as easy (which is not all that easy for most of us) as the solution 16.4. So the classic analytic solutions of celestial mechanics are usually expressed in terms of polar coordinates.

Path coordinates. When there is no force, $\vec{F} = m\vec{a}$ is expressed in path coordinates as

$$\underbrace{F_t}_{0}\ \hat{e}_t + \underbrace{F_n}_{0}\ \hat{e}_n = m(\dot{v}\hat{e}_t + \underbrace{(v^2/\rho)\hat{e}_n}_{v^2\vec{\kappa}}).$$

That is,

$$\dot{v} = 0 \quad \text{and} \quad v^2/\rho = 0.$$

So the speed v must be constant and the radius of curvature ρ of the path infinite. That is, the particle moves at constant speed in a straight line.

Relative to a rotating reference frame Let's look at the equations using a frame $\mathcal{B}$ that shares an origin with $\mathcal{F}$ but is rotating at a constant rate $\vec{\omega}_{\mathcal{B}} = \omega\hat{k}$ relative to $\mathcal{F}$. Thus

$$\vec{\alpha}_{\mathcal{B}} = \vec{0} \quad \text{and} \quad \vec{a}_{0'/0} = \vec{0}$$

and we have that $\vec{F} = m\vec{a}$ is written as

$$\begin{aligned}
\vec{F} &= m\vec{a} \\
\vec{0} &= m\,\{\underbrace{\vec{a}_{0'}}_{\vec{0}} \quad -\omega_{\mathcal{B}}^2\vec{r} \quad + \underbrace{\vec{\alpha}_{\mathcal{B}}}_{\vec{0}} \times\vec{r} \quad +\vec{a}_{/\mathcal{B}} \quad + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{/\mathcal{B}}\,\} \\
&= -\omega^2\vec{r} + \vec{a}_{/\mathcal{B}} + 2\omega\hat{k} \times \vec{v}_{/\mathcal{B}}. \qquad (16.4)
\end{aligned}$$

Now, using the rotating base vectors $\hat{\imath}'$ and $\hat{\jmath}'$, we have that $F_{x'}\hat{\imath}' + F_{y'}\hat{\jmath}' = 0\hat{\imath}' + 0\hat{\jmath}'$ and

$$\vec{r} = x'\hat{\imath}' + y'uj', \quad \vec{v}_{\mathcal{B}} = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}', \quad \text{and} \quad \vec{a}_{\mathcal{B}} = \ddot{x}'\hat{\imath}' + \ddot{y}'\hat{\jmath}'$$

so eqn. (16.4) can be rewritten as

$$\vec{0} \;=\; -\omega^2(x'\hat{\imath}' + y'uj') + (\ddot{x}'\hat{\imath}' + \ddot{y}'\hat{\jmath}') + 2\omega\hat{k} \times (\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}')$$

which in turn can be broken into components and written as:

$$\begin{aligned}
\ddot{x}' &= \omega^2 x' + 2\omega\dot{y}' \qquad (16.5) \\
\ddot{y}' &= \omega^2 y' - 2\omega\dot{x}'
\end{aligned}$$

which makes up a pair of second order linear differential equations. With some work, someone good at ODEs can solve this with pencil and paper. But most of us would use a computer for such a system. If, in some consistent units, we had

$$y'(0) = 0, \dot{y}'(0) = 0, x'(0) = 0, \dot{x}'(0) = 1, \omega = 1$$

then the solution turns out to be

$$\begin{aligned}
x' &= \quad t\cos(\omega t) \\
y' &= -t\sin(\omega t)
\end{aligned}$$

as you can check by substituting into eqn. (16.6). That is, a particle which goes in a straight line away from the origin goes, in spirals as seen in the rotating frame①.

①And the straight line motion could have been more complicated yet, as seen in the rotating frame, with variable rate rotation and acceleration.This is why we are lucky the earth is not spinning fast or at highly variable rates. Otherwise the motion we would see, us humans here on the rotating earth, would be particles with no force on them spiraling around all over the place. And that's what Isaac Newton would have seen. And he would have written "A particle in motion tends to go in crazy spirals, and a particle that is initially at rest also goes bonkers" and the equations we use through out this book would have been much harder to discover.

Constrained motion

A particle in a plane has 2 degrees of freedom. There is basically only one kind of constraint — to a path. When constrained to a path the particle has one remaining degree of freedom so its configuration can be described with one variable. For a given problem you must think about

- What force(s) constrain the motion to the path?
- What do you want to use for a configuration variable?

You use these ideas to

- draw an appropriate free body diagram, and then
- calculate the velocity and acceleration in terms of the configuration variable and its derivatives.

After these key first steps you plug into equations of motion and solve for what you are interested in. Of course the needed math could be difficult or impossible, but the work is somewhat routine from a mechanics point of view.

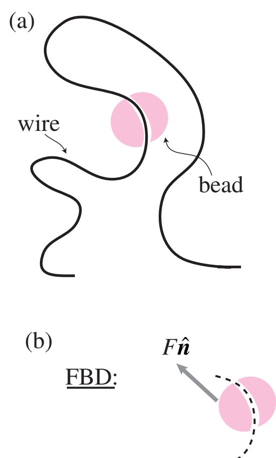

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Figure 16.2: A point-mass bead slides on a rigid immobile frictionless wire. The free body diagram shows that the only force on the bead is in the direction normal to the wire.

Example: Bead on frictionless wire

A bead slides on a frictionless wire with a crazy but smooth shape. No forces are applied to the bead besides the constraint force (see fig. 16.2).

Fig. 16.2b shows a free body diagram where $\hat{\boldsymbol{n}}$ is the normal to the wire at the point of interest. It doesn't matter if you use for $\hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_n$, or $\hat{\boldsymbol{n}} =$ a vector always, say, to the left, just so long as you know what you mean by $\hat{\boldsymbol{n}}$. The free body diagram shows that you know that the constraint force is normal to the path (the frictionless wire) but that you don't know how big it is (F is an unknown scalar).

For some purposes, especially general problems like this where no specific path is given, the most appropriate configuration variable is s, the arc length along the path. If the path is given we assume we know the position at any given arc length by the functions

$$x(s) \quad \text{and} \quad y(s).$$

So

$$\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}(s), \vec{\boldsymbol{v}} = \dot{s}\hat{\boldsymbol{e}}_t, \quad \text{and} \quad \vec{\boldsymbol{a}} = (\dot{s}^2/\rho)\hat{\boldsymbol{e}}_n + \ddot{s}\hat{\boldsymbol{e}}_t.$$

Now we can write linear momentum balance

$$\begin{aligned} && \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ && \{\, F\hat{\boldsymbol{n}} &= m\left((\dot{s}^2/\rho)\hat{\boldsymbol{e}}_n + \ddot{s}\hat{\boldsymbol{e}}_t \right)\} \\ \{\}\cdot\hat{\boldsymbol{e}}_t &\Rightarrow & \dot{v} &= 0 \quad \text{and} \\ \{\}\cdot\hat{\boldsymbol{e}}_n &\Rightarrow & F &= mv^2/\rho. \end{aligned} \tag{16.6}$$

Eqn. 16.6 tells us that the bead moves at constant speed, no matter what the shape of the wire. It also tells us that the more curved the wire, the bigger the constraint force needed to keep the bead on the wire.

Because this is a 1 DOF system, any one equation of motion should give us the result. Instead of linear momentum balance we could have used power balance to get the same result like this:

$$\begin{aligned} P &= \dot{E}_{\text{K}} \\ \Rightarrow \quad \vec{\boldsymbol{F}}_{\text{tot}}\cdot\vec{\boldsymbol{v}} &= \frac{d}{dt}\left\{\frac{1}{2}mv^2\right\} \\ \Rightarrow \quad F\hat{\boldsymbol{e}}_n\cdot v\hat{\boldsymbol{e}}_t &= mv\dot{v} \\ \Rightarrow \quad 0 &= \dot{v}. \end{aligned}$$

This is natural enough. The only force on the particle is perpendicular to its motion, so does no work. So the particle must have constant kinetic energy and its speed must be constant.

The example above doesn't seem that applicable; how often does one see beads on frictionless wires? It is a bit more useful than it appears. For example, if a car is coasting on an open road and tire resistance and sideslip can be ignored, the reasoning above shows that the car is neither speeded nor slowed by turning. Similarly, an idealization of an airplane wing is as something which only causes force perpendicular to motion. So in quick manuevers where the gravity force is relatively small, the plane maintains its speed.

Example: The hard brachistochrone problem.
Here is a puzzle proposed by Johann Bernoulli on January 1, 1669 ②: given that a roller coaster has to coast from rest at one place to another place that is no higher, what shape should the track be to make the trip as quick as possible.

The solution. Finding the solution, or even verifying it, is a problem in the calculus of variations, *i.e.,* too advanced for this book. The solution turns out to be the brachistochrone curve that obeys the following relationship between arc-length s from the origin and vertical position y (drawn accurately in fig. 16.3):

$$y = \frac{1}{2}cs^2 \qquad (\text{for}|s| \leq 1/c). \tag{16.7}$$

Starting at the origin this curve is close to $y = cx^2/2$ but gets a bit higher (bigger y) because for a given value of x, s is greater than x. The curve terminates at vertical tangents at $s = \pm 1/c$ where $y = 1/2c$. (see fig. 16.3). To solve the puzzle this curve is scaled (by choosing a value of c) and displaced so that it has a vertical tangent at A and also so the curve goes through B. The idea that the hard math seems to be expressing, is that the particle should first build up as much speed as it can (by going straight down) and then head off in the right direction (see fig. 16.4).

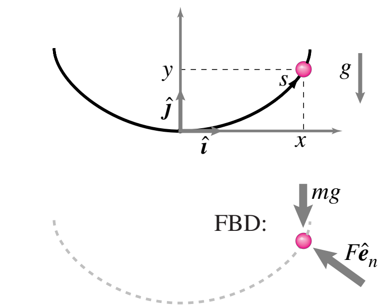

Figure 16.3: A bead slides on a frictionless wire on the curve implicitly defined by $y = cs^2/2$, where s is arc-length along the curve measured from the origin.

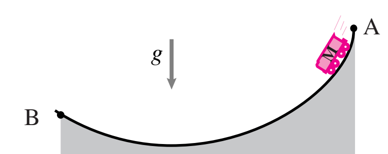

Figure 16.4: The roller coaster that gets from A to B the fastest is the one with a track in the shape of the brachistochrone $y = cs^2/2$.

On the other hand, here is an easier problem that is a virtual setup for the techniques now at hand.

Example: The easy brachistochrone problem
How long does it take for a particle to slide back and forth on a frictionless wire with $y = cs^2/2$ as driven by gravity? (see fig. 16.3)

Let's use s as our configuration variable. The power balance equation is:

$$\begin{aligned}
& & P &= \dot{E}_K \\
(F\hat{e}_n)\cdot(v\hat{e}_t) = 0 \quad &\Rightarrow & -mg\hat{\jmath}\cdot(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) &= \frac{d}{dt}\left\{\frac{1}{2}mv^2\right\} \\
&\Rightarrow & -mg\dot{y} &= m\frac{d}{dt}\frac{\dot{s}^2}{2} \\
&\Rightarrow & -mgcs\dot{s} &= m\dot{s}\ddot{s} \\
\text{Assuming } \dot{s} \neq 0 \quad &\Rightarrow & \ddot{s} + gcs = 0. &
\end{aligned}$$

This, remarkably, is the simple harmonic oscillator equation with general solution

$$s = A\cos(\sqrt{gc}\,t) + B\sin(\sqrt{gc}\,t).$$

Thus the period of oscillation (T such that $\sqrt{gc}T = 2\pi$) is

$$T = \frac{2\pi}{\sqrt{gc}}$$

which is independent of the amplitude of oscillation ③.

The key to this quick solution was using a configuration variable that made the expression for the velocity simple, and using an equation of motion that didn't involve the unknown reaction force F which we also didn't care about. We could have got the same equation of motion by writing $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and eliminated $F\hat{\boldsymbol{e}}_n$ by dotting both sides with a convenient vector orthogonal to $F\hat{\boldsymbol{e}}_n$, say $\vec{\boldsymbol{v}}$.

The brachistochrone is a famous curve that has various interesting properties (*e.g.,* Box 16.1 on page 914).

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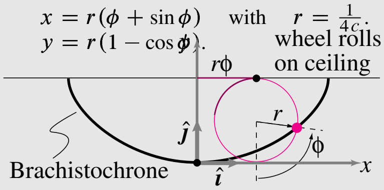
Example: A collar on two rotating rods

Consider a pair of collars hinged together as a point mass m at P. Each slides frictionlessly on a rod about whose rotation everything is known (see fig. 16.5. What is the force of rod 1 on the mass? For this 2 degree of freedom system lets use configuration variables θ_1 and θ_2, and two sets of rotating base vectors: $\hat{\boldsymbol{\lambda}}_1\hat{\boldsymbol{n}}_1$ and $\hat{\boldsymbol{\lambda}}_2\hat{\boldsymbol{n}}_2$. These rotating base vectors can be written in terms of the θs, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in the standard manner. Assume we know ℓ_1 and ℓ_2 in this configuration. First find

16.1 Some brachistochrone curiosities

The brachistochrone is a cycloid. There is no straightforward way to draw the curve $y = cs^2/2$ because the formula doesn't tell you the x coordinates of the points. You could find them by integrating $dx = \sqrt{ds^2 - dy^2}$ numerically or with calculus tricks. But it turns out (see below) that the curve with $y = cs^2/2$ is described by the parametric equations

$$x = r(\phi + \sin\phi) \quad \text{with} \quad r = \frac{1}{4c}.$$
$$y = r(1 - \cos\phi).$$



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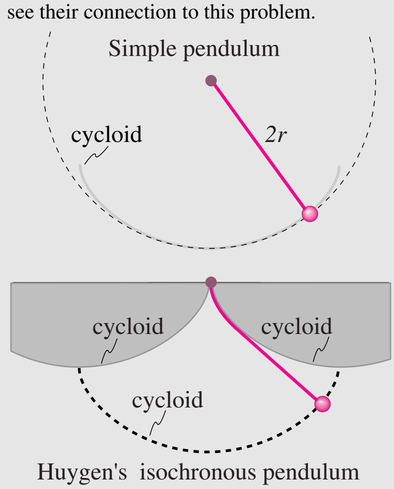
This is the path of a particle on the perimeter of a wheel that rolls against a horizontal ceiling a distance $2r$ above the origin, as you can verify by adding up distances in the picture above (see page 847). We will show below that the upside down cycloid and the curve $y = cs^2/2$ are one and the same.

Note that the osculating circle of this cycloid at its lowest point has radius $4r = 1/c$, just the length of a simple pendulum that, for small oscillations, has the same frequency of oscillation as the bead on the brachistochrone. A point mass swinging on a string is like a bead on a frictionless circular wire and this, in turn, is close to the motion of a bead on a brachistochrone wire for small oscillations.

Galileo (1564-1642). Well before Bernoulli's challenge, Galileo was interested in things rolling and sliding on ramps. He knew that the shortest distance between two points is a straight line, and had noted that a ball rolling down an appropriately curved ramp gets to its destination faster than a ball traveling the shortest route. A ball going on a straight ramp just doesn't pick up much speed, and when it finally has its greatest speed the trip is over. Imagine sliding straight sideways; it takes forever on a straight-line route. Better, he must have reasoned, to get the ball rolling fast at the start and then go fast for most of its journey, possibly slowing at the end. Galileo thought the best shape was the bottom of a circle (or fraction thereof), which isn't far off either in shape or concept, but isn't quite right. Galileo was apparently obsessed with cycloids for other reasons but didn't see their connection to this problem.



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A constant period pendulum. For clock time keeping, a pendulum is better than a bead on a wire because the friction of sliding is avoided. Unfortunately, a simple pendulum has a period which is longer if the amplitude is bigger. Not much longer, 18% if the swinging is $\pm 90^\circ$ and only 1.7% longer if the amplitude is $\pm 30^\circ$,

(continued...)

$\dot{\ell}_1$ and $\dot{\ell}_2$ by thinking of the velocity of the collar two different ways:

$$\begin{array}{rcl} \vec{v} & = & \vec{v} \\ \{\ \dot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \dot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 & = & \dot{\ell}_2\hat{\boldsymbol{\lambda}}_2 + \dot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\ \} \\ \{\}\cdot\hat{\boldsymbol{n}}_2 & \Rightarrow & \dot{\ell}_1 = \dfrac{\dot{\theta}_2\ell_2 - \dot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2}{\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{n}}_2} \\ \{\}\cdot\hat{\boldsymbol{n}}_1 & \Rightarrow & \dot{\ell}_2 = \dfrac{\dot{\theta}_1\ell_1 - \dot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_1}{\hat{\boldsymbol{\lambda}}_2\cdot\hat{\boldsymbol{n}}_1}. \end{array} \tag{16.8}$$

Having found $\dot{\ell}_1$ and $\dot{\ell}_2$ we can find the velocity $\vec{v}$ by evaluating either side of eqn. (16.8). Now we apply identical reasoning with the acceleration. The result looks messy, but the approach is straightforward:

$$\begin{array}{rcl} \vec{a} & = & \vec{a} \\ \{\ \ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1 & = & \ddot{\ell}_2\hat{\boldsymbol{\lambda}}_2 + \ddot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2 + 2\dot{\ell}_2\dot{\theta}_2\hat{\boldsymbol{n}}_2\ \} \end{array} \tag{16.9}$$

$$\{\}\cdot\hat{\boldsymbol{n}}_2 \quad \Rightarrow \quad \ddot{\ell}_1 = \frac{\ddot{\theta}_2\ell_2 + 2\dot{\ell}_2\dot{\theta}_2 - \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2 - 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{n}}_2}{\hat{\boldsymbol{\lambda}}_1\cdot\hat{\boldsymbol{n}}_2}$$

$$\{\}\cdot\hat{\boldsymbol{n}}_1 \quad \Rightarrow \quad \ddot{\ell}_2 = \frac{\ddot{\theta}_1\ell_1 + 1\dot{\ell}_1\dot{\theta}_1 - \ddot{\theta}_2\ell_2\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_1 - 2\dot{\ell}_2\dot{\theta}_2\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{n}}_1}{\hat{\boldsymbol{\lambda}}_2\cdot\hat{\boldsymbol{n}}_1}$$

16.1 Some brachistochrone curiosities (continued)

but enough to annoy clock designers. A bead sliding frictionlessly on the path $y = s^2/2$ has the nice property that the period does not depend on the amplitude. But any real bead sliding on any real wire has substantial friction. So, at first blush the brachistochrone curve, despite its nice constant-period property, cannot be used to keep time.

But Huygens, one of the smart old timers, looked for a curve that, when a string wraps around it, makes the end follow the brachistochrone curve. To this day you can see fancy old clocks with this wrapping device, a solid piece with the cuspoidal shape of neighboring cycloids, near the hinge of the swinging ("isochronous" or "tautochrone") pendulum which wraps around it.

Geometry. The two key features discussed above, that the curve $y = cs^2/2$ is a cycloid, and that a cycloid can be generated by wrapping a string around another cycloid, can be found from the geometric construction below. Two cycloids are shown, one from wheel 1 rolling under line L_1 and another from wheel 2 rolling under line L_2 a distance $2r$ below. Both wheels have radius r. Imagine that the cycloids A_1M_1B and A_2M_2B are drawn by wheels always arranged with vertically aligned rolling contact points C_1 and C_2 and with points M_1 and M_2 initially aligned vertically a distance $4r$ apart at A_1 and A_2.

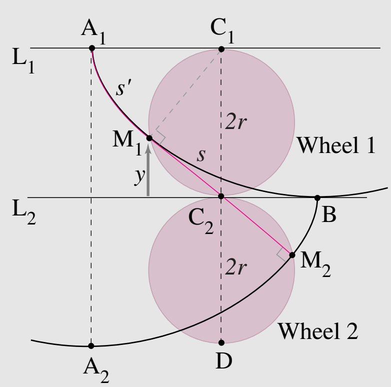
The two cycloids are thus the same shape but are displaced with one being $2r$ below and πr sideways from the other.

Because both wheels have rolled the same distance (A_1C_1) they have rotated the same amount and M_2 is as far forward of C_1C_2D as M_1 is behind. Similarly M_1 is as far above L_2 as M_2 is below. So the line M_1M_2 is bisected by the point C_2.

Because C_1C_2 is the diameter of a circle with M_1 on the perimeter, angle $C_1M_1C_2$ is a right angle. Because material point C_1 on the wheel has zero velocity the velocity of M_1 (and thus the tangent to the curve) is orthogonal to C_1M_1. Thus the line M_1M_2 is tangent to the upper cycloid.

The rolling of wheel 2 instantaneously rotating about C_2 makes the tangent to the lower cycloid orthogonal to M_1M_2, the condition for the motion of M_2 to be from the wrapping of an inextensible line around the curve A_1M_1B. This shows that cycloid A_2M_2B is generated by the wrapping of a line anchored at A_1 about the upper cycloid. And this is Huygen's wrapping mechanism for making a pendulum bob follow a cycloid. Because of this wrapping generation, the arc-length $s' + s$ of A_1M_1B must be $4r$ and the arc length s of M_1B is M_1M_2 so the length M_1C_2 is $s/2$. By the similarity of the two right triangles that share the length $s/2$ of M_1C_2:

$$\frac{y}{s/2} = \frac{s/2}{2r} \quad \Rightarrow \quad y = \frac{1}{4r}\frac{s^2}{2} = c\frac{s^2}{2}$$



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which shows that the upper cycloid is the curve $y = cs^2/2$ if $c = 1/4r$, where s is measured from B. This was the equation used to show the constant period nature of the sliding motion of a bead on a frictionless cycloidal curve using power balance.

② **The original brachistochrone (least time) puzzle:**
"I, Johann Bernoulli, greet the most clever mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem whose possible solution will bestow fame and remain as lasting monument. Following the example set by Pascal, Fermat, etc., I hope to earn the gratitude of the entire scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall then publicly declare him worthy of praise.

"*Let two points A and B be given in a vertical plane. Find the curve that a point M, moving on a path AMB must follow such that, starting from A, reaches B in the shortest time under its own gravity.*"

Newton. Besides Bernoulli's son Dan, one of the people to solve the puzzle was 55 year old Isaac Newton. He attempted to keep his solution, said to have been worked out in one evening while he also invented the calculus of variations, anonymous. But Bernoulli supposedly saw through this deception, commenting "I recognize the lion by his paw print", which was presumably not a comment about Newton's handwriting.

③ Actually, the amplitude can't be arbitrarily large. The solution to the defining eqn. (16.7) only makes sense for $|s| < 1/c$. For $|s| > 1/c$ there is no curve satisfying eqn. (16.7).

We use the results from eqn. (16.8) for $\dot{\ell}_1$ and $\dot{\ell}_2$ to evaluate the right hand sides of the expressions for $\ddot{\ell}_2$ and $\ddot{\ell}_3$ in eqn. (16.9). So now either the left hand side or the right hand side of the second of Eqns. 16.9 can be used to evaluate the acceleration $\vec{\boldsymbol{a}}$, all the terms in both expressions have been found.

To find the forces we use linear momentum balance and the free body diagram

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{tot}} &= m\vec{\boldsymbol{a}} \qquad (16.10)\\
\{\, F_1\hat{\boldsymbol{n}}_1 + F_2\hat{\boldsymbol{n}}_2 &= m\{\ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\}\,\}\\
\{\}\cdot\hat{\boldsymbol{\lambda}}_2 \quad \Rightarrow \quad F_2 &= \frac{\{\ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\}\cdot\hat{\boldsymbol{\lambda}}_2}{\hat{\boldsymbol{n}}_1\cdot\hat{\boldsymbol{\lambda}}_2}\\
\{\}\cdot\hat{\boldsymbol{\lambda}}_1 \quad \Rightarrow \quad F_2 &= \frac{\{\ddot{\ell}_1\hat{\boldsymbol{\lambda}}_1 + \ddot{\theta}_1\ell_1\hat{\boldsymbol{n}}_1 + 2\dot{\ell}_1\dot{\theta}_1\hat{\boldsymbol{n}}_1\}\cdot\hat{\boldsymbol{\lambda}}_1}{\hat{\boldsymbol{n}}_2\cdot\hat{\boldsymbol{\lambda}}_1}
\end{aligned}$$

When actually evaluating the expressions above one can write the base vectors in terms of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ or use geometry.

Often when working out a problem it is best to not substitute numbers until the end of a problem. This example shows the opposite. If we left the expressions for $\dot{\ell}_1$ and $\dot{\ell}_2$ with letters and substituted that into the expressions for $\ddot{\ell}_1$ and $\ddot{\ell}_2$ and left those expressions intact while substituting for the acceleration $\vec{\boldsymbol{a}}$ we would have large expressions for the force components F_1 and F_2. On the other hand, by using numbers as the calculation progresses the formulas do not grow so much in complexity.

As is the case with most mechanism-mechanics problems, the hard work in getting the dynamics equations is in the kinematics. Generally there are no great short-cuts. There are alternative methods. In this case the location of the base points and the two angles determine the base and two angles of a triangle. This triangle can be solved for the location of the point P. Once that position is known in terms of θ_1 and θ_2 the velocity and acceleration can be found by differentiation.

As a robot manipulator, this design has the advantage that no motors need to be displaced. It has the disadvantage of requiring good sliding joints.

An alternative solution of the kinematics of this problem would be to use trigonometry to find the position of point P in terms of the angles θ_1 and θ_2. Then the acceleration of point P is found by taking two time derivatives. The result is approximately equal in the complexity of its appearance to the results used above. That method requires more cleverness at the start (solving an angle-side-angle triangle) and then just brute force differentiation using the chain rule and the product rule.

Example: Inverted pendulum with a vibrating base

Assume that the base $0'$ of an inverted point-mass pendulum of length ℓ is vibrated according to (see fig. 16.6)

$$\vec{\boldsymbol{r}}_{0'} = d\,\sin\omega t\,\hat{\boldsymbol{\imath}}.$$

The point P thus has acceleration

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{P}} &= \vec{\boldsymbol{a}}_{0'} + \vec{\boldsymbol{a}}_{\text{P}/0'}\\
&= -d\omega^2\sin\omega t\,\hat{\boldsymbol{\imath}} + \ddot{\theta}\ell\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2\ell\hat{\boldsymbol{e}}_R
\end{aligned}$$

Now apply linear momentum balance as

$$\begin{aligned}
\vec{\boldsymbol{F}}_{\text{tot}} &= m\vec{\boldsymbol{a}}_{\text{P}}\\
\{\, -mg\hat{\boldsymbol{\imath}} + -T\hat{\boldsymbol{e}}_R &= m\{-d\omega^2\sin\omega t\,\hat{\boldsymbol{\imath}} + \ddot{\theta}\ell\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2\ell\hat{\boldsymbol{e}}_R\}\,\}\\
\{\}\cdot\hat{\boldsymbol{e}}_\theta \quad \Rightarrow \quad -g\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{e}}_\theta &= -d\omega^2\sin\omega t\,\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{e}}_\theta + \ddot{\theta}\ell\\
\Rightarrow \quad g\sin\theta &= d\omega^2\sin\omega t\sin\theta + \ddot{\theta}\ell
\end{aligned}$$

which you write as

$$\ddot{\theta} + (d\omega^2 \sin\omega t - g)\sin\theta/\ell \quad \text{or} \quad \ddot{\theta} = (g - d\omega^2 \sin\omega t)\sin\theta/\ell$$

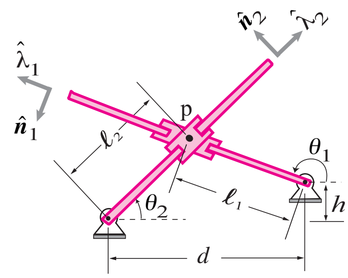
depending on whether you are analytically or numerically inclined. This is a second order non-linear ordinary differential equation. If $\omega = 0$ or $d = 0$ then this is the classic inverted pendulum equation and has solutions that show that the pendulum doesn't stay near upright. But, you can find by analytic cleverness or numerical integration that for some values of d and ω that the pendulum does not fall down! Just shaking the base keeps the pendulum up ($\omega^2 d > g$ for all cases where this is possible). This isn't just academic nonsense, the device can be built and the balancing demonstrated.

One alternative to using linear momentum balance in the equations above would be to use angular momentum balance about the point 0'. The resulting vector equation

$$\ell\hat{\boldsymbol{e}}_R \times (-mg\hat{\boldsymbol{\imath}}) = \vec{\boldsymbol{r}}_{P/0'} \times (m\vec{\boldsymbol{a}})$$

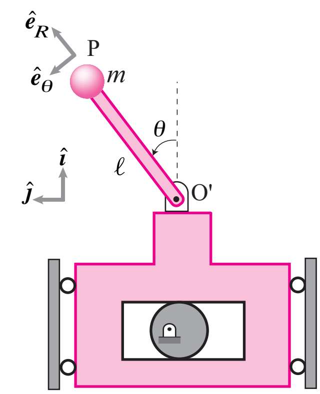
yields the same second order scalar ODE.

The vibrating mechanism shown is a "Scotch yoke". An eccentric disk is mounted to the shaft of a constant angular velocity motor. The rectangular slot moves up and down sinusoidally as the disk wobbles.



Filename:tfigure-collaron2rods

Figure 16.5: A point-mass collar slides simultaneously on 2 rods.



Filename:tfigure-vibratingpend

Figure 16.6: The base of a pendulum is vertically vibrated.

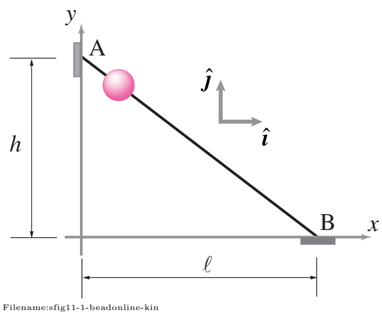


Figure 16.7:

SAMPLE 16.1 A bead on a straight wire. A straight wire is hung between points A and B in the xy plane as shown in the figure. A bead slides down the wire from point A. Write the geometric constraint equation for the bead's motion and derive the conditions on velocity and acceleration components of the bead due to the constraint.

Solution The constraint on the bead's motion is that its path must be along the wire, *i.e.,* a straight line between points A and B. Thus the geometric constraint on the motion is expressed by the equation of the path which is

$$y = h - \frac{h}{\ell}x.$$

Since the bead is constrained to move on this path, its velocity and acceleration vectors are also constrained to be directed along AB. This imposes conditions on their x and y components that are easily derived by differentiating the geometric constraint equation with respect to t. Thus,

$$\begin{aligned} \dot{y} &= -\frac{h}{\ell}\dot{x}, \\ \ddot{y} &= -\frac{h}{\ell}\ddot{x}. \end{aligned}$$

$$y = h - \tfrac{h}{\ell}x, \quad \dot{y} = -\tfrac{h}{\ell}\dot{x}, \quad \ddot{y} = -\tfrac{h}{\ell}\ddot{x}$$

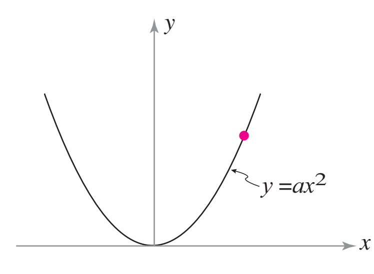


Figure 16.8:

SAMPLE 16.2 A particle sliding on a parabolic path. A particle slides on a parabolic trough given by $y = ax^2$ where a is a constant. Write the geometric constraints of motion (on the path, velocity, and acceleration) of the particle. Write the velocity and acceleration of the particle at a generic location (x, y) on its path.

Solution The geometric constraint on the path of the particle is already given, $y = ax^2$. DIfferentiating the path constraint with respect to time, we get the constraint on velocity and acceleration components.

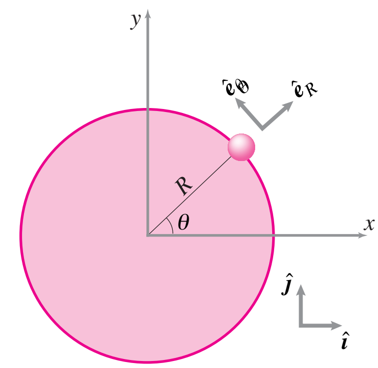
$$\begin{aligned} \dot{y} &= 2ax\dot{x}, \\ \ddot{y} &= 2ax\ddot{x} + 2a\dot{x}^2. \end{aligned}$$

Now, at a point (x, y), we can write the velocity and acceleration of the particle as

$$\begin{aligned} \vec{\boldsymbol{v}} &= \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\hat{\boldsymbol{\jmath}}, \\ \vec{\boldsymbol{a}} &= \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} = \ddot{x}\hat{\boldsymbol{\imath}} + (2ax\ddot{x} + 2a\dot{x}^2)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\hat{\boldsymbol{\jmath}}, \quad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + (2ax\ddot{x} + 2a\dot{x}^2)\hat{\boldsymbol{\jmath}}$$

SAMPLE 16.3 Circular motion of a particle. A particle is constrained to move on a frictionless circular path of radius R_0 with constant angular speed $\dot{\theta}$. There is no gravity. Find the equation of motion of the particle in the x-direction and show that this motion is simple harmonic.



Filename:sfig11-1-circle

Figure 16.9:

Solution This is simple problem that you have solved before, probably a few times. Here, we do this problem again just to show how it works out with the constraint machinery in evidence. The geometric constraint on the path of the particle is $R = R_0$ (in polar coordinates). This constraint gives us $\dot{R} = 0$ and $\ddot{R} = 0$. Then the acceleration of the particle (in polar coordinates), $\vec{a} = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_t$ reduces to $\vec{a} = -R\dot{\theta}^2\hat{e}_R$, (of course).

The free-body diagram of the particle shows that there is only one force acting on the particle, the normal reaction N of the path acting in the $\hat{e}_R$ direction. Therefore, the linear momentum balance gives,

$$N\hat{e}_R = m\vec{a} = m(-R\dot{\theta}^2\hat{e}_R) \quad \Rightarrow \quad N = -mR\dot{\theta}^2.$$

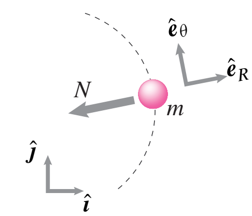


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Figure 16.10:

But, to write the equation of motion in the x-direction, we need to write the linear momentum balance in the x-direction. We can write $\sum \vec{F} = m\vec{a}$ using mixed basis vectors as $N\hat{e}_R = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$. Dotting this equation with $\hat{\imath}$, we get

$$\ddot{x} = \frac{N}{m}(\hat{e}_R \cdot \hat{\imath}) = \frac{-mR\dot{\theta}^2}{m}\cos\theta = -\dot{\theta}^2(R\cos\theta) = -\dot{\theta}^2 x$$

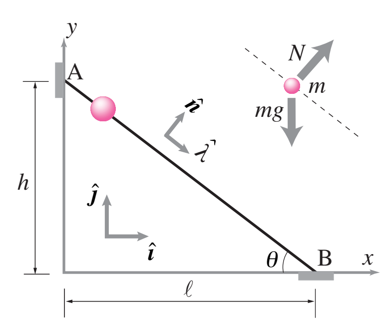
or, $\ddot{x} + \dot{\theta}^2 x = 0$, which is the equation of simple harmonic motion in x. You can easily show that the motion in the y-direction is also simple harmonic ($\ddot{y} + \dot{\theta}^2 y = 0$).

SAMPLE 16.4 A bead slides on a straight wire. Consider the problem of the bead sliding on a straight, inclined, frictionless wire of Sample 16.1 again. Find the position of the bead $x(t)$ and $y(t)$ assuming it slides under gravity starting from rest at A.

Solution To find the position of the bead, we need to write the equation of motion and solve it. This is single DOF system and, therefore, one scalar equation of motion should suffice.

The free-body diagram of the bead is shown in fig. 16.11. Using basis vectors $(\hat{\boldsymbol{\lambda}}, \hat{\boldsymbol{n}})$ and $(\hat{\imath}, \hat{\jmath})$ we write the LMB for the bead as

$$-mg\hat{\jmath} + N\hat{n} = m\vec{a} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}).$$



Filename:sfig11-1-beadonline-a

Figure 16.11:

We can easily eliminate the constraint force N from this equation by dotting this equation with $\hat{\boldsymbol{\lambda}}$, which gives

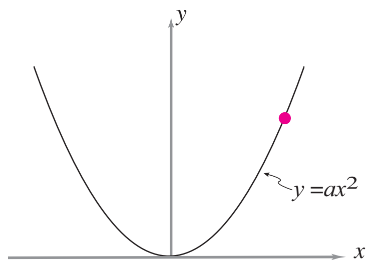
$$\begin{aligned} -\cancel{m}g(\hat{\jmath} \cdot \hat{\boldsymbol{\lambda}}) &= \cancel{m}[\ddot{x}(\hat{\imath} \cdot \hat{\boldsymbol{\lambda}}) + \ddot{y}(\hat{\jmath} \cdot \hat{\boldsymbol{\lambda}})] \\ \Rightarrow \quad g\sin\theta &= \ddot{x}\cos\theta - \ddot{y}\sin\theta. \end{aligned}$$

But, from the geometric constraint $y = h - (h/\ell)x$, we have $\ddot{y} = -(h/\ell)\ddot{x} = -(\tan\theta)\ddot{x}$. Therefore,

$$g\sin\theta = \ddot{x}\cos\theta + \ddot{x}\tan\theta\sin\theta \quad \Rightarrow \quad \ddot{x} = g\sin\theta\cos\theta.$$

Since $g\sin\theta\cos\theta$ is constant, we integrate the equation of motion easily to find $x(t) = \frac{1}{2}g\sin\theta\cos\theta t^2$ since $x(0) = 0, \dot{x}(0) = 0$. And, since $y = h - x\tan\theta$, we have $y(t) = h - \frac{1}{2}g\sin^2\theta t^2$.

$$x(t) = \tfrac{1}{2}ght^2\sin\theta\cos\theta, \qquad y(t) = h - \tfrac{1}{2}gt^2\sin^2\theta$$



Filename:sfig11-1-parabola

Figure 16.12:

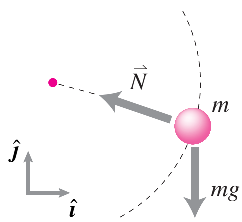


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Figure 16.13:

SAMPLE 16.5 A bead sliding down a parabolic trough. Consider the problem of Sample 16.2 again. Find the equation of motion of the bead.

Solution This is, again, a one DOF system. Therefore, we will get a single scalar equation of motion. The free-body diagram shown in fig. 16.13 shows two forces acting on the bead. The constraint force N acts normal to the path. Let $\hat{\boldsymbol{e}}_t$ and $\hat{\boldsymbol{e}}_n$ be unit vectors tangential and normal to the path, respectively. Then the linear momentum balance gives

$$-mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{e}}_n = m\vec{\boldsymbol{a}} = m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}).$$

To eliminate the unknown constraint force N from this equation, we can take a dot product of this equation with $\hat{\boldsymbol{e}}_t$. However, we must first find $\hat{\boldsymbol{e}}_t$. Now $\hat{\boldsymbol{e}}_t$ is the unit tangent vector. So, we can find it by finding a tangent vector to the path (remember gradient of a function ∇f?) and then dividing it by the length of the vector. That is doable but a little complicated. All we need here is the dot product with a vector *normal* to $\hat{\boldsymbol{e}}_n$. Why not use the velocity vector $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$? The velocity vector is always tangential to the path. Furthermore, we know that from the geometric constraint ($y = ax^2$), $\dot{y} = 2ax\dot{x}$ and $\ddot{y} = 2ax\ddot{x} + 2a\dot{x}^2$. Therefore, $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + 2ax\dot{x}\hat{\boldsymbol{\jmath}}$. Now, dotting the LMB equation with $\vec{\boldsymbol{v}}$ we get,

$$\begin{aligned}
-\cancel{m}g\overbrace{(\hat{\boldsymbol{\jmath}}\cdot\vec{\boldsymbol{v}})}^{2ax\dot{x}} &= \cancel{m}\left[\ddot{x}\overbrace{(\hat{\boldsymbol{\imath}}\cdot\vec{\boldsymbol{v}})}^{\dot{x}} + \ddot{y}\overbrace{(\hat{\boldsymbol{\jmath}}\cdot\vec{\boldsymbol{v}})}^{2ax\dot{x}}\right] \\
-2gax\cancel{\dot{x}} &= \ddot{x}\cancel{\dot{x}} + 2a\ddot{y}\cancel{\dot{x}} \\
&= \ddot{x} + 2a\underbrace{(2ax\ddot{x} + 2a\dot{x}^2)}_{\ddot{y}} \\
&= \ddot{x}(1 + 4a^2x) + 4a^2\dot{x}^2.
\end{aligned}$$

Rearranging the terms above, we get the required equation of motion:

$$\ddot{x} + \frac{4a^2}{1+4a^2x}\dot{x}^2 + \frac{2gax}{1+4a^2x} = 0.$$

As you can see, this is a nonlinear ODE. Analytical solution of this equation is rather difficult. We can, however, always solve it numerically. Note that a solution of this equation only gives you $x(t)$, *i.e.,* the x coordinate of the position of the bead. But, you can always find the y coordinate since $y = ax^2$.

$$\boxed{\ddot{x} + \frac{4a^2}{1+4a^2x}\dot{x}^2 + \frac{2gax}{1+4a^2x} = 0}$$

Comment: Note that if we consider x and $\dot{x}$ to be very small so that we can ignore the $\dot{x}^2$ term completely and take $1 + 4a^2x \approx 1$, then the equation of motion becomes

$$\ddot{x} + (2ga)x = 0$$

which is the equation of simple harmonic motion with frequency $\sqrt{2ga}$. Thus, if we consider a shallow parabola, and release the bead close to the origin, it executes simple harmonic motion, much like a simple pendulum. This is an intuitively realizable motion.

SAMPLE 16.6 Constrained motion of a pin. During a small interval of its motion, a pin of 100 grams is constrained to move in a groove described by the equation $R = R_0 + k\theta$ where $R_0 = 0.3\,\text{m}$ and $k = 0.05\,\text{m}$. The pin is driven by a slotted arm AB and is free to slide along the arm in the slot. The arm rotates at a constant speed $\omega = 6\,\text{rad/s}$. Find the magnitude of the force on the pin at $\theta = 60^\circ$.

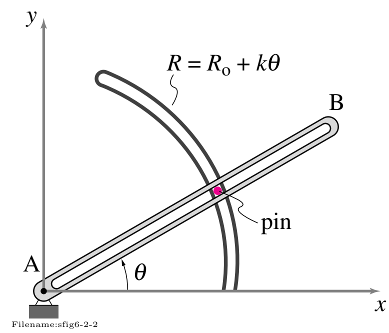


Figure 16.14: A pin is constrained to move in a groove and a slotted arm.

Solution Let $\vec{F}$ denote the net force on the pin. Then from the linear momentum balance

$$\vec{F} = m\vec{a}$$

where $\vec{a}$ is the acceleration of the pin. Therefore, to find the force at $\theta = 60^\circ$ we need to find the acceleration at that position.

From the given figure, we assume that the pin is in the groove at $\theta = 60^\circ$. Since the equation of the groove (and hence the path of the pin) is given in polar coordinates, it seems natural to use polar coordinate formula for the acceleration. For planar motion, the acceleration is

$$a = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta.$$

We are given that $\dot{\theta} \equiv \omega = 6\,\text{rad/s}$ and the radial position of the pin $R = R_0 + k\,\theta$. Therefore, ④

④ Note that R is a function of θ and θ is a function of time, therefore R is a function of time. Although we are interested in finding $\dot{R}$ and $\ddot{R}$ at $\theta = 60^\circ$, we cannot first substitute $\theta = 60^\circ$ in the expression for R and then take its time derivatives (which will be zero).

$$\begin{aligned}
\ddot{\theta} &= \frac{d\dot{\theta}}{dt} = 0 \quad (\text{ since } \dot{\theta} = \text{constant}) \\
\dot{R} &= \frac{d}{dt}(R_0 + k\theta) = k\dot{\theta} \quad \text{and} \\
\ddot{R} &= k\ddot{\theta} = 0.
\end{aligned}$$

Substituting these expressions in the acceleration formula and then substituting the numerical values at $\theta = 60^\circ$, (remember, θ must be in radians!), we get

$$\begin{aligned}
\vec{a} &= -\overbrace{(R_0 + k\theta)\dot{\theta}^2}^{R\dot{\theta}^2}\,\hat{e}_R + \overbrace{2k\dot{\theta}^2}^{2\dot{R}\dot{\theta}}\,\hat{e}_\theta \\
&= -(0.3\,\text{m} + 0.05\,\text{m}\cdot\frac{\pi}{3})\cdot(6\,\text{rad/s})^2\,\hat{e}_R + 2\cdot 0.3\,\text{m}\cdot(6\,\text{rad/s})^2\,\hat{e}_\theta \\
&= -13.63\,\text{m/s}^2\hat{e}_R + 21.60\,\text{m/s}^2\hat{e}_\theta.
\end{aligned}$$

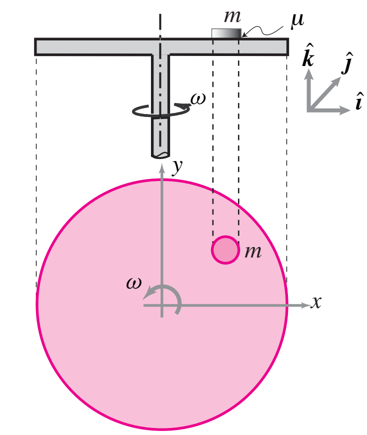
Therefore the net force on the pin is

$$\begin{aligned}
\vec{F} &= m\vec{a} \\
&= 0.1\,\text{kg}\cdot(-13.63\hat{e}_R + 21.60\hat{e}_\theta)\,\text{m/s}^2 \\
&= (-1.36\hat{e}_R + 2.16\hat{e}_\theta)\,\text{N}
\end{aligned}$$

and the magnitude of the net force is

$$F = |\vec{F}| = \sqrt{(1.36\,\text{N})^2 + (2.16\,\text{N})^2} = 2.55\,\text{N}.$$

$$\boxed{F = 2.55\,\text{N}}$$



Filename:sfig11-1-puckontable

Figure 16.15:

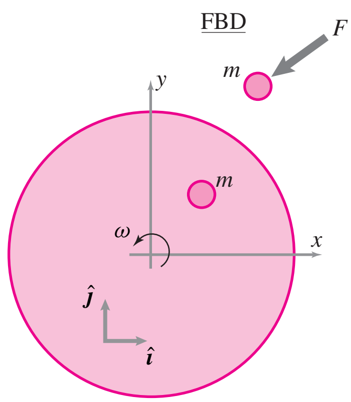


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Figure 16.16: A partial free-body diagram of the puck. For linear momentum balance we need to consider only the forces acting in the plane of motion.

SAMPLE 16.7 A puck sliding on a rough rotating table. A horizontal turntable rotates with constant angular speed $\omega = 100\,\text{rpm}$. A puck of mass $m = 0.1\,\text{kg}$ gently placed on the rotating turntable. The puck begins to slide. The coefficient of friction between the puck and the turntable is 0.25. Find the equation of motion of the puck using

1. A fixed reference frame with cartesian coordinates
2. A rotating reference frame with cartesian coordinates

Solution

1. **Equation of motion using a fixed reference frame:** The puck has two DOF on the turntable. So, we will need two configuration variables, say x and y, and we will have to find equation of motion for each variable.

   Let us use a fixed cartesian coordinate system with the origin at the center of the turntable. Let $\vec{r}_P = x\hat{\imath} + y\hat{\jmath}$ be the position of the puck at some instant t, so that its velocity is $\vec{v}_P = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$ and acceleration is $\vec{a}_P = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$.

   The free-body diagram of the puck should show three forces — the force of gravity (in $-\hat{k}$ direction), the normal reaction of the turntable (in $\hat{k}$ direction) and the friction force $\vec{F}$. Since, there is no motion in the vertical direction, we know that $N = mg$ and that $F = |\vec{F}| = \mu N = \mu mg$. But, what is the direction of the friction force? Well, we know that it acts in the opposite direction of the *relative slip*, so that

$$\vec{F} = -\mu N \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|}.$$

   So, we need to find $\vec{v}_{\text{rel}}$. Now $\vec{v}_{\text{rel}}$ is the velocity of the puck relative to the turntable, or more precisely, relative to the point on the turntable just underneath the puck. Let us denote that point by P'. Clearly, P' goes in circles with constant speed, so that its velocity is

$$\vec{v}_{P'} = \vec{\omega} \times \vec{r}_{P'} = \dot{\theta}\hat{k} \times (x\hat{\imath} + y\hat{\jmath}) = \dot{\theta}x\hat{\jmath} - \dot{\theta}y\hat{\imath}.$$

   Therefore, the relative velocity, $\vec{v}_{\text{rel}}$ is

$$\vec{v}_{\text{rel}} = \vec{v}_P - \vec{v}_{P'} = (\dot{x} + \dot{\theta}y)\hat{\imath} + (\dot{y} - \dot{\theta}x)\hat{\jmath}$$

   Now the linear momentum balance for the puck in the xy plane gives

$$\begin{aligned}
-\mu \cancel{m} g \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|} &= \cancel{m}(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \\
\Rightarrow \quad \ddot{x} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|}(\vec{v}_{\text{rel}} \cdot \hat{\imath}) = -\frac{\mu g(\dot{x} + \dot{\theta}y)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}} \\
\text{and} \quad \ddot{y} &= \frac{-\mu g}{|\vec{v}_{\text{rel}}|}(\vec{v}_{\text{rel}} \cdot \hat{\jmath}) = -\frac{\mu g(\dot{y} - \dot{\theta}x)}{\sqrt{(\dot{x} + \dot{\theta}y)^2 + (\dot{y} - \dot{\theta}x)^2}}.
\end{aligned}$$

   These are coupled nonlinear ODEs that represent the equations of motion of the puck.

$$\ddot{x} = -\frac{\mu g(\dot{x}+\dot{\theta}y)}{\sqrt{(\dot{x}+\dot{\theta}y)^2+(\dot{y}-\dot{\theta}x)^2}}, \quad \ddot{y} = -\frac{\mu g(\dot{y}-\dot{\theta}x)}{\sqrt{(\dot{x}+\dot{\theta}y)^2+(\dot{y}-\dot{\theta}x)^2}}$$

   Note that these equations are valid only as long as there is relative slip between the puck and the turntable. If the puck stops sliding due to friction, it simply goes in circles with the turntable and, therefore, its equations of motion then are $\ddot{x} = -\dot{\theta}^2 x$, $\ddot{y} = -\dot{\theta}^2 y$.

2. **Equation of motion using a rotating reference frame:** Now we derive the equations of motion using a rotating reference frame, $\mathcal{B}$, with (x', y') coordinate axes, fixed to the rotating turntable. Let the position of the puck in the rotating frame be $\vec{r}_{P/O'} = x'\hat{\imath}' + y'\hat{\jmath}'$. Note that the velocity of the puck in the rotating reference frame is $\vec{v}_{P/\mathcal{B}} = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'$, and acceleration is $\vec{a}_{P/\mathcal{B}} = \ddot{x}'\hat{\imath}' + \ddot{y}\hat{\jmath}'$.

   Now, from the linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the puck, we get

   $$-\mu \cancel{m} g \frac{\vec{v}_{\text{rel}}}{|\vec{v}_{\text{rel}}|} = \cancel{m}(\vec{a}_{P'} + \vec{a}_{P/\mathcal{B}} + 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}})$$

   where we have used the three term acceleration formula for $\vec{a}_P$. Here,

   $$\begin{aligned} \vec{a}_{P'} &= -\dot{\theta}^2(x'\hat{\imath}' + y'\hat{\jmath}') \\ \vec{a}_{P/\mathcal{B}} &= \ddot{x}'\hat{\imath}' + \ddot{y}'\hat{\jmath}' \\ 2\vec{\omega}_{\mathcal{B}} \times \vec{v}_{P/\mathcal{B}} &= -2\dot{\theta}\dot{y}'\hat{\imath}' + 2\dot{\theta}\dot{x}'\hat{\jmath}' \end{aligned}$$

   Note that the point P', coincident with P and fixed on the turntable, is stationary with respect to the rotating frame. Therefore, the relative velocity of P as observed in the rotating frame is $\vec{v}_{\text{rel}} = \vec{v}_{P/\mathcal{B}} = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'$. Substituting these terms in the LMB equation above, we have

   $$-\mu g \frac{\dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} = (-\dot{\theta}^2 x' + \ddot{x}' - 2\dot{\theta}\dot{y}')\hat{\imath}' + (-\dot{\theta}^2 y' + \ddot{y}' + 2\dot{\theta}\dot{x}')\hat{\jmath}'$$

   Dotting this equation with $\hat{\imath}'$ and $\hat{\jmath}'$, respectively, we get

   $$\begin{aligned} \ddot{x}' &= \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} + 2\dot{\theta}\,\dot{y}' \\ \ddot{y}' &= \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2 + \dot{y}'^2}} - 2\dot{\theta}\,\dot{x}' \end{aligned}$$

   These are the required equations of motion for the puck in the rotating frame.

   $$\ddot{x}' = \dot{\theta}^2 x' - \frac{\mu g \dot{x}'}{\sqrt{\dot{x}'^2+\dot{y}'^2}} + 2\dot{\theta}\,\dot{y}', \quad \ddot{y}' = \dot{\theta}^2 y' - \frac{\mu g \dot{y}'}{\sqrt{\dot{x}'^2+\dot{y}'^2}} - 2\dot{\theta}\,\dot{x}'$$

   Once we find a solution $x'(t)$ and $y'(t)$ of these equations, we can find the solution in the fixed frame by transforming (x', y') to (x, y) through

   $$\left\{ \begin{array}{c} x(t) \\ y(t) \end{array} \right\} = \left[\begin{array}{cc} \cos(\dot{\theta}t) & -\sin(\dot{\theta}t) \\ \sin(\dot{\theta}t) & \cos(\dot{\theta}t) \end{array} \right] \left\{ \begin{array}{c} x'(t) \\ y'(t) \end{array} \right\}$$

   Also, note that when the solution of the equations of motion in the rotating reference frame brings the puck to halt, the puck stops with respect to the rotating turntable. To an observer in the fixed frame, the puck will be going in circles with a constant $\dot{\theta}$.

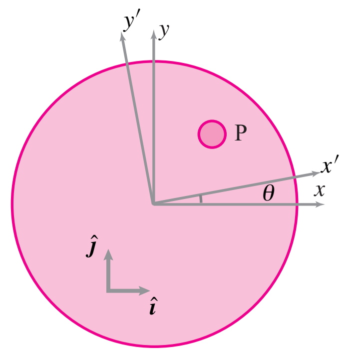

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Figure 16.17: Axes (x', y') represent the rotating frame $\mathcal{B}$ fixed to the rotating turntable; ie (x', y') rotate with angular velocity $\vec{\omega}_{\mathcal{B}} = \dot{\theta}\hat{k}$.

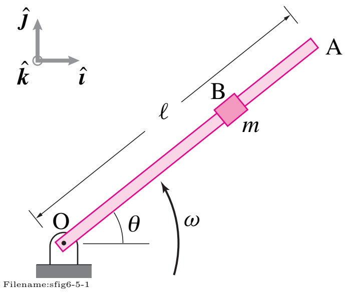

Figure 16.18: A collar slides on a rough bar and finally shoots off the end of the bar as the bar rotates with constant angular speed.

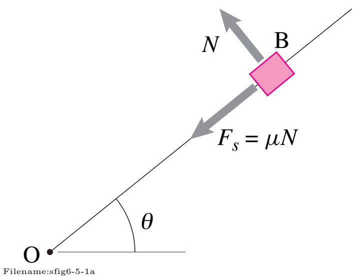

Figure 16.19: Free-Body Diagram of the collar. The only forces on the collar are the interaction forces of the bar, which are the normal force N and the friction force $F_s = \mu N$.

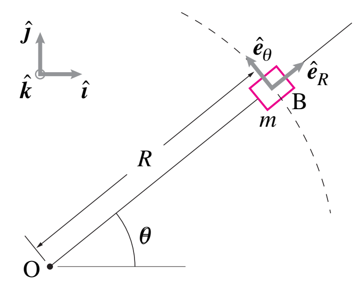

Figure 16.20: Geometry of the collar position at an arbitrary time during its slide on the rod.

SAMPLE 16.8 A collar sliding on a rough rod. A collar of mass $m = 0.5$ lb slides on a massless rigid rod OA of length $\ell = 8$ ft. The rod rotates counterclockwise with a constant angular speed $\dot{\theta} = 5$ rad/s. The coefficient of friction between the rod and the collar is $\mu = 0.3$. At time $t = 0$ s, the bar is horizontal and the collar is at rest at 1 ft from the center of rotation O. Ignore gravity.

1. How does the position of the collar change with time (*i.e.,* what is the equation of motion of the collar)?
2. Plot the path of the collar starting from $t = 0$ s till the collar shoots off the end of the bar.
3. How long does it take for the collar to leave the bar?

Solution

1. First, we draw a Free-Body Diagram of the the collar at a general position (R, θ). The FBD is shown in Fig. 16.19 and the geometry of the position vector and basis vectors is shown in Fig. 16.20. In the Free-Body Diagram there are only two forces acting on the collar (forces exerted by the bar) — the normal force $\vec{N} = N\hat{e}_\theta$ acting normal to the rod and the force of friction $\vec{F}_s = -\mu N \hat{e}_R$ acting along the rod. Now, we can write the linear momentum balance for the collar:

$$\begin{aligned} \sum \vec{F} &= m\vec{a} \\ -\mu N \hat{e}_R + N\hat{e}_\theta &= m[(\ddot{R} - R\dot{\theta}^2)\,\hat{e}_R + (2\dot{R}\dot{\theta} + R\underbrace{\ddot{\theta}}_{0})\,\hat{e}_\theta \end{aligned} \qquad (16.11)$$

Note that $\ddot{\theta} = 0$ because the rod is rotating at a constant rate. Now dotting both sides of eqn. (16.11) with $\hat{e}_R$ and $\hat{e}_\theta$ we get

$$\begin{aligned} [\text{Eqn. (16.11)}] \cdot \hat{e}_R \quad \Rightarrow \quad -\mu N &= m(\ddot{R} - R\dot{\theta}^2) \\ \text{or} \quad \ddot{R} - R\dot{\theta}^2 &= -\frac{\mu N}{m} \\ [\text{Eqn. (16.11)}] \cdot \hat{e}_\theta \quad \Rightarrow \quad N &= 2m\dot{R}\dot{\theta}. \end{aligned}$$

Eliminating N from the last two equations we get

$$\ddot{R} + 2\mu\dot{\theta}\,\dot{R} - \dot{\theta}^2\,R = 0.$$

Since $\dot{\theta} = \omega$ is constant, the above equation is of the form

$$\ddot{R} + C\,\dot{R} - \omega^2\,R = 0 \qquad (16.12)$$

where $C = 2\mu\omega$ and $\omega = \dot{\theta}$.

Note that we can derive eqn. (16.12) from eqn. (16.11) in a single step by taking a dot product of eqn. (16.11) with $\hat{e}_R + \mu\hat{e}_\theta$. This dot product is motivated by looking at the left hand side of eqn. (16.11). The net reaction force is $N(-\mu\hat{e}_R + \hat{e}_\theta)$ acting in the direction $(-\mu\hat{e}_R + \hat{e}_\theta)$. If we dot it with a vector normal to its direction, we get rid of this unknown force. A vector normal to $(-\mu\hat{e}_R + \hat{e}_\theta)$ is $\hat{e}_R + \mu\hat{e}_\theta$ and hence we can use it to find the component of the vector equation, eqn. (16.11), normal to the unknown reaction force:

$$\begin{aligned} [\text{Eqn. (16.11)}] \cdot (\hat{e}_R + \mu\hat{e}_\theta) \quad \Rightarrow \quad 0 &= m[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + 2\dot{R}\dot{\theta}\hat{e}_\theta] \cdot (\hat{e}_R + \mu\hat{e}_\theta) \\ \Rightarrow \quad 0 &= \ddot{R} + 2\mu\dot{\theta}\,\dot{R} - \dot{\theta}^2\,R \end{aligned}$$

which is the same equation as eqn. (16.12).

Solution of equation (16.12): The characteristic equation associated with Eqn. (16.12) (time to pull out your math books and see the solution of ODEs) is

$$\begin{aligned}\lambda^2 + C\lambda - \omega^2 &= 0 \\ \Rightarrow \qquad \lambda &= \frac{-C \pm \sqrt{C^2 + 4\omega^2}}{2} \\ &= \omega(-\mu \pm \sqrt{\mu^2 + 1}).\end{aligned}$$

Therefore, the solution of Eqn. (16.12) is

$$\begin{aligned}R(t) &= Ae^{\lambda_1 t} + Be^{\lambda_2 t} \\ &= Ae^{(-\mu+\sqrt{\mu^2+1})\omega t} + Be^{(-\mu-\sqrt{\mu^2+1})\omega t}.\end{aligned}$$

Substituting the given initial conditions: $R(0) = 1$ ft and $\dot{R}(0) = 0$ we get

$$R(t) = \frac{1\,\text{ft}}{2}\left[e^{(-\mu+\sqrt{\mu^2+1})\omega t} + e^{(-\mu-\sqrt{\mu^2+1})\omega t}\right]. \qquad (16.13)$$

$$R(t) = \tfrac{1\,\text{ft}}{2}\left[e^{(-\mu+\sqrt{\mu^2+1})\omega t} + e^{(-\mu-\sqrt{\mu^2+1})\omega t}\right].$$

2. To draw the path of the collar we need both R and θ. SInce $\dot{\theta} = 5\,\text{rad/s} = \text{constant}$,

$$\theta = \dot{\theta}\,t = (5\,\text{rad/s})\,t.$$

ow we can take various values of t from 0 s to, say, 1 s, and calculate values of θ and R. Plotting all these values of R and θ, however, does not give us an entirely correct path of the collar, since the equation for $R(t)$ is valid only till R = length of the bar = 8 ft. We, therefore, need to find the final time t_f such that $R(t_f) = 8$ ft. Equation (16.13) is a nonlinear algebraic equation which is hard to solve for t. We can, however, solve the equation iteratively on a computer, or with some patience, even on a calculator using trial and error. One way to find t_f would be to simply plot $R(t)$ and find the intersection with $R = 8$ (see fig. 16.21) and read the corresponding value of t. Either by refining the time interval around the intersection or by interpolation, we can find t_f. Following this method, we find that $t_f = 0.74518$ s here. Now, we can plot the path of the collar by computing R and θ from $t = 0$ to $t = t_f$ and making a polar plot on a computer as follows (pseudocode).

```
tf = 0.74518                    % final value of t
t = 0:tf/100:tf;                % take 101 points in [0 tf]
R0 = 1; w = 5; mu = .3;         % initialize variables
f1 = -mu + sqrt(mu^2 +1);       % first partial exponent
f2 = -mu - sqrt(mu^2 +1);       % second partial exponent
R = 0.5*R0*(exp(f1*w*t) + exp(f2*w*t));  % calculate R
theta = w*t;                    % calculate theta
polarplot(theta,r)
```

The plot produced thus is shown in Fig. 16.22.

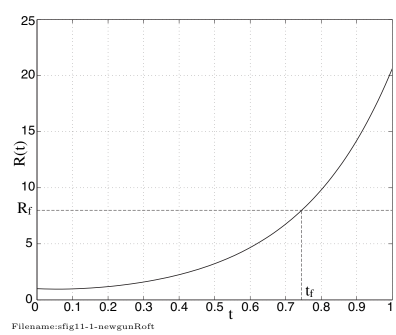

Figure 16.21: N

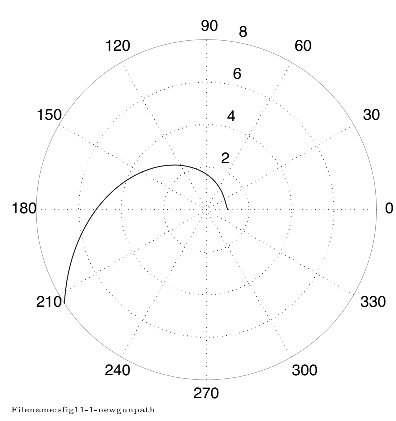

Figure 16.22: Plot of the path of the collar till it leaves the rod.

3. The time t_f computed above was

$$t_f = 0.7452\,\text{s}.$$

By plugging this value in the expression for $R(t)$ (Eqn. (16.13) we get, indeed,

$$R = 8\,\text{ft}.$$

$$t_f = 0.7452\,\text{s}$$

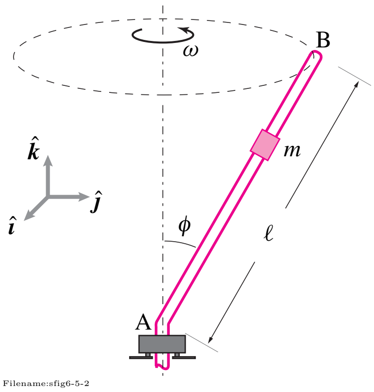

Figure 16.23: A collar of mass m slides on a massless and frictionless bar, bent at an angle ϕ with the vertical axis. The bar is rotating at a constant angular speed ω about the z-axis.

SAMPLE 16.9 A collar sliding on a rotating rod in 3-D. A massless and frictionless rod AB is rotating about the vertical axis through point A. The rod is bent at an angle ϕ from the vertical and is rotating with a constant angular speed ω about the vertical axis. A small collar of mass m slides on the rod. Assume that at time $t = 0$ the collar is released from a rest position with respect to the rod at a distance R_0 from the axis of rotation. There is no gravity.

1. Find the equation of motion for the collar (a differential equation for the position of the collar).
2. How does the distance of the collar from the vertical axis change with time?
3. For $\phi = \pi/2$, show that the solution obtained in (ii) above is the same as that obtained in Sample 16.8 for $\mu = 0$.

Solution Since the bar is bent and it rotates about the vertical axis, it sweeps a conical surface about the axis of rotation. As the collar slides on the rod, it traces a path on this surface. Let (R, θ, z) be the cylindrical coordinates of the collar at any general time t (Fig. 16.24(a)). Since the rod is frictionless and there is no gravity, the only force acting on the collar is the normal reaction from the rod. This force is shown in the free-body diagram of the collar in Fig. 16.24(b). Note that there are a lot of possible directions for a normal to the rod at the collar. In fact, any vector in the plane perpendicular to the rod at the collar is normal to the rod. So, at this point let us write

$$\vec{N} = N\hat{\boldsymbol{n}}$$

where $\hat{\boldsymbol{n}}$ is a unit normal to the rod. Thus, if $\hat{\boldsymbol{\lambda}}$ is a unit vector along the rod, then

$$\hat{\boldsymbol{\lambda}} \cdot \hat{\boldsymbol{n}} = 0 \tag{16.14}$$

where

$$\hat{\boldsymbol{\lambda}} = \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \sin\phi\,\hat{\boldsymbol{e}}_R + \cos\phi\,\hat{\boldsymbol{k}}.$$

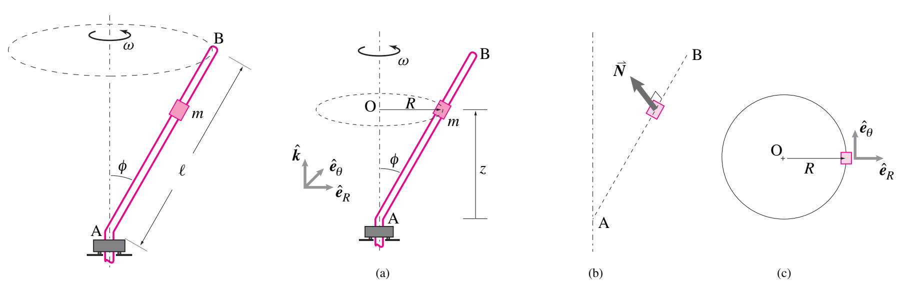

Figure 16.24: (a) Cylindrical coordinates R and z of the collar (θ is not shown) and the orientation of the cylindrical basis vectors. (b) Free-body diagram of the collar (there is no gravity). (c) Instantaneous R-$theta$ plane of the collar.

1. **Equation of Motion:** We now write the linear momentum balance for the collar:

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

where

$$\begin{aligned}\sum \vec{F} &= N\hat{\boldsymbol{n}} \\ \vec{a} &= (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (2\dot{R}\dot{\theta} + R\underbrace{\ddot{\theta}}_{0})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}\end{aligned}$$

Dotting both sides of the equation $\sum \vec{F} = m\vec{a}$ by $\hat{\boldsymbol{\lambda}}$ we get

$$\begin{aligned}N\underbrace{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\lambda}}}_{0} &= m[(\ddot{R} - R\dot{\theta}^2)\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{\lambda}}}_{\sin\phi} + 2\dot{R}\dot{\theta}\underbrace{\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{\lambda}}}_{0} + \ddot{z}\underbrace{\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{\lambda}}}_{\cos\phi}] \\ \Rightarrow \quad & (\ddot{R} - R\dot{\theta}^2)\sin\phi + \ddot{z}\cos\phi = 0.\end{aligned}$$

This expression is *an* equation of motion of the collar. However, it is only a single equation in terms of derivatives of two variables R and z. Now, from geometry

$$R = z\tan\phi \qquad \Rightarrow \qquad \ddot{z} = \ddot{R}/\tan\phi.$$

Substituting this relationship in the equation of motion we get

$$\ddot{R}(\sin\phi + \frac{\cos\phi}{\tan\phi}) - R\dot{\theta}^2 = 0.$$

Noting that $\dot{\theta} = \omega =$ a constant, we may write the above equation, with some trigonometric simplifications, as

$$\ddot{R} + (\omega\ \sin\phi)^2\, R = 0 \tag{16.15}$$

which is an equation of motion of the collar in terms of its distance R from the vertical axis.

2. **Solution of the equation of motion:** The solution of Eqn. (16.15) is given by ⑤

$$R(t) = C_1 e^{(\omega\ \sin\phi)t} + C_2 e^{-(\omega\ \sin\phi)t}$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions. Substituting the given initial conditions: $R(0) = R_0$ and $\dot{R}(0) = 0$ we get

$$C_1 = C_2 = \frac{R_0}{2}.$$

Therefore, the solution may be written as

$$R(t) = \frac{R_0}{2}\left[e^{(\omega\ \sin\phi)t} + e^{-(\omega\ \sin\phi)t}\right]. \tag{16.16}$$

3. **Special case, $\phi = \pi/2$:** Substituting $\phi = \pi/2$ in Eqn. (16.16) we get

$$R(t) = \frac{R_0}{2}\left[e^{\omega t} + e^{-\omega t}\right]$$

which is the same solution as obtained in Eqn. (16.16) in Sample 16.8 for $\mu = 0$ and $R_0 = 1$ ft.

⑤ You may find the solution by solving the corresponding *characteristic equation* (see the solution of Eqn. (16.15) in Sample 16.8, for example), or by looking up the table of "The Simplest ODEs and Their Solutions" on page 22 of the text, or by guessing a solution yourself if you have some experience with ODEs.

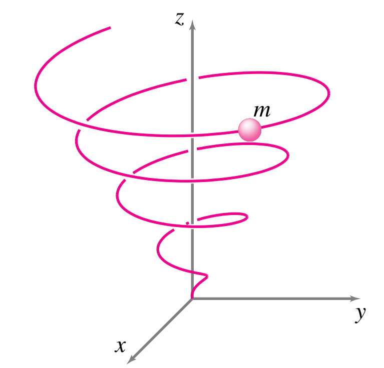

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Figure 16.25: A motor driven bead slides down a helical wire-frame described by the equations $R = R_0\theta,\ z = 2R_0\theta$.

SAMPLE 16.10 Osculating circle in 3-D. A small bead is driven down a a wire-frame bent in the shape of a conical helix by a tiny motor imbedded in the bead. The combined mass of the bead and the motor is $m = 200$ gm. The shape of the helix is given: $R = R_0\theta,\ z = 2R_0\theta$ and where $R_0 = 0.3\,\text{m}$. At the instant when $\theta = 2$ radians, the angular speed and the angular acceleration of the bead are $\dot{\theta} = 1\,\text{rad/s}$ and $\ddot{\theta} = 2\,\text{rad/s}^2$. Find

1. the *net* normal force on the bead and
2. the radius of the osculating circle at the instant given.

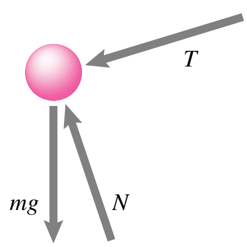

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Figure 16.26: Free body diagram of the bead. Note that the *net* normal force is $F_n = \sum \vec{F} \cdot \hat{e}_n$ where $\sum \vec{F} = N\hat{e}_n + T\hat{e}_t - mg\hat{k}$.

Solution

1. We can find the net normal force on the bead from the linear momentum balance of the bead (see the free body diagram of the bead):

$$\begin{aligned}
\sum \vec{F} &= m\vec{a} \\
N\hat{e}_n + T\hat{e}_t - mg\hat{k} &= m(a_t\hat{e}_t + a_n\hat{e}_n) \\
\Rightarrow \quad F_n &\equiv \sum \vec{F} \cdot \hat{e}_n = ma_n.
\end{aligned}$$

Thus we need to find the normal acceleration of the bead. We can write the position of the bead as

$$\begin{aligned}
\vec{r} &= \underbrace{R}_{R_0\theta}\hat{e}_R + \underbrace{z}_{2R_0\theta}\hat{k} \\
\Rightarrow \quad \vec{v} &\equiv \dot{R}\hat{e}_R + R\dot{\theta}\hat{e}_\theta + \dot{z}\hat{k} \\
&= R_0\dot{\theta}\hat{e}_R + R_0\theta\dot{\theta}\hat{e}_\theta + 2R_0\dot{\theta}\hat{k} \\
&= R_0\dot{\theta}(\hat{e}_R + \theta\hat{e}_\theta + 2\hat{k}), \\
\text{and} \quad \vec{a} &\equiv (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta + \ddot{z}\hat{k} \\
&= (R_0\ddot{\theta} - R_0\theta\dot{\theta}^2)\hat{e}_R + (2R_0\dot{\theta}^2 + R_0\theta\ddot{\theta})\hat{e}_\theta + 2R_0\ddot{\theta}\hat{k} \\
&= R_0[(\ddot{\theta} - \theta\dot{\theta}^2)\hat{e}_R + (\dot{\theta}^2 + \theta\ddot{\theta})\hat{e}_\theta + 2\ddot{\theta}\hat{k}]
\end{aligned}$$

Substituting the given numerical values for θ, $\dot{\theta}$, R_0 and $\ddot{\theta}$ in the above expressions for $\vec{v}$ and $\vec{a}$ we get the velocity and the acceleration of the bead at the moment of interest:

$$\begin{aligned}
\vec{v} &= 0.3\,\text{m/s}(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k}) \\
\vec{a} &= 1.2\,\text{m/s}^2(2\hat{e}_\theta + \hat{k}).
\end{aligned}$$

In path coordinates,

$$\vec{a} = \vec{a}_t + \vec{a}_n = a_t\hat{e}_t + a_n\hat{e}_n$$

where

$$\hat{e}_t = \frac{\vec{v}}{|\vec{v}|} \tag{16.17}$$

$$= \frac{0.3\,\text{m/s}(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k})}{0.3\sqrt{1+4+4}\ \text{m/s}} \tag{16.18}$$

$$= \frac{1}{3}(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k}). \tag{16.19}$$

Therefore,

$$\begin{aligned}
\vec{a}_t &= (\vec{a}\cdot\hat{e}_t)\hat{e}_t \\
&= (\frac{4.8+2.4}{3}\,\text{m/s}^2)\hat{e}_t \\
&= 0.8\,\text{m/s}^2(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k}) \\
\vec{a}_n &= \vec{a} - \vec{a}_t \\
&= [2.4\hat{e}_\theta + 1.2\hat{k} - 0.8\,\text{m/s}^2(\hat{e}_R + 2\hat{e}_\theta + 2\hat{k})] \\
&= (-0.8\hat{e}_R + 0.8\hat{e}_\theta - 0.4\hat{k})\,\text{m/s}^2 \\
a_n &= |\vec{a}_n| = 1.2\,\text{m/s}^2 \\
\hat{e}_n &= \frac{\vec{a}_n}{a_n} \\
&= -\frac{2}{3}\hat{e}_R + \frac{2}{3}\hat{e}_\theta - \frac{1}{3}\hat{k}.
\end{aligned}$$

Hence, the net normal force on the bead

$$\begin{aligned}
\vec{F}_n &= m a_n \hat{e}_n \\
&= 0.2\,\text{kg}\cdot 1.2\,\text{m/s}^2\cdot\frac{1}{3}(-2\hat{e}_R + 2\hat{e}_\theta - \hat{k}) \\
&= (-0.16\hat{e}_R + 0.16\hat{e}_\theta - 0.08\hat{k})\,\text{N}.
\end{aligned}$$

$$\boxed{\vec{F}_n = 0.08\,\text{N}(-2\hat{e}_R + 2\hat{e}_\theta + \hat{k})}$$

2. For calculating the radius ρ of the osculating circle, we note that

$$\begin{aligned}
a_n &= \frac{v^2}{\rho} \\
\Rightarrow \quad \rho &= \frac{v^2}{a_n} = \frac{(0.9\,\text{m/s})^2}{1.2\,\text{m/s}^2} = 0.675\,\text{m}.
\end{aligned}$$

$$\boxed{\rho = 0.675\,\text{m}}$$

16.2 Mechanics of one-degree-of-freedom 2-D mechanisms

A one-degree-of-freedom mechanism is a collection of parts linked together so that they can move in only one way①. The word "freedom" has to be taken lightly here because in practice even the one "freedom" is often controlled or restricted. Frankly, most machine designers don't trust the laws of mechanics to enforce motions that they want. Instead they choose kinematic restrictions that enforce the desired motions and then use a motor, a computer controlled actuator or big flywheel to keep that motion moving at a prescribed rate.

①No physical system can move in only one way. The idea that a machine has one degree of freedom is an idealization that takes the rigid-object description of the parts and the ideal nature of the connections literally. In fact, all parts deform at least a little, and no connections are so precise as to be exact geometric constraints. It may be reasonable to respect the standard rigid-object idealizations and consider a machine as one-degree-of-freedom for basic analysis, as we do in this section. When, for example, trying to figure out why a machine vibrates in an undesirable way it may also be reasonable to relax some of the assumptions we make here and consider more degrees of freedom.

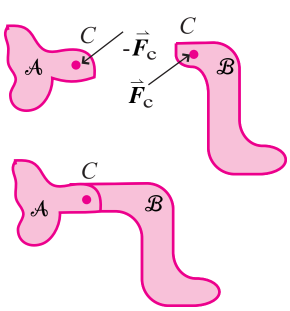
We consider here machines that can move in just one way, whether or not that one motion is free. So in this sense, "one"-degree-of-freedom machines include machines with no freedom at all, just so long as they move in only one way.

Some familiar examples of one-degree-of-freedom mechanisms are a 1-D spring and mass, a pendulum, a slider-crank, a grounded 4-bar linkage, and a gear train.

Most ideal constraints are workless constraints

A fruitful equation for studying one-degree-of-freedom mechanisms is power balance, or for conservative systems, energy balance. The reason these equations are so useful is because most ideal connections are workless. That is:

> the net work of the interaction forces (and moments) of a pair of parts that are connected with the standard ideal connections is zero. This includes welds, frictionless hinges, frictionless sliding contact, rolling contact, or parts connected by a massless inextensible link.



Filename:tfigure-hingeisworkless

Figure 16.27: A frictionless hinge is a workless constraint. The *net* fork of the interaction force on the two contacting bodies is zero.

Example: A frictionless hinge is a workless constraint

Body $\mathcal{A}$ is connected to body $\mathcal{B}$ by a frictionless hinge at C (see fig. 16.27). The force on body $\mathcal{B}$ at C from $\mathcal{A}$ is $\vec{F}_C$ and the force on $\mathcal{A}$ from body $\mathcal{B}$ at C is $-\vec{F}_C$. The power of the interaction force on body B is $P_{\mathcal{B}\mathrm{on}\mathcal{A}} = \vec{F}_C \cdot \vec{v}_C$. This power contributes to the increase in the kinetic energy of $\mathcal{B}$. The power of the interaction force on $\mathcal{A}$ is $P_{\dashv\mathrm{on}\mathcal{B}} = -\vec{F}_C \cdot \vec{v}_C = -P_{\mathcal{B}}$. So the contribution to the increase in the kinetic energy of body $\mathcal{A}$ is minus the contribution to body $\mathcal{B}$ and the net power on the system of two bodies is zero. Writing this out,

$$
\begin{aligned}
\overbrace{P_{\text{total}}}^{\boxed{\text{The net power of the pair of interaction forces on the pair of bodies}}} &= P_{\mathcal{B}\text{on}\mathcal{A}} + P_{\mathcal{A}\text{on}\mathcal{B}} \\
&= \vec{F}_C \cdot \vec{v}_{\mathrm{C}} + (-\vec{F}_C) \cdot \vec{v}_{\mathrm{C}} \\
&= (\vec{F}_C - \vec{F}_C) \cdot \vec{v}_{\mathrm{C}} \\
&= 0.
\end{aligned}
$$

Basically the same situation holds for all the standard ideal connections as explained in the box on page 931.

If one of two interacting bodies is known to be stationary, like the ground, then the work of the constraint forces is zero on both of the bodies. Thus the work of the hinge force on a pendulum, and the ground reaction forces on a frictionlessly sliding body or the ground force on a perfectly rolling body is zero. But be careful with the words “workless constraint forces”, however.

> The workless constraint connecting moving bodies $\mathcal{A}$ and $\mathcal{B}$ is likely to do positive work on one of the bodies and negative work on the other.

It is just the net work on the two bodies which is zero.

Energy method: single degree of freedom systems

Although linear and angular momentum balance apply to a single degree of freedom system and all of its parts, often one finds what one wants with a single scalar equation, namely energy or power balance.

Imagine a complex machine that only has one degree of freedom, meaning the position of the whole machine is determined by a single configuration variable, call it q. Further assume that the machine has no motion when

16.2 Ideal constraints and workless constraints

All of the ideal constraints we consider are interactions between two bodies $\mathcal{A}$ and $\mathcal{B}$. One of these could be the ground. Let’s take the interaction force $\vec{F}$ and moment $\vec{M}$ to be the force and moment of $\mathcal{A}$ on $\mathcal{B}$. The point of interaction is A on $\mathcal{A}$ and B on $\mathcal{B}$. By the principle of action and reaction, the net power of the interaction force on the two bodies is

$$
\begin{aligned}
P &= \vec{F} \cdot \vec{v}_{\mathrm{B}} + \vec{M} \cdot \vec{\omega}_{\mathcal{B}} + (-\vec{F}) \cdot \vec{v}_{\mathrm{A}} + (-\vec{M}) \cdot \vec{\omega}_{\mathcal{A}} \\
&= \vec{F} \cdot \vec{v}_{\mathrm{B/A}} + \vec{M} \cdot \vec{\omega}_{\mathcal{B}/\mathcal{A}}.
\end{aligned}
$$

All of our ideal constraints are designed to exactly make these dot products zero. The ideal hinge is considered in the text. Another example is perfect rolling. In that case the interaction moment is assumed to be zero. The no-slip condition means that $\vec{v}_{\mathrm{B/A}}$. On the other hand for frictionless sliding there $\vec{v}_{\mathrm{B/A}}$ can have a component tangent to the surfaces. But that is exactly the direction where the friction force is assumed to be zero.

And so it is for all of the ideal “workless” constraints.

Examples of non-workless constraints, that is, interactions that contribute to the energy equations are: sliding with non-zero friction, joints with non-zero friction torques, joints with motors, or interactions mediated by springs, dampers or actuators.

$\dot{q} = 0$. The variable q could be, for example, the angle of one of the linked-together machine parts. Also, assume that the machine has no dissipative parts: no friction, no collisions, no inelastic deformation. Because q characterizes the position of all of the parts of the system we can, in principal, calculate the potential energy of the system as a function of q,

$$E_P = E_P(q).$$

We find this function by adding up the potential energies of all the springs in the machine and the gravitational potential energies of the parts. Similarly we can write the system's kinetic energy in terms of q and its rate of change $\dot{q}$. Because at any configuration the velocity of every point in the system is proportional to $\dot{q}$ we can write the kinetic energy as:

$$E_K = M(q)\dot{q}^2/2$$

where $M(q)$ is a function that one can determine by calculating the machine's total kinetic energy in terms of q and $\dot{q}$ and then factoring $\dot{q}^2$ out of the resulting expression.

Now, if we accept the equation of mechanical energy conservation we have

$$\begin{aligned}
\text{constant} &= E_T \quad \text{conservation of energy,}\\
\Rightarrow \quad 0 &= \frac{d}{dt}E_T \quad \text{taking one time derivative,}\\
&= \frac{d}{dt}[E_P + E_K] \quad \text{total energy is potential plus kinetic}\\
&= \frac{d}{dt}[E_P(q) + \frac{1}{2}M(q)\dot{q}^2] \quad \text{substituting from paragraphs above}\\
\Rightarrow \quad 0 &= \frac{d}{dq}[E_P(q)]\,\dot{q} + \frac{1}{2}\frac{d}{dq}[M(q)]\dot{q}\dot{q}^2 + M(q)\dot{q}\ddot{q}\\
0 &= \frac{d}{dq}[E_P(q)] + \left(\frac{1}{2}\frac{d}{dq}[M(q)]\right)\dot{q}^2 + M(q)\,\ddot{q} \quad \text{cancelling } \dot{q}\\
0 &= f_1(q) + f_2(q)\,\dot{q}^2 + f_3(q)\,\ddot{q} \qquad (16.20)\\
&\text{with } f_1(q) \equiv \frac{d}{dq}[E_P(q)],\\
&f_2(q) \equiv \frac{1}{2}\frac{d}{dq}[M(q)], \quad \text{and}\\
&f_3(q) \equiv M(q).
\end{aligned}$$

The cancellation of $\dot{q}$ above lacks mathematical rigor, but doesn't cause problems②.The equation of motion is complicated because when we take the time derivative of a function of $M(q)$ and $E_P(q)$ we have to use the chain rule. Also, because we have products of terms, we had to use the product rule. Eqn. 16.20 is the general equation of motion of a conservative one-degree-of-freedom system. It is really just a special case of the equation of motion for one-degree-of-freedom systems found from power balance. Rather than

②The cancellation of the factor $\dot{q}$ from equation 16.20 depends on $\dot{q}$ being other than zero. While moving, $\dot{q}$ is not zero. Strictly we cannot cancel the $\dot{q}$ term from the equation at the instants when $\dot{q} = 0$. However, to say that a differential equation is true except for certain instants in time is, in practice to say that it is always true, at least if we make reasonable assumptions about the smoothness of the motions.

memorizing eqn. (16.20) it is probably best to look at its derivation as an algorithm to be reproduced on a problem by problem basis.

Example: Spring and mass

Although the motion of a spring and mass system can be found easily enough from linear momentum balance, it is also a good example for energy balance (see fig. 16.28). Using conservation of energy for the spring and mass system:

$$\begin{aligned}
E_{\mathrm{T}} &= constant \\
0 &= \frac{d}{dt}E_{\mathrm{T}} \\
&= \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} \\
&= \frac{d}{dt}(mv^2/2) + \frac{d}{dt}(kx^2/2) \\
&= mv\dot{v} + kx\dot{x} \\
v = \dot{x} \quad \Rightarrow \quad 0 &= m\ddot{x} + kx.
\end{aligned}$$

Similarly power balance could have been used to get the same result, looking at just the mass

$$\begin{aligned}
P &= \frac{d}{dt}E_{\mathrm{K}} \\
(-kx)(\dot{x}) &= \frac{d}{dt}(mv^2/2) = mv\dot{v} \\
\Rightarrow \quad 0 &= kx + m\ddot{x}
\end{aligned}$$

as before.

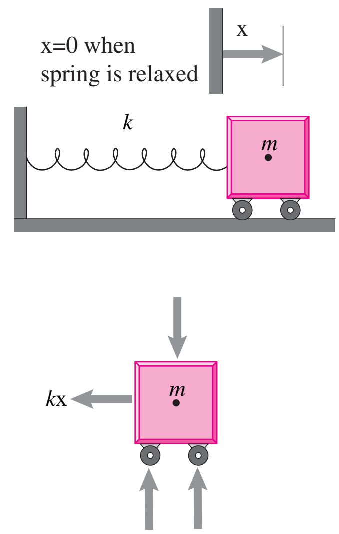


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Figure 16.28: The familiar one degree of freedom spring and mass system.

Example: Pendulum

Consider a rigid body with mass m and moment of inertia I^o about a hinge which is a distance ℓ from the center-of-mass (see fig. 16.29). The familiar simple pendulum is another single degree of freedom system for which the equation of motion can be found from conservation of energy.

$$\begin{aligned}
E_{\mathrm{T}} &= constant \\
\rightarrow 0 &= \frac{d}{dt}E_{\mathrm{T}} = \dot{E}_{\mathrm{K}} + \dot{E}_{\mathrm{P}} \\
&= \frac{d}{dt}(I^o\omega^2/2) + \frac{d}{dt}(-mg\ell\cos\theta) \\
&= I^o\omega\dot{\omega} + -mg\ell(-\sin\theta)\dot{\theta} \\
\omega = \dot{\theta} \quad \Rightarrow \quad 0 &= \ddot{\theta} + \frac{mg\ell}{I^o}\sin\theta.
\end{aligned}$$

the pendulum equation that we have derived before by this and other means (angular momentum balance about point o).

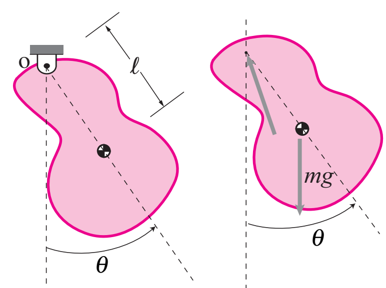

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Figure 16.29: A rigid body suspended from a frictionless hinge is an energy conserving one-degree-of-freedom mechanism.

The above examples are old friends which are handled easily with other techniques. Here is a problem which is much more difficult without the energy method.

Example: Three bars act like a simple pendulum

Assume all three bars in the structure shown in fig. 16.30are of equal length ℓ and have mass m uniformly distributed along their length. It is intuitively obvious that this device swings back and forth something like a simple pendulum. But how can we get the laws of mechanics to tell us this? One approach, which will work in the end, is to draw free body diagrams of all the parts, write linear and angular momentum balance for each, and then add and subtract equations to eliminate the unknown constraint forces at the various hinges.

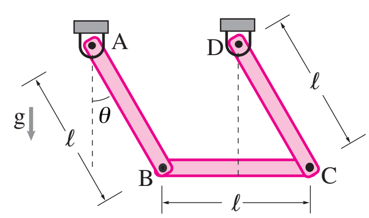

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Figure 16.30: Three identical uniform bars are pinned with frictionless hinges and swing, obviously, something like a simple pendulum.

The more direct approach is to write the energy equation, adding up the potential and kinetic energies of the parts, all evaluated in terms of the single configuration variable θ. Taking the potential energy to be zero at $\theta = \pi/2$ (when all centers of mass are at hinge height) we have

$$\begin{aligned}
E_\mathrm{T} &= constant \\
0 &= \frac{d}{dt} E_\mathrm{T} = \dot{E}_\mathrm{P} + \dot{E}_\mathrm{K} \\
&= \frac{d}{dt}\left((I^o\omega^2/2) + (I^o\omega^2/2) + (m(\ell\omega)^2/2) \right) \\
&\quad + \frac{d}{dt}\left((-gm(\ell/2)\cos\theta \right. \\
&\quad \left. -gm(\ell/2)\cos\theta) - gm\ell\cos\theta \right) \\
I^o = m\ell^2/3 \quad \Rightarrow \quad 0 &= \frac{d}{dt}\left(5m\ell^2\omega^2/6 \right) + \frac{d}{dt}\left(-2gm\ell\cos\theta \right) \\
&= (5m\ell^2\omega\dot{\omega}/3 + 2gm\ell\omega\sin\theta) \\
\Rightarrow \quad 0 &= 5\ell\dot{\omega}/3 + 2g\sin\theta \\
\Rightarrow \quad 0 &= \ddot{\theta} + \frac{6g}{5\ell}\sin\theta
\end{aligned}$$

which is the same governing equation as for a point-mass pendulum with length $5\ell/6$. This is just half way between the following two cases. If the side links had no mass the equation would have been the same as for a point mass pendulum with length ℓ

$$0 = \ddot{\theta} + \frac{g}{\ell}\sin\theta$$

and if the bottom link had no mass the equation would be the same as a stick hanging from one end which goes back and forth like a point mass pendulum with length $2\ell/3$ according to

$$0 = \ddot{\theta} + \frac{3g}{2\ell}\sin\theta.$$

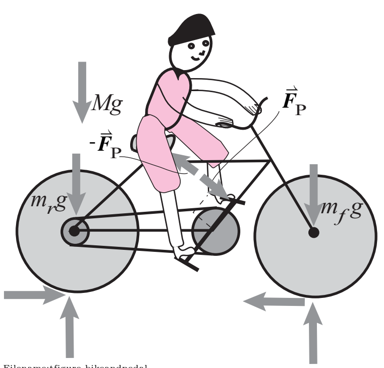

Figure 16.31: A person rides a bike. The pedaling leg is idealized as a pair of equal and opposite forces acting on the seat and pedal.

Example: One on the rim is like two on the frame.

A bicycle transmission is such that the speed of the bike relative to the ground is n times the speed pedal relative to the frame :

$$v_\mathrm{bike} = n v_\mathrm{pedal/bike}$$

Assume the kinetic energy of the relative motion of a rider's legs can be neglected, as can be the weight of the rider's leg. At the moment in question the velocity of the pedal is parallel to the direction from the seat to the leg. Thus the free body diagram of the bike/person system, leaving out the pedaling leg is as shown in fig. 16.31. Let's assume the bike and rider have mass M and that the wheels have mass m_r and m_f concentrated on the rim (the hubs are considered part of the frame and the spokes are neglected). Neglecting air resistance *etc.*the power balance equation is:

$$P = \dot{E}_\mathrm{K} \tag{16.21}$$

Let's do some side calculations for evaluating the terms in eqn. (16.21). First, the only forces that do work on the system as drawn are the force on the pedal and the force on the seat.

$$\begin{aligned}
P &= -\vec{F}_\mathrm{P}\cdot\vec{v}_\mathrm{seat} + \vec{F}_\mathrm{P}\cdot\vec{v}_\mathrm{pedal} \\
&= -\vec{F}_\mathrm{P}\cdot\vec{v}_\mathrm{seat} + \vec{F}_\mathrm{P}\cdot\left(\vec{v}_\mathrm{bike} + \vec{v}_\mathrm{pedal/bike}\right) \\
&= \underbrace{-\vec{F}_\mathrm{P}\cdot\vec{v}_\mathrm{seat} + \vec{F}_\mathrm{P}\cdot\vec{v}_\mathrm{bike}}_{0} + \underbrace{\vec{F}_\mathrm{P}\cdot\vec{v}_\mathrm{pedal/bike}}_{F_P v_\mathrm{pedal/bike}} \\
&= F_P v_\mathrm{bike}/n
\end{aligned}$$

The net power of the leg is expressed by the compression it carries times its extension rate. The kinetic energy of the wheel comes from both its rotation and its translation. The moment of inertia of a hoop about its center is $I = mr^2$. For rolling contact $|\omega R| = v$ so, for *one* wheel:

$$\begin{aligned} E_{\mathrm{K\,wheel}} &= m v_{\mathrm{bike}}^2/2 + I\omega^2/2 \\ &= m v_{\mathrm{bike}}^2/2 + (mR^2)(v/R)^2/2 \\ &= m v_{\mathrm{bike}}^2. \end{aligned}$$

The kinetic energy of a rolling hoop is twice that of a point mass moving at the same speed. Putting these results back in to eqn. (16.21) we have

$$\begin{aligned} P &= \dot{E}_{\mathrm{K}} && (16.22) \\ F_P v_{\mathrm{bike}}/n &= \frac{d}{dt}\left(M v_{\mathrm{bike}}^2/2 + (m_r + m_f) v_{\mathrm{bike}}^2 \right) && (16.23) \\ &= \left(M\dot{v}_{\mathrm{bike}} + 2(m_r + m_f)\dot{v}_{\mathrm{bike}} \right) v_{\mathrm{bike}} && (16.24) \\ \Rightarrow \quad F_P/n &= (M + 2(m_r + m_f))\dot{v}_{\mathrm{bike}} && (16.25) \\ \Rightarrow \quad \dot{v}_{\mathrm{bike}} &= \frac{F_P}{n(M + 2(m_r + m_f))}. && (16.26) \end{aligned}$$

The bigger the pedal force, the bigger the acceleration, obviously. The higher the gear ratio, the less the acceleration; the faster gears let you pedal slower for a given bike speed, but demand more pedal force for a given acceleration. A heavier bike accelerates less. But the contribution to slowing a bike is twice as much for mass added to the rim as for mass added to the frame or body.

Some comments. n typically ranges from about 1.7 to 8 for a new 21 speed bike and is about 5 for an adult European, Indian or Chinese 1-speed. For a given speed of bicycle riding your feet go n times slower relative to your body than for walking or running at that speed. This calculation is for accelerating a bike on level ground with no wind and rolling resistance. The net speed of a bike in a bike race is not so dependent on weight, because the main enemy is wind resistance. To the extent that weight is a problem it is for steady uphill travel. In this case the mass on the rim makes the same contribution as mass on the frame.

Vibrations

The preponderance of systems where vibrations occur is not due to the fact that so many systems look like a spring connected to a mass, a simple pendulum, or a torsional oscillator. Instead there is a general class of systems which can be expected to vibrate sinusoidally near some equilibrium position. These systems are one-degree -of-freedom (one DOF) near an energy minimum.

In detail why this works out is explained in Box 16.3.

Examples of 1 DOF harmonic oscillators

In the previous section, we have shown that any non-dissipative one-degree-of-freedom system that is near a potential energy minimum can be expected to have simple harmonic motion. Besides the three examples we have given so far, namely,

- a spring and mass,

- a simple pendulum, and
- a rigid body and a torsional spring,

there are examples that are somewhat more complex, such as

- a cylinder rolling near the bottom of a valley,
- a cart rolling near the bottom of a valley, and a
- a four bar linkage swinging freely near its energy minimum.

The restriction of this theory to systems with only one-degree-of-freedom is not so bad as it seems at first sight. First of all, it turns out that simple harmonic motion is important for systems with multiple-degrees-of-freedom. We will discuss this generalization in more detail later with regard to normal modes. Secondly, one can also get a good understanding of a vibrating system with multiple-degrees-of-freedom by modeling it as if it has only one-degree-of-freedom.

Example: Cylinder rolling in a valley
Consider the uniform cylinder with radius r rolling without slip in an cylindrical 'ideal' valley of radius R.

16.3 One degree of freedom systems near a potential energy minimum are harmonic oscillators

In order to specialize to the case of oscillations, we want to look at a one degree of freedom system near a stable equilibrium point, a potential energy minimum.

At a potential energy minimum we have, as you will recall from 'max-min' problems in calculus, that $dE_\mathrm{P}(q)/dq = 0$. To keep our notation simple, let's assume that we have defined q so that $q = 0$ at this minimum. Physically this means that q measures how far the system is from its equilibrium position. That means that if we take a Taylor series approximation of the potential energy the expression for potential energy can be expressed as follows:

$$E_\mathrm{P} \approx \text{const} + \underbrace{\frac{dE_\mathrm{P}}{dq}}_{0} \cdot q + \frac{1}{2}\underbrace{\frac{d^2E_\mathrm{P}}{dq^2}}_{K_\text{equiv}} \cdot q^2 + \dots$$

$$\Rightarrow \quad \frac{dE_\mathrm{P}}{dq} \approx K_\text{equiv} \cdot q \tag{16.27}$$

Applying this result to equation(16.20) we get:

$$0 = K_\text{equiv} q + \frac{1}{2}\frac{d}{dq}[M(q)]\dot{q}^2 + M(q)\ddot{q}. \tag{16.28}$$

We now write $M(q)$ in terms of its Taylor series. We have

$$M(q) = M(0) + dM/dq|_0 \cdot q + \dots \tag{16.29}$$

and substitute this result into equation 16.28. We have not finished using our assumption that we are only going to look at motions that are close to the equilibrium position $q = 0$ where q is small. The nature of motion close to an equilibrium is that when the deflections are small, the rates and accelerations are also small. Thus, to be consistent in our approximation we should neglect any terms that involve products of $q, \dot{q}$, or $\ddot{q}$. Thus the middle term involving $\dot{q}^2$ is negligibly smaller than other terms. Similarly, using the Taylor series for $M(q)$, the last term is well approximated by $M(0)\ddot{q}$, where $M(0)$ is a constant which we will call M_equiv. Now we have for the equation of motion:

$$0 = \frac{d}{dt}E_\mathrm{T} \quad \Rightarrow \quad 0 = K_\text{equiv} q + M_\text{equiv}\ddot{q}, \tag{16.30}$$

which you should recognize as the harmonic oscillator equation. So we have found that for any energy conserving one degree of freedom system near a position of stable equilibrium, the equation governing small motions is the harmonic oscillator equation. The effective stiffness is found from the potential energy by $K_\text{equiv} = d^2E_\mathrm{P}/dq^2$ and the effective mass is the coefficient of $\dot{q}^2/2$ in the expansion for the kinetic energy E_K. The displacement of any part of the system from equilibrium will thus be given by

$$A\sin(\lambda t) + B\cos(\lambda t) \tag{16.31}$$

with $\lambda^2 = K_\text{equiv}/M_\text{equiv}$, and A and B determined by the initial conditions. So we have found that *all stable non-dissipative one-degree-of-freedom systems oscillate when disturbed slightly from equilibrium* and we have found how to calculate the frequency of vibration.

For this problem we can calculate E_K and E_P in terms of θ. Briefly,

$$\begin{aligned} E_P &= -mg(R-r)\cos\theta \\ E_K &= \frac{1}{2}\left(\frac{3}{2}mr^2\right)\left(\frac{\dot\theta(R-r)}{r}\right)^2 \\ &= \frac{3}{4}m(R-r)^2\dot\theta^2 \end{aligned}$$

So we can derive the equation of motion using the fact of constant total energy.

$$\begin{aligned} 0 &= \frac{d}{dt}(E_T) = \frac{d}{dt}(E_K + E_P) \\ &= \frac{d}{dt}\left(\underbrace{-mg(R-r)\cos\theta}_{E_P} + \underbrace{\frac{3}{4}m(R-r)^2\dot\theta^2}_{E_K}\right) \\ &= (mg(R-r)\sin\theta)\dot\theta + \frac{3}{2}(R-r)^2\dot\theta\ddot\theta \\ \Rightarrow \quad 0 &= mg(R-r)\sin\theta + \frac{3}{2}(R-r)^2 m\ddot\theta \end{aligned}$$

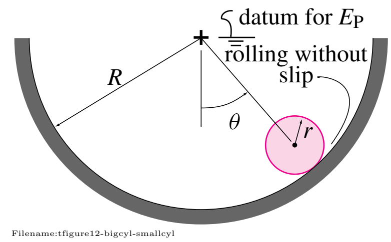

Figure 16.32: Cylinder rolling without slip in a cylinder. This gives the pendulum equation, but the effective pendulum length does not tend to R as $r \to 0$ (see text).

Now, assuming small angles, so $\theta \approx \sin\theta$, we get

$$g(R-r)\theta + \frac{3}{2}(R-r)^2\ddot\theta = 0 \qquad (16.32)$$

$$\ddot\theta + \underbrace{\left(\frac{2}{3}\frac{g}{(R-r)}\right)}_{\lambda^2}\theta = 0 \qquad (16.33)$$

This is, naturally, our old friend the harmonic oscillator equation. The period is a funny combination of terms. If $r \ll R$ it looks like a point mass pendulum with length $3R/2$, more than R. That is, the rolling effect doesn't go away and make the roller act like a point mass even when the radius goes to zero. See page 800 for the angular momentum approach to this problem.

Although, in some abstract way the energy approach always works, practically speaking it has limitations for systems where the configuration is not easily found from one configuration variable.

Example: A four bar linkage using with energy methods

Although probably not usually the best approach, energy methods can be used to find the motions of a 4-bar linkage. Take a four-bar linkage with one bar grounded. Assume the bars all have different lengths. This is a one-DOF system which can use the angle θ of one of the links as a configuration variable. But finding the potential energy as a single formula in terms of all of the links in terms of θ is more trigonometry than most of us like. And then finding the kinetic energy in terms of θ and $\dot\theta$ is close enough to impossible that people don't do it.

So, though it is true that there are functions $E_P(\theta)$ and $E_K(\theta, \dot\theta)$ and that the equations of motion could be written in terms of them, it is really not practical to do so.

How do you find the motions of a 4-bar linkage in practice? It's more tricky. One approach is to solve the kinematics by integrating kinematic differential equations, as in Sample 15.5 on page 898. Then you set up and solve the balance equations of the separate parts as in Sample 16.21 on page 958

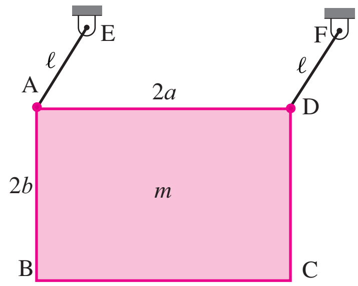

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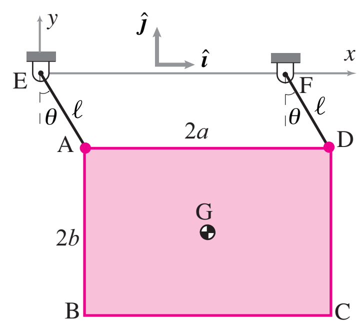
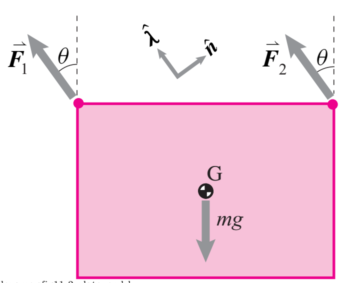
Figure 16.33:



Filename:sfig11-2-platepend-a

Figure 16.34:



Filename:sfig11-2-platepend-b

Figure 16.35:

SAMPLE 16.11 A plate pendulum. A $2a \times 2b$ rectangular plate of mass m hangs from two parallel, massless links EA and FD of length ℓ each. The links are hinged at both ends so that when the plate swings, its edges AD and BC remain horizontal at all times. The only driving force present is gravity. Find the equation of motion of the plate.

Solution The given system is a single DOF system. So, we need just one configuration variable and the equation of motion be just one scalar equation in this variable. Let us take angle θ (fig. 16.34) as our configuration variable.

The free-body diagram of the plate is shown in fig. 16.35. Note that the link forces $\vec{F}_1$ and $\vec{F}_2$ act along the links because massless links are two force bodies. Let (x, y) be the coordinates of the center-of-mass. Then the linear momentum balance for the plate gives

$$\vec{F}_1 + \vec{F}_2 - mg\hat{\jmath} = m\vec{a} = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}).$$

Now, we can eliminate both the unknown link forces from this equation by dotting the equation with $\hat{n} = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$, a unit vector normal to the links. Then, we have

$$\begin{aligned} -mg(\hat{\jmath}\cdot\hat{n}) &= m\big[\ddot{x}(\hat{\imath}\cdot\hat{n}) + \ddot{y}(\hat{\jmath}\cdot\hat{n})\big] \\ -g\sin\theta &= \ddot{x}\cos\theta + \ddot{y}\sin\theta. \end{aligned} \tag{16.34}$$

Now we need to find a relationship between x and θ, and y and θ, so that we can write $\ddot{x}$ and $\ddot{y}$ in terms of our configuration variable θ and its derivatives. From fig. 16.34, we have

$$\begin{aligned} \vec{r}_G &= \overbrace{(\ell\sin\theta + a)}^{x}\,\hat{\imath} + \overbrace{(-\ell\cos\theta - b)}^{y}\,\hat{\jmath} \\ \Rightarrow\quad \ddot{x} &= \ell(\cos\theta\cdot\ddot{\theta} - \sin\theta\cdot\dot{\theta}^2) \\ \ddot{y} &= \ell(\sin\theta\cdot\ddot{\theta} + \cos\theta\cdot\dot{\theta}^2) \end{aligned}$$

Substituting these expressions for $\ddot{x}$ and $\ddot{y}$ in eqn. (16.34) , we get

$$\begin{aligned} -g\sin\theta &= \ell\ddot{\theta} \\ \Rightarrow\quad \ddot{\theta} + \frac{g}{\ell}\sin\theta &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \tfrac{g}{\ell}\sin\theta = 0}$$

This is the equation of a simple pendulum! Well, the plate does behave just like a simple pendulum in the given mechanism. From the expressions for the x and y coordinates of the center-of-mass, we have

$$\begin{aligned} x - a &= \ell\sin\theta \\ y + b &= -\ell\cos\theta \\ \Rightarrow\quad (x-a)^2 &+ (y+b)^2 = \ell^2 \end{aligned}$$

that is, the center-of-mass follows a circle of radius ℓ centered at $(-a, b)$. Since the orientation of the plate never changes (AD and BC always remain horizontal), the plate has no angular velocity. Thus the motion of the plate is equivalent to the motion of a particle of mass m going in a circle centered at $(-a, b)$ and driven by gravity. That is the simple pendulum.

SAMPLE 16.12 Equation of motion from power balance. A slider crank mechanism is shown in fig. 16.36 where the crank is a uniform wheel of mass m and radius R anchored at the center and the connecting rod AB is a massless rod of length ℓ. The rod is driven by a piston at B with a known force $F(t) = F_0 \cos \Omega t$. There is no gravity. Find the equation of motion of the wheel.

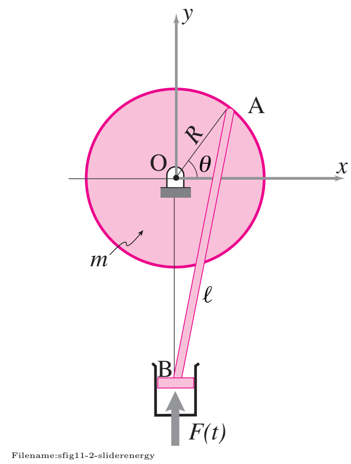


Figure 16.36:

Solution The given mechanism is a one DOF system. So, let us choose a single configuration variable, θ for specifying the configuration of the system and derive an equation that determines θ. Since the applied force is given and the point of application of the force has a simple motion (vertical), it will be easy to calculate power of this force. Also, the connecting rod is massless, so it does not enter into dynamic calculations. The wheel rotates about its center and, therefore, it is easy to calculate its kinetic energy. So, we use the power balance, $\dot{E}_K = P$, here to find the equation of motion of the wheel. Since $E_K = \frac{1}{2} I_{zz}^{cm} \dot{\theta}^2$ and $P = \vec{F} \cdot \vec{v}_B$, we have

$$I_{zz}^{cm} \ddot{\theta}\, \dot{\theta} = F(t) \hat{\jmath} \cdot v_B \hat{\jmath} = F(t) v_B \qquad (16.35)$$

Now, we need to find v_B and express it using the configuration variable θ and its derivatives. There are several ways we could find v_B. Vectorially, we could write, $\vec{v}_B \equiv v_B \hat{\jmath} = \vec{v}_A + \vec{\omega}_{AB} \times \vec{r}_{B/A}$ where $\vec{\omega}_{AB} = \dot{\phi} \hat{k}$. Dotting both sides of this equation with $\hat{\imath}$ and $\hat{\jmath}$ we can find ϕ in terms of θ and v_B in terms of θ and $\dot{\theta}$. But, for a change, let us use geometry here.

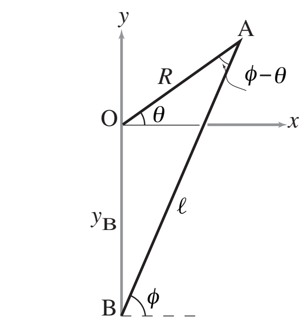


Figure 16.37:

See fig. 16.37. From triangle ABO, we have

$$\begin{aligned}
\frac{R}{\sin(90^o - \phi)} &= \frac{\ell}{\sin(90^o + \theta)} \\
\Rightarrow \quad \ell \cos \phi &= R \cos \theta \\
\Rightarrow \quad -\ell \sin \phi \cdot \dot{\phi} &= -R \sin \theta \cdot \dot{\theta} \\
\dot{\phi} &= \frac{R \sin \theta}{\ell \sin \phi} \dot{\theta}
\end{aligned}$$

Now,

$$\begin{aligned}
y_B &= R \sin \theta - \ell \sin \phi \\
\Rightarrow \quad v_B &\equiv \dot{y}_B = R \cos \theta \cdot \dot{\theta} - \ell \cos \phi \cdot \dot{\phi} \\
&= R\dot{\theta} \cos \theta - R \cos \theta \cdot \frac{R \sin \theta}{\ell \sin \phi} \dot{\theta} \\
&= R\dot{\theta} \left(\cos \theta - \frac{R \sin \theta \cos \theta}{\sqrt{\ell^2 - R^2 \cos^2 \theta}} \right)
\end{aligned}$$

Substituting this expression for v_B in power balance eqn. (16.35), we get

$$\begin{aligned}
I_{zz}^{cm} \ddot{\theta}\, \cancel{\dot{\theta}} &= F(t) \cdot R \cancel{\dot{\theta}} \left(\cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right) \\
\ddot{\theta} &= \frac{R F_0 \cos \Omega t \cdot}{\frac{1}{2} m R^2} \left(\cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right).
\end{aligned}$$

This is the required equation of motion. As is evident, it is a nonlinear ODE which requires numerical solution on a computer if we would like to plot $\theta(t)$.

$$\ddot{\theta} = \frac{2 F_0 \cos \Omega t \cdot}{m R} \left(\cos \theta - \frac{\sin 2\theta}{2\sqrt{(\ell/R)^2 - \cos^2 \theta}} \right)$$

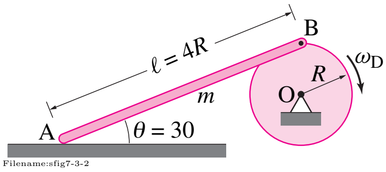


Figure 16.38: End A of bar AB is free to slide on the frictionless horizontal surface while end B is going in circles with a disk rotating at a constant rate.

SAMPLE 16.13 Instantaneous dynamics of slider crank. A uniform rigid rod AB of mass m and length $\ell = 4R$ has one of its ends pinned to the rim of a disk of radius R. The other end of the bar is free to slide on a frictionless horizontal surface. A motor, connected to the center of the disk at O, keeps the disk rotating at a constant angular speed ω_D. At the instant shown, end B of the rod is directly above the center of the disk making θ to be 30^o.

1. Find all the forces acting on the rod.
2. Is there a value of ω_D which makes end A of the rod lift off the horizontal surface when $\theta = 30^o$?

Solution The disk is rotating at constant speed. Since end B of the rod is pinned to the disk, end B is going in circles at constant rate. The motion of end B of the rod is completely prescribed. Since end A can only move horizontally (assuming it has not lifted off yet), the orientation (and hence the position of each point) of the rod is completely determined at any instant during the motion. Therefore, the rod represents a zero degree of freedom system.

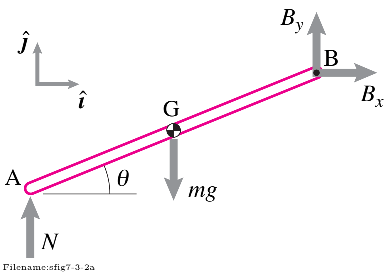


Figure 16.39: Free-body diagram of the bar.

1. **Forces on the rod:** The free-body diagram of the rod is shown in Fig. 16.39. The pin at B exerts two forces B_x and B_y while the surface in contact at A exerts only a normal force N because there is no friction. Now, we can write the momentum balance equations for the rod. The linear momentum balance ($\sum \vec{F} = m\vec{a}$) for the rod gives

$$B_x\hat{\imath} + (B_y + N - mg)\hat{\jmath} = m\vec{a}_G. \tag{16.36}$$

The angular momentum balance about the center-of-mass G ($\sum \vec{M}_{/G} = \dot{\vec{H}}_{/G}$) of the rod gives

$$\vec{r}_{A/G} \times N\hat{\jmath} + \vec{r}_{B/G} \times (B_x\hat{\imath} + B_y\hat{\jmath}) = I_{zz/G}\,\alpha_{rod}\hat{k}. \tag{16.37}$$

From these two vector equations we can get three scalar equations (the Angular Momentum Balance gives only one scalar equation in 2-D since the quantities on both sides of the equation are only in the $\hat{k}$ direction), but we have six unknowns — B_x, B_y, N, $\vec{a}_G$ (counts as two unknowns), and α_{rod}. Therefore, we need more equations. We have already used the momentum balance equations, hence, the extra equations have to come from kinematics.

$$\begin{aligned}
\vec{v}_A &= \vec{v}_B + \overbrace{\vec{\omega}_{rod} \times \vec{r}_{A/B}}^{\vec{v}_{A/B}} \\
\text{or} \qquad v_A\hat{\imath} &= \omega_D R\hat{\imath} + \omega_{rod}\hat{k} \times \ell(-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= (\omega_D R + \omega_{rod}\ell\sin\theta)\hat{\imath} - \omega_{rod}\ell\cos\theta\hat{\jmath}
\end{aligned}$$

Dotting both sides of the equation with $\hat{\jmath}$ we get

$$0 = \omega_{rod}\ell\cos\theta \qquad \Rightarrow \qquad \omega_{rod} = 0.$$

Also,

$$\begin{aligned}
\vec{a}_A &= \vec{a}_B + \overbrace{\dot{\vec{\omega}} \times \vec{r}_{A/B} + \underbrace{\vec{\omega}_{rod}}_{\vec{0}} \times (\vec{\omega}_{rod} \times \vec{r}_{A/B})}^{\vec{a}_{A/B}} \\
\text{or} \qquad a_A\hat{\imath} &= -\omega_D^2 R\hat{\jmath} + \dot{\omega}_{rod}\hat{k} \times \ell(-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= -(\omega_D^2 R + \dot{\omega}_{rod}\ell\cos\theta)\hat{\jmath} + \dot{\omega}_{rod}\ell\sin\theta\hat{\imath}.
\end{aligned}$$

Dotting both sides of this equation by $\hat{\jmath}$ we get

$$\dot{\omega}_{rod} = -\frac{\omega_D^2 R}{\ell\cos\theta}. \tag{16.38}$$

Now, we can find the acceleration of the center-of-mass:

$$
\begin{aligned}
\vec{a}_G &= \vec{a}_B + \overbrace{\dot{\vec{\omega}} \times \vec{r}_{G/B} + \underbrace{\vec{\omega}_{rod}}_{\vec{0}} \times (\vec{\omega}_{rod} \times \vec{r}_{G/B})}^{\vec{a}_{G/B}} \\
&= -\omega_D^2 R\hat{\jmath} + \dot{\omega}_{rod}\hat{k} \times \frac{1}{2}\ell(-\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= -(\omega_D^2 R + \frac{1}{2}\dot{\omega}_{rod}\ell\cos\theta)\hat{\jmath} + \frac{1}{2}\dot{\omega}_{rod}\ell\sin\theta\hat{\imath}.
\end{aligned}
$$

Substituting for $\dot{\omega}_{rod}$ from eqn. (16.38) and 30^o for θ above, we obtain

$$
\vec{a}_G = -\frac{1}{2}\omega_D^2 R(\frac{1}{\sqrt{3}}\hat{\imath} + \hat{\jmath}).
$$

Substituting this expression for $\vec{a}_G$ in eqn. (16.36) and dotting both sides by $\hat{\imath}$ and then by $\hat{\jmath}$ we get

$$
\begin{aligned}
B_x &= -\frac{1}{2\sqrt{3}}m\omega_D^2 R, \\
B_y + N &= -\frac{1}{2}m\omega_D^2 R + mg
\end{aligned}
\tag{16.39}
$$

From eqn. (16.37)

$$
\begin{aligned}
\frac{1}{2}\ell[(B_y - N)\cos\theta - B_x\sin\theta]\hat{k} &= \frac{1}{12}m\ell^2(-\frac{\omega_D^2 R}{\ell\cos\theta})\hat{k} \\
\text{or} \qquad B_y - N &= -\frac{1}{6}\frac{m\omega_D^2 R}{\cos^2\theta} + B_x\tan\theta \\
&= -\frac{2}{9}m\omega_D^2 R - \frac{1}{6}m\omega_D^2 R \\
&= -\frac{7}{18}m\omega_D^2 R
\end{aligned}
\tag{16.40}
$$

From eqns. (16.39) and (16.40)

$$
B_y = \frac{1}{2}\left(mg - \frac{8}{9}m\omega_D^2 R\right)
$$

and

$$
N = \frac{1}{2}\left(mg - \frac{1}{9}m\omega_D^2 R\right).
$$

2. **Lift off of end A:** End A of the rod loses contact with the ground when normal force N becomes zero. From the expression for N from above, this condition is satisfied when

$$
\begin{aligned}
\frac{2}{9}m\omega_D^2 R &= mg \\
\Rightarrow \qquad \omega_D &= 3\sqrt{\frac{g}{R}}.
\end{aligned}
$$

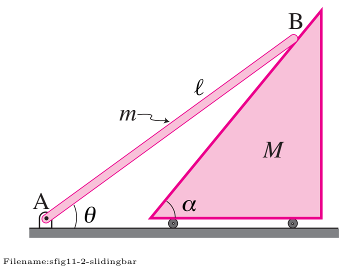


Figure 16.40:

SAMPLE 16.14 A bar sliding on a sliding wedge. A bar AB of mass m and length ℓ is hinged at end A rests on a wedge of mass M at the other end B. The contact at B is frictionless. The wedge is free to slide horizontally without any friction. The motion of the system is driven only by gravity. Find the equation of motion of the bar using

1. momentum balance
2. energy (or power) balance.

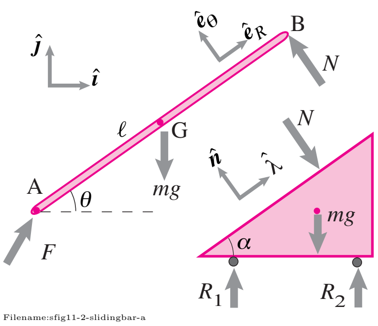


Figure 16.41:

Solution (a) **Momentum Balance:** The bar and the wedge make up a single DOF system. To derive the equations of motion of the bar, let us choose θ as the configuration variable. The free-body diagram of the bar and the wedge are shown in fig. 16.41. Note that the normal reaction at B is normal to the wedge surface, *i.e.,* $\vec{N} = N\hat{\boldsymbol{n}}$. Now, the angular momentum balance for the bar about point A gives,

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{\mathrm{A}} &= \sum \vec{\boldsymbol{M}}_{\mathrm{A}} \\
I_{zz}^{A}\ddot{\theta}\hat{\boldsymbol{k}} &= \frac{\ell}{2}\hat{\boldsymbol{e}}_R \times (-mg\,\hat{\boldsymbol{\jmath}}) + \ell\hat{\boldsymbol{e}}_R \times N\hat{\boldsymbol{n}} \\
&= -\frac{1}{2}mg\ell\cos\theta\,\hat{\boldsymbol{k}} + N\ell\cos(\alpha-\theta)\hat{\boldsymbol{k}} \qquad (16.41)
\end{aligned}$$

where the last line follows from the fact that $\hat{\boldsymbol{e}}_R = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$, the unit normal $\hat{\boldsymbol{n}} = -\sin\alpha\,\hat{\boldsymbol{\imath}} + \cos\alpha\,\hat{\boldsymbol{\jmath}}$, and so, $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{\jmath}} = \cos\theta\,\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{n}} = \cos(\alpha-\theta)\hat{\boldsymbol{k}}$. We now need to eliminate the unknown normal reaction N from the above equation. Since the wedge is constrained to move only horizontally, we can write the linear momentum balance for the wedge as

$$N\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\imath}} = M\ddot{x} \quad\Rightarrow\quad N = \frac{M\ddot{x}}{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{\imath}}} = \frac{M\ddot{x}}{\sin\alpha} \qquad (16.42)$$

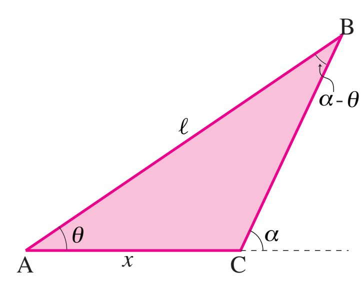


Figure 16.42:

Thus, we have found N in terms of $\ddot{x}$ that we can use in eqn. (16.41) to get rid of N. But, we now need to express $\ddot{x}$ in terms of our configuration variable θ and its derivatives. Consider the triangle ABC formed by the bar and the slanted edge of the wedge. Let x = AC denote the horizontal position of the wedge. Then, from the law of sines, we have $\frac{x}{\sin(\alpha-\theta)} = \frac{\ell}{\sin\alpha}$, so that

$$\begin{aligned}
x &= \frac{\ell}{\sin\alpha}\sin(\alpha-\theta) \\
\Rightarrow\quad \dot{x} &= \frac{\ell}{\sin\alpha}\cos(\alpha-\theta)\cdot(-\dot{\theta}) \qquad (16.43)\\
\Rightarrow\quad \ddot{x} &= \frac{\ell}{\sin\alpha}\big[\ddot{\theta}\cos(\alpha-\theta) + \dot{\theta}^2\sin(\alpha-\theta)\big]. \qquad (16.44)
\end{aligned}$$

Now, substituting for N in eqn. (16.41) from eqn. (16.42), using the expression for $\ddot{x}$ from above, and dotting the resulting equation with $\hat{\boldsymbol{k}}$, we get

$$\begin{aligned}
\frac{1}{3}m\ell^2\ddot{\theta} &= -\frac{mg\ell}{2}\cos\theta + \frac{M\ell^2\cos(\alpha-\theta)}{\sin^2\alpha}\big[\ddot{\theta}\cos(\alpha-\theta) + \dot{\theta}^2\sin(\alpha-\theta)\big] \\
\Rightarrow\quad \ddot{\theta} &= -\frac{3mg\sin^2\alpha\cos\theta + 3M\ell\dot{\theta}^2\sin2(\alpha-\theta)}{2m\ell\sin^2\alpha + 6M\ell\cos^2(\alpha-\theta)} \\
&= -\frac{3\big[(g/\ell)\sin^2\alpha\cos\theta + (M/m)\dot{\theta}^2\sin2(\alpha-\theta)\big]}{2\big[\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)\big]} \qquad (16.45)
\end{aligned}$$

$$\boxed{\ddot{\theta} = -\frac{3\big[(g/\ell)\sin^2\alpha\cos\theta + (M/m)\dot{\theta}^2\sin2(\alpha-\theta)\big]}{2\big[\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)\big]}}$$

(b) **Power balance:** Now we derive the equation of motion for the bar using power balance $\dot{E}_K = P$. For power balance, we have to consider both the bar and the wedge. The bar rotates about the fixed point A, therefore, its kinetic energy is $(1/2)I_z^A z\dot{\theta}^2$. The wedge moves with horizontal speed $\dot{x}$, therefore, its kinetic energy is $(1/2)M\dot{x}^2$. Thus, $E_K = 1/2)I_z^A z\dot{\theta}^2 + (1/2)M\dot{x}^2$. The only force that contributes to power is the force of gravity on the rod because $\vec{v}_G \cdot (-mg\hat{\jmath})$ is non-zero. The sliding contact at B is frictionless and hence the net power due to the contact force there is zero. Now, $\vec{v}_G = \frac{1}{2}\ell\dot{\theta}\hat{e}_\theta$. So, $P = -\frac{1}{2}mg\ell\dot{\theta}(\hat{e}_\theta \cdot \hat{\jmath}) = -\frac{1}{2}mg\ell\dot{\theta}\cos\theta$. Thus the power balance for the system gives

$$I_{zz}^A\ddot{\theta}\dot{\theta} + M\ddot{x}\dot{x} = -\frac{1}{2}mg\ell\dot{\theta}\cos\theta. \tag{16.46}$$

We can simplify this equation further. Note that, $\ddot{\theta}\dot{\theta} = \frac{d}{dt}(\frac{1}{2}\dot{\theta}^2)$ and $\ddot{x}\dot{x} = \frac{d}{dt}(\frac{1}{2}\dot{x}^2)$. So that we can write the above equation as

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}I_{zz}^A\dot{\theta}^2\right) + \frac{d}{dt}\left(\frac{1}{2}M\dot{x}^2\right) &= -\frac{1}{2}mg\ell\dot{\theta}\cos\theta \\
\text{or} \qquad \int d\left(I_{zz}^A\dot{\theta}^2 + M\dot{x}^2\right) &= -mg\ell\int\cos\theta\, d\theta \\
\Rightarrow \qquad I_{zz}^A\dot{\theta}^2 + M\dot{x}^2 &= C - mg\ell\sin\theta
\end{aligned}$$

where C is a constant of integration to be determined from initial conditions. For example, when the bar begins to slide from rest, we have, at $t = 0$, $\dot{x} = 0$ and $\dot{\theta} = 0$. At that instant, if $\theta(0) = \theta_0$, then $C = mg\ell\sin\theta_0$. So, we can write

$$I_{zz}^A\dot{\theta}^2 + M\dot{x}^2 = mg\ell(\sin\theta_0 - \sin\theta).$$

Now, replacing $\dot{x}$ in this equation with the expression we obtained in eqn. (16.43), we have

$$\begin{aligned}
I_{zz}^A\dot{\theta}^2 + M\ell^2\dot{\theta}^2\frac{\cos^2(\alpha-\theta)}{\sin^2\alpha} &= mg\ell(\sin\theta_0 - \sin\theta) \\
\Rightarrow \quad \dot{\theta}^2 &= \frac{mg\ell(\sin\theta_0 - \sin\theta)}{I_{zz}^A + M\ell^2\frac{\cos^2(\alpha-\theta)}{\sin^2\alpha}} \\
&= \frac{mg\ell\sin^2\alpha(\sin\theta_0 - \sin\theta)}{(1/3)m\ell^2\sin^2\alpha + M\ell^2\cos^2(\alpha-\theta)} \\
\Rightarrow \quad \dot{\theta} &= \sqrt{\frac{3g}{\ell}}\sin\alpha\sqrt{\frac{\sin\theta_0 - \sin\theta}{\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)}}.
\end{aligned}$$

This is a first order, nonlinear, ODE compared to the second order equation we got in eqn. (16.45). However, again, we need to resort to numerical solution if we wish to solve for $\theta(t)$, in which case, this reduction to the first order equation does not save much work.

$$\boxed{\dot{\theta} = \sqrt{\frac{3g}{\ell}}\sin\alpha\sqrt{\frac{\sin\theta_0 - \sin\theta}{\sin^2\alpha + 3(M/m)\cos^2(\alpha-\theta)}}}$$

16.3 Dynamics of multi-degree-of-freedom 2-D mechanisms

A typical machine has many parts. They may work in concert as a one degree-of-freedom system or they may be designed to move independently.

One degree-of-freedom machines. Some mechanisms are designed so the parts work together to do something, one thing, well. A car going straight down a road has dozens of moving parts: cylinders, connecting rods, crank shaft, cam shaft, transmission gears, drive shaft, differential, wheel axles, wheels and the car itself. All of these parts move, each in its own different way, approximately like a rigid object. But in the simplest description① they move as a one-degree-of freedom mechanism towards the end of fighting friction and moving the car down the road. If the crankshaft rotates a small amount, each part moves a corresponding amount; the amount of rotation of the crank shaft determines the position of all of the parts. And all of the velocities and accelerations of all of the bits of mass in the car can be determined by the rotation θ_{crank} and its derivatives ($dot\theta_{\text{crank}}$ and $ddot\theta_{\text{crank}}$). Thus a car, even with lots of pieces moving this way and that, might be well-modeled for some purposes as a one degree-of-freedom mechanism. For one-degree-of-freedom mechanisms the methods of Section 16.2 may be appropriate.

①**A car has one degree-of-freedom?** For studying the acceleration of a car going straight one might ignore the flexibility of the drive shaft and the bounce of the suspension, *etc.*thus leading to a 1 DOF model. But if you want to see how acceleration causes the car to tip back, or how the car oscillates when the engine lugs, then a 2, or more, DOF model would be appropriate.

Multi DOF systems In some modeling of machines it is important to keep track of multiple degrees of freedom (See fig. 15.51 on page 882 for examples of simple mechanisms). There are various reasons that a multi-DOF analysis is relevant:

- Some machines intentionally have various ways of moving that are controlled by various motors. Classic examples include:
  - Robots,
  - Animal bodies.
- Some systems intentionally have various degrees of freedom so as to respond smoothly to disturbances, for example the suspensions on the bottoms of cars and washing machines.
- Some systems have undesirable degrees of freedom do to the parts which are nominally rigid actually being elastic. Thus a machine which is intended to have one degree of freedom might have various undesirable vibration modes.

Mechanical analysis of multi-DOF systems. To solve problems with multiple degrees of freedom the basic strategy is, as described at the start of the chapter:

1. Draw FBDs of each object,

2. Pick configuration variables,
3. Write linear and angular momentum balance equations
4. Solve the equations for variables of interest (usually forces and second derivatives of the configuration variables).
5. Set up and solve the resulting differential equations (if you are trying to find the motion).

There are two basic approaches to these multi-object problems which, for lack of better language we call "brute-force" and "clever".

A. In the **brute force** approach you write three times as many scalar balance equations as you have objects (limiting attention to 2D where each object has 3 DOFs and 3 independent momentum balance equations). That is, for example, for each free body diagram you write linear momentum balance and angular momentum balance about the center of mass. Then you take this set of $3n$ equations and add and subtract them to solve for variables of interest.

   This approach is quite suitable for computers so most commercial general purpose dynamic simulators use a variant of this approach. For individual use with packaged software, the brute-force approach is generally both more reliable and more time consuming.

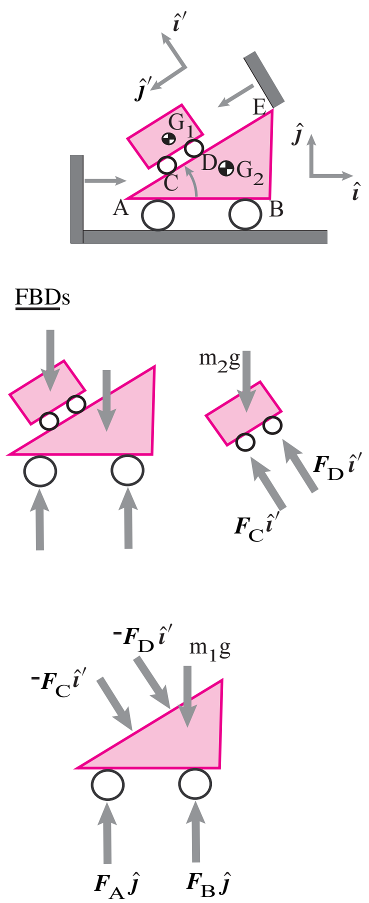
B. In the clever approach you write as many scalar momentum balance equations as you have unknowns. For example, if you have 2 degrees of freedom and you are concerned with motions and not reaction forces, you write 2 equations. You do this by finding momentum balance equations that do not include the variables you are not interested in. Usually this involves using angular momentum balance about hinge points, or linear momentum balance orthogonal to sliding contacts.

   The **clever** approach does not always work; the four-bar linkage is the classic problem case. However the desire to find such minimal sets of equations of motion it is historically important②.

② Attempts to automate the clever approach, that is to quickly find minimal equations of motion, led to Lagrange equations which led to Hamilton's equations which led to quantum mechanics (but we won't be that clever here).

At this point in the subject there are no quick simple problems. All problems are involved, especially if taken from start all the way to plotting solutions to the differential equations. The examples that follow emphasize getting to the equations of motion. The skills for numerically solving the differential equations and plotting the solutions are the same as from the start of dynamics so are not discussed until the sample problems which show all the work from setup to solution.

Example: Block sliding on sliding block: clever approach
Block 1 with mass m_1 rolls without friction on ideal massless wheels at A and B (see fig. 16.43). Block 2 with mass m_2 rolls down the tipped top of block 1 on ideal massless rollers at C and D. The locations of G_1 relative to A and B, and of G_2 relative to C and D are known. How do blocks 1 and 2 move?



Filename:tfigure-blockslidesonblock

Figure 16.43: Block 2 rolls on block 1 which rolls on the ground. All rollers are ideal (frictionless and massless). The moving $\hat{\imath}'\hat{\jmath}'$ frame moves with the lower block and is oriented with the slope.

First look at the free body diagram of the system and not that there are no unknown forces in the $\hat{\imath}$ direction. So, for the system

$$\begin{aligned} \left\{ \sum \vec{F}_i \right. &= \left. \dot{\vec{L}} \right\} \cdot \hat{\imath} \\ \Rightarrow \quad 0 &= m_1\ddot{x} + m_2(\ddot{x}\hat{\imath} + \ddot{y}'\hat{\jmath}')\cdot\hat{\imath} \\ &= (m_1+m_2)\ddot{x} - \ddot{y}' m_2 \cos\theta. \end{aligned} \tag{16.47}$$

Looking at the free body diagram of mass 2 note that there are no unknown forces in the $\hat{\jmath}'$ direction, so

$$\begin{aligned} \left\{ \sum \vec{F}_i \right. &= \left. \dot{\vec{L}} \right\} \cdot \hat{\jmath}' \\ \Rightarrow \quad m_2 g \sin\theta &= m_2(\ddot{x}\hat{\imath} + \ddot{y}'\hat{\jmath}')\cdot\hat{\jmath}' \\ m_2 g \sin\theta &= m_2(-\cos\theta\,\ddot{x} + \ddot{y}'). \end{aligned} \tag{16.48}$$

Eqns. 16.47 and 16.47 are a system of two equations in the two unknowns $\ddot{x}$ and $\ddot{y}'$

$$\begin{aligned} (m_1+m_2)\,\ddot{x} \quad -m_2\cos\theta\,\ddot{y}' &= 0 \\ -m_2\cos\theta\,\ddot{x} \quad +m_2\,\ddot{y}' &= m_2 g\sin\theta \end{aligned}$$

which can be solved for $\ddot{x}$ and $\ddot{y}$ by hand or on the computer. Finding $x(t)$ and $y'(t)$ is then easy because both $\ddot{x}$ and $\ddot{y}'$ are constants.

Now we look at the same example, but proceed in a more naive manner.

Example: Block sliding on sliding block: brute force approach

Now we look at the free body diagrams of the two separate blocks. We will use 3 balance equations from each free body diagram, taking account of the kinematic constraints.

For the lower block we have

$$\text{AMB}_{G_1} \quad \Rightarrow \quad \sum \vec{M}_{/G_1} = \dot{\vec{H}}_{/G_1} \tag{16.49}$$

$$\begin{aligned} \text{where} \quad \sum \vec{M}_{/G_1} = \; & \vec{r}_{D/G_1} \times (-F_D\hat{\imath}') \\ & + \vec{r}_{C/G_1} \times (-F_C\hat{\imath}') \\ & + \vec{r}_{B/G_1} \times F_B\hat{\jmath} \\ & + \vec{r}_{A/G_1} \times F_A\hat{\jmath} \end{aligned}$$

$$\text{and} \quad \dot{\vec{H}}_{/G_1} = \vec{0} \qquad (\vec{\omega}_1 = \vec{0})$$

$$\text{and LMB} \quad \Rightarrow \quad \sum \vec{F}_i = \dot{\vec{L}} \tag{16.50}$$

$$\text{where} \quad \sum \vec{F}_i = -F_D\hat{\imath}' - F_C\hat{\imath}' + F_B\hat{\jmath} + F_A\hat{\jmath} - m_1 g\hat{\jmath}$$

$$\text{and} \quad \dot{\vec{L}} = m_1\ddot{x}\hat{\imath}$$

Similarly for the upper block:

$$\text{AMB}_{G_2} \quad \Rightarrow \quad \sum \vec{M}_{/\mathrm{G}_2} = \dot{\vec{H}}_{/\mathrm{G}_2} \tag{16.51}$$

$$\text{where} \quad \sum \vec{M}_{/\mathrm{G}_1} = \vec{r}_{\mathrm{D}/\mathrm{G}_2} \times F_D \hat{\imath}' + \vec{r}_{\mathrm{C}/\mathrm{G}_2} \times F_C \hat{\imath}'$$

$$\text{and} \quad \dot{\vec{H}}_{/\mathrm{G}_2} = \vec{0} \qquad (\vec{\omega}_1 = \vec{0})$$

$$\text{and LMB} \quad \Rightarrow \quad \sum \vec{F}_{\mathrm{i}} = \dot{\vec{L}} \tag{16.52}$$

$$\text{where} \quad \sum \vec{F}_{\mathrm{i}} = F_D \hat{\imath}' + F_C \hat{\imath}' - m_2 g \hat{\jmath}'$$

$$\text{and} \quad \dot{\vec{L}} = m_2(\ddot{x}\hat{\imath} + \ddot{y}'\hat{\jmath}')$$

Eqns. 16.49-16.52 can be written as scalar equations by dotting the LMB equations with $\hat{\imath}$ and $\hat{\jmath}$ and the AMB equations with $\hat{k}$. All is known in these 6 equations but the six scalars: F_A, F_B, F_C, F_D, $\ddot{x}$, and $\ddot{y}'$. These could be set up as a matrix equation and solved on the computer, or you could try to find your way through by adding and subtracting equations. In any case you could solve for $\ddot{x}$ and $\ddot{y}'$ and thus have differential equations to solve to find the motions.

One quick inference one can make is from looking at the equations: no term in the coefficients of the unknowns depends on $x, \dot{x}, y$, or $\dot{y}'$. So all of the reactions F_A, F_B, F_C, and F_D as well as the accelerations $\ddot{x}$ and $\ddot{y}'$ are constants in time (until the upper mass hits the ground).

Example: Block sliding on sliding block: even more brute force

This "multi-body" problem can be solved in an even more naive and more brute force manner. The method is the same as shown in Section 12.1 on page 607 for a one dimensional problem.

We would use 6 configuration variables: the x and y coordinates of G_1 and G_2 and the rotations of the two bodies:

$$x_1, y_1, \theta_1, x_2, y_2, \quad \text{and} \quad \theta_2$$

That the two bodies don't rotate would be expressed indirectly by noting that the velocities of points A and B on the lower mass must have acceleration in the $\hat{\imath}$ direction. These two equations would be added to the 6 linear and angular momentum balance equations. Similar constraint equations would be written for the interactions at C and D. Altogether there are now 6 configuration variables and 4 constraint forces. But there are 6 differential equations of motion and 4 constraint equations. Thus at one instant in time a set of 10 simultaneous equations needs to be solved. Then these are used to evaluate the right hand sides in the differential equations.

These 10 equations are an impractical mess for solving one simple problem like this. But these equations lend themselves to easy automation and this is closest to the approach used by general purpose dynamics simulators.

The example above is particularly simple because block 1 moves in a straight line without rotating and block 2 moves in a straight line without rotating relative to block 1. Even for a system of just two objects the situation could be much more complex if the first object had a complex motion and the second a complex motion relative to the first. But because of the preponderance of hinges in the world, circular motion, and motion relative to circular motion, is the most complex motion that need be considered by many engineers. Here is a version of the most common example of that class.

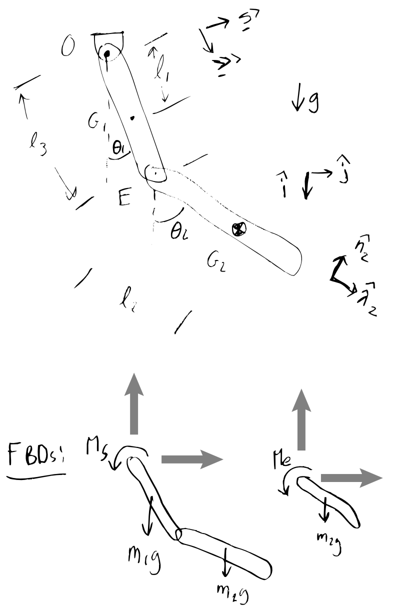


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Figure 16.44: A two link robot arm. Free body diagrams are shown of the whole system (including the motor torque at the shoulder M_s) and of the fore-arm (including the motor torque at the elbow M_e) .

Example: A two link robot arm: finesse the finding of reactions

A robot arm has two links. There are motors that apply known torque M_s at the shoulder (reacted by the base and a torque M_e at the elbow (reacted by the upper arm). Dimensions are as marked. This system has 2 degrees of freedom. So we need 2 configuration variables and 2 independent balance equations to find the motion.

The natural configuration variables are the angles of the upper arm relative to a fixed reference and the angle of the lower arm relative to a fixed reference. It would also be natural to instead use the angle of the lower arm relative to the upper arm. This leads to simpler equations in the end, but more work in set up.

Angular momentum balance for the system about the shoulder contains no unknown reaction forces, nor does angular momentum balance of the fore-arm about the hinge. So we base our work on these two equations:

$$\text{System AMB}_{/O} \quad \Rightarrow \quad \sum \vec{\boldsymbol{M}}_{/\text{O}} \; = \; \dot{\vec{\boldsymbol{H}}}_{/\text{O}} \tag{16.53}$$

$$\text{Forearm AMB}_{/E} \quad \Rightarrow \quad \sum \vec{\boldsymbol{M}}_{/\text{E}} \; = \; \dot{\vec{\boldsymbol{H}}}_{/\text{E}}. \tag{16.54}$$

The goal, equations of motion, is reached by evaluating the left and right sides of these equations in terms of known geometric and mass quantities as well as the configuration variables When we write $\sum \vec{\boldsymbol{M}}_{/\text{O}}$ $\dot{\vec{\boldsymbol{H}}}_{/\text{O}}$ we implicitly mean for the whole system. Likewise $\sum \vec{\boldsymbol{M}}_{/\text{E}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\text{E}}$ apply to the forearm.

At each step in the calculations below imagine the results can be substituted into the later steps. We don't do that here because the expressions grow in size. Further, if the angles and their rates of change are known, as they are when doing most dynamics problems, the intermediate calculations will result in numbers, and these do not become more numerous as the calculation proceeds.

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_1 &= \cos\theta_1 \hat{\boldsymbol{\imath}} + \sin\theta_1 \hat{\boldsymbol{\jmath}} && \text{and} && \hat{\boldsymbol{\lambda}}_2 = \cos\theta_2 \hat{\boldsymbol{\imath}} + \sin\theta_2 \hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{r}}_{\text{G}_1/\text{O}} &= \ell_1 \hat{\boldsymbol{\lambda}}_1,\ \vec{\boldsymbol{r}}_{\text{E}/\text{O}} = \ell_3 \hat{\boldsymbol{\lambda}}_1 && \text{and} && \vec{\boldsymbol{r}}_{\text{G}_2/\text{E}} = \ell_2 \hat{\boldsymbol{\lambda}}_2 \\
& && \text{and} && \vec{\boldsymbol{r}}_{\text{G}_2/\text{O}} = \vec{\boldsymbol{r}}_{\text{E}/\text{O}} + \vec{\boldsymbol{r}}_{\text{G}_2/\text{E}}.
\end{aligned}$$

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\text{G}_1/\text{O}} &= -\dot{\theta}_1^2 \vec{\boldsymbol{r}}_{\text{G}_1/\text{O}} + \ddot{\theta}_1 \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\text{G}_1/\text{O}} && , \\
\vec{\boldsymbol{a}}_{\text{E}/\text{O}} &= -\dot{\theta}_1^2 \vec{\boldsymbol{r}}_{\text{E}/\text{O}} + \ddot{\theta}_1 \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\text{E}/\text{O}} && , \\
\vec{\boldsymbol{a}}_{\text{G}_2/\text{E}} &= -\dot{\theta}_2^2 \vec{\boldsymbol{r}}_{\text{G}_2/\text{E}} + \ddot{\theta}_2 \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\text{G}_2/\text{E}} && \text{and} && \vec{\boldsymbol{a}}_{\text{G}_2/\text{O}} = \vec{\boldsymbol{a}}_{\text{E}/\text{O}} + \vec{\boldsymbol{a}}_{\text{G}_2/\text{E}}.
\end{aligned}$$

These terms are all we need to evaluate the 4 terms in Eqns. 16.53 and 16.54.

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/\text{O}} &= \vec{\boldsymbol{r}}_{\text{G}_1/\text{O}} \times (m_1 g \hat{\boldsymbol{\imath}}) + \vec{\boldsymbol{r}}_{\text{G}_2/\text{O}} \times (-m_2 g \hat{\boldsymbol{\jmath}}) + M_s \hat{\boldsymbol{k}}, \\
\sum \vec{\boldsymbol{M}}_{/\text{E}} &= \vec{\boldsymbol{r}}_{\text{G}_2/\text{E}} \times (m_2 g \hat{\boldsymbol{\imath}}) + M_e \hat{\boldsymbol{k}},
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\text{O}} &= m_1 \vec{\boldsymbol{r}}_{\text{G}_1/\text{O}} \times \vec{\boldsymbol{a}}_{\text{G}_1/\text{O}} + \ddot{\theta}_1 I_1 \hat{\boldsymbol{k}} + m_2 \vec{\boldsymbol{r}}_{\text{G}_2/\text{O}} \times \vec{\boldsymbol{a}}_{\text{G}_2/\text{O}} + \ddot{\theta}_2 I_2 \hat{\boldsymbol{k}}, \\
\text{and} \quad \dot{\vec{\boldsymbol{H}}}_{/\text{E}} &= m_2 \vec{\boldsymbol{r}}_{\text{G}_2/\text{E}} \times \vec{\boldsymbol{a}}_{\text{G}_2/\text{O}} + \ddot{\theta}_2 I_2 \hat{\boldsymbol{k}}.
\end{aligned}$$

Once these are substituted into Eqns. 16.53 and 16.54 one has 2 vector equations with only $\hat{\boldsymbol{k}}$ components. In other words we have two scalar equations in the two unknowns $\ddot{\theta}_1$ and $\ddot{\theta}_2$. One can go through the algebra and solve for them explicitly, but the expressions are quite complex, even when simplified. At given values of θ_1, $\dot{\theta}_1$, θ_2, and $\dot{\theta}_2$ however these are just two linear equations in two unknowns.

Closed kinematic chains

When a series of mechanical links is *open* you can not go from one link to the next successively and get back to your starting point. Such chains include a pendulum (1 link), a double pendulum (2 links), a 100 link pendulum, and a model of the human body (so long as only one foot is on the ground). A *closed* chain has at least one loop in it. You can go from link to next and get back to where you started. A slider-crank, a 4-bar linkage, and a person with two feet on the ground are closed chains.

Closed chains are kinematically difficult because they have fewer degrees of freedom than they have joints. So some of the joint angles depend on the others. The values of any minimal set of configuration variables, say some of the joint angles, determines all of the joint angles, but by geometry that is difficult or impossible to express with formulas.

Example: Four bar linkage.
It is impractically difficult to write the positions velocities and accelerations of a 4-bar linkage in terms of θ, $\dot{\theta}$ and $\ddot{\theta}$ of any one of its joints. Why? Because finding all of the bar angles from one angle is a trigonometric mess. And differentiating that mess once (for velocities) and then once again (for accelerations) makes a huge mess.

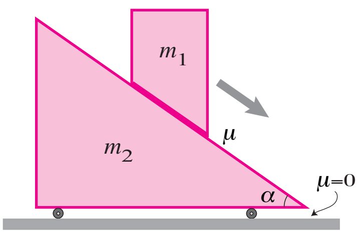

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Figure 16.45:

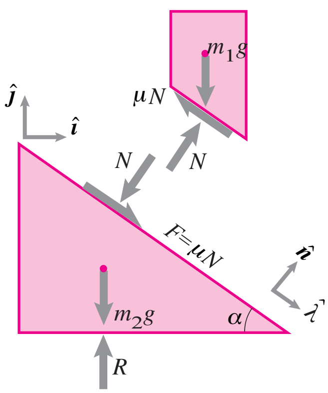

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Figure 16.46:

SAMPLE 16.15 Dynamics of sliding wedges. A wedge shaped body of mass m_2 sits on a frictionless ground. Another wedge shaped body of mass m_1 is gently placed on the inclined face of the stationary wedge. The top wedge starts to slide down. The coefficient of friction between the two wedges is μ. Find the sliding acceleration of the top wedge along the incline (*i.e.,* the relative acceleration of m_1 with respect to m_2).

Solution The free-body diagrams of the two wedges are shown in fig. 16.46. Note that the friction force is μN since the wedges are sliding with respect to each other (if they were not sliding already then the friction force is an unknown force $F \leq \mu N$). Let the absolute acceleration of m_2 be $\vec{\boldsymbol{a}}_2 = a_2\hat{\boldsymbol{\imath}}$. Then, the absolute acceleration of m_1 is $\vec{\boldsymbol{a}}_1 = \vec{\boldsymbol{a}}_2 + \vec{\boldsymbol{a}}_{1/2} = a_2\hat{\boldsymbol{\imath}} + a_{\text{rel}}\hat{\boldsymbol{\lambda}}$. Now, we can write the linear momentum balance for m_1 and m_2 as follows.

$$
\begin{aligned}
N\hat{\boldsymbol{n}} - m_1 g\hat{\boldsymbol{\jmath}} - \mu N\hat{\boldsymbol{\lambda}} &= m_1(a_2\hat{\boldsymbol{\imath}} + a_{\text{rel}}\hat{\boldsymbol{\lambda}}) && (16.55)\\
(R - m_2 g)\hat{\boldsymbol{\jmath}} - N\hat{\boldsymbol{n}} + \mu N\hat{\boldsymbol{\lambda}} &= m_2 a_2\hat{\boldsymbol{\imath}} && (16.56)
\end{aligned}
$$

where $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\boldsymbol{\imath}} - \sin\alpha\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{n}} = \sin\alpha\hat{\boldsymbol{\imath}} + \cos\alpha\hat{\boldsymbol{\jmath}}$. Here, we have 4 independent scalar equations (from the two 2-D vector equations) in four unknowns N, R, a_2, and a_{rel}. Thus, we can certainly solve for them. We are, however, only interested in a_{rel}. So, we should try to find the answer with fewer calculations. Dotting eqn. (16.55) with $\hat{\boldsymbol{\lambda}}$, we have

$$m_1 a_{\text{rel}} = -m_1 a_2 \cos\alpha - \mu N + m_1 g \sin\alpha \tag{16.57}$$

So, to find a_{rel}, we need a_2 and N. Dotting eqn. (16.55) with $\hat{\boldsymbol{n}}$, we have

$$m_1 a_2 \sin\alpha = N - m_1 g\cos\alpha, \tag{16.58}$$

and dotting eqn. (16.56) with $\hat{\boldsymbol{\imath}}$, we have

$$m_2 a_2 = -N\sin\alpha + \mu N\cos\alpha. \tag{16.59}$$

Solving eqn. (16.58) and (16.59) simultaneously, and using new variables (for convenience) $M = m_1/m_2$, $C = \cos\alpha$, and $S = \sin\alpha$, we get

$$a_2 = \frac{MC(\mu C - S)}{1 - MS(\mu C - S)}\,g, \quad N = m_1 g\,\frac{C}{1 - MS(\mu C - S)}$$

Substituting these expression in eqn. (16.57), we get

$$a_{\text{rel}} = gS - \frac{gC\left[MC(\mu C - S) - \mu\right]}{1 - MS(\mu C - S)}$$

$$\boxed{a_{\text{rel}} = g\sin\alpha - \frac{g\cos\alpha\left[\frac{m_1}{m_2}\cos\alpha(\mu\cos\alpha - \sin\alpha) - \mu\right]}{1 - \frac{m_1}{m_2}\sin\alpha(\mu\cos\alpha - \sin\alpha)}}$$

Note that when there is no friction ($\mu = 0$), the expression for a_{rel} reduces to

$$a_{\text{rel}} = gS + \frac{gMC^2S}{1 + MS^2}$$

and if we let $m_2 \to \infty$ (*i.e.,* m_2 represents fixed ramp) so that $M \to 0$, then $a_{\text{rel}} = g\sin\alpha$ which is the acceleration of a point mass down a frictionless ramp of slope $\tan\alpha$.

SAMPLE 16.16 Dynamics of a new gun. A new gun consists of a uniform rod AB of mass m_1 and a small collar C of mass m_2 that slides freely on the rod. A motor at A rotates the rod with constant torque T.

1. Find the equations of motion of the collar.
2. Show that if $T = 0$ then the equations of motion imply conservation of angular momentum about point A.

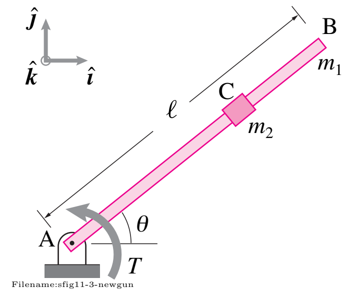


Figure 16.47:

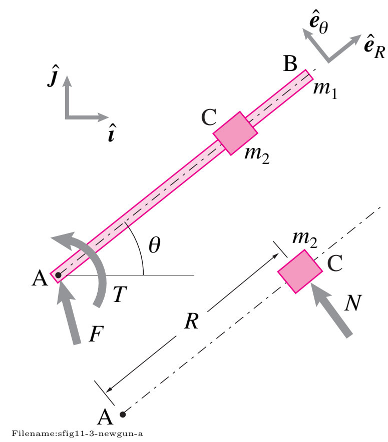


Figure 16.48:

Solution

1. Let us denote the configuration of the collar with R, the radial distance from the fixed point A along the rod, and θ, the angular displacement of the rod. We need to find differential equations that determine R and θ as functions of time. The free-body diagram of the whole system (rod and collar together) and that of the collar is shown in fig. 16.48. We can write angular momentum balance for the whole system about point A so that the unknown reaction force F at A does not enter the equations. Noting that the acceleration of the collar is $\vec{a}_C = (\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta$, and letting $I_1 \equiv I_z^A z$ be the moment of inertia of the rod about A, we have,

$$\begin{aligned}
\sum \vec{M}_A &= \dot{\vec{H}}_A \\
T\hat{k} &= I_1\ddot{\theta}\hat{k} + R\hat{e}_R \times m_2\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta\right] \\
&= I_1\ddot{\theta}\hat{k} + m_2R^2\ddot{\theta}\hat{k} + 2m_2R\dot{R}\dot{\theta}\hat{k}
\end{aligned}$$

Dotting this equation with $\hat{k}$, we have

$$\ddot{\theta} = \frac{T}{I_1 + m_2R^2} - \frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \tag{16.60}$$

Thus we have obtained the equation of motion for θ. Now we consider the free-body diagram of the collar alone and write the linear momentum balance for it in the $\hat{e}_R$ direction, *i.e.*, $\hat{e}_R \cdot (\sum \vec{F} = m\vec{a})$, so that we do not have to care about the unknown normal reaction $\vec{N}$. So, we have,

$$\begin{aligned}
0 &= \hat{e}_R \cdot m_2\left[(\ddot{R} - R\dot{\theta}^2)\hat{e}_R + (2\dot{R}\dot{\theta} + R\ddot{\theta})\hat{e}_\theta\right] \\
&= m_2(\ddot{R} - R\dot{\theta}^2) \\
\Rightarrow \quad \ddot{R} &= R\dot{\theta}^2. \qquad (16.61)
\end{aligned}$$

Thus we have the required equations of motion. Note that eqn. (16.60) and (16.61) are coupled nonlinear differential equations. So, to find $\theta(t)$ and $R(t)$ we need to solve them numerically.

$$\boxed{\ddot{\theta} = \frac{T}{I_1+m_2R^2} - \frac{2m_2R}{I_1+m_2R^2}\dot{R}\dot{\theta}, \qquad \ddot{R} = R\dot{\theta}^2}$$

2. Now we set $T = 0$ in our equations of motion. Note that the equation for R is independent of T. The equation for θ becomes

$$\ddot{\theta} = -\frac{2m_2R}{I_1 + m_2R^2}\dot{R}\dot{\theta}. \quad \Rightarrow \quad (I_1 + m_2R^2)\ddot{\theta} + 2m_2R\dot{R}\dot{\theta} = 0$$

But the last expression is simply $\dot{H}_A$ for the system. Thus we have $\dot{H}_A = 0$ which implies that $H_A =$ constant. That is conservation of angular momentum about point A.

SAMPLE 16.17 Numerical solutions of new gun equations. Consider Sample 16.16 again. Set up the equations of motion for numerical solution. Take $T = 1\,\text{N·m}$, $\ell = 1\,\text{m}$, $m_2 = 1\,\text{kg}$, and $m_1 = m_2/3$. Carry out numerical solutions for the following cases.

1. Let the system start from rest at $\theta = 0$ and $R = 0.1\,\text{m}$. Find the solution from $t = 0$ to $t = 1\,\text{s}$. Plot $R(t)$, $\theta(t)$ and $R(\theta)$ (in polar coordinates).
2. Find the solution till the collar leaves the rod. What is the speed of the collar at this instant?
3. Compute and plot the total energy of the system as a function of time. Also, plot the work done by the torque as a function of time and show that the work done is equal to the total energy of the system at each instant.
4. Vary torque T and carry out solutions for several values of T. Find the terminal value of θ_f (when the collar leaves the rod) for each T. Justify your observation about θ_f by plotting $\dot{R}/\dot{\theta}$ as a function of T.

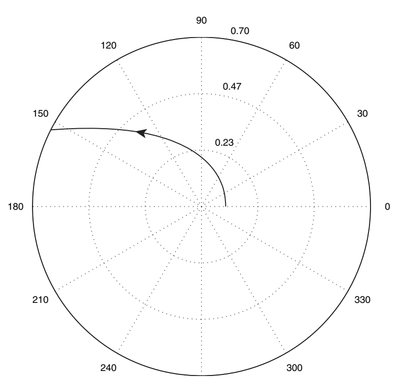
Solution We first need to write the equations of motion, eqn. (16.60) and (16.61), as a set of first order ODEs. We can easily do so by introducing new variables $\omega \equiv \dot{\theta}$ and $v_R \equiv \dot{R}$, so that we have,

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \\ \dot{R} \\ \dot{v}_R \end{pmatrix} = \begin{pmatrix} \omega \\ \frac{T}{I_1+m_2R^2} - \frac{2m_2R}{I_1+m_2R^2}v_R\omega \\ v_R \\ R\omega^2 \end{pmatrix}$$

Given the values of all constants, we only need to specify the initial conditions for θ, ω, R, and v_R for solving these equations numerically.

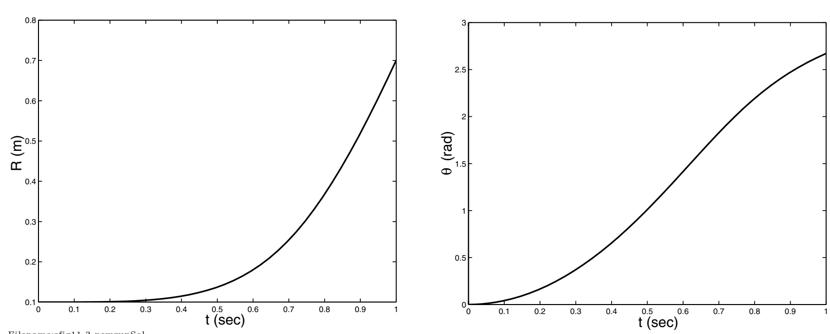
1. We use the following pseudocode to carry out the numerical solution.

```
Set T = 1, L = 1, m2 = 1, m1 = m2/3
Let I1 = m1*L^2/3, I2 = m2*R^2,
ODEs = {thetadot = w,
        wdot = (T-2*m2*R*vR*w)/(I1+I2),
        Rdot = vR,
        vRdot = R*w^2}
IC = {theta = 0, w = 0, R = 0.1, vR = 0}
Solve ODEs with IC for t=0 to t=1
Plot t vs R, Plot t vs theta,
Polarplot theta vs R
```



Filename:sfig11-3-newgunRth

Figure 16.49:



Filename:sfig11-3-newgunSol

Figure 16.50:

The $R(t)$ and $\theta(t)$ plots obtained from the numerical solution are shown in fig. 16.50 and the polar plot of $R(\theta)$ is shown in fig. 16.49.

2. We do not know apriori the value of t at which the collar leaves the rod. So, we have to carry out the solution for some assumed t_f which gives us $R(t_f) > \ell$ so that we know the collar has gone past the end of the rod. We then plot $R(t)$, including the unreal value of $R(t_f) > \ell$, and find the time t at which $R(t) = \ell$, either by zooming into the graph or by interpolation (although, there are various sophisticated algorithms to find this t). Following the method of zooming into the graph (see fig. 16.51) we find the terminal value of t to be 1.147 s. We carry out the numerical solution again from $t = 0$ to $t_f = 0.147$ s and find that $R(t_f) = 1$ m, $v_R(t_f) = 2.13$ m/s, and $\omega(t_f) = 1.03$ rad/s, so that $v_f = \sqrt{\dot{R}^2 + (R\dot{\theta})^2} = 2.37$ m/s. This is the terminal speed of the collar.

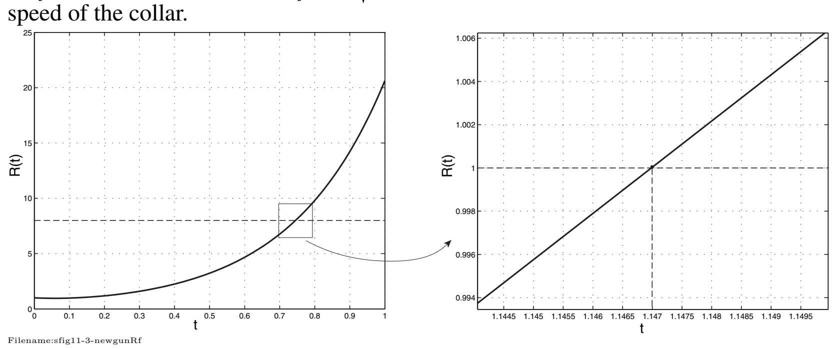


Figure 16.51: Finding the time t at which the collar leaves the rod from the graph of $R(t)$.

3. The work done by the torque is $W = T\theta$ at any instant. The system only possesses kinetic energy. So the energy of the system at any instant is $E = E_1 + E_2$ where $E_1 = \frac{1}{2}I_1\omega^2$ and $E_2 = \frac{1}{2}m_2(\dot{R}^2 + R^2\omega^2)$. Computing these quantities for the solution obtained above, we plot W and E vs t as shown in fig. 16.52. Clearly, $W = E$.

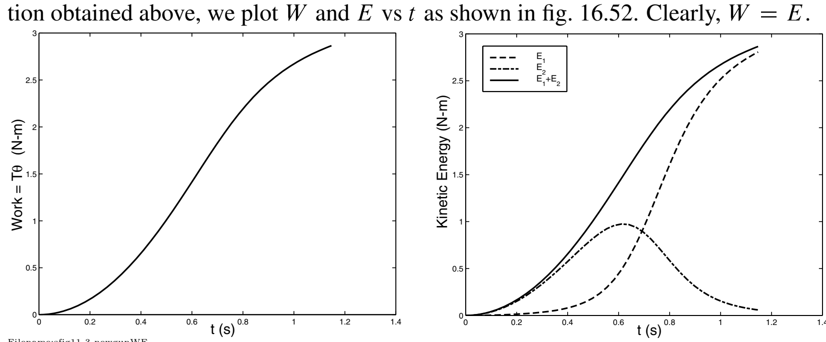


Figure 16.52: Work done by the torque and the kinetic energy of the system.

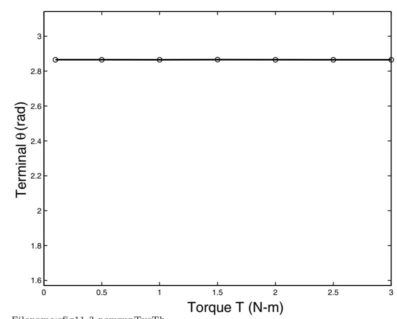


Figure 16.53: Angle θ at which the collar leaves the rod vs torque T.

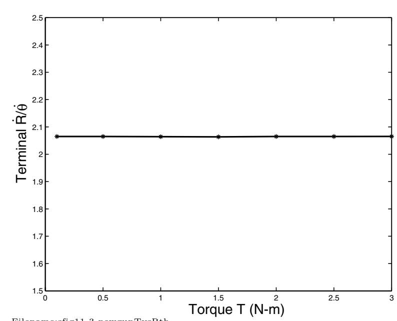
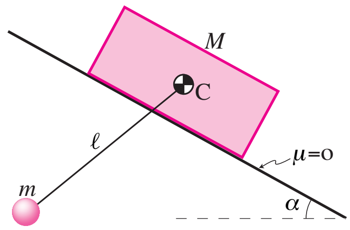


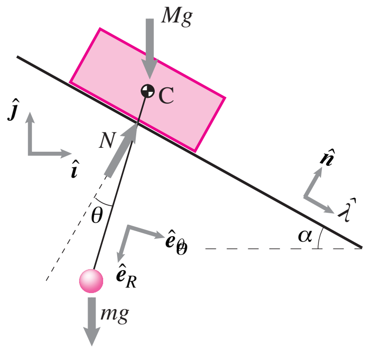
Figure 16.54: The ratio of terminal $\dot{R}/\dot{\theta}$ vs torque T.

4. Now we take several values of T (0.1, 0.5, 1, 1.5, 2, 2.5, and 3) and carry out the numerical solutions for each T. We note the terminal values of $\theta, \omega(= \dot{\theta})$, and $v_R(= \dot{R})$. By plotting the terminal value of θ against T (fig. 16.53), we see that the collar leaves the rod at exactly the same $\theta = 2.86$ rad for each T! But this is possible only if $\dot{R}$ and $\dot{\theta}$ both change in the same proportion for each T. So, plot the ratio $\dot{R}/\dot{\theta}$ just for the terminal values against T and find that the ratio is indeed constant (see fig. 16.54).



Filename:sfig11-3-slidingpend

Figure 16.55:



Filename:sfig11-3-slidingpend-a

Figure 16.56:

SAMPLE 16.18 Dynamics of a sliding-base pendulum. A cart of mass M slides down a frictionless inclined plane as shown in the figure. A simple pendulum of mass m and length ℓ hangs from the center-of-mass of the cart. Find the equation of motion of the pendulum.

Solution Let us measure the angular displacement of the pendulum with respect to the cart with angle θ measured anticlockwise from the normal to the inclined plane. Let s be the position of the cart along the inclined plane from some reference point. Then, the acceleration of the cart can be written as $\vec{a}_C = \ddot{s}\hat{\lambda}$ and the acceleration of the pendulum mass as $\vec{a} = \vec{a}_C + \vec{a}_{rel} = \ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R$.

The free-body diagram of the cart and the pendulum system is shown in fig. 16.56. Writing angular momentum balance of the system about point C, we get

$$\begin{aligned} \vec{M}_C &= \dot{\vec{H}}_C \\ \ell\hat{e}_R \times (-mg\hat{\jmath}) &= \ell\hat{e}_R \times m(\ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R) \\ -mgl\sin(\theta-\alpha)\hat{k} &= m\ddot{s}\ell\cos\theta\hat{k} + m\ell^2\ddot{\theta}\hat{k} \\ \Rightarrow \quad \ddot{\theta} &= -\frac{g}{\ell}\sin(\theta-\alpha) - \frac{\ddot{s}}{\ell}\cos\theta. \end{aligned} \tag{16.62}$$

To find $\ddot{s}$, we write the linear momentum balance for the whole system in the $\hat{\lambda}$ (so that we do not involve the unknown normal reaction N) direction.

$$\begin{aligned} \hat{\lambda}\cdot(-Mg\hat{\jmath} - mg\hat{\jmath}) &= \hat{\lambda}\cdot\left[M\ddot{s}\hat{\lambda} + m(\ddot{s}\hat{\lambda} + \ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R)\right] \\ \Rightarrow \quad (M+m)g\sin\sin\alpha &= (M+m)\ddot{s} + m\ell\ddot{\theta}\cos\theta - m\ell\dot{\theta}^2\sin\theta \\ \Rightarrow \quad \ddot{s} &= g\sin\alpha - \frac{m\ell}{M+m}(\ddot{\theta}\cos\theta + \dot{\theta}^2\sin\theta) \end{aligned} \tag{16.63}$$

Substituting eqn. (16.63) in eqn. (16.62) and rearranging terms, we get

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta\frac{(1+\frac{m}{M})\cos\alpha}{1+\frac{m}{M}\sin^2\theta} + \frac{\frac{m}{M}\dot{\theta}^2\sin\theta\cos\theta}{1+\frac{m}{M}\sin^2\theta}$$

$$\boxed{\ddot{\theta} = -\frac{g}{\ell}\sin\theta\frac{(1+\frac{m}{M})\cos\alpha}{1+\frac{m}{M}\sin^2\theta} + \frac{\frac{m}{M}\dot{\theta}^2\sin\theta\cos\theta}{1+\frac{m}{M}\sin^2\theta}}$$

Note that if we set $\alpha = 0$ and let $M \to \infty$ so that the cart behaves like a fixed ground, then we recover the equation of simple pendulum, $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$, from the equation of motion above. It is a good practice to carry out such simple checks wherever possible.

Remarks: We could write the equations of motion, eqn. (16.62) and eqn. (16.63) in the coupled form as

$$\begin{bmatrix} \ell & \cos\theta \\ m\ell\cos\theta & M+m \end{bmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} -\frac{g}{\ell}\sin(\theta-\alpha) \\ (M+m)g\sin\alpha - m\ell\dot{\theta}^2\sin\theta \end{pmatrix}$$

and leave it at that, since for most computational purposes, it is enough. It is not so hard to find expressions for $\ddot{\theta}$ and $\ddot{s}$ from here by solving the matrix equation, even by hand.

SAMPLE 16.19 Resonant capture. A slightly unbalanced motor mounted on an elastic machine part is modeled as a spring mass system with a simple pendulum of mass m and length ϵ driven by a constant torque T as shown in the figure. The spring has stiffness k and the motor has mass M. There is no friction between mass M and the horizontal surface.

1. Find the equation of motion of the system.
2. Take $M = m = 1\,\text{kg}$, $k = 1\,\text{N/m}$, $T = 15 \times 10^{-3}\,\text{N·m}$. Solve (numerically) the equations of motion with zero initial conditions and plot $x(t)$ and $\theta(t)$ for $t = 0$ to $100\,\text{s}$.

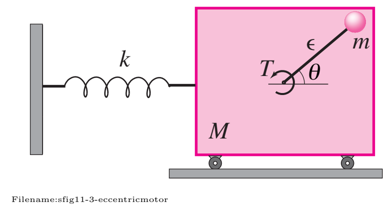


Figure 16.57:

Solution

1. The free-body diagram of the system is shown in fig. 16.58. Let the angular displacement of the eccentric mass m at some instant t be θ. At the same instant, let the displacement of the motor be x from the relaxed state of the spring. Then we can write the acceleration of the motor as $\ddot{x}\hat{\imath}$ and that of the eccentric mass as $\vec{a}_P = \ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R$. Now, we can write the angular momentum balance for the system about point C (fixed in the stationary frame of reference but instantly coincident with the center-of-mass of motor M) as

$$\begin{aligned}
T\hat{k} &= \epsilon\hat{e}_R \times m(\ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R) \\
&= m\epsilon\ddot{x}(\hat{e}_R \times \hat{\imath}) + m\epsilon^2\ddot{\theta}(\hat{e}_R \times \hat{e}_\theta) \\
&= m\epsilon(-\ddot{x}\sin\theta + \epsilon\ddot{\theta})\hat{k} \\
\Rightarrow \quad \epsilon\ddot{\theta} - \sin\theta\,\ddot{x} &= \frac{T}{m\epsilon} \qquad (16.64)
\end{aligned}$$

This is just one scalar equation in $\ddot{\theta}$ and $\ddot{x}$. We need one more independent equation $\ddot{\theta}$ and $\ddot{x}$ without involving any other unknowns. So, we write the linear momentum balance for the system in the x-direction:

$$\begin{aligned}
-kx &= M\ddot{x} + m(\ddot{x}\hat{\imath} + \epsilon\ddot{\theta}\hat{e}_\theta - \epsilon\dot{\theta}^2\hat{e}_R)\cdot\hat{\imath} \\
&= (M+m)\ddot{x} - m\epsilon\ddot{\theta}\sin\theta - m\epsilon\dot{\theta}^2\cos\theta \\
\Rightarrow \quad \epsilon\sin\theta\,\ddot{\theta} - \left(\frac{M+m}{m}\right)\ddot{x} &= \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta. \qquad (16.65)
\end{aligned}$$

Thus we have the required equations of motion. We can write eqn. (16.64) and (16.65) compactly as

$$\begin{bmatrix} \epsilon & -\sin\theta \\ \epsilon\sin\theta & -\frac{m+M}{M} \end{bmatrix} \begin{pmatrix} \ddot{\theta} \\ \ddot{x} \end{pmatrix} = \begin{pmatrix} \frac{T}{m\epsilon} \\ \frac{k}{m}x - \epsilon\dot{\theta}^2\cos\theta. \end{pmatrix}$$

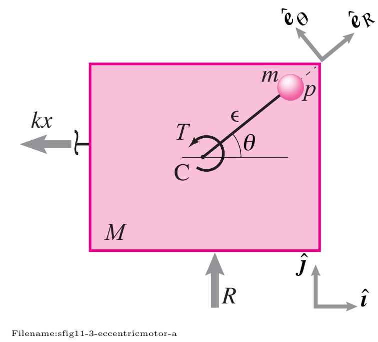


Figure 16.58:

2. We use the following pseudocode to solve the equations of motion. Note that we first convert the two second order ODEs into four first order ODEs by introducing new variables $\omega = \dot{\theta}$ and $u = \dot{x}$.

```
Set T = 0.015, m = 1, M = 1, k = 1, e = 1
A=[e -sin(theta); e*sin(theta) -(1+M/m)];
b = [T/(m*e); k/m*x-e*omega^2*cos(theta)];
solve A*acln = b for acln  % acln = accelerations
ODEs = {omega = thetadot, u  = xdot,
        omegadot = acln(1), udot = acln(2)}
IC   = {theta = 0, x = 0, omega = 0, u = 0}
Solve ODEs with IC for t=0 to t=100
```

The plots of $x(t)$ and $\theta(t)$ obtained from the numerical solution are shown in fig. 16.59 and fig. 16.60, respectively. Note the resonance of M for the given values of the system.

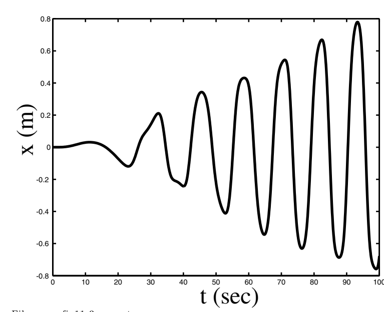


Figure 16.59:

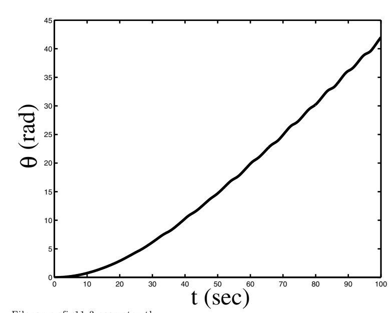


Figure 16.60:

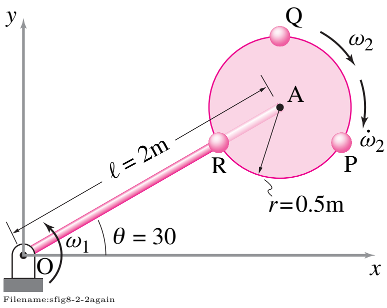


Figure 16.61:

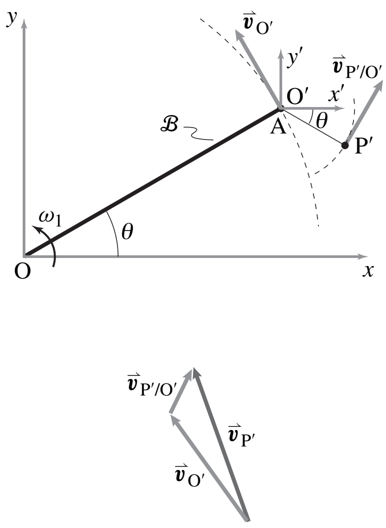


Figure 16.62: The velocity of point P' is the sum of two terms: the velocity of O' and the velocity of P' relative to O'.

SAMPLE 16.20 Dynamics using a rotating and translating coordinate system. Consider the rotating wheel of Sample 15.11 which is shown here again in Figure 16.61. At the instant shown in the figure find

1. the linear momentum of the mass P and
2. the net force on the mass P.

For calculations, use a frame $\mathcal{B}$ attached to the rod and a coordinate system in $\mathcal{B}$ with origin at point A of the rod OA.

Solution We attach a frame $\mathcal{B}$ to the rod. We choose a coordinate system $x'y'z'$ in this frame with its origin O′ at point A. We also choose the orientation of the primed coordinate system to be parallel to the fixed coordinate system xyz (see Fig. 16.62), *i.e.,* $\hat{\imath}' = \hat{\imath}$, $\hat{\jmath}' = \hat{\jmath}$, and $\hat{k}' = \hat{k}$.

1. **Linear momentum of P:** The linear momentum of the mass P is given by

$$\vec{L} = m\vec{v}_P.$$

Clearly, we need to calculate the velocity of point P to find $\vec{L}$. Now,

$$\vec{v}_P = \vec{v}_{P'} + \vec{v}_{\text{rel}} = \underbrace{\vec{v}_{O'} + \vec{v}_{P'/O'}}_{\vec{v}_{P'}} + \vec{v}_{\text{rel}}.$$

Note that O′ and P′ are two points on the same (imaginary) rigid body OAP′. Therefore, we can find $\vec{v}_{P'}$ as follows:

$$\begin{aligned}
\vec{v}_{P'} &= \overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}}^{\vec{v}_{O'}} + \overbrace{\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}}^{\vec{v}_{P'/O'}} \\
&= \omega_1 \hat{k} \times \ell(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + \omega_1 \hat{k} \times r(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) \\
&= \omega_1[(\ell + r)\cos\theta\hat{\jmath} - (\ell - r)\sin\theta\hat{\imath}] \\
&= 3\,\text{rad/s} \cdot [2.5\,\text{m} \cdot \cos 30^\circ\hat{\jmath} - 1.5\,\text{m} \cdot \sin 30^\circ\hat{\imath}] \\
&= (6.50\hat{\jmath} - 2.25\hat{\imath})\,\text{m/s} \quad \text{(same as in Sample 15.11.)},
\end{aligned}$$

$$\begin{aligned}
\vec{v}_{\text{rel}} &= \vec{v}_{P/\mathcal{B}} \\
&= -\omega_2 \hat{k}' \times r(\cos\theta\hat{\imath}' - \sin\theta\hat{\jmath}') \\
&= -\omega_2 r(\cos\theta\hat{\jmath}' + \sin\theta\hat{\imath}') \\
&= -(2.16\hat{\jmath}' + 1.25\hat{\imath}')\,\text{m/s} \\
&= -(2.16\hat{\jmath} + 1.25\hat{\imath})\,\text{m/s}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{v}_P &= \vec{v}_{P'} + \vec{v}_{\text{rel}} \\
&= (4.34\hat{\jmath} - 3.50\hat{\imath})\,\text{m/s} \qquad \text{and} \\
\vec{L} &= m\vec{v}_P \\
&= 0.5\,\text{kg} \cdot (4.34\hat{\jmath} - 3.50\hat{\imath})\,\text{m/s} \\
&= (-1.75\hat{\imath} + 2.17\hat{\jmath})\,\text{kg·m/s}.
\end{aligned}$$

$$\boxed{\vec{L} = (-1.75\hat{\imath} + 2.17\hat{\jmath})\,\text{kg·m/s}}$$

2. **Net force on P:** From the

$$\sum \vec{F} = m\vec{a}$$

for the mass P we get $\sum \vec{F} = m\vec{a}_P$. Thus to find the net force $\sum \vec{F}$ we need to find $\vec{a}_P$. The calculation of $\vec{a}_P$ is the same as in Sample 15.11 except that $\vec{a}_{P'}$ is now calculated from

$$\vec{a}_{P'} = \vec{a}_{O'} + \vec{a}_{P'/O'}$$

where

$$\begin{aligned}
\vec{a}_{O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{O'/O}) \\
&= -\omega_1^2 \vec{r}_{O'/O} \\
&= -\omega_1^2 \ell (\cos\theta \hat{\imath} + \sin\theta \hat{\jmath}) \\
&= -(3\,\mathrm{rad/s})^2 \cdot 2\,\mathrm{m} \cdot (\cos 30^\circ \hat{\imath} + \sin 30^\circ \hat{\jmath}) \\
&= -(15.59\hat{\imath} + 9.00\hat{\jmath})\,\mathrm{m/s^2},
\end{aligned}$$

$$\begin{aligned}
\vec{a}_{P'/O'} &= \vec{\omega}_{\mathcal{B}} \times (\vec{\omega}_{\mathcal{B}} \times \vec{r}_{P'/O'}) \\
&= -\omega_1^2 \vec{r}_{P'/O'} \\
&= -\omega_1^2 r (\cos\theta \hat{\imath} - \sin\theta \hat{\jmath}) \\
&= -(3\,\mathrm{rad/s})^2 \cdot 0.5\,\mathrm{m} \cdot (\cos 30^\circ \hat{\imath} - \sin 30^\circ \hat{\jmath}) \\
&= -(3.90\hat{\imath} - 2.25\hat{\jmath})\,\mathrm{m/s^2}.
\end{aligned}$$

Thus,

$$\vec{a}_{P'} = -(19.49\hat{\imath} + 6.75\hat{\jmath})\,\mathrm{m/s^2}$$

which, of course, is the same as calculated in Sample 15.11. The other two terms, $\vec{a}_{\mathrm{cor}}$ and $\vec{a}_{\mathrm{rel}}$, are exactly the same as in Sample 15.11. Therefore, we get the same value for $\vec{a}_P$ by adding the three terms:

$$\vec{a}_P = -(17.83\hat{\imath} + 3.63\hat{\jmath})\,\mathrm{m/s^2}.$$

The net force on P is

$$\begin{aligned}
\sum \vec{F} &= m\vec{a}_P \\
&= 0.5\,\mathrm{kg} \cdot (-17.83\hat{\imath} - 3.63\hat{\jmath})\,\mathrm{m/s^2} \\
&= -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\mathrm{N}.
\end{aligned}$$

$$\boxed{\sum \vec{F} = -(8.92\hat{\imath} + 1.81\hat{\jmath})\,\mathrm{N}}$$

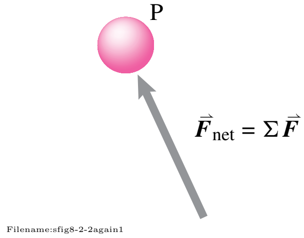
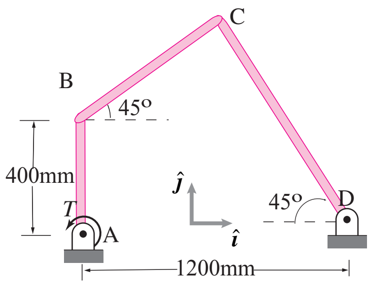


Figure 16.63:



Filename:sfig11-4-fourbar

Figure 16.64:

SAMPLE 16.21 Inverse dynamics of a four bar mechanism. A four bar mechanism ABCD consists of three uniform bars AB, BC, and CD of length ℓ_1, ℓ_2, ℓ_3, and mass m_1, m_2, m_3, respectively. The mechanism is driven by a torque T applied at A such that bar AB rotates at constant angular speed. Write equations to find the torque T at some instant t.

Solution This is an inverse dynamics problem, that is, we are given the motion and we are supposed to find the forces (torque T in this case) that cause that motion. We are given that rod AB rotates at constant angular speed, say $\dot{\theta}$. From kinematics, we can find out angular velocities and angular accelerations of the other two bars as well as the accelerations of center-of-mass of each rod. Then we can write the momentum balance equations and compute the forces and moments required to generate this motion. So, in contrast to what we usually do, let us do the kinematics first. Please see Sample 15.5 on page 898. We found the angular velocities, $\dot{\beta}$ (eqn. (15.69)) and $\dot{\phi}$ (eqn. (15.68)), of rods BC and CD, respectively, in terms of $\dot{\theta}$. We can rewrite those equations as

$$\begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix} \dot{\beta} \\ \dot{\phi} \end{pmatrix} = \dot{\theta}\begin{pmatrix} \ell_1\sin\theta \\ \ell_1\cos\theta \end{pmatrix} \tag{16.66}$$

We wrote this equation in matrix form to make it easier for us to find the angular accelerations which we do by simply differentiating this equation once:

$$\begin{aligned} \begin{bmatrix} -\ell_2\cos\beta\,\dot{\beta} & \ell_3\cos\phi\,\dot{\phi} \\ \ell_2\sin\beta\,\dot{\beta} & -\ell_3\sin\phi\,\dot{\phi} \end{bmatrix}\begin{pmatrix} \dot{\beta} \\ \dot{\phi} \end{pmatrix} &+ \begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix} \ddot{\beta} \\ \ddot{\phi} \end{pmatrix} \\ &= \ell_1\begin{pmatrix} \ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta \\ \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta \end{pmatrix} \end{aligned}$$

Rearranging terms we get

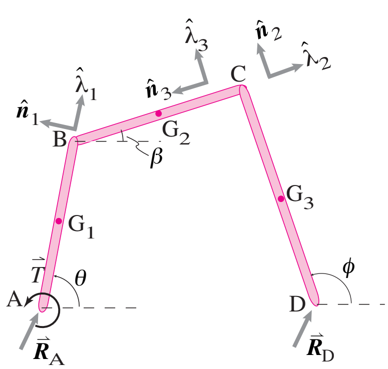
$$\begin{aligned} \begin{bmatrix} -\ell_2\sin\beta & \ell_3\sin\phi \\ -\ell_2\cos\beta & \ell_3\cos\phi \end{bmatrix}\begin{pmatrix} \ddot{\beta} \\ \ddot{\phi} \end{pmatrix} &= -\begin{bmatrix} -\ell_2\cos\beta & \ell_3\cos\phi \\ \ell_2\sin\beta & \ell_3\sin\phi \end{bmatrix}\begin{pmatrix} \dot{\beta}^2 \\ \dot{\phi}^2 \end{pmatrix} \\ &\quad +\ell_1\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{pmatrix} \ddot{\theta} \\ \dot{\theta}^2 \end{pmatrix} \end{aligned} \tag{16.67}$$

Thus, we can find the angular accelerations of BC and CD, $\ddot{\beta}$ and $\ddot{\phi}$, because the quantities on the right hand side are known ($\ddot{\theta}(=0)$ and $\dot{\theta}$ are given, and $\dot{\beta}$ and $\dot{\phi}$ are determined by eqn. (16.66)). Now, we can find the accelerations of center-of-mass of each rod as follows.

$$\vec{a}_{G_1} = -\frac{\ell_1}{2}\dot{\theta}^2\hat{\boldsymbol{\lambda}}_1 \tag{16.68}$$

$$\vec{a}_{G_2} = \vec{a}_B + \vec{a}_{G_2/B} = -\ell_1\dot{\theta}^2\hat{\boldsymbol{\lambda}}_1 - \frac{\ell_2}{2}\dot{\beta}^2\hat{\boldsymbol{\lambda}}_2 + \ell_2\ddot{\beta}\hat{\boldsymbol{n}}_2 \tag{16.69}$$

$$\vec{a}_{G_3} = -\frac{\ell_3}{2}\dot{\phi}^2\hat{\boldsymbol{\lambda}}_3 \tag{16.70}$$



Filename:sfig11-4-fourbar-fbd1

Figure 16.65:

We are now ready to write momentum balance equations. Since we are only interested in finding the torque T, we should try to write equations involving minimum number of unknown forces. So, we draw free-body diagrams of the whole mechanism, of part BCD, and of bar CD alone; and write angular momentum balance equations about appropriate points so that we involve only the unknown torque T and the unknown reaction $\vec{R}_D$ at D. Thus, we will have only three scalar unknowns T, R_{D_x} and R_{D_y} (since $\vec{R}_D = R_{D_x}\hat{\imath} + R_{D_y}\hat{\jmath}$). So, we will need only three independent equations.

Consider the free-body diagram of the whole mechanism. We cab write angular momentum balance about point A for the whole mechanism as

$$T\hat{\boldsymbol{k}} + \vec{r}_{D/A}\times\vec{R}_D = \dot{\vec{H}}_A = \dot{\vec{H}}_{1/A} + \dot{\vec{H}}_{2/A} + \dot{\vec{H}}_{3/A} \tag{16.71}$$

where

$$\begin{aligned}
\dot{\vec{H}}_{1/\mathrm{A}} &= I_1\ddot{\theta}\hat{\boldsymbol{k}} + \vec{r}_{\mathrm{G}_1} \times m_1\vec{\boldsymbol{a}}_{\mathrm{G}_1} \\
\dot{\vec{H}}_{2/\mathrm{A}} &= I_2\ddot{\beta}\hat{\boldsymbol{k}} + \vec{r}_{\mathrm{G}_2} \times m_2\vec{\boldsymbol{a}}_{\mathrm{G}_2} = I_2\ddot{\beta}\hat{\boldsymbol{k}} + (\vec{r}_{\mathrm{B}} + \vec{r}_{\mathrm{G}_2/\mathrm{B}}) \times m_2\vec{\boldsymbol{a}}_{\mathrm{G}_2} \\
\dot{\vec{H}}_{3/\mathrm{A}} &= I_3\ddot{\phi}\hat{\boldsymbol{k}} + \vec{r}_{\mathrm{G}_3} \times m_3\vec{\boldsymbol{a}}_{\mathrm{G}_3} = I_3\ddot{\phi}\hat{\boldsymbol{k}} + (\vec{r}_{\mathrm{D}} + \vec{r}_{\mathrm{G}_3/\mathrm{D}}) \times m_3\vec{\boldsymbol{a}}_{\mathrm{G}_3}.
\end{aligned}$$

Similarly, the angular momentum balance about point B for BCD gives

$$\vec{r}_{\mathrm{D/B}} \times \vec{\boldsymbol{R}}_{\mathrm{D}} = \dot{\vec{H}}_{2/\mathrm{B}} + \dot{\vec{H}}_{3/\mathrm{B}} \tag{16.72}$$

where

$$\begin{aligned}
\dot{\vec{H}}_{2/\mathrm{B}} &= I_2\ddot{\beta}\hat{\boldsymbol{k}} + \vec{r}_{\mathrm{G}_2/\mathrm{B}} \times m_2\vec{\boldsymbol{a}}_{\mathrm{G}_2} \\
\dot{\vec{H}}_{3/\mathrm{B}} &= I_3\ddot{\phi}\hat{\boldsymbol{k}} + \vec{r}_{\mathrm{G}_3/\mathrm{B}} \times m_3\vec{\boldsymbol{a}}_{\mathrm{G}_3}
\end{aligned}$$

and angular momentum balance of bar CD about point C gives

$$\vec{r}_{\mathrm{D/C}} \times \vec{\boldsymbol{R}}_{\mathrm{D}} = \dot{\vec{H}}_{3/\mathrm{C}} = I_3\ddot{\phi}\hat{\boldsymbol{k}} + \vec{r}_{\mathrm{G}_3/\mathrm{C}} \times m_3\vec{\boldsymbol{a}}_{\mathrm{G}_3}. \tag{16.73}$$

Note that we can easily write the position vectors in terms of ℓ_1, ℓ_2, ℓ_3 and the unit vectors $(\hat{\boldsymbol{\lambda}}_1, \hat{\boldsymbol{n}}_1)$, $(\hat{\boldsymbol{\lambda}}_2, \hat{\boldsymbol{n}}_2)$ and $(\hat{\boldsymbol{\lambda}}_3, \hat{\boldsymbol{n}}_3)$ where

$$\begin{aligned}
\hat{\boldsymbol{\lambda}}_1 &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}, & \hat{\boldsymbol{n}}_1 &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} \\
\hat{\boldsymbol{\lambda}}_2 &= \cos\beta\hat{\boldsymbol{\imath}} + \sin\beta\hat{\boldsymbol{\jmath}}, & \hat{\boldsymbol{n}}_2 &= -\sin\beta\hat{\boldsymbol{\imath}} + \cos\beta\hat{\boldsymbol{\jmath}} \\
\hat{\boldsymbol{\lambda}}_1 &= \cos\phi\hat{\boldsymbol{\imath}} + \sin\phi\hat{\boldsymbol{\jmath}}, & \hat{\boldsymbol{n}}_1 &= -\sin\phi\hat{\boldsymbol{\imath}} + \cos\phi\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

We can put all the three angular momentum balance equations, (16.71), (16.72), and (16.73), in one matrix equation by dotting both sides of the equations with $\hat{\boldsymbol{k}}$ and assembling them as follows.

$$\begin{bmatrix} 1 & 0 & \ell_4 \\ 0 & \hat{\boldsymbol{k}}\cdot(\ell_2\hat{\boldsymbol{\lambda}}_2 - \ell_3\hat{\boldsymbol{\lambda}}_3)\times\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{k}}\cdot(\ell_2\hat{\boldsymbol{\lambda}}_2 - \ell_3\hat{\boldsymbol{\lambda}}_3)\times\hat{\boldsymbol{\jmath}} \\ 0 & \hat{\boldsymbol{k}}\cdot(-\ell_3\hat{\boldsymbol{\lambda}}_3\times\hat{\boldsymbol{\imath}}) & \hat{\boldsymbol{k}}\cdot(-\ell_3\hat{\boldsymbol{\lambda}}_3\times\hat{\boldsymbol{\jmath}}) \end{bmatrix} \begin{pmatrix} T \\ R_{D_x} \\ R_{D_y} \end{pmatrix} = \begin{pmatrix} \dot{H}_{123/A} \\ \dot{H}_{23/B} \\ \dot{H}_{3/C} \end{pmatrix} \tag{16.74}$$

where $\dot{H}_{123/A} = \hat{\boldsymbol{k}}\cdot(\dot{\vec{H}}_{1/\mathrm{A}} + \dot{\vec{H}}_{2/\mathrm{A}} + \dot{\vec{H}}_{3/\mathrm{A}})$, $\dot{H}_{23/B} = \hat{\boldsymbol{k}}\cdot(\dot{\vec{H}}_{2/\mathrm{B}} + \dot{\vec{H}}_{3/\mathrm{B}})$, and $\dot{H}_{3/C} = \hat{\boldsymbol{k}}\cdot\dot{\vec{H}}_{3/\mathrm{C}}$.

Note that we know the $\dot{\vec{H}}$'s on the right hand side and the matrix on the left side can be evaluated for any given (θ, β, ϕ). Thus we can solve for T, R_{D_x}, and R_{D_y}.

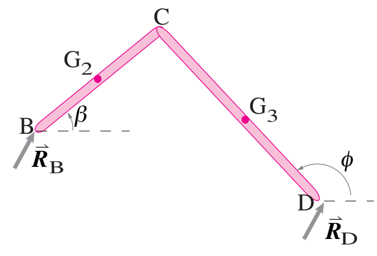


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Figure 16.66:

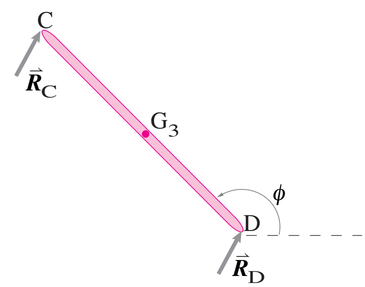

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Figure 16.67:

SAMPLE 16.22 Numerical solution of the inverse dynamics problem. Consider Sample 16.21 again. Using numerical solutions on a computer, find and plot torque T against θ for one complete cycle of the drive arm AB. Take $m_1 = m_2 = m_3 = 1\,\text{kg}$ and $\ell_1 = 400\,\text{mm}$, $\ell_2 = 400\sqrt{2}\,\text{mm}$, $\ell_3 = 800\sqrt{2}\,\text{mm}$, and $\ell_4 = 1200\,\text{mm}$.

Solution

Since we have to plot T against θ for one complete revolution, we need to find angular velocities, angular accelerations and center-of-mass accelerations for several values of θ and then solve for T for each of those θ's. We can do this several ways. One way would be to first solve kinematic equations to find $\theta(t)$, $\beta(t)$, and $\phi(t)$ at discrete times over one complete cycle and then compute all other quantities at each $(\theta(t_i), \beta(t_i), \phi(t_i))$ where t_i represents a discrete time. So, let us follow this method step by step with pseudocodes. Here, we assume that we have vector functions called `dot` and `cross` that compute the dot product and the cross product of two vectors that are given as input arguments.

Step-1: solve for angular positions. Specify the given geometry

```
L1=0.4, L2=0.4*sqrt(2), L3=0.8*sqrt(2), L4=1.2
```

and use the pseudocode of Sample 15.5 to find $\theta(t_i), \beta(t_i), \phi(t_i)$ for, say, 100 values of t_i between 0 and 1 sec. Now, for each triad of $(\theta(t_i), \beta(t_i), \phi(t_i))$, follow all the steps below.

Step-2: solve for angular velocities. Since $\dot{\theta} = 2\pi\,\text{rad/s}$ is given, we only need to solve for $\dot{\beta}$ and $\dot{\phi}$. We use eqn. (15.69) and eqn. (15.68) to compute $\dot{\beta}$ and $\dot{\phi}$ as follows (or modify the pseudocode of Sample 15.5 to save $\dot{\beta}$ and $\dot{\phi}$ along with the values for β and ϕ.

```
define thdot=thetadot, bdot=betadot, pdot=phidot
thdot = 2*pi   % this is given
set th = theta(ti), b = beta(ti), p = phi(ti)
set unit vectors
  l1=[cos(th) sin(th) 0]', n1=[-sin(th) cos(th) 0]'
  l2=[cos(b) sin(b) 0]', n2=[-sin(b) cos(b) 0]'
  l3=[cos(p) sin(p) 0]', n3=[-sin(p) cos(p) 0]'
bdot = -(L1/L2)*(cross(n1,l3)/cross(n2,l3))*thdot
pdot = (L1/L3)*(cross(n1,l2)/cross(n3,l2))*thdot
```

Step-3: solve for angular accelerations. Now that we have (θ, β, ϕ) and the corresponding values of $(\dot{\theta}, \dot{\beta}, \dot{\phi})$, we can use eqn. (16.67) to calculate $\ddot{\beta}$ and $\ddot{\phi}$ (we are given $\ddot{\theta} = 0$).

```
define thddot=thetaddot, bddot=betaddot,
       pddot=phiddot
thddot = 0   % this is given
B = [-L2*sin(b) L3*sin(p); -L2*cos(b) L3*cos(p)]
C = L1*[sin(th) cos(th); cos(th) -sin(th)]
D = [-cos(b) cos(p); sin(b) -sin(p)]
c = [thddot thdot^2]',  d = [L2*bdot^2 L3*pdot^2]'
assume w = [bddot pddot]'
solve B*w = C*c + D*d for w
```

So, now we know $\ddot{\theta}, \ddot{\beta}, \ddot{\phi}$ also. We are now ready to compute $\dot{\vec{H}}$'s required for dynamic calculations.

Step-4: set up equations and solve for unknown forces. We need to set up and solve eqn. (16.74). Note that we need to compute several quantities for this equation but the vector computations are more or less straightforward.

```
% set mass and inertia properties
m1 = 1, m2 = 1, m3 = 1
I1 = m1*L1^2/12, I2 = m2*L2^2/12, I3 = m3*L3^2/12
% set fixed unit vectors

i = [1 0 0]', j = [0 1 0]', k = [0 0 1]'
% compute position vectors
rA = [0;0;0],  rB = rA+L1*l1
rC = rB+L2*l2, rD = L4*l4
rG1 = L1/2*l1
rG2 = rB+L2/2*l2
rG3 = rD+L3/2*l3
% compute center-of-mass accelerations
aG1 = 0.5*L1*(tddot*n1-tdot^2*l1)
aG2 = 2*aG1+0.5*L2*(bddot*n2-bdot^2*l2)     % aB = 2*aG1
aG3 = 0.5*L3*(pddot*n3-pdot^2*l3)

% compute Hdot_cms

Hdot_cm1 = I1*tddot*uk
Hdot_cm2 = I2*bddot*uk
Hdot_cm3 = I3*pddot*uk
% compute Hdots
Hdot_123_A = Hdot_cm1 + cross(rG1, m1*aG1)
              + Hdot_cm2 + cross(rG2, m2*aG2)
                + Hdot_cm3 + cross(rG3, m3*aG3)
Hdot_23_B = Hdot_cm2 + cross(rG2-rB, m2*aG2)
              + Hdot_cm3 + cross(rG3-rB, m3*aG3)
Hdot_3_C = Hdot_cm3 + cross(rG3-rC, m3*aG3)
% set up the linear eqns for torque and RD
b = [dot(Hdot_123_A,k) dot(Hdot_23_B,k) dot(Hdot_3_C,k)]
A = [1  dot(k,cross(rD,i))      dot(k,cross(rD,j))
     0  dot(k,cross(rD-rB,i))   dot(k,cross(rD-rB,j))
     0  dot(k,cross(rD-rC,i))   dot(k,cross(rD-rC,j))]
% let forces = [T  RDx  RDy]'
solve A*forces = b for forces
```

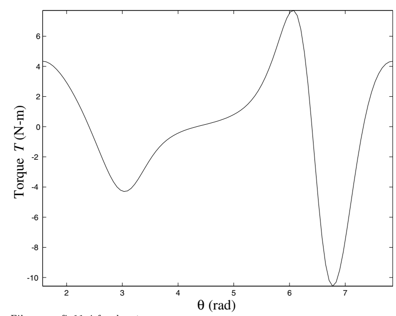


Figure 16.68: Torque T as a functions of θ over one complete cycle of motion ($\theta(0) = \pi/2, \theta(1) = 5\pi/2$).

Step-5, repeat calculations. Now repeat Step-2 – Step-4 for each triad (θ, β, ϕ) obtained in Step-1 and save the corresponding values of T in a vector. Finally,

```
plot T vs theta
```

The plot thus obtained is shown in fig. 16.68. We can also plot T vs time (as shown in fig. 16.69), and, of course, expect to see the same graph of T since θ is just a linear function of t. Note that the area under the graph of T over one complete cycle must equal zero since the net impulse must be zero over one cycle.

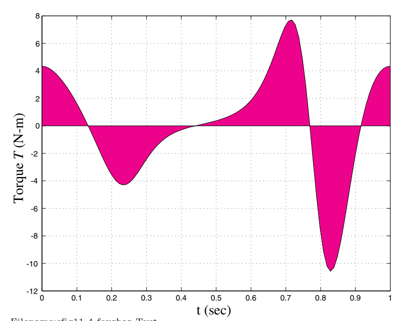


Figure 16.69: Torque T as a functions of time over one complete cycle of motion.

Problems for Chapter 16

Constrained particles and rigid objects

16.1 Mechanics of a constrained particle

16.1.1 A bead slides on a frictionless circular hoop . The mass of the bead $m_{\text{bead}} = 2$ grams, mass of hoop $m_{\text{hoop}} = 1$ kg and radius of hoop $R_{\text{hoop}} = 3$ m. Neglect gravity. The center of the circular hoop is the origin O of a fixed (Newtonian) coordinate system $Oxyz$. The hoop is on the xy plane. The hoop is kept from moving by little angels who let the bead slide by unimpeded. At $t = 0$, the speed of the bead is $4\,\text{m/s}$, it is traveling counterclockwise (looking down the z-axis), and it is on the $+x$-axis. There are no other external forces applied.

a) At $t = 0$ what is the bead's kinetic energy ?

b) At $t = 0$ what is the bead's linear momentum ?

c) At $t = 0$ what is the bead's angular momentum about the origin?

d) At $t = 0$ what is the bead's acceleration?

e) At $t = 0$ what is the radius of the osculating circle of the bead's path ρ?

f) At $t = 0$ what is the force of the hoop on the bead?

g) At $t = 0$ what is the net force of the angel's hands on the hoop?

h) At $t = 27s$ what is the x-component of the bead's linear momentum?

16.1.2 A warehouse operator wants to move a crate of weight $W = 100$ lb from a 6 ft high platform to ground level by means of a roller conveyer as shown. The rollers on the inclined plane are well lubricated and thus assumed frictionless; the rollers on the horizontal conveyer are frictional, thus providing an effective friction coefficient μ. Assume the rollers are massless.

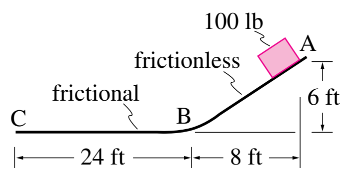
a) What are the kinetic and potential energies(pick a suitable datum) of the crate at the elevated platform, point A?

b) What are the kinetic and potential energies of the cart at the end of the inclined plane just before it moves onto the horizontal conveyor, point B.

c) Using conservation of energy, calculate the maximum speed attained by the crate at point B, assuming it starts moving from rest at point A.

d) If the crate slides a distance x, say, on the horizontal conveyor, what is the energy lost to sliding in terms of μ, W, and x?

e) Calculate the value of the coefficient of friction, μ, such that the crate comes to rest at point C.



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Problem 16.1.2

16.1.3 On a wintry evening, a student of mass m starts down the steepest street in town, *Steep Street*, which is of height h and slope θ. At the top of the slope she starts to slide (coefficient of dynamic friction μ).

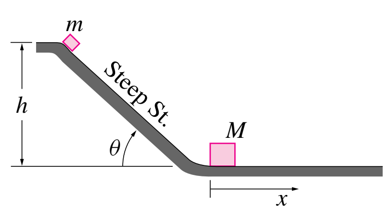
a) What is her initial kinetic and potential energy?

b) What is the energy lost to friction as she slides down the hill?

c) What is her velocity on reaching the bottom of the hill? Ignore air resistance and all cross streets (i.e. assume the hill is of constant slope).

d) If, upon reaching the bottom of the street, she collides with another student of mass M and they embrace, what is their instantaneous mutual velocity just after the embrace?

e) Assuming that friction still acts on the level flats on the bottom, how much time will it take before they come to rest?



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Problem 16.1.3

16.1.4 Reconsider the system of blocks in problem 3.1.14, this time with equal mass, $m_1 = m_2 = m$. Also, now, both blocks are frictional and sitting on a frictional surface. Assume that both blocks are sliding to the right with the top block moving faster. The coefficient of friction between the two blocks is μ_1. The coefficient of friction between the lower block and the floor is μ_2. It is known that $\mu_1 > 2\mu_2$.

a) Draw free body diagrams of the blocks together and separately.

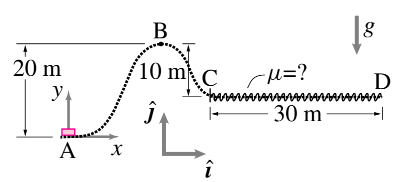
b) Write the equations of linear momentum balance for each block.

c) Find the acceleration of each block for the case of frictionless blocks. For the frictional blocks, find the acceleration of each block. What happens if $\mu_1 >> \mu_2$?

16.1.5 An initially motionless roller-coaster car of mass 20 kg is given a horizontal impulse $\int \vec{F}\,dt = P\hat{\imath}$ at position A, causing it to move along the track, as shown below.

a) Assuming that the track is perfectly frictionless from point A to C and that the car never leaves the track, determine the magnitude of the impulse I so that the car "just makes it" over the hill at B.

b) In the ensuing motion, assume that the horizontal track C-D is frictional and determine the coefficient of friction required to bring the car to a stop at D.



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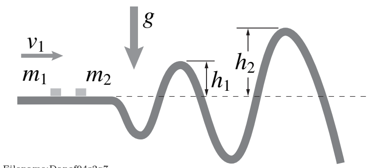
Problem 16.1.5

16.1.6 Masses $m_1 = 1$ kg and m_2 kg move on the frictionless varied terrain shown. Initially, m_1 has speed $v_1 = 12\,\text{m/s}$ and m_2 is at rest. The two masses collide on the flat section. The coefficient of restitution in the collision is $e = 0.5$. Find the speed of m_2 at the top of the first hill of elevation $h_1 = 1$ m. Does m_2 make it over the second hill of elevation $h_2 = 4$ m?



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Problem 16.1.6

16.1.7 A **Scotch yoke** is a device for converting rotary motion into linear motion. In this case it is used to make a horizontal platform go up and down. The motion of the platform is $y = A\sin(\omega t)$ with constant ω. A dead bug with mass m is standing on the platform. Her feet have no glue on them. There is gravity.

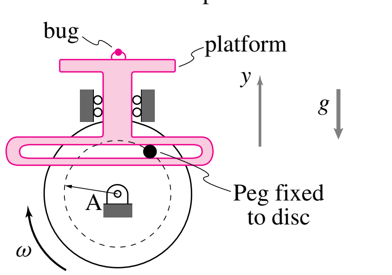
a) Draw a free-body diagram of the bug.

b) What is the acceleration of the platform and, hence, the bug?

c) Write the equation of linear momentum balance for the bug.

d) What condition must be met so that the bug does not bounce the platform?

e) What is the maximum spin rate ω that can be used if the bug is not to bounce off the platform?



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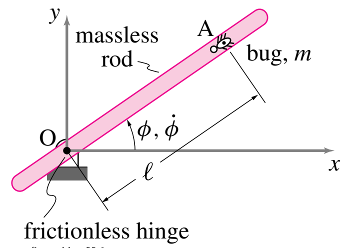
Problem 16.1.7

16.1.8 A bug of mass m walks straight-forwards with speed v_A and rate of change of speed $\dot{v}_A$ on a straight light (assumed to be massless) stick. The stick is hinged at the origin so that it is always horizontal but is free to rotate about the z axis. Assume that the distance the bug is from the origin, ℓ, the angle the stick makes with the x axis, ϕ, and its rate of change, $\dot{\phi}$, are known at the instant of interest. Ignore gravity. Answer the following questions in terms of ϕ, $\dot{\phi}$, ℓ, m, v_A, and $\dot{v}_A$.

a) What is $\ddot{\phi}$?

b) What is the force exerted by the rod on the support?

c) What is the acceleration of the bug?



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Problem 16.1.8

16.1.9 A new kind of gun. Assume $\omega = \omega_0$ is a constant for the rod in the figure. Assume the mass is free to slide. At $t = 0$, the rod is aligned with the x-axis and the bead is one foot from the origin and has no radial velocity ($dR/dt = 0$).

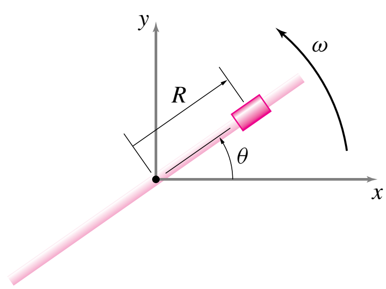
a) Find a differential equation for $R(t)$. *

b) Turn this equation into a differential equation for $R(\theta)$. *

c) How far will the bead have moved after one revolution of the rod? How far after two? *

d) What is the speed of the bead after one revolution of the rod (use $\omega_0 = 2\pi$ rad/s $= 1$ rev/sec)? *

e) How much kinetic energy does the bead have after one revolution and where did it come from? *



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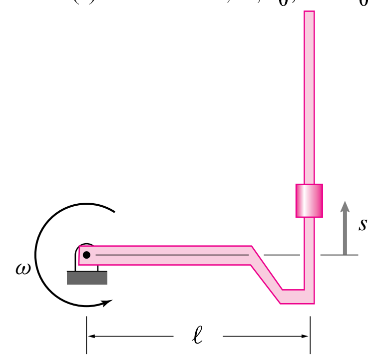
Problem 16.1.9

16.1.10 The new gun gets old and rusty. Reconsider the bead on a rod in problem 16.1.9. This time, friction cannot be neglected. The friction coefficient is μ. At the instant of interest the bead with mass m has radius R_0 with rate of change $\dot{R}_0$. The angle θ is zero and ω is a constant. Neglect gravity.

a) What is $\ddot{R}$ at this instant? Give your answer in terms of any or all of R_0, $\dot{R}_0$, ω, m, μ, $\hat{\boldsymbol{e}}_\theta$, $\hat{\boldsymbol{e}}_R$, $\hat{\boldsymbol{\imath}}$, and $\hat{\boldsymbol{\jmath}}$. *

b) After a very long time it is observed that the angle ϕ between the path of the bead and the rod/trough is nearly constant. What, in terms of μ, is this ϕ? *

16.1.11 A newer kind of gun. As an attempt to make an improvement on the 'new gun' demonstrated in problem 16.1.9, a person adds a length ℓ to the shaft on which the bead slides. Assume there is no friction between the bead (mass m) and the wire. Assume the bead starts at $s = 0$ with $\dot{s} = v_0$. The rigid rod, on which the bead slides, rotates at a constant rate $\omega = \omega_0$. Find $s(t)$ in terms of ℓ, m, v_0, and ω_0.



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Problem 16.1.11

16.1.12 Slippery bead on straight rotating stick. A long stick with mass m_s rotates at a constant rate and a bead, modeled as a point mass, slides on the stick. The stick rotates at constant rate $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. Neglect gravity. The bead has mass m_b. Initially the mass is at a distance R_0 from the hinge point on the stick and has no radial velocity ($\dot{R} = 0$). The initial angle of the stick is $\theta = 0$, measured counterclockwise from the positive x axis.

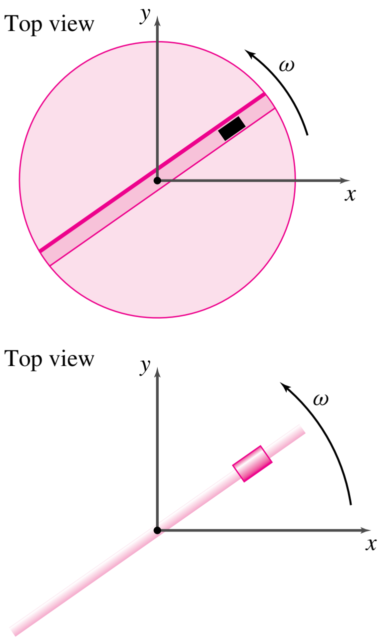
a) What is the torque, as a function of the net angle the stick has rotated, θ, required in order to keep the stick rotating at a constant rate ?

b) What is the path of the bead in the xy-plane? (Draw an accurate picture showing about one half of one revolution.)

c) How long should the stick be if the bead is to fly in the negative x-direction when it gets to the end of the stick?

d) **Add friction.** How does the speed of the bead over one revolution depend on μ, the coefficient of friction between the bead and the wire? Make a plot of $|\vec{\boldsymbol{v}}_{\text{one revolution}}|$ versus μ. [Note, you have been working with $\mu = 0$ in the problems above.]

16.1.13 Mass on a lightly greased slotted turntable or spinning uniform rod. Assume that the rod/turntable in the figure is massless and also free to rotate. Assume that at $t = 0$, the angular velocity of the rod/turntable is 1 rad/s, that the radius of the bead is one meter, and that the radial velocity of the bead, dR/dt, is zero. The bead is free to slide on the rod. Where is the bead at $t = 5$ sec?



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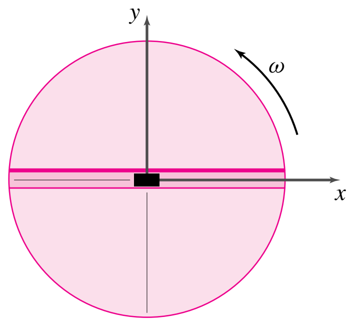
Problem 16.1.13

16.1.14 A bead slides in a frictionless slot in a turntable. The turntable spins a constant rate ω. The slot is straight and goes through the center of the turntable. If the bead is at radius R_o with $\dot{R} = 0$ at $t = 0$, what are the components of the acceleration vector in the directions normal and tangential to the path of the bead after one revolution? Neglect gravity.

16.1.15 A bead of mass $m = 1$ gm bead is constrained to slide in a straight frictionless slot in a disk which is spinning counterclockwise at constant rate $\omega = 3$ rad/s per second. At time $t = 0$, the slot is parallel to the x-axis and the bead is in the center of the disk moving out (in the plus x direction) at a rate $\dot{R}_o = .5$ m/s. After a net rotation θ of one and one eighth (1.125) revolutions, what is the force $\vec{F}$ of the disk on the bead? Express this answer in terms of $\hat{\imath}$ and $\hat{\jmath}$. Make the unreasonable assumption that the slot is long enough to contain the bead for this motion.



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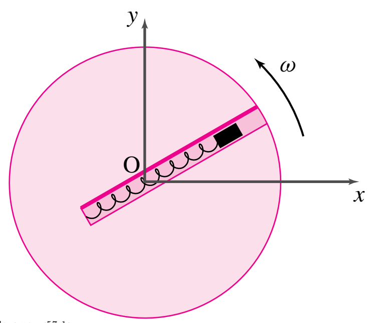
Problem 16.1.15

16.1.16 Bead on springy leash in a slot on a turntable. The bead in the figure is held by a spring that is relaxed when the bead is at the origin. The constant of the spring is k. The turntable speed is controlled by a strong stiff motor.

a) Assume $\omega = 0$ for all time. What are possible motions of the bead?

b) Assume $\omega = \omega_0$ is a constant. What are possible motions of the bead? Notice there are two cases depending on the value of ω_0. What is going on here?



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Problem 16.1.16: Mass in a slot.

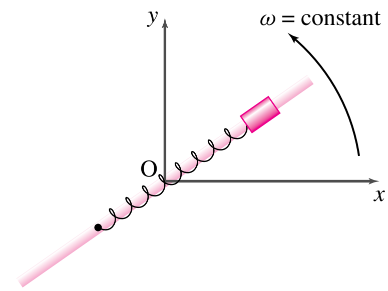
16.1.17 A small bead with mass m slides without friction on a rigid rod which rotates about the z axis with constant ω (maintained by a stiff motor not shown in the figure). The bead is also attached to a spring with constant K, the other end of which is attached to the rod. The spring is relaxed when the bead is at the center position. Assume the bead is pulled to a distance d from the center of the rod and then released with an initial $\dot{R} = 0$. If needed, you may assume that $K > m\omega^2$.

a) Derive the equation of motion for the position of the bead $R(t)$.

b) How is the motion affected by large versus small values of K?

c) What is the magnitude of the force of the rod on the bead as the bead passes through the center position? Neglect gravity. Answer in terms of m, ω, d, K.

d) Write an expression for the Coriolis acceleration. Give an example of a situation in which this acceleration is important and explain why it arises.



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Problem 16.1.17

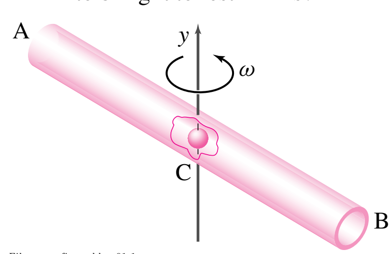
16.1.18 A small ball of mass $m_b = 500$ grams may slide in a slender tube of length $l = 1.2$ m. and of mass $m_t = 1.5$ kg. The tube rotates freely about a vertical axis passing through its center C. (Hint: Treat the tube as a slender rod.)

a) If the angular velocity of the tube is $\omega = 8$ rad/s as the ball passes through C with speed relative to the tube, v_0, calculate the angular velocity of the tube just before the ball leaves the tube. *

b) Calculate the angular velocity of the tube just after the ball leaves the tube. *

c) If the speed of the ball as it passes C is $v = 1.8$ m/s, determine the transverse and radial components of velocity of the ball as it leaves the tube. *

d) After the ball leaves the tube, what constant torque must be applied to the tube (about its axis of rotation) to bring it to rest in 10 s? *



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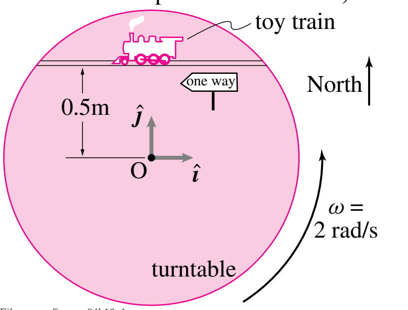
Problem 16.1.18

16.1.19 Toy train car on a turn-around. The 0.1 kg toy train's speedometer reads a constant $1 m/s$ when, heading west, it passes due north of point O. The train is on a level straight track which is mounted on a spinning turntable whose center is 'pinned' to the ground. The turntable spins at the constant rate of 2 rad/s. What is the force of the turntable on the train? (Don't worry about the z-component of the force.) *



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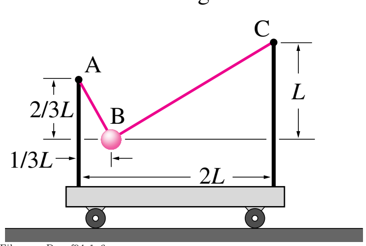
Problem 16.1.19

16.1.20 Due to forces not shown, the cart moves to the right with constant acceleration a_x. The ball B has mass m_B. At time $t = 0$, the string AB is cut. Find

a) the tension in string BC before cutting, *

b) the absolute acceleration of the mass at the instant of cutting, *

c) the tension in string BC at the instant of cutting. *



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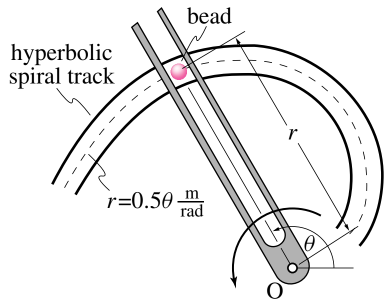
Problem 16.1.20

16.1.21 The forked arm mechanism pushes the bead of mass 1 kg along a frictionless hyperbolic spiral track given by $r = 0.5\theta(\mathrm{m}/\mathrm{rad})$. The arm rotates about its pivot point at O with constant angular acceleration $\ddot{\theta} = 1\,\mathrm{rad/s}^2$ driven by a motor (not shown). The arm starts from rest at $\theta = 0^{\circ}$.

a) Determine the radial and transverse components of the acceleration of the bead after 2 s have elapsed from the start of its motion.

b) Determine the magnitude of the net force on the mass at the same instant in time.



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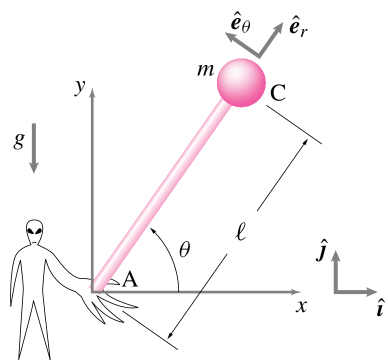
Problem 16.1.21

16.1.22 A rod is on the palm of your hand at point A. Its length is ℓ. Its mass m is assumed to be concentrated at its end at C. Assume that you know θ and $\dot{\theta}$ at the instant of interest. Also assume that your hand is accelerating both vertically and horizontally with $\vec{a}_{\text{hand}} = a_{hx}\hat{\imath} + a_{hy}\hat{\jmath}$. Coordinates and directions are as marked in the figure.

a) Draw a Free Body Diagram of the rod.

b) Assume that the hand is stationary. Solve for $\ddot{\theta}$ in terms of g, ℓ, m, θ, and $\dot{\theta}$.

c) If the hand is not stationary but $\ddot{\theta}$ has been determined somehow, find the vertical force of the hand on the rod in terms of $\ddot{\theta}$, $\dot{\theta}$, θ, g, ℓ, m, a_{hx}, and a_{hy}.



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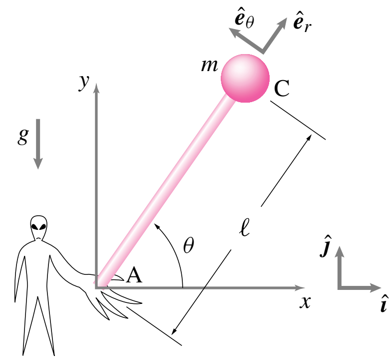
Problem 16.1.22: Rod on a Moving Support(2-D).

16.1.23 Balancing a broom. Assume the hand is accelerating to the right with acceleration $\boldsymbol{a} = a\hat{\imath}$. What is the force of the hand on the broom in terms of m, ℓ, θ, $\dot{\theta}$, a, $\hat{\imath}$, $\hat{\jmath}$, and g? *(You may not have any $\hat{e}_R$ or $\hat{e}_\theta$ in your answer.)*



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Problem 16.1.23

16.2 One-degree-of-freedom 2-D mechanisms

16.2.1 A conservative vibratory system has the following equation of conservation of energy.

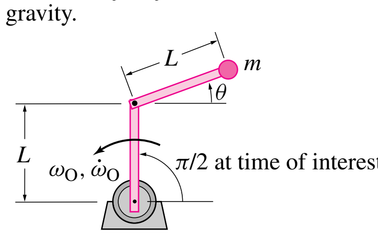
$$m(\ell\dot{\theta})^2 - mg\ell(1-\cos\theta) + K(a\theta)^2 = E_0$$

where E_0 is a constant.

a) Obtain the differential equation of motion of this system by differentiating this energy equation with respect to t.

b) Determine the circular frequency of small oscillations of the system in part (a). HINT: (Let $\sin \cong \theta \cong$ and $\cos\theta \cong 1 - \theta^2/2$).

16.2.2 A motor at O turns at rate ω_o whose rate of change is $\dot{\omega}_o$. At the end of a stick connected to this motor is a frictionless hinge attached to a second massless stick. Both sticks have length L. At the end of the second stick is a mass m. For the configuration shown, what is $\ddot{\theta}$? Answer in terms of ω_o, $\dot{\omega}_o$, L, m, θ, and $\dot{\theta}$. Ignore gravity.



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Problem 16.2.2

16.2.3 In problem 15.3.4, find

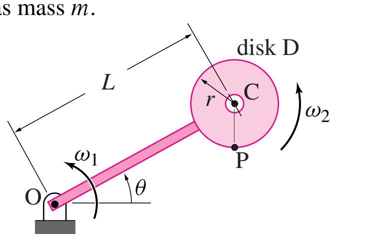
a) the angular momentum about point O, *

b) the rate of change of angular momentum of the disk about point O, *

c) the angular momentum about point C, *

d) the rate of change of angular momentum of the disk about point C. *

Assume the rod is massless and the disk has mass m.



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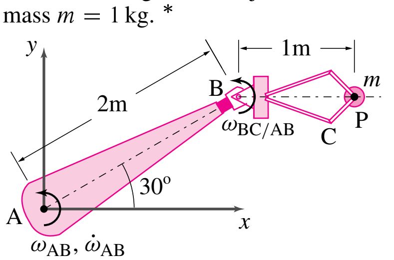
Problem 16.2.3

16.2.4 Robot arm, 2-D . The robot arm AB is rotating about point A with $\omega_{AB} = 5\,\text{rad/s}$ and $\dot{\omega}_{AB} = 2\,\text{rad/s}^2$. Meanwhile the forearm BC is rotating at a constant angular speed *with respect to AB* of $\omega_{BC/AB} = 3\,\text{rad/s}$. Gravity cannot be neglected. At the instant shown, find the *net* force acting on the object P which has mass $m = 1\,\text{kg}$. *



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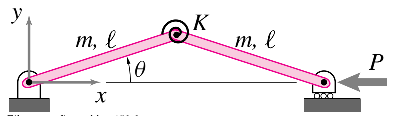
Problem 16.2.4

16.2.5 A crude model for a column, shown in the figure, consists of two identical rods of mass m and length ℓ with hinge connections, a linear torsional spring of stiffness K attached to the center hinge (the spring is relaxed when $\theta = 0$), and a load P applied at the top end.

a) Obtain the exact nonlinear equation of motion.

b) Obtain the squared natural frequency for small motions θ.

c) Check the stability of the straight equilibrium state $\theta = 0$ for all $P \geq 0$ via minimum potential energy. How do these results compare with those from part(b)?



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Problem 16.2.5

16.2.6 A thin uniform rod of mass m rests against a frictionless wall and on a frictionless floor. There is gravity.

a) Draw a free body diagram of the rod.

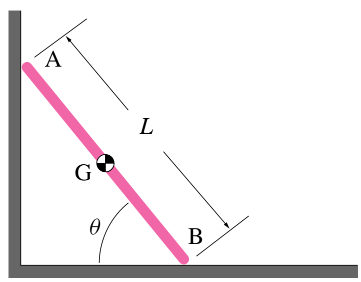
b) The rod is released from rest at $\theta = \theta_0 \neq 0$. Write the equation of motion of the rod. *

c) Using the equation of motion, find the initial angular acceleration, $\dot{\omega}_{AB}$, and the acceleration of the center of mass, $\vec{a}_G$, of the rod. *

d) Find the reactions on the rod at points A and B. *

e) Find the acceleration of point B. *

f) When $\theta = \frac{\theta_0}{2}$, find $\vec{\omega}_{AB}$ and the acceleration of point A. *



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Problem 16.2.6

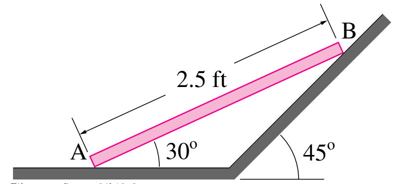
16.2.7 Bar leans on a crooked wall. A uniform 3 lbm bar leans on a wall and floor and is let go from rest. Gravity pulls it down.

a) Draw a Free Body Diagram of the bar.

b) Kinematics: find the velocity and acceleration of point B in terms of the velocity and acceleration of point A

c) Using equations of motion, find the acceleration of point A.

d) Do the reaction forces at A and B add up to the weight of the bar? Why or why not? (You do not need to solve for the reaction forces in order to answer this part.)



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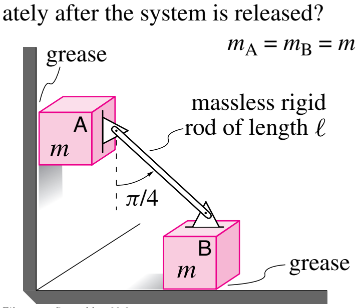
Problem 16.2.7

16.2.8 Two blocks, each with mass m, slide without friction on the wall and floor shown. They are attached with a rigid massless rod of length ℓ that is pinned at both ends. The system is released from rest when the rod makes an angle of 45°. What is the acceleration of the block B immediately after the system is released?



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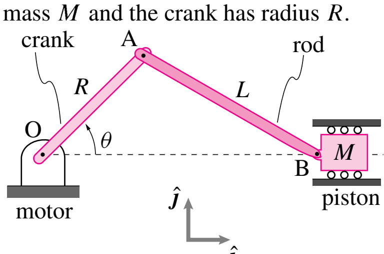
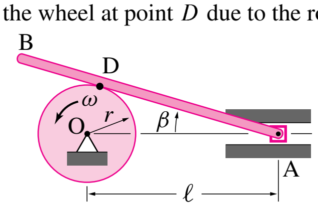
Problem 16.2.8

16.2.9 Slider Crank. 2-D . No gravity. Refer to the figure in problem 15.4.2. What is the tension in the massless rod AB (with length L) when the slider crank is in the position with $\theta = 0$ (piston is at maximum extent)? Assume the crankshaft has constant angular velocity ω, that the connecting rod AB is massless, that the cylinder walls are frictionless, that there is no gas pressure in the cylinder, that the piston has mass M and the crank has radius R.



Filename:slidercrank-tension
Problem 16.2.9

16.2.10 In problem 15.4.8, find the force on the wheel at point D due to the rod.



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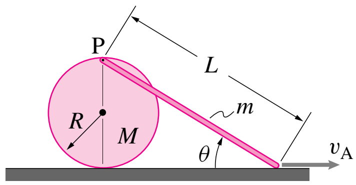
Problem 16.2.10

16.2.11 In problem 15.4.9, find the force on the rod at point P.



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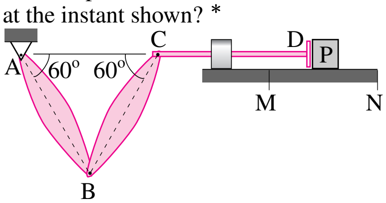
Problem 16.2.11

16.2.12 Slider-Crank mechanism. The slider crank mechanism shown is used to push a 2 lbm block P. Arm AB and BC are each 0.5 ft long. Given that arm AB rotates counterclockwise at a constant 2 revolutions per second, what is the force on P at the instant shown? *



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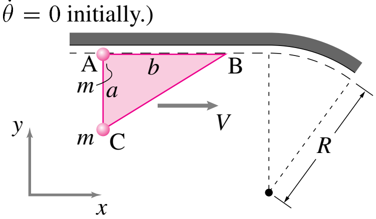
Problem 16.2.12

16.2.13 The large masses m at C and A were supported by the light triangular plate ABC, whose corners follow the guide. B enters the curved guide with velocity V. Neglecting gravity, find the vertical reactions (in the y direction) at A and B. (Hint: for the rigid planar body ABC, find $\ddot{y}_A$ and $\ddot{y}_B$ in terms of a_y^{cm} and $\ddot{\theta}$, assuming $\dot{\theta} = 0$ initially.)



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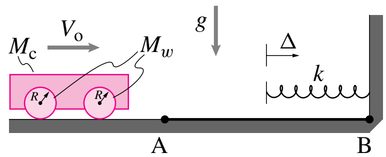
Problem 16.2.13

16.2.14 An idealized model for a car comprises a rigid chassis of mass M_c and four identical rigid disks (wheels) of mass M_w and radius R, as shown in the figure. Initially in motion with speed v_0, the car is momentarily brought to rest by compressing an initially uncompressed spring of stiffness k. Assume no frictional losses.

a) Assuming no wheel slip, determine the compression Δ of the spring required to stop the car. *

b) While the car is in contact with the spring but still moving forward, in which direction is the *tangential* force on any one of the wheels (due to contact with the ground) pointing? Why? (Illustrate with a sketch). *

c) Repeat part (a) assuming now that the ground is perfectly frictionless from points A to B shown in the figure. *



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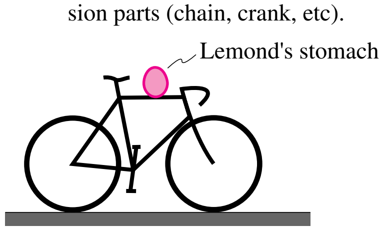
Problem 16.2.14

16.2.15 Assume Greg Lemond's riding "tuck" was so good that you can neglect air resistance when you think about him and his bike. Further, you can regard his and the bike's combined mass (all 70 kg)) as concentrated at a point in his stomach somewhere. Greg's left foot has just fallen off the pedal so he is only pedaling with his right foot, which at the moment in question is at its lowest point in the motion. You note that, relative to the ground the right foot is only going 3/4 as fast as the bike (since it is going backwards relative to the bike), though you can't make out all the radii of his frictionless gears and rigid round wheels. Greg, ever in touch with his body, tells you he is pushing back on the pedal with a force of 70 N. You would like to know Greg's acceleration.

a) In your first misconceived experiment you set up a 70 kg bicycle in your laboratory and balance it with strings that cause no fore or aft forces. You tie a string to the vertically down right pedal and pull back with a force of 70 N. What acceleration do you measure?

b) What is Greg's actual acceleration? [Hint: Greg's massless leg is pushing forward on his body with a force of 70 N] You can neglect the mass of the wheels and other transmission parts (chain, crank, etc).



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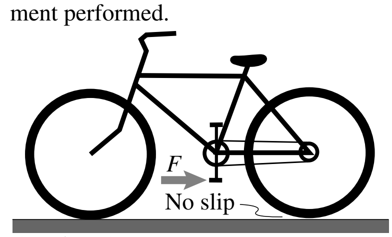
Problem 16.2.15

16.2.16 Which way does the bike accelerate? A bicycle with all frictionless bearings is standing still on level ground. A horizontal force F is applied on one of the pedals as shown. There is no slip between the wheels and the ground. The bicycle is gently balanced from falling over sideways. It is heavy enough so that both wheels stay on the ground. Does the bicycle accelerate forward, backward, or not at all? Make any reasonable assumptions about the dimensions and mass. Justify your answer as clearly as you can, clearly enough to convince a person similar to yourself but who has not seen the experiment performed.



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Problem 16.2.16

16.3 Multi-degree-of-freedom 2-D mechanisms

16.3.1 Particle on a springy leash. A particle with mass m slides on a rigid horizontal frictionless plane. It is held by a string which is in turn connected to a linear elastic spring with constant k. The string length is such that the spring is relaxed when the mass is on top of the hole in the plane. The position of the particle is $\vec{r} = x\hat{\imath} + y\hat{\jmath}$. For each of the statements below, state the circumstances in which the statement is true (assuming the particle stays on the plane). Justify your answer with convincing explanation and/or calculation.

a) The force of the plane on the particle is $mg\hat{\boldsymbol{k}}$.

b) $\ddot{x} + \frac{k}{m}x = 0$

c) $\ddot{y} + \frac{k}{m}y = 0$

d) $\ddot{r} + \frac{k}{m}r = 0$, where $r = |\vec{r}|$

e) $r =$ constant

f) $\dot{\theta} =$constant

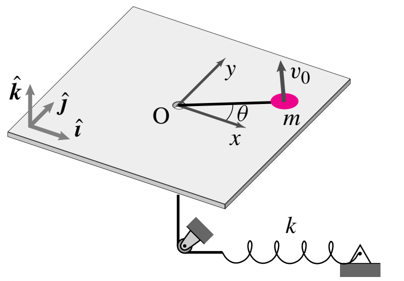
g) $r^2\dot{\theta} =$ constant.

h) $m(\dot{x}^2 + \dot{y}^2) + kr^2 =$ constant

i) The trajectory is a straight line segment.

j) The trajectory is a circle.

k) The trajectory is not a closed curve.



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Problem 16.3.1: Particle on a springy leash.

16.3.2 "Yo-yo" mechanism of satellite de-spinning. A satellite, modeled as a uniform disk, is "de-spun" by the following mechanism. Before launch two equal length long strings are attached to the satellite at diametrically opposite points and then wound around the satellite with the same sense of rotation. At the end of the strings are placed 2 equal masses. At the start of de-spinning the two masses are released from their position wrapped against the satellite. Find the motion of the satellite as a function of time for the two cases: (a) the strings are wrapped in the same direction as the initial spin, and (b) the strings are wrapped in the opposite direction as the initial spin.

16.3.3 A particle with mass m is held by two long springs each with stiffness k so that the springs are relaxed when the mass is at the origin. Assume the motion is planar. Assume that the particle displacement is much smaller than the lengths of the springs.

a) Write the equations of motion in cartesian components. *

b) Write the equations of motion in polar coordinates. *

c) Express the conservation of angular momentum in cartesian coordinates. *

d) Express the conservation of angular momentum in polar coordinates. *

e) Show that (a) implies (c) and (b) implies (d) even if you didn't note them *a-priori*. *

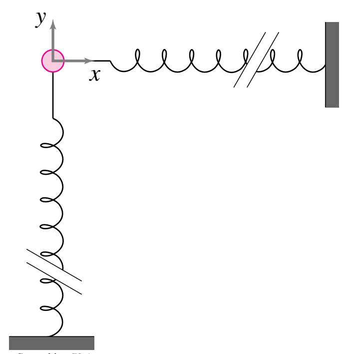
f) Express the conservation of energy in cartesian coordinates. *

g) Express the conservation of energy in polar coordinates. *

h) Show that (a) implies (f) and (b) implies (g) even if you didn't note them *a-priori*. *

i) Find the general motion by solving the equations in (a). Describe all possible paths of the mass. *

j) Can the mass move back and forth on a line which is not the x or y axis? *



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Problem 16.3.3

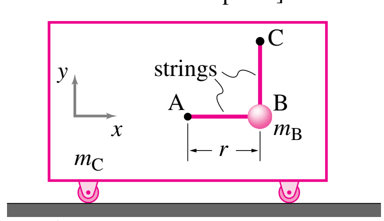
16.3.4 Cart and pendulum A mass m_B = 6 kg hangs by two strings from a cart with mass m_C = 12 kg. Before string BC is cut string AB is horizontal. The length of string AB is r = 1 m. At time $t = 0$ all masses are stationary and the string BC is cleanly and quietly cut. After some unknown time t_{vert} the string AB is vertical.

a) What is the net displacement of the cart x_C at $t = t_{\text{vert}}$? *

b) What is the velocity of the cart v_C at $t = t_{\text{vert}}$? *

c) What is the tension in the string at $t = t_{\text{vert}}$? *

d) (**Optional**) What is $t = t_{\text{vert}}$? [You will either have to leave your answer in the form of an integral you cannot evaluate analytically or you will have to get part of your solution from a computer.]



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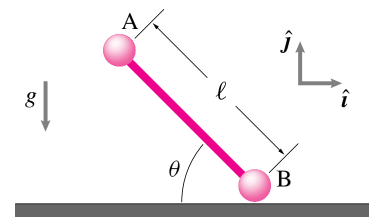
Problem 16.3.4

16.3.5 A dumbbell slides on a floor. Two point masses m at A and B are connected by a massless rigid rod with length ℓ. Mass B slides on a frictionless floor so that it only moves horizontally. Assume this dumbbell is released from rest in the configuration shown. *[Hint: What is the acceleration of A relative to B?]*

a) Find the acceleration of point B just after the dumbbell is released.

b) Find the velocity of point A just before it hits the floor



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Problem 16.3.5

16.3.6 After a winning goal one second before the clock ran out a psychologically stunned hockey player (modeled as a uniform rod) stands nearly vertical, stationary and rigid. The players perfectly slippery skates start to pop out from under her as she falls. Her height is ℓ, her mass m, her tip from the vertical θ, and the gravitational constant is g.

a) What is the path of her center of mass as she falls? (Show clearly with equations, sketches or words.)

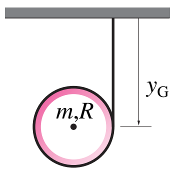
b) What is her angular velocity just before she hits the ice, a millisecond before she sticks our her hands and brakes her fall (first assume her skates remain in contact the whole time and then check the assumption)?

c) Find a differential equation that only involves θ, its time derivatives, m, g, and ℓ. (This equation could be solved to find θ as a function of time. It is a non-linear equation and you are not being asked to solve it numerically or otherwise.)

16.3.7 Falling hoop. A bicycle rim (no spokes, tube, tire, or hub) is idealized as a hoop with mass m and radius R. G is at the center of the hoop. An inextensible string is wrapped around the hoop and attached to the ceiling. The hoop is released from rest at the position shown at $t = 0$.

a) Find y_G at a later time t in terms of any or all of m, R, g, and t.

b) Does G move sideways as the hoop falls and unrolls?



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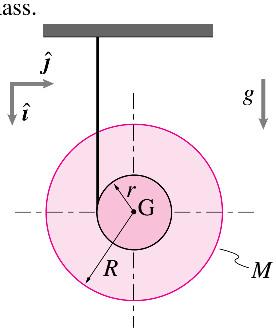
Problem 16.3.7

16.3.8 A model for a yo-yo consists of a thin disk of mass M and radius R and a light drum of radius r, rigidly attached to the disk, around which a light inextensible cable is wound. Assuming that the cable unravels without slipping on the drum, determine the acceleration a_G of the center of mass.



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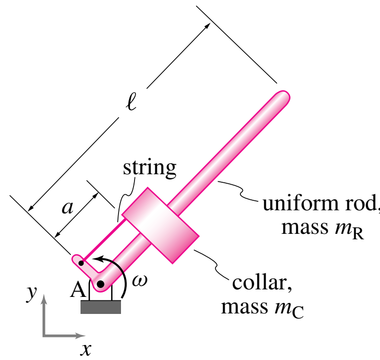
Problem 16.3.8

16.3.9 A uniform rod with mass m_R pivots without friction about point A in the xy-plane. A collar with mass m_C slides without friction on the rod after the string connecting it to point A is cut. There is no gravity. Before the string is cut, the rod has angular velocity ω_1.

a) What is the speed of the collar after it flies off the end of the rod? Use the following values for the constants and initial conditions: $m_R = 1\,\text{kg}$, $m_C = 3\,\text{kg}$, $a = 1\,\text{m}$, $\ell = 3\,\text{m}$, and $\omega_1 = 1\,\text{rad/s}$

b) Consider the special case $m_R = 0$. Sketch (approximately) the path of the motion of the collar from the time the string is cut until some time after it leaves the end of the rod.



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Problem 16.3.9

16.3.10 Assume the rod in the figure for problem 15.1.5 has polar moment of inertia $Iozz$. Assume it is free to rotate. The bead is free to slide on the rod. Assume that at $t = 0$ the angular velocity of the rod is 1 rad/s, that the radius of the bead is one meter and that the radial velocity of the bead, dR/dt, is zero.

a) Draw separate free body diagrams of the bead and rod.

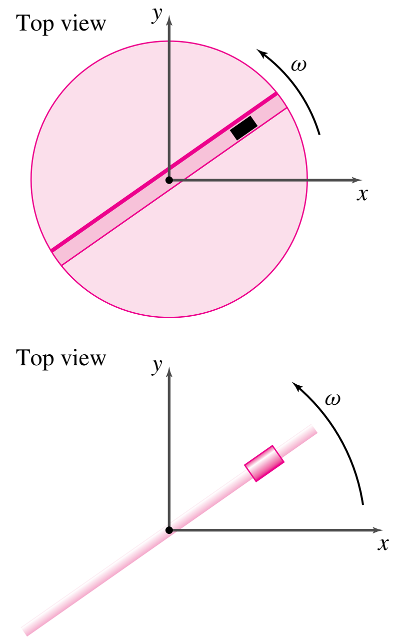
b) Write equations of motion for the system. *

c) Use the equations of motion to show that angular momentum is conserved. *

d) Find one equation of motion for the system using: (1) the equations of motion for the bead and rod and (2) conservation of angular momentum. *

e) Write an expression for conservation of energy. Let the initial total energy of the system be, say, E_0. *

f) As t goes to infinity does the bead's distance go to infinity? Its speed? The angular velocity of the turntable? The net angle of twist of the turntable? *



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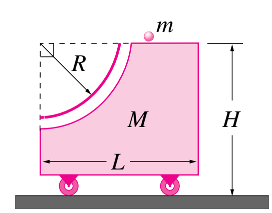
Problem 16.3.10: **Coupled motion of bead and rod and turntable.**

16.3.11 A primitive gun rides on a cart (mass M) and carries a cannon ball (of mass m) on a platform at a height H above the ground. The cannon ball is dropped through a frictionless tube shaped like a quarter circle of radius R.

a) If the system starts from rest, compute the horizontal speed (relative to the ground) that the cannon ball has as it leaves the bottom of the tube. Also find the cart's speed at the same instant.

b) Compute the velocity of the cannon ball relative to the cart.

c) If two balls are dropped simultaneously through the tube, what speed does the cart have when the balls reach the bottom? Is the same final speed also achieved if one ball is allowed to depart the system entirely before the second ball is released? Why/why not?



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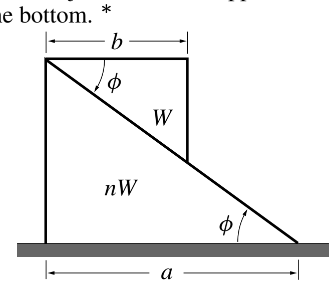
Problem 16.3.11

16.3.12 Numerically simulate the coupled system in problem 16.3.10. Use the simulation to show that the net angle of the turntable or rod is finite.

a) Write the equations of motion for the system from part (b) in problem 16.3.10 as a set of first order differential equations. *

b) Numerically integrate the equations of motion.

16.3.13 Two frictionless prisms of similar right triangular sections are placed on a frictionless horizontal plane. The top prism weighs W and the lower one nW. The prisms are held in the initial position shown and then released, so that the upper prism slides down along the lower one until it just touches the horizontal plane. The center of mass of a triangle is located at one-third of its height from the base. Compute the **velocities** of the two prisms at the moment just before the upper one reaches the bottom. *



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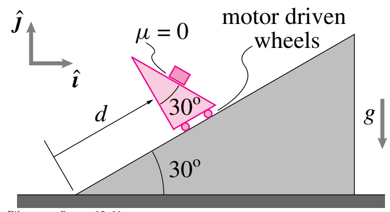
Problem 16.3.13

16.3.14 Mass slides on an accelerating cart. 2D. A cart is driven by a powerful motor to move along the 30° sloped ramp according to the formula: $d = d_0 + v_0 t + a_0 t^2/2$ where d_0, v_0, and a_0 are given constants. The cart is held from tipping over. The cart itself has a 30° sloped upper surface on which rests a mass (given mass m). The surface on which the mass rests is frictionless. Initially the mass is at rest with regard to the cart.

a) What is the force of the cart on the mass? [in terms of $g, d_0, v_0, t, a_0, m, g, \hat{\imath}$, and $\hat{\jmath}$.] *

b) For what values of d_0, v_0, t and a_0 is the acceleration of the mass exactly vertical (i.e., in the $\hat{\jmath}$ direction)? *



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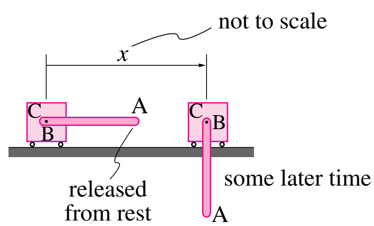
Problem 16.3.14

16.3.15 A thin rod AB of mass $W_{AB} = 10\,\text{lbm}$ and length $L_{AB} = 2\,\text{ft}$ is pinned to a cart C of mass $W_C = 10\,\text{lbm}$, the latter of which is free to move along a frictionless horizontal surface, as shown in the figure. The system is released from rest with the rod in the horizontal position.

a) Determine the angular speed of the rod as it passes through the vertical position (at some later time). *

b) Determine the displacement x of the cart at the same instant. *

c) After the rod passes through vertical, it is momentarily horizontal but on the left side of the cart. How far has the cart moved when this configuration is reached?



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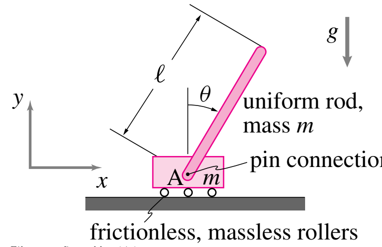
Problem 16.3.15

16.3.16 As shown in the figure, a block of mass m rolls without friction on a rigid surface and is at position x (measured from a fixed point). Attached to the block is a uniform rod of length ℓ which pivots about one end which is at the center of mass of the block. The rod and block have equal mass. The rod makes an angle θ with the vertical. Use the numbers below for the values of the constants and variables at the time of interest:

$$
\begin{aligned}
\ell &= 1\,\text{m} \\
m &= 2\,\text{kg} \\
\theta &= \pi/2 \\
d\theta/dt &= 1\,\text{rad/s} \\
d^2\theta/dt^2 &= 2\,\text{rad/s}^2 \\
x &= 1\,\text{m} \\
dx/dt &= 2\,\text{m/s} \\
d^2x/dt^2 &= 3\,\text{m/s}^2
\end{aligned}
$$

a) What is the kinetic energy of the system?

b) What is the linear momentum of the system (momentum is a vector)?



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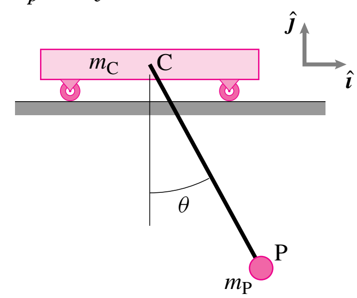
Problem 16.3.16

16.3.17 A pendulum of length ℓ hangs from a cart. The pendulum is massless except for a point mass of mass m_p at the end. The cart rolls without friction and has mass m_c. The cart is initially stationary and the pendulum is released from rest at an angle θ. What is the acceleration of the cart just after the mass is released? [Hints: $\mathbf{a}_P = \vec{a}_C + \vec{a}_{P/C}$. The answer is $\vec{a}_C = (g/3)\mathbf{i}$ in the special case when $m_p = m_c$ and $\theta = \pi/4$.]



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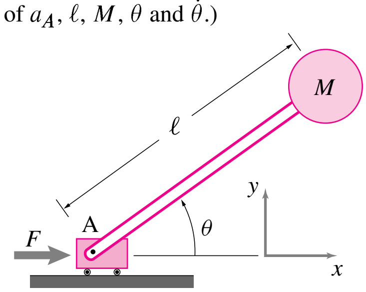
Problem 16.3.17

16.3.18 Due to the application of some unknown force F, the base of the pendulum A is accelerating with $\vec{a}_A = a_A \imath$. There is a frictionless hinge at A. The angle of the pendulum θ with the x axis and its rate of change $\dot{\theta}$ are assumed to be known. The length of the massless pendulum rod is ℓ. The mass of the pendulum bob M. There is no gravity. What is $\ddot{\theta}$? (Answer in terms of a_A, ℓ, M, θ and $\dot{\theta}$.)



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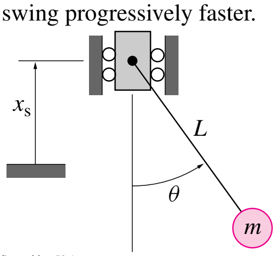
Problem 16.3.18

16.3.19 Pumping a Swing Can a swing be pumped by moving the support point up and down?

For simplicity, neglect gravity and consider the problem of swinging a rock in circles on a string. Let the rock be mass m attached to a string of fixed length ℓ. Can you speed it up by moving your hand up and down? How? Can you make a quantitative prediction? Let x_S be a function of time that you can specify to try to make the mass swing progressively faster.



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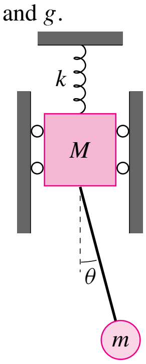
Problem 16.3.19

16.3.20 Using free body diagrams and appropriate momentum balance equations, find differential equations that govern the angle θ and the vertical deflection y of the system shown. Be clear about your datum for y. Your equations should be in terms of θ, y and their time-derivatives, as well as M, m, ℓ, and g.



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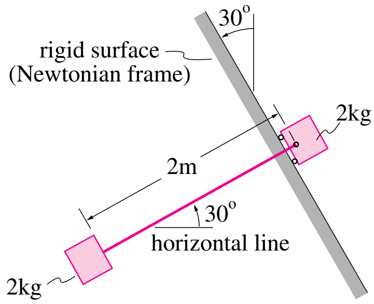
Problem 16.3.20

16.3.21 The two blocks shown are released from rest at $t = 0$. There is no friction and the cable is initially taut. (a) What is the tension in the cable immediately after release? (Use any reasonable value for the gravitational constant). (b) What is the tension after 5 s?



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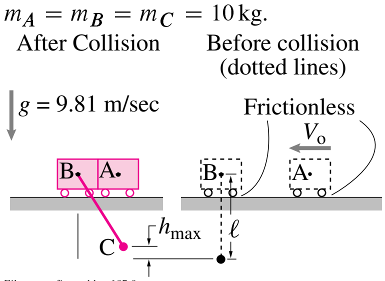
Problem 16.3.21

16.3.22 Carts A and B are free to move along a frictionless horizontal surface, and bob C is connected to cart B by a massless, inextensible cord of length ℓ, as shown in the figure. Cart A moves to the left at a constant speed $v_0 = 1\,\mathrm{m/s}$ and makes a perfectly plastic collision ($e = 0$) with cart B which, together with bob C, is stationary prior to impact. Find the maximum vertical position of bob C, h_{max}, after impact. The masses of the carts and pendulum bob are $m_A = m_B = m_C = 10\,\mathrm{kg}$.



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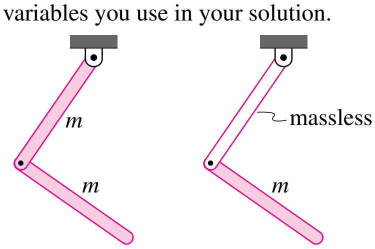
Problem 16.3.22

16.3.23 A double pendulum is made of two uniform rigid rods, each of length ℓ. The first rod is massless. Find equations of motion for the second rod. Define any variables you use in your solution.



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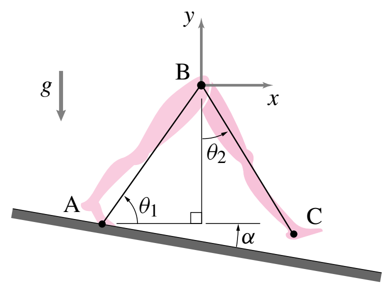
Problem 16.3.23

16.3.24 A model of walking involves two straight legs. During the part of the motion when one foot is on the ground, the system looks like the picture in the figure, confined to motion in the $x - y$ plane. Write two equations from which one could find $\ddot{\theta}_1$, and $\ddot{\theta}_2$ given $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ and all mass and length quantities.

Hint:
$$\begin{bmatrix} \sum \vec{M}_{/A} & = & \dot{\vec{H}}_{/A} & \text{whole system} \\ \sum \vec{M}_{/B} & = & \dot{\vec{H}}_{/B} & \text{for bar BC} \end{bmatrix}$$



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Problem 16.3.24

Advanced problems in 2D motion

16.3.25 A pendulum is hanging from a moving support in the xy-plane. The support moves in a known way given by $\vec{r}(t) = r(t)\hat{\imath}$. For the following cases, find a differential equation whose solution would determine $\theta(t)$, measured clockwise from vertical; then find an expression for $d^2\theta/dt^2$ in terms of θ and $r(t)$:

a) with no gravity and assuming the pendulum rod is massless with a point mass of mass m at the end,

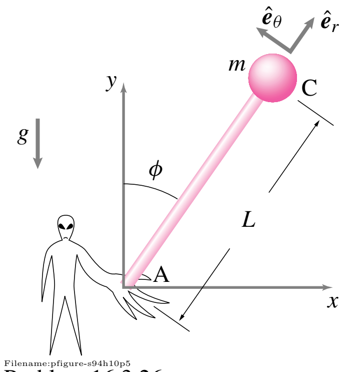
b) as above but with gravity,

c) assuming the pendulum is a uniform rod of mass m and length ℓ.

16.3.26 Robotics problem: balancing a broom stick by sideways motion. Try to balance a broom stick by moving your hand horizontally. Model your hand contact with the broom as a hinge. You can model the broom as a uniform stick or as a point mass at the end of a stick — your choice.

a) **Equation of motion.** Given the acceleration of your hand (horizontal only), the current tip, and the rate of tip of the broom, find the angular acceleration of the broom. *

b) **Control?** Can you find a hand acceleration in terms of the tip and the tip rate that will make the broom balance upright? *



Problem 16.3.26

16.3.27 Balancing the broom again: vertical shaking works too. You can balance a broom by holding it at the bottom and applying appropriate torques, as in problem 13.6.45 or by moving your hand back and forth in an appropriate manner, as in problem 16.3.26. In this problem, you will to try to balance the broom differently. The lesson to be learned here is more subtle, and you should probably just wonder at it rather than try to understand it in detail.

In the previous balancing schemes, you used knowledge of the state of the broom (θ and $\dot{\theta}$) to determine what corrective action to apply. Now balance the broom by moving it in a way that ignores what the broom is doing. In the language of robotics, what you have been doing before is 'closed-loop feedback' control. The new strategy, which is simultaneously more simple minded and more subtle, is an 'open loop' control. Imagine that your hand connection to the broom is a hinge.

a) **Picture and model.** Assume your hand oscillates sinusoidally up and down with some frequency and some amplitude. The broom is instantaneously at some angle from vertical. Draw a picture which defines all the variables you will use. Use any mass distribution that you like.

b) **FBD.** Draw a FBD of the broom.

c) **Momentum balance.** Write the equation of angular momentum balance about the point instantaneously coinciding with the hinge.

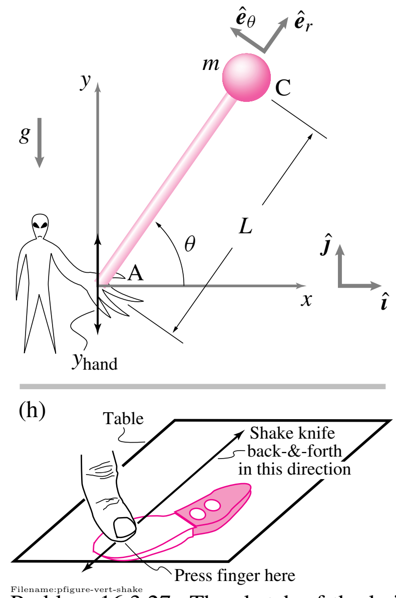
d) **Kinematics.** Use any geometry and kinematics that you need to evaluate the terms in the angular momentum balance equation in terms of the tip angle and its time derivatives and other known quantities (take the vertical motion of your hand as 'given'). [Hint: There are many approaches to this problem.]

e) **Equation of motion.** Using the angular momentum balance equation, write a governing differential equation for the tip angle. *

f) **Simulation.** Taking the hand motion as given, simulate on the computer the system you have found.

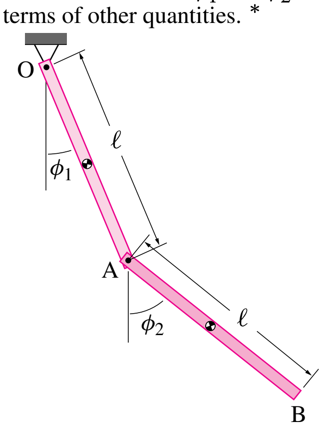
g) **Stability?** Can you find an amplitude and frequency of shaking so that the broom stays upright if started from a near upright position? You probably *cannot* find linear equations to solve that will give you a control strategy. So, this problem might best be solved by guessing on the computer. Successful strategies require the hand acceleration to be quite a bit bigger than g, the gravitational constant. The stability obtained is like the stability of an undamped uninverted pendulum — oscillations persist. You improve the stability a little by including a little friction in the hinge.

h) **Dinner table experiment for nerdy eaters.** If you put a table knife on a table and put your finger down on the tip of the blade, you can see that this experiment might work. Rapidly shake your hand back and forth, keeping the knife from sliding out from your finger but with the knife sliding rapidly on the table. Note that the knife aligns with the direction of shaking (use scratch-resistant surface). The knife is different from the broom in some important ways: there is no gravity trying to 'unalign' it and there is friction between the knife and table that is much more significant than the broom interaction with the air. Nonetheless, the experiment should convince you of the plausibility of the balancing mechanism. Because of the large accelerations required, you *cannot* do this experiment with a broom.



Problem 16.3.27: The sketch of the knife on the table goes with part (h).

16.3.28 Double pendulum. The double pendulum shown is made up of two uniform bars, each of length ℓ and mass m. The pendulum is released from rest at $\phi_1 = 0$ and $\phi_2 = \pi/2$. Just after release what are the values of $\ddot{\phi}_1$ and $\ddot{\phi}_2$? Answer in terms of other quantities. *

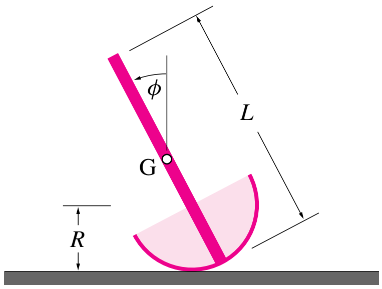


Problem 16.3.28

16.3.29 A rocker. A standing dummy is modeled as having massless rigid circular feet of radius R rigidly attached to their uniform rigid body of length L and mass m. The feet do not slip on the floor.

a) Given the tip angle ϕ, the tip rate $\dot{\phi}$ and the values of the various parameters (m, R, L, g) find $\ddot{\phi}$. [You may assume ϕ and $\dot{\phi}$ are small.] *

b) Using the result of (a) or any other clear reasoning find the conditions on the parameters (m, R, L, g) that make vertical passive dynamic standing stable. [Stable means that if the person is slightly perturbed from vertically up that their resulting motion will be such that they remain nearly vertically up for all future time.] *



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Problem 16.3.29

16.3.30 Consider a rigid spoked wheel with no rim. Assume that when it rolls a spoke hits the ground and doesn't bounce. The body just swings around the contact point until the next spoke hits the ground. The uniform spokes have length R. Assume that the mass of the wheel is m, and that the polar moment of inertia about its center is I (use $I = mR^2/2$ if you want to get a better sense of the solution). Assume that just before collision number n, the angular velocity of the wheel is ω_{n-}, the kinetic energy is T_{n-}, the potential energy (you must clearly define your datum) is U_n-. Just after collision n the angular velocity of the wheel is ω_{n+}. The Kinetic Energy is T_{N+}, the potential energy (you must clearly define your datum) is U_{n+}. The wheel has k spokes (pick $k = 4$ if you have trouble with abstraction). This problem is not easy. It can be answered at a variety of levels. The deeper you get into it the more you will learn.

a) What is the relation between ω_{n-} and T_{n-}?

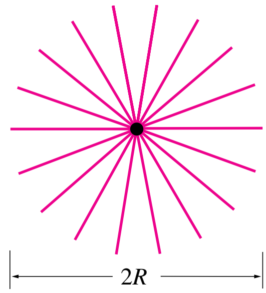
b) What is the relation between ω_{n-} and ω_{n+}?

c) Assume 'rolling' on level ground. What is the relation between ω_{n+} and ω_{n+1}?

d) Assume rolling down hill at slope θ. What is the relation between ω_{n+} and ω_{n+1}?

e) Can it be true that $\omega_{n+} = \omega_{(n+1)+}$? About how fast is the wheel going in this situation?

f) As the number of spokes m goes to infinity, in what senses does this wheel become like an ordinary wheel?



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Problem 16.3.30

APPENDIX **A**

Units and dimensions

Some things that are important but don't fit in the flow of a homework-driven course. This is the first appendix about one of those things.

This chapter concerns issues related to ***units and dimensions****. Most important is this: a quantity is the product of a number and a unit. Thus units are part of a calculation. Some simple advice follows: a) balance units, b) carry units and c) check units. Rules for changing units also follow.*

Many engineering texts have, somewhere near the front, a tedious and pedantic section about units and dimensions. This book is completely different. That section is here at the back. We don't want to diminish the importance of the topic, but put it here in the back because students are immune to preaching. The only way a student will get good at managing units is by imitation, or when forced to do so in a time of panic, or at a moment of idle curiosity. As for imitation, we have tried to set a good example in the whole of the book. As for panic and curiosity, this section is here for you.

Not everyone will take the care with units that we advise for you. You will find, both in school and at work, that there are a variety of ways in which people use and abuse units, all within the context of productive engineering. So you will have to be aware and tolerant of the various conventions, even if they sometimes seem vague and imprecise.

A.1 Balancing and carrying units

The central rules which we advise that you follow are:

a) Balance your units and b) Carry your units.

Where do these rules come from?

Physical quantities that are dimensional are represented by a number multiplying a unit.

Thus $d = 7\,\mathrm{m}$ means $7\times$(one meter). The 7 and the 'm' are of equal status in any math you do.

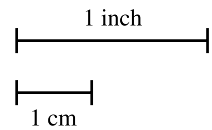

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Figure A.1: Relative size of an inch and a centimeter.

Balance your units

Every line of every calculation should be dimensionally sensible. That is, the dimensions on the left of the equal sign should be consistent with the dimensions on the right the same way numbers have to balance. Otherwise the equations are not equations. For example, if two bicycles tied in a race you could say they were in some way equal. But even if you noticed that the weight difference between these equivalent bicycles was 10% over 2 pounds you would not write

$$8\,\mathrm{kg} = 9\,\mathrm{kg}.$$

The equivalence between the two bikes in race times does not make eight kilograms equal to nine kilograms. In this same way it would be wrong to write

$$1\,\mathrm{in} = 1\,\mathrm{s}.$$

if you noticed that it takes a bug about a minute (60 seconds) to walk the length of your body (say about 60 inches). The passing of a second corresponds to the passing of an inch, so for some purposes an inch is equivalent to a second. But that does not mean that an inch *is* a second. An inch has dimensions of length which cannot be equal to a second which has dimensions of time. Length can equal time no more than 8 can equal 9.

But it is correct to write that

$$5.08\,\mathrm{cm} = 2\,\mathrm{in}.$$

Both centimeters and inches have dimensions of length and one inch is equivalent to 2.54 centimeters always (fig. A.1). An equation where the units on both sides of the equation are the same physical quantities (length in the example above) is *balanced* with regard to units.

Carry your units

When you go from one line of a calculation to the next you should carry (keep written track of) the units with as much care as any other numerical or algebraic quantities. When you do arithmetic and don't forget any terms you have 'carried' the numbers from one line of calculation to the next. Similarly, carrying the units just means not forgetting them *in* your calculations①.

①**Caution:** Students commonly put units next to their equations with a vague notion that the units apply to the equation. This sometimes works out in the end and sometimes doesn't. Because the rules for manipulating the units are the same as those for manipulating numbers things necessarily work out if units are part of the equations.

Example: Dividing meters by seconds.
A bicycle goes 7 meters in 2 seconds so

$$v = \frac{d}{t} = \frac{7\,\mathrm{m}}{2\,\mathrm{s}} = 3.5\,\mathrm{m/s}.$$

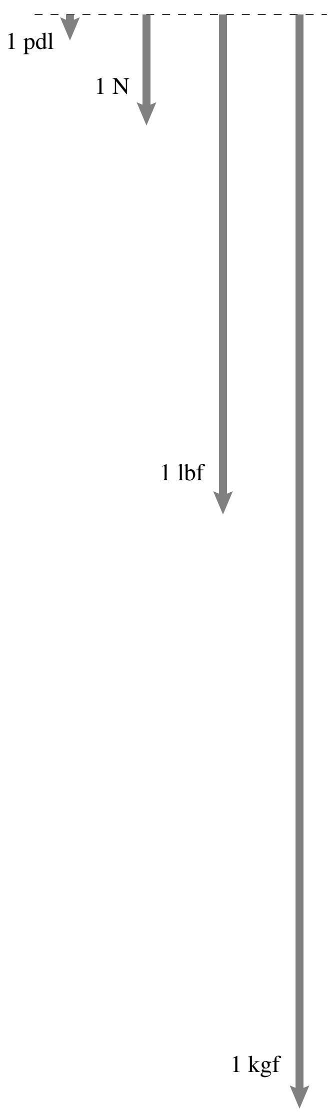
Here we have divided 7 by 2 and also divided m by s. But a meter (m) is not a number, nor is a second (s). So the ratio m/s cannot be reduced more. In particular, the m/s is not sitting next to the equation but is part of the equation: the velocity is not $v = 3.5$ but rather $v = 3.5\,\mathrm{m/s}$.

The rest of this section is, more or less, a discussion of how and why to 'carry your units.'

A.2 Dimensions, units and changing units

Distance has dimensions of length $[L]$ that can be measured with various units — centimeters (cm), yards (yd), or furlongs (an obsolete unit equal to 1/8 mile). A meter is the standard unit of length in the SI system. In answer to the question 'What is the length of a bicycle crank ℓ?' we say 'ℓ is seven inches' and write $\ell = 7$ in or say 'ℓ is seventeen point seven centimeters' and write $\ell = 17.7\,\mathrm{cm}$. In each case, a number multiplies a dimensional unit.

Force has dimension of mass times acceleration $[m \cdot a]$. Because acceleration itself has dimensions of length over time squared $[L/T^2]$, force also has dimensions of mass times length divided by time squared $[M \cdot L/T^2]$. Because force has such a central role in mechanics, it is often convenient to think of force as having its own units. Force then has dimensions of, simply, force $[F]$. The most common units for force are Newton (N) and the pound (lbf). The 'f' in the notation for the pound lbf is to distinguish a pound force lbf from the pound mass lbm, $1\,\mathrm{lbf} = \mathrm{lbm} \cdot g \approx 1\,\mathrm{lbm} \cdot 32.2\,\mathrm{ft/s^2} \approx 32.2\,\mathrm{lbm} \cdot \mathrm{ft/s^2}$. Some people use lb to mean pound force or pound mass, depending on context. To avoid confusion we use lbm for pound mass and lbf for pound force.



Filename:tfigure1-b

Figure A.2: Relative size of a poundal pdl, a Newton N, a pound force lbf, and a kilogram force kgf. $1\,\mathrm{N} = 7.24\,\mathrm{pdl}$, $1\,\mathrm{lbf} = 4.45\,\mathrm{N}$, and $1\,\mathrm{kgf} = 1\,\mathrm{kilopond} = 9.81\,\mathrm{N} = 2.2\,\mathrm{lbf}$.

Changing units

We can say 'The typical force of a seated racing bicyclist on a bicycle pedal is thirty pounds,' and write any of the following:

$$
\begin{aligned}
F &= 30\ \mathrm{lbf} \\
F &= 30\ \mathrm{lbf} \cdot (1) \\
F &= 30\ \mathrm{l\cancel{b}f} \cdot \underbrace{\left(\frac{4.45\ \mathrm{N}}{1\ \mathrm{l\cancel{b}f}}\right)}_{1} \\
F &= 133.5\ \mathrm{N}.
\end{aligned}
$$

Here we have shown one way to change units. Multiply the expression of interest by one (1) and then make an appropriate substitution for one. Any table of units will tell us that 1 lbf is approximately 4.45 N. So we can write $1 = (4.45\,\mathrm{Newtons}/1\,\mathrm{lbf})$ and multiply any part of an equation by it without affecting the equation's validity. (See fig. A.2 to get a sense of the relation between a pound force, a Newton, and the less used force units, the poundal and the kilogram-force.)

What if we had made a mistake and instead multiplied the right hand side by the reciprocal expression $1 = (1\,\text{lbf}/4.45\,\text{Newton})$? No problem. We would then have

$$F \;=\; 30\ \text{lbf} \;=\; 30\ \text{lbf}\cdot\frac{1\ \text{lbf}}{4.45\ \text{Newton}} \;=\; \frac{30}{4.45}\ \text{lbf}^2/\,\text{N}.$$

This expression is admittedly weird, but it is correct. If you should end up with such a weird but correct solution you can compensate by multiplying by one again and again until the units cancel in a way that you find pleasing. In this case we could get an answer in a more conventional form by multiplying the right hand side by 1^2 using $1 = (4.45\,\text{N}/\,\text{lbf})$:

$$F = \frac{30}{4.45}\,\frac{\text{lbf}^2}{\text{N}}\cdot 1^2 = \frac{3.0}{4.45}\,\frac{\cancel{\text{lbf}}^2}{\text{N}}\cdot\left(\frac{4.45\,\text{N}}{\cancel{\text{lbf}}}\right)^2 = 133.5\,\text{N} \qquad \text{(as expected).}$$

A trivial but surprisingly useful observation is that $F = F$. A quantity is equal to itself no matter how it is represented. That is, $30\,\text{lbf} = 133.5\,\text{N}$ even though $30 \neq 133.5$. To summarize:

> Units are manipulated in any and all calculations as if they were numbers or algebraic symbols. For example, canceling equal units from the top and bottom of a fraction is the same as canceling numbers or algebraic symbols.

An advertisement for careful use of units

Units and dimensions are part of scientific notation just as spelling, punctuation, and grammar are parts of English composition. If used properly, they aid both thinking and the communication of these thoughts to others. If units and dimensions are used improperly they can impede communication, even with oneself, and convey the wrong meaning.

Example: Breaking load
A gadget that breaks with a 300 N (300 Newtons) load instead of a needed 300 lbf (300 pounds force) load is exactly as bad as one that breaks with a 67 lbf load instead of a needed 300 lbf load. An unsatisfied consumer will not be placated by learning that the engineer's calculation was 'numerically correct'.

If anybody is ever to use your calculation, giving them the wrong units is just as bad as giving them the wrong numerical value.

Although using units properly often seems annoyingly tedious, it also often pays. If units are carried through honestly, *not just tagged on to the end of an equation for appearance*, you can check your work for dimensional consistency. If you are trying to find a speed and your answer comes out $13\,\text{kg}\cdot\text{m}/\,\text{s}$, you know you have made a mistake — $\text{kg}\cdot\text{m}/\,\text{s}$ just isn't a speed.

Such dimensional errors in a calculation often reveal corresponding algebraic or conceptual mistakes. Also, if a problem is based on data with mixed units, such as cm and meters, or pound force and pound mass, you may often not know the units of your answer unless you properly 'carry' your units①.

① You can easily generate errors of approximately a factor of 1000 with English units if you wishfully multiply or divide answers by 32.2 (the value of g in ft/s^2) at the end of a sloppy calculation. If you do it wrong you get an error of a factor of 32.2^2 which is 3% greater than 1000. Following sloppiness with unscrupulousness, some are tempted to then slide a decimal point three places to the right or left to 'fix' things. The decrepit insecurity that provokes such crimes is avoided by going to a church, temple or mosque regularly, or just by carrying units.

Three ways to be fussy about units.

People are most pleased if you speak their language, speak correctly, and make sense. Similarly, scientists and engineers with whom you communicate will be most comfortable if you use the units they use and use them with correct notation. But most importantly, you should use units in a way that makes physical sense. Just as the United Nations argues over which language to use for communication, educators, editors, and makers of standards have argued for decades over conventions for units: whether they should come in multiples of 10, whether they should use the standard international scientific conventions, and whether they will be clear to someone who has worked in the stock room of a supplier of $\frac{1}{2}$-inch bolts for 35 years and thinks SI might be a friend of his cousin Amil.

Even if you are not fluent in someone's favorite language, you can still say sensible things. Similarly, no matter what you or your work place's choice of units (SI, English, or hodge-podge), no matter whether you use upper case and lower case correctly, you should make sense. Physically sensible units — that is, balanced units — should be used to make your equations dimensionally correct. Then you should work on refining your notation so as to be more professional.

So, in order of importance,

1. use balanced units.
2. use units of the type that are liked by your colleagues.
3. spell and punctuate these units correctly.

If you are in a situation where your only problem is the third item on the list you are doing fine, unless you are *really* fussy, or work for someone who is *really* fussy. (*e.g.,* the authors of this book only hope to be good at the first two items on this list.)

A.3 Using units in practice

Units with calculators and computers

Calculators and computers generally do not keep track of units for you. In order for your numerical calculations to make sense you have the following choices.

Use dimensionless variables. Using dimensionless variables is the preferred method of scientists and theoretical engineers. The approach requires that you define a new set of dimensionless variables in terms of your original dimensional variables.

Use a consistent unit system. Express all quantities in terms of units that are consistent①. For example, all lengths should be in the same units and the unit of force should equal the unit of mass times the unit of distance divided by the unit of time squared. Each row of the table below defines a consistent set of units for mechanics.

Name	length	mass	force	time	angle
mks	meter	kilogram	Newton	second	radian
cgs	centimeter	gram	dyne	second	radian
English1	foot	lbm	poundal	second	radian
English2	foot	slug	lbf	second	radian

The radian is the unit of angle in all consistent unit systems. Whether or not a radian is a proper unit or not is an issue of some philosophical debate. Practically speaking, you can generally replace 1 radian with the number 1.

①**Caution:** Doing a computer calculation using quantities from an inconsistent unit system can easily lead to wrong results. To be safe make sure that all quantities are expressed in terms of only one row of the table shown.

Use numerical equations. If you are using the computer to evaluate a formula that you trust, and you have balanced the units in a way that makes you secure, you can have the computer do the arithmetic part of the calculation. It is easy to make mistakes, however, unless the formula is expressed in consistent units.

Example: Force units conversion

What, in the SI system, is the net braking force when a 2000 lbm car skids to a stop on level ground? For this units problem we skip the careful mechanics and just work with the formula

$$F = \mu m g$$

where m is the mass of the car, g is the local gravitational constant and μ is the coefficient of friction for sliding between the tire and the road. We won't be off by more than a quarter of a percent using the standard rather than the local value of the gravitational constant, $g = 32.2\,\mathrm{ft/s^2}$. The coefficient of friction for rubber and dry road is about one, so we use $\mu = 1$. We proceed by plugging in values into the formula and then multiplying by 1 until things are in standard SI (Systémé Internationale) form. We use a table of units to make the various substitutions for 1. A few of the detailed steps could be contracted. The approach below is only one, albeit an awkward one, of many routes to the answer.

$$\begin{aligned}
F &= \mu m g \\
F &= (1)\cdot(2000\,\mathrm{lbm})\cdot(32.2\,\mathrm{ft/s^2}) \\
&= (2000{\cdot}32.2)\frac{\mathrm{lbm\cdot ft}}{\mathrm{s^2}} \\
&= (2000{\cdot}32.2)\frac{\cancel{\mathrm{lbm}}\cdot\cancel{\mathrm{ft}}}{\mathrm{s^2}}\cdot\underbrace{\left(\frac{1\,\mathrm{kg}}{2.2\,\cancel{\mathrm{lbm}}}\right)}_{1}\cdot\underbrace{\left(\frac{30.48\,\cancel{\mathrm{cm}}}{1\,\cancel{\mathrm{ft}}}\right)}_{1}\cdot\underbrace{\left(\frac{1\,\mathrm{m}}{100\,\cancel{\mathrm{cm}}}\right)}_{1} \\
&= 8917\frac{\cancel{\mathrm{kg}}\cdot\cancel{\mathrm{m}}}{\cancel{\mathrm{s^2}}}\cdot\underbrace{\left(\frac{1\,\cancel{\mathrm{N}}}{1\,\cancel{\mathrm{kg}}\cdot\cancel{\mathrm{m}}/\cancel{\mathrm{s^2}}}\right)}_{1}\cdot\underbrace{\left(\frac{1\,\mathrm{kN}}{1000\,\cancel{\mathrm{N}}}\right)}_{1} \\
&= 8.92\ \mathrm{kN}
\end{aligned}$$

The net braking force is 8.92 kN. In each step of the calculation we accumulate what we had from the previous step and then multiply by 1, where 1 is the ratio of two quantities that have the same dimensions but different units.

Repeating, in engineering we do math not just with numbers, but with dimensional quantities. The bad habits of many of us not withstanding, there are good and useful standards for how to deal with units in calculations②.

②An excellent description of good practice is the "Guide for the Use of the International System of Units (SI)" by Barry Taylor, 1998, NIST (National Institute of Standards and Technology) publication # 811.

Use of units in old-style handbooks.

Many standard empirical formulas, formulas based on experience and not theory, are presented in an undimensional or numerical form. The units are not part of the equations. We present the approach here, not because we want to promote it, we don't. But we don't want the more formal approach to units we advocate here to stop you from reading and using empirical sources.

For example, Mark's *Handbook for Mechanical Engineers (8th edition, page 8-138)* presents the following useful formula to describe the working life of commercially manufactured ball bearings:

$$L_{10} = \frac{16,700}{N}\left(\frac{C}{P}\right)^K,$$

A.1 Examples of advised and ill-advised use of units

Good use of units Say a car has a constant speed of $v = 50$ mi/ hr for half an hour. The following is true and expressed correctly.

The distance traveled in time t is $x = vt$, so

$$\begin{aligned} x &= vt \\ &= (50\,\text{mi/hr})(30\,\text{min}) = 50 \cdot 30\,\text{mi} \cdot \text{min/hr} \\ &\qquad\qquad \text{(Awkward but true!)} \\ &= 50 \cdot 30\,\text{mi} \cdot \cancel{\text{m}}\text{in}/\cancel{\text{h}}\text{r} \underbrace{\left(\frac{1\,\cancel{\text{h}}\text{r}}{60\,\cancel{\text{m}}\text{in}}\right)}_{1} \\ &= 25\,\text{mi} \end{aligned}$$

That is, unsurprisingly, the distance covered in half an hour is 25 mi.

Another good use of units. If we start with the dimensionally correct formula $x = (50\,\text{mi/hr})t$ we can differentiate to get

$$v = \frac{dx}{dt} = 50\,\text{mi/hr}.$$

The answer is dimensionally correct without having to think about the units. v is speed and contains its units, x is distance and contains its units. In any formula that contains t, x or v we can substitute *any* time, distance or speed. How far does the car go in one minute? As in the previous example,

$$\begin{aligned} x &= vt \\ &= (50\,\text{mi/hr})(1\,\text{min}) \\ &= (50\,\text{mi}/\cancel{\text{h}}\text{r})(1\,\cancel{\text{m}}\text{in}) \underbrace{\left(\frac{1\,\cancel{\text{h}}\text{r}}{60\,\cancel{\text{m}}\text{in}}\right)}_{1} \\ &= \frac{5}{6}\,\text{mi} \end{aligned}$$

Not such good use of units It is common practice to write sentences like 'the distance the car travels is

$$x = 50t,$$

where x is the distance in miles and t is the time of travel in hours', although we discourage it. Why? Because the variables x and t are ambiguously defined. We would like to use the fact that speed v is the derivative of distance with respect to time:

$$v = \frac{dx}{dt} = \frac{d}{dt}(50t) = 50.$$

But now we have a speed equal to a pure number, 50, rather than a dimensional quantity. In this simple example, common sense tells us that the speed v is measured in mi/hr. But if we want to think of v as a speed, a variable with dimensions of length divided by time, the formula misleads us and requires us to add the units. For this simple example it is not much of a problem to determine what units to add.

But better is if units are included correctly in the equations; then they take care of themselves whenever they are needed. The 'not such good' use of units above is sometimes called using numerical equations, that is equations that have numbers in them only. The good use of units uses quantity equations, that is equations that use dimensional quantities.

where

L_{10} = the number of hours that pass before 10% of the bearings fail,
N = the rotational speed in revolutions per minute
C = the rated load capacity of the bearing in lbf,
P = the actual load on the bearing in lbf, and
K = 3 for ball bearings, $10/3$ for roller bearings.

In this approach the idea of dimensional consistency has been disguised for the sake of brevity. L_{10}, N, C, and P are just numbers. Such an equation is sometimes called a 'numerical equation'. It is a relation between numerical quantities. If you happen to know the rotation speed of the shaft in radians per second instead of revolutions per minute you will have to first convert before plugging in the formula. Unlike a dimensional formula, the formula does not help you to convert these units. An alternative to this 'numerical formula' approach for empirical formulas is in box A.2 on page 981.

Units with calculators and computers

Unfortunately, most calculators and computers are not equipped to carry units. They are only equipped to carry numbers. How do we handle this problem? The best and clearest option is only to do calculations with dimensionless variables.

The simplest way to use dimensionless variables, though not necessarily the best, is to do something that involves notational compromise. For example, let x represent dimensionless distance rather than distance. That is, x represents distance divided by 1 mi. Similarly, t is time divided by 1 hr. And dx/dt is dimensionless distance differentiated with respect to dimensionless

A.2 An improvement to the old-style handbook approach

An alternative to the standard approach to empirical formulas is to write a formula that makes sense with any dimensional variables. The bearing life formula would be replaced with the formula below:

$$l_{10} = \frac{16,700}{n}\left(\frac{c}{p}\right)^K \text{hr}\cdot\text{rev/min}$$

where

l_{10} = the time that passes before 10% of the bearings fail,
n = the rotational speed,
c = the rated load capacity of the bearing,
p = the actual load on the bearing, and
K = 3 for ball bearings, $10/3$ for roller bearings.

and the variables l_{10}, n, c, and p are dimensional quantities. One can use any dimensions one wants for all of the variables. For example, using

n = 50 rev/sec
c = 1 kN
p = 100 lbf, and
K = 3 for the given ball bearing,

we can calculate the life of the bearing by plugging these values in to the formula directly.

$$\begin{aligned} l_{10} &= \frac{16,700}{50\,\frac{\text{rev}}{\cancel{\text{sec}}}}\left(\frac{1\,\text{kN}}{100\,\text{lbf}}\right)^3 \text{hr}\cdot\text{rev/min} \\ &= \frac{16,700}{50\,\text{rev}/\cancel{\text{sec}}}\underbrace{\left(\frac{60\text{sec}}{\text{min}}\right)}_{1}\cdot \\ &\quad \left(\frac{1\,\cancel{\text{k}}\text{N}}{100\,\cancel{\text{lb}}\text{f}}\underbrace{\left(\frac{1\,\cancel{\text{lb}}\text{f}}{4.448\,\cancel{\text{N}}}\right)}_{1}\underbrace{\left(\frac{1\,\cancel{\text{N}}}{1000\,\cancel{\text{k}}\text{N}}\right)}_{1}\right)^3 \text{hr}\cdot\text{min/rev} \\ &\approx 45000\,\text{hr} \end{aligned}$$

This approach has the advantage of precision if mixed units are used. Any of the quantities can be measured with any units and the answer always comes out right. Furthermore, the user is free to measure all the quantities in those units which work out best, in this case using the same units for c and p and measuring n in rev/min. But the user is also free to use any units.

time, which is, evidently, dimensionless speed. In this example, recovering the dimensional speed is common sense: speed is in mi/hr. The notational compromise is that v is being used to represent both dimensional and dimensionless speed, with the precise meaning depending on context

Example: Table of values.
Using notational compromise we can use the formula $x = vt$ with $v = 50\,\text{mi/hr}$ to do a set of calculations. Say we want to know the distance x every quarter of an hour for two hours. So we multiply 50 by .25, .5, .75, ... and thus make a table with two columns labeled t (hr) and x (mi).

t (hr)	x (mi)
0	0
.25	12.5
.5	25
.75	37.5

This approach has some ambiguity to some eyes. Here is a more clear way to make the same table.

Example: Less ambiguous table of values.
The exact meaning of the columns in the above example are a little ambiguous. We can make it more precise by labeling the columns as follows

$t/(\text{hr})$	$x/(\text{mi})$
0	0
.25	12.5
.5	25
.75	37.5

That is, the columns of numbers are dimensionless. The first column, is the time divided by one hour the second is distance divided by one mile.

Finally, the way things are most often done in science, and sometimes in engineering practice, is to only use clearly defined and distinct dimensionless variables (ie, not to use v both for the speed and for the speed as measured in m/s. This approach is more precise, if cumbersome, than using v to be both dimensional and dimensionless depending on context.

Example: Dimensionless table of values.
If we take x to be dimensional distance, t to be dimensional time, and v to be dimensional speed, we can define new dimensionless variables. $t^* = t/(1\,\text{hr})$, $x^* = x/(1\,\text{mi})$, and $v^* = v/(1\,\text{mi/hr})$. Now there is no ambiguity: x is dimensional and x^* is dimensionless. Dividing the equation $x = vt$ on both sides by one mile, and multiplying the right side by 1, in the form of $1 = (1\,\text{hr}/1\,\text{hr})$ we get:

$$\frac{x}{1\,\text{mi}} = \frac{v}{1\,\text{mi/hr}} \cdot \frac{t}{1\,\text{hr}}$$

which is, using the dimensionless variables,

$$x^* = v^* t^*.$$

Because v is $50\,\text{mi/hr}$, $v^* = 50$ as we can show formally as follows:

$$v^* = v/(1\,\text{mi/hr}) = (50\,\text{mi/hr})/(1\,\text{mi/hr}) = 50.$$

The dimensionless speed v^* is just the dimensionless number 50. Now we can make a table by multiplying 50 by .25, .5, .75, ... The columns of the table can be labeled t^* and x^* and all variables are clearly defined.

t^*	x^*
0	0
.25	12.5
.5	25
.75	37.5

Most people will not often go to the trouble of defining a whole set of dimensionless variables unless they have got confused with the difference between a pound force and a pound mass, or from some variables being measured in meters and others in feet, etc.

A.3 Force, Weight and English Units

The force of gravity on an object is its weight — well, almost. A given object has different weight on different parts of the earth, with up to 0.5% variation. That is, g, the earth's gravitational 'constant,' varies from about $9.78\,\mathrm{m/s^2}$ at the equator to about $9.83\,\mathrm{m/s^2}$ at the North Pole. The official value of the 'constant' g is in between at exactly $9.80665\,\mathrm{m/s^2}$ (this is about $32.1740486\,\mathrm{ft/s^2}$). Multiplying the official g by the mass m will give you almost exactly the force it takes to hold it up if you are in exactly the official place, somewhere in Potsdam. Outside of Potsdam you have to accept an error of up to $1/4\%$ when calculating gravitational forces, unless you happen to know the value of g in your neighborhood.

Historically, people understood weight before they understood mass: bigger things are harder to hold up so were said to have more weight. And comparisons were made with balances. Weight is an easier concept for the pre-Newtonian mind than that bigger things are harder to accelerate, i.e., have more mass. So people defined the quantity of matter by weight. 'How much flour?' one would ask. 'A pound of flour,' meaning one pound weight, might be the answer. A one pound weight is pulled with a 1 lbf by gravity, or in the older notation where one did not worry about mass, by 1 lb. People didn't notice that it was a little harder, i.e.. would stretch a given spring more, to hold something up on the north pole than at the top of Mount Everest, so the earth's gravity force on an object was a fine measure of quantity.

When it became important to talk about mass, as opposed to weight, the pound mass was defined as the mass of something that weighed a pound. That is,

$$1\,\mathrm{lbm} \equiv 1\,\mathrm{lbf}/g.$$

Then people thought 'what is the mass that accelerates one foot per second squared if a one-pound force is applied?' They found

$$\begin{aligned} m &= F/a \\ &= (1\,\mathrm{lbf})/(1\,\mathrm{ft/s^2}) = 1\left(\mathrm{lbf/\,ft/s^2}\right)\underbrace{\left(\frac{32.174\,\mathrm{lbm\,ft/s^2}}{\mathrm{lbf}}\right)}_{1} \\ &= 32.174\,\mathrm{lbm}. \end{aligned}$$

But this 32.174 was awkward. People felt that if a unit force causes something to accelerate at a unit rate that thing should have a unit mass. So they invented the slug. $1\,\mathrm{slug} \equiv 1\,\mathrm{lbf}/(1\,\mathrm{ft/s^2})$. So what do we get for the mass in the previous equation?

$$\begin{aligned} m &= F/a \\ &= (1\,\mathrm{lbf})/(1\,\mathrm{ft/s^2}) = 1\left(\mathrm{lbf/ft/s^2}\right)\underbrace{\left(\frac{32.174\,\mathrm{lbm\,ft/s^2}}{\mathrm{lbf}}\right)}_{1} \\ &\stackrel{def}{=} 1\,\mathrm{slug} \end{aligned}$$

That is, 1 slug accelerates $1\,\mathrm{ft/s^2}$ when 1 lbf is applied. How much does a slug weigh? The force of gravity on a slug, in Potsdam, is $32.174\,\mathrm{lbf}$.

Now the invention of the slug did not make people happy enough. They thought, 'what is the force required to accelerate 1 lbm at an acceleration of $1\,\mathrm{ft/s^2}$?' It is

$$\begin{aligned} F &= ma \\ &= (1\,\mathrm{lbm})(1\,\mathrm{ft/s^2}) = 1\left(\cancel{\mathrm{lb}}\mathrm{m\,ft}\cancel{/}\mathrm{s^2}\right)\underbrace{\left(\frac{1\,\mathrm{lbf}}{32.174\,\cancel{\mathrm{lb}}\mathrm{m\,ft}\cancel{/}\mathrm{s^2}}\right)}_{1} \\ &= \left(\frac{1}{32.174}\right)\mathrm{lbf}. \end{aligned}$$

People found *this* $1/32.174$ awkward also, so in order to simplify some arithmetic and confuse many generations of engineers, they invented the poundal. They defined the poundal to be the force it takes to accelerate one pound mass at one foot per second squared. So they got

$$\begin{aligned} F &= ma \\ &= (1\,\mathrm{lbm})(1\,\mathrm{ft/s^2}) \stackrel{def}{=} 1\,\mathrm{pdl}. \end{aligned}$$

So, because scientists and engineers of old liked the number 1 better than both the number 32.174 and the number $1/32.174$ they left us two new units to worry about: the poundal $= 1\,\mathrm{lbm\,ft/s^2} = (1/32.174)\,\mathrm{lbf}$, and the slug $= 1\,\mathrm{lbf}/(\mathrm{ft/s^2}) = 32.174\,\mathrm{lbm}$. If you are used to the internationally acceptable units for force and mass $1\,\mathrm{pdl} = .138255\,\mathrm{N}$ and $1\,\mathrm{slug} = 14.5939\,\mathrm{kg}$. Fortunately, the slug and the poundal are used less and less as the decades roll by. Now days here are far more people who laugh at their confusion about slugs and poundals than there are people who use them seriously.

Don't laugh if you are from Europe Unfortunately for dimensional purists, engineers using the SI system have copied one of the confusing traditions that the SI system was designed to avoid. They invented the kilogram-force, kgf, also called a kilopond, which is 1 kg times the official value of g. That is $1\,\mathrm{kgf} = 1\,\mathrm{kilopond} = 9.80665\,\mathrm{N}$. A kilopond is the force of gravity on a kilogram, exactly so somewhere in Potsdam — well, almost.

Well, almost. Why do we say 'well, almost' about 'g' being the acceleration due to gravity? Because, confusingly, mg is not the force due to gravity. It is the force of the spring which holds up the mass on a rotating earth! What is called g is the 'effective' gravity which is the acceleration due to gravity minus a centripetal term due to the earth's rotation.

APPENDIX B

Friction: perspectives on friction laws

Here we include various friction topics too advanced for the main text. First, what approximations are used in Coulomb's law of friction? Second, why is the concept of having a static friction coefficient higher than the dynamic coefficient a problematic concept? Third, we show an alternative way of writing the equations governing Coulomb friction.

This appendix is not about how to solve friction mechanics problems but about the equations that describe friction. Unless one knows better, we have advised using Coulomb's law with a single friction coefficient μ with

$$\mu = \mu_k = \mu_d = \mu_s.$$

That is, use the approximation that one number μ is the dynamic coefficient μ_d, the kinetic coefficient μ_k and the static coefficient μ_s. Of course friction does vary, even between a given pair of surfaces. One could write a PhD thesis about that. But for this book, a single coefficient is a good enough approximation.

Here are three equivalent descriptions of Coulomb's law of friction. We assume isotropic (same in all directions) friction.

1. Given this μ Coulomb's law of friction for body $\mathcal{B}$ contacting $\mathcal{A}$ with normal force N and possibly sliding with relative velocity $\vec{\boldsymbol{v}}_{\mathcal{B}/\mathcal{A}}$ is this:

$$\text{if } \vec{\boldsymbol{v}}_{\mathcal{B}/\mathcal{A}} \neq \vec{\mathbf{0}} \quad \Rightarrow \quad \vec{\boldsymbol{F}}_{\text{friction on } \mathcal{B} \text{ from } \mathcal{A}} = -\mu N \frac{\vec{\boldsymbol{v}}_{\mathcal{B}/\mathcal{A}}}{|\mathcal{B}/\mathcal{A}|} \tag{B.1}$$

$$\text{if } \vec{\boldsymbol{v}}_{\mathcal{B}/\mathcal{A}} = \vec{\mathbf{0}} \quad \Rightarrow \quad |\vec{\boldsymbol{F}}_{\text{friction on } \mathcal{B} \text{ from } \mathcal{A}}| \leq \mu N \tag{B.2}$$

2. Friction resists relative motion with magnitude μN. The strength of a non-sliding contact is also μN but the actual frictional force at a non-sliding contact can be anything with magnitude up to μN.

3. In 2D, the friction force F the normal force N and the relative sliding speed $\dot{\delta}$ must all lie on the curve of fig. 3.29 on page 172.

Most mechanics books at this level teach that static friction is higher than dynamic (or kinetic) friction. This idea, while catching a sometimes important aspect of real friction, is problematic.

B.1 A problem with the concept of static friction

A commonly used static friction law assumes that the friction force instantly jumps from the static value $\mu_s N$ to a lower dynamic value $\mu_d N$ when slip starts. We do not use that law in this book. Why not?

If the contacting surfaces have more than one contact point this jump from static to dynamic friction implicitly makes use of two simultaneous limits.

1. That the body is *infinitely* stiff.
2. That the coefficient of friction *instantly* drops from the static value μ_s to the dynamic value μ_k when slip starts.

The problem with simultaneous use of these limits is highlighted by considering a body that has finite stiffness and for which the friction gradually drops as slip starts. We should then *hope* to recover the concept of static friction and rigid body slip as a limit of this model. But, there is trouble.

Let's get a little more specific.

Slip weakening friction law

The friction law we will employ is a 'slip-weakening' friction law described by the graph below. Although this description is obviously incomplete if slip occurs more than once or reverses direction, it suffices for our considerations. Here the describing equations are:

$$
\begin{aligned}
F/N &\leq \mu_s && \text{if no slip has occurred} \\
F/N &= \mu_s - (\mu_s - \mu_d)\delta/d && \text{if } \delta \leq d \\
F/N &= \mu_d && \text{if } \delta \geq d
\end{aligned}
$$

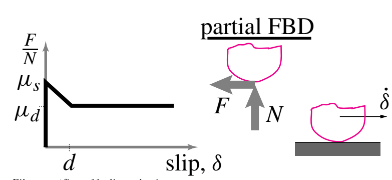


Figure B.1: Friction as a function of displacement, starting from rest.

If we keep μ_s and μ_d constant and look at the limit as $d \to 0$ this friction law becomes the classic 'static-dynamic' friction law which we are now critiquing. There are other friction laws, such as those with rate dependence, we could use that reduce to static-dynamic friction in some limit, but these laws also would lead to problems something like those we discuss below.

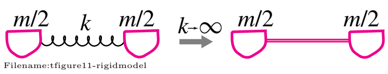


Figure B.2:

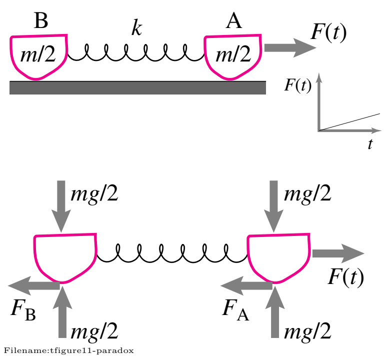


Figure B.3:

Model of a rigid body

The model for a rigid body that we employ is: two point masses are connected with a spring. In the limit as the spring constant k goes to infinity this model becomes, at least in the common sense of the words, a 'rigid body.'

A sliding problem

Now let's consider the problem of initial slip of the system shown. For simplicity and definiteness let's assume the spring is relaxed in the configuration shown. Now, we very slowly increase the applied force F until both masses end up sliding. Here is the question:

What force does it take to start the pair of masses sliding?

There is no force causing block B to slide except the spring tension. But the spring tension does not build until the spring stretches due to the slip of A. So F increases until $F = \mu_s mg/2$ at which time block A starts to slide. When the tension in the spring reaches $\mu_s mg/2$ then block B starts to slide. What is the value of F at this time?

We could work out the details for every value of the parameters, but we need not do this generality to make our point. First let's take the limit that the body is rigid $k \to \infty$.

Rigid body, gradual friction drop

In this case the two masses always move together. As slip starts the two masses both have a friction force of $\mu_s mg/2$ and the force required to cause slip of both masses is

$$F = \mu_s mg \text{ at slip,}$$

as expected. As motion progresses this force gradually reduces to $\mu_d mg$ at a rate that depends on the frictional slip weakening distance d. But for any finite value of d the applied force must first reach $\mu_s mg$ before slip proceeds.

Compliant body, sudden friction drop

Now let's take the limit the other way. Let's assume that the spring has fixed stiffness, possibly very high, and look at the limit $d \to 0$, the limit which reduces the friction law to the classical law. In this case block A breaks entirely free before there is any tension in the spring. Exactly when block B will start to slip depends on the details of all the parameters, so it turns out that finding the start of slip of block B is a genuinely complicated problem. But, no matter what, the spring stretches some before block A comes to rest. Block A may slip several times before the spring stretch is enough to cause the slip of block B, again the details depend on the relative values of μ_s and μ_d. But eventually block B will be excited into sliding. This slip will most likely start when block A is already sliding. Thus the applied force need only overcome the *dynamic* friction of block A $\mu_d mg/2$ and the static friction $\mu_s mg/2$ of block B. Due to the complex dynamics of the situation, it turns out that the two blocks can sometimes end up sliding if the applied force is just a hair above $\mu_d mg$, even when K is very large (but still finite).

$$F = ? \text{ (something less than } \mu_s mg\text{)}$$

The rigid-body static-friction paradox

If we take the limit $k \to \infty$ and then $d \to 0$ we get an overall effective coefficient of static friction μ_s for the whole body. If instead we take the limit $d \to 0$ and then $k \to \infty$ the effective static friction limit does not exist, but for some arbitrarily large values of k it can be as low as μ_d. That is,

the problem of initial slip of a rigid body with more than one point of contact and with static-dynamic friction is ill-defined

This paradox can be resolved a number of ways. One is to assume it away, effectively taking the $k \to \infty$ limit first. Another more complex solution, beyond this book and beyond the level of detail that most people want to deal with, is to only use more sophisticated friction laws and to keep track of solid deformation.

Compromise

To avoid these issues by users of this text we just use one coefficient of friction $\mu_s = \mu_d = \mu$. May the user beware if using a more complex law than this one.

B.2 A critique of Coulomb friction

This section is an aside, not needed for homework, about the place of Coulomb friction amongst more general friction laws. First (below) a quick outline and then more details (further below).

The good. In short, Coulomb's law of friction is good because

- Coulomb's law of friction is simple.
- Coulomb's law usefully predicts many phenomena.
- It has some of the right trends: the friction force
  - *is* roughly independent of slip rate, and
  - *is* roughly proportional to the normal force.
- Other candidate laws sometimes suffer more from complexity than they gain in accuracy or usefulness.

The bad. On the other hand,

- The friction coefficient is not stable, it may vary from day to day or between samples of nearly identical materials.

- Coulomb's law, without a separate static coefficient of friction or an explicit dependence on rate of slip, cannot be used to explain frictional phenomena such as
  - the squeaking of doors,
  - the excitement of a violin string by a bow, and
  - earthquakes from sliding rocks.
- For some materials the friction is noticeably not proportional to the normal force. Rubber on road, for example, has more friction force per unit normal force when the normal force is low; for a given normal force, increasing the area increases the friction. This is why racing cars have fat tires.

We expand on some of these points below.

The friction coefficient is not a stable property John Jaeger's third empirical friction friction law is said to be:

A friction experiment will make a monkey out of you.

For any pair of objects and any given experiment to measure the friction coefficient, the measured value will likely vary from day to day. What's the source of this apparent violation of determinacy? It's probably because friction involves the interaction of surfaces. The chemistry of a surface can be dramatically changed by very small quantities of material (a surface is a very small volume!). So any change in humidity, or perhaps a random finger touch, or a slight spray from here or there can dramatically change the surface chemistry and hence the friction.

This problem of the non-constancy of friction from day to day or sample to sample cannot be overcome by a better friction law. Unless one understands the materials and their chemical environment extremely well, all friction laws, however sophisticated, are doomed to large inaccuracy.

Coulomb's friction law and "static" friction Most simple treatments of friction immediately introduce two coefficients of friction: the dynamic coefficient μ_d i (also sometimes called the kinetic coefficient μ_k) and the 'static' coefficient of friction μ_s. According to standard lore, each pair of bodies has friction which is described by μ_s and μ_d, with the understanding that the static coefficient of friction is greater than the dynamic coefficient of friction, $\mu_s > \mu_d$.

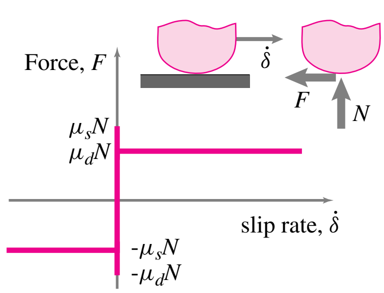


Figure B.4: A graph of friction *vs* slip rate shows a small peak at the start. This is 'static' friction.

Static-Dynamic Friction. The relation between friction velocity and friction force is such that at all times the pair of values is found on the dark line shown. This description is useful for roughly characterizing the following phenomenon:

It is harder to start something sliding than it is to keep it sliding.

If this drop in friction force is important in your problem, the static-dynamic friction law is one attractive candidate. But be forewarned: although this law is great for qualitatively explaining how a bow excites a violin string, or why anti-lock brakes work better than all out skidding, it is not very accurate.

Experiments trying to learn in more detail how the friction force drops from a higher value to (as slip starts) a lower one, reveal more subtle phenomena that are not well captured with two simple friction coefficients. Further, using two coefficients of friction leads to various paradoxes and indeterminacies when one studies slightly more complex problems. (See page 987.)

Friction is not always proportional to normal force The Coulomb friction equation, applicable during slip or at impending slip,

$$F = \mu N$$

is most directly translated into English as:

the friction force is proportional to the normal force.

This proportionality is, as far as we know, not fundamental, but rather is often a reasonable approximation. Two effects that contribute to this proportionality: 1) when rough surfaces touch (at a micro-scale all surfaces are rough) the area of micro-contact grows as they are squeezed together, and 2) each micro-contact grows in area as it yields to the contact pressure.

In some books you will see an additional law of friction stated as:

The friction force is independent of the area of contact.

By *'area of contact'* is meant the area you would measure macroscopically. For a 4 in $\times$ 8 in brick sliding on a pavement the area of contact is 32 in^2

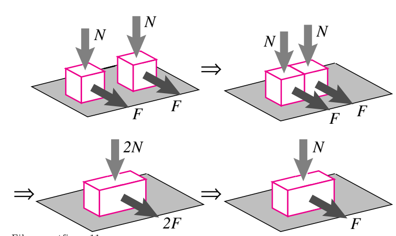


Figure B.5: If the friction force is proportional to normal force it must be independent of area. Why? Imagine two blocks side by side each with the same normal force and same friction force. Glue these blocks together and the total of the forces should not change. But if the friction force is proportional to the normal force on this new big block we can divide both by two. Now we have a new big block with the same normal force as the original block but with twice the area. So area changed and the friction force did not.

F independent of $A \Leftrightarrow \mu$ independent of N That the friction force does not depend on area is actually equivalent to the proportionality of friction force with normal force. Here's the gist of the argument. Imagine two identical blocks side by side on a plane as in the figure above. The force pushing down on each is N and the friction force to cause slip is $F = \mu N$. The act of glueing the two together side-by-side should have no effect. Now we have one bigger block with twice the normal force, twice the friction force and twice the area of contact. Assuming that friction force is proportional to normal force, means that if we cut the normal force in half then the friction force will be cut in half. But now we have a new block with twice the area of contact as each of the original blocks with the same normal force and the same friction force. Thus the friction force is unchanged by doubling the area of contact.

Dependence of μ on area and N. But in fact, some materials have friction force which does depend on the normal force, or for a given normal force, does depend on the area of contact. One example is the friction between rubber and pavement. For a given weight car, a larger friction force can be generated with a fat tire than a narrow one. That is, for rubber on pavement the ratio of the friction force to normal force *decreases* as the normal force increases and *increases* as the area increases.

[Real area of contact. Another concept of area of contact is the actual area of contact at all the little micro-bumps called asperities. This definition of area of contact is useful for tribologists (people who study friction) but is of little concern to people interested in the mechanics of macroscopic things.]

Alternatives to Coulomb's law

Of course there are situations which one may want to understand where the transition from static to dynamic friction is essential. For these cases a static-dynamic friction model might provide some insight, but it may also cause basic modeling problems.

Static-dynamic friction As mentioned in the discussion above, the fact that it is generally harder to start sliding something than to keep it sliding is most simply described using static μ_s and dynamic μ_d friction. However, be warned

Static/dynamic friction is somewhat pathological

See, for example, box B.1 on page987.

Velocity-dependent friction One problem with Coulomb friction is that in computations one needs to have separate cases for sliding and not-sliding. One approximation, more motivated by convenience than data, is to write the friction law as

$$F = -\mu N f(v)$$

where f(v) is a function of velocity with these properties:

- $f(0) = 0$, when there is no sliding the force is zero,
- $f(v) = -f(-v)$ for all v, sliding forwards has the same resistance as sliding backwards, and
- $f(v \to \infty) \to 1$, the net friction force approaches μN as the sliding speed gets high.

One candidate function for this purpose is the arctangent function

$$f(v) = \frac{2}{\pi} \arctan(v/v_c).$$

An alternative is the logistic function (offset and scaled):

$$f(v) = \frac{1 - e^{v/v_c}}{1 + e^{v/v_c}}.$$

For either the arctangent or logistic function the constant v_c has to be chosen to be small enough so that for practical purposes the behavior is close to that of Coulomb friction, but large enough so as to not cause numerical issues (*e.g.,* instability or 'stiff' equations). Note that with this velocity dependent friction things don't hold together in the long run, everything creeps; a block on even a slight slope slowly slides down.

Stribeck velocity-dependent friction In the same spirit as above, where a velocity dependence mimics Coulomb friction, Stribeck friction does the same with the static-to-dynamic drop. Stribeck friction is again of the form

$$F = -\mu N f(v)$$

and again

- $f(0) = 0$
- $f(v) = -f(-v)$ for all v,
- $f(v \to \infty) \to 1$.

but now the function $f(v)$ rather than being monotonic has a local maximum at some small speed (and a corresponding minimum at minus that speed). The descending branch, the part of the curve where the friction force decreases with increasing speed, mimics the drop from dynamic to static friction.

All things considered, Coulomb's law, $F = \mu N$, is alright

Coulomb's law, the version with one coefficient of friction, is the simplest dry friction constitutive law. It is reasonably accurate, considering that it is one out of a mediocre crowd, and more elaborate laws often introduce complexity with little or no gain in accuracy. Most often it is reasonable to assume that static friction is close enough to dynamic friction that it is not worth the trouble to distinguish them. So we generally use just one coefficient of friction μ in this book ($\mu_d = \mu_s = \mu$).

B.3 Another expression for Coulomb friction: an advanced aside

The law of Coulomb friction is both simple and confusing. Part of the confusion comes from the requirement of calculating the friction force with one equation during slip and then not being able to find the friction force, at least from the friction law, when there is no slip.

We would like to confuse the issue a little further now for the case of slip on a plane. That is, slip can be in any direction on, say, the xy-plane, not just to the right or to the left. If we define $\vec{v}$ to be the sliding velocity of the point of contact of the body of interest relative to its partner in friction, and $\vec{F}$ to be the tangential contact force that it causes on its partner then we could write the friction law with a pair of inequalities which must both be satisfied at all times.

$$\begin{aligned} \vec{v} \cdot (\vec{F} - \vec{F}^*) &\geq 0 \\ |\vec{F}| &\leq \mu N \end{aligned}$$

where $\vec{v}$ and $\vec{F}$ are such that these inequalities are satisfied for every possible $\vec{F}^*$ that is tangent to the slip surface and has magnitude $|\vec{F}^*| \leq \mu N$. The force $\vec{F}^*$ is not any actual force in the problem. It is just a label for the set of all possible friction forces consistent with the friction law.

The meaning of these inequalities can be seen in the figure below. If you are a normal reader you will have two questions:

- How is this inequality the same as the Coulomb friction law described in the text? And,
- why bother to write the friction law in this strange way?

The answer to the first question, how are these inequalities an expression of Coulomb friction, is found by considering the various cases of slip and no-slip. To show that these inequalities imply that friction directly opposes motion during slip takes a little thought. The reason is that $\vec{F}^*$ can be any point on or inside the circle shown. By considering the two cases that $\vec{F}^*$ is on the limit-circle just clockwise and just counterclockwise from the actual friction force you can see that for the inequality $(\vec{F} - \vec{F}^*) \cdot \vec{v} \geq 0$ for both cases, $\vec{v}$ must be perpendicular to the circle.

The answer to the second question, why bother, is: The pair of inequalities shown allow the proof of various theorems about frictional sliding, allow a simple description of friction on distributed contacts, and also allows a simple generalization to friction that is anisotropic, that is, of different magnitude in different directions of slip. For those who are going on to study advanced solid mechanics, this expression for the friction law shows one of the connections between friction and classical plasticity.

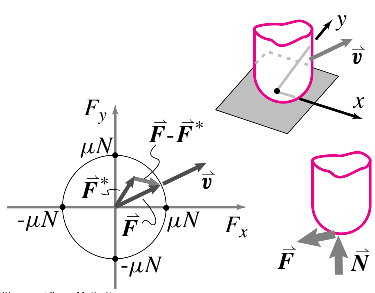


Figure B.6: For a given slip velocity $\vec{v}$ actual friction force is that point on or in the friction circle that maximizes the dissipation.

APPENDIX C

Center of mass theorems for systems of particles

*The **center of mass** allows simplifications for expressions for momentum, angular momentum, and kinetic energy. Furthermore, the energy equations for systems of particles provide foreshadowing for the first law of thermodynamics.*

C.1 Velocity and acceleration of the center-of-mass of a system of particles

The average position of mass in a system is at a point called the center-of-mass. The position of the center-of-mass is

$$\vec{\boldsymbol{r}}_{cm} = \frac{\sum \vec{\boldsymbol{r}}_i \, m_i}{m_{\text{tot}}}.$$

Multiplying through by m_{tot}, we get

$$\vec{\boldsymbol{r}}_{cm} m_{\text{tot}} = \sum \vec{\boldsymbol{r}}_i \, m_i.$$

By taking the time derivatives of the equation above, we get

$$\begin{aligned}\vec{v}_{cm}m_{\text{tot}} &= \sum \vec{v}_i\ m_i \quad \text{and}\\ \vec{a}_{cm}m_{\text{tot}} &= \sum \vec{a}_i\ \mathrm{m}_i \quad .\end{aligned}$$

for the velocity and acceleration of the center-of-mass. The results above are useful for simplifying various momenta and energy expressions. Note, for example, that

$$\begin{aligned}\vec{L} = \sum \vec{v}_i\ \mathrm{m}_i &= \vec{v}_{cm}m_{\text{tot}}\\ \dot{\vec{L}} = \sum \vec{a}_i\ \mathrm{m}_i &= \vec{a}_{cm}m_{\text{tot}}.\end{aligned}$$

Linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$

One of our three basic dynamics equations is linear momentum balance:

$$\sum \vec{F} = \dot{\vec{L}}.$$

The first quantity of interest in this section is the linear momentum $\vec{L}$ ① whose derivative, $\dot{\vec{L}}$, with respect to a Newtonian frame is so important. Linear momentum is a measure of the translational motion of a system.

$$\underbrace{\vec{L}}_{\text{linear momentum}} \equiv \underbrace{\sum m_i \vec{v}_i}_{\substack{\text{summed over}\\ \text{all the mass particles}}} = m_{\text{tot}}\vec{v}_{cm} \qquad \text{(C.1)}$$

①In Isaac Newton's language: *'The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly'*. In other words, Newton's dynamics equations for a particle were based on the product of $\vec{v}$ and m. This quantity, $m\vec{v}$, is now called $\vec{L}$, the linear momentum of a particle.

Example: Center of Mass position, velocity, and acceleration

A particle of mass $m_A = 3\,\text{kg}$ and another point of mass $m_B = 2\,\text{kg}$ have positions, respectively,

$$\vec{r}_A(t) = \left[3\hat{\imath} + 5\left(\frac{t}{\text{s}}\right)\hat{\jmath}\right]\text{m}, \text{ and } \vec{r}_B(t) = \left[6\left(\frac{t^2}{\text{s}^2}\right)\hat{\imath} - 4\hat{\jmath}\right]\text{m}$$

due to forces that we do not discuss here. The position of the center-of-mass of the system of particles, according to equation 2.49 on page 121, is

$$\begin{aligned}\vec{r}_{cm}(t) &= \frac{\overbrace{m_A\vec{r}_A(t) + m_B\vec{r}_B(t)}^{\sum m_i\vec{r}_i}}{\underbrace{(m_A + m_B)}_{m_{\text{tot}}=5\,\text{kg}}}\\ \vec{r}_{cm}(t) &= \left[\left(\frac{9}{5} + \frac{12}{5}\left(\frac{t^2}{\text{s}^2}\right)\right)\hat{\imath} + \left(3\left(\frac{t}{\text{s}}\right) - \frac{8}{5}\right)\hat{\jmath}\right]\text{m}.\end{aligned}$$

To get the velocity and acceleration of the center-of-mass, we differentiate the position of the center-of-mass once and twice, respectively, to get②

$$\vec{v}_{cm}(t) = \dot{\vec{r}}_{cm}(t) = \left[\frac{24}{5}\left(\frac{t}{\text{s}^2}\right)\hat{\imath} + \frac{3}{\text{s}}\hat{\jmath}\right]\text{m} = \left[\frac{24}{5}\left(\frac{t}{\text{s}}\right)\hat{\imath} + 3\hat{\jmath}\right]\text{m/s}$$

and

$$\vec{a}_{cm}(t) = \dot{\vec{v}}_{cm}(t) = \ddot{\vec{r}}_{cm}(t) = \left[\frac{24}{5}\left(\frac{1}{\text{s}^2}\right)\hat{\imath}\right]\text{m} = \left(\frac{24}{5}\right)\text{m/s}^2\hat{\imath}.$$

②That is, particle A travels on the line $x = 3\,\text{m}$ with constant speed $\dot{r}_{A_y} = 5\,\text{m/s}$ and particle B travels on the line $y = -4\,\text{m}$ at changing speed $\dot{r}_{B_x} = 12t(\text{m/s}^2)$.

In this example, the center-of-mass turns out to have constant acceleration in the x-direction.

The second part of equation C.1 follows from the definition of the center-of-mass (see box C.1 on page 996).③ The total linear momentum of a system is the same as that of a particle that is located at the center-of-mass and which has mass equal to that of the whole system. The linear momentum is also given by

$$\vec{L} = \frac{d}{dt}\left(m_{\text{tot}}\vec{r}_{cm}\right).$$

We only consider systems of fixed mass, $\frac{d}{dt}(m_{\text{tot}}) = 0$. Thus, for a fixed mass system, the linear momentum of the system is equal to the total mass of the system times the derivative of the center-of-mass position.

Finally, since the sum defining linear momentum can be grouped any which way (the associative rule of addition) the linear momentum can be found by dividing the system into parts and using the mass of those parts and the center-of-mass motion of those parts. That is, the sum $\sum m_i\vec{v}_i$ can be interpreted as the sum over the center-of-mass velocities and masses of the various subsystems, say the parts of a machine.

Example: System Momentum
See fig. C.1 for a schematic example of the total momentum of system being made of the sum of the momenta of its two parts.

The reasoning for this allowed subdivision is similar to that for the center-of-mass in box 2.15 on page 126.

The quantity $\dot{\vec{L}}$ figures a little more directly in our presentation of dynamics than just plain $\vec{L}$ ④The rate of change of linear momentum, $\dot{\vec{L}}$, is

$$\begin{aligned}
\dot{\vec{L}} &= \frac{d}{dt}\vec{L} \\
&= \frac{d}{dt}\sum m_i\vec{v}_i \\
&= m_{\text{tot}}\frac{d\vec{v}_{\text{cm}}}{dt} \\
\dot{\vec{L}} &= m_{\text{tot}}\vec{a}_{\text{cm}}
\end{aligned}$$

The last three equations could be thought of as the *definition* of $\dot{\vec{L}}$. That $\dot{\vec{L}}$ turns out to be $\frac{d}{dt}(\vec{L})$ is, then, a derived result. Again, using the definition of center-of-mass,

> the total rate of change of linear momentum is the same as that of a particle that is located at the center of mass which has mass equal to that of the whole system.

The rate of change of linear momentum is also given by

$$\dot{\vec{L}} = \frac{d}{dt}(m_{\text{tot}}\vec{v}_{\text{cm}}) = \frac{d^2}{dt^2}(m_{\text{tot}}\vec{r}_{\text{cm}}).$$

③Some books use the symbol $\vec{P}$ for linear momentum. Because $\vec{P}$ is often used to mean force or impulse and P for power we use $\vec{L}$ for linear momentum.

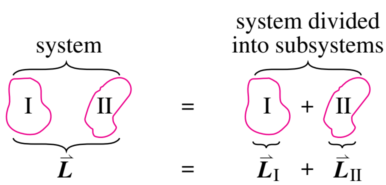


Figure C.1: System composed of two parts. The momentum of the whole is the sum of the momentums of the two parts.

④No slight of Sir Isaac is intended.

The momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ can be expressed in terms of the total mass of a system and the motion of the center-of-mass. This simplification holds for any system, however complex, and any motion, however contorted and wild.

Angular momentum $\vec{H}$ and its rate of change $\dot{\vec{H}}$

After linear momentum balance, the second basic mechanics principle is angular momentum balance:

$$\sum \vec{M}_{\mathrm{C}} = \dot{\vec{H}}_{/\mathrm{C}},$$

where C is any point, preferably one that is fixed in a Newtonian frame.If you choose your point C to be a moving point you may have the confusing problem that the quantity we would like to call $\dot{\vec{H}}_{/\mathrm{C}}$ is not the time derivative of $\vec{H}_{/\mathrm{C}}$. The first quantity of interest in this sub-section is the angular momentum with respect to some point C, $\vec{H}_{/\mathrm{C}}$, whose rate of change $\dot{\vec{H}}_{/\mathrm{C}} = d\vec{H}_{/\mathrm{C}}/dt$ is so important.

$$\underbrace{\vec{H}_{/\mathrm{C}}}_{\text{angular momentum.}} \equiv \underbrace{\sum \vec{r}_{i/C} \times m_i \vec{v}_i}_{\substack{\textit{summed over all}\\ \textit{the mass particles}}}$$

A useful theorem about angular momentum is the following (see box 13.11 on page 707), applicable to all systems

$$\vec{H}_{/\mathrm{C}} = \overbrace{\vec{r}_{cm/C} \times \vec{v}_{cm} m_{\mathrm{tot}}}^{\substack{\text{angular momentum due to}\\ \text{center-of-mass motion}}} + \overbrace{\sum \underbrace{\vec{r}_{i/cm}}_{\substack{\text{position of } m_i \text{ relative to the center-of-mass}\\ \vec{r}_{i/cm} \equiv \vec{r}_i - \vec{r}_{cm}}} \times \underbrace{\vec{v}_{i/cm}}_{\substack{\text{velocity of } m_i \text{ relative to the center-of-mass}\\ \vec{v}_{i/cm} \equiv \vec{v}_i - \vec{v}_{cm}}} m_i}^{\substack{\text{angular momentum relative}\\ \text{to the center-of-mass}}} . \tag{C.2}$$

A system of particles is shown in fig. C.2. The angular momentum of any system is the same as that of a particle at its center-of-mass *plus* the angular momentum associated with motion relative to the center-of-mass.

The angular momentum about point C is a measure of the average rotation rate of the system about point C. Angular momentum is not so intuitive as linear momentum for a number of reasons:

- First, recall that linear momentum is the derivative of the total mass times the center-of-mass position. Unfortunately, in general,

  angular momentum is not the derivative of anything.

- Second, the angular momentum of a given system at a given time depends on the reference point C. So there is not one single quantity that is *the* angular momentum. For different points C_1, C_2, etc., the same system has different angular momentums.
- Finally, calculation of angular momentum involves a vector cross product and many beginning dynamics students are intimidated by vector cross products.

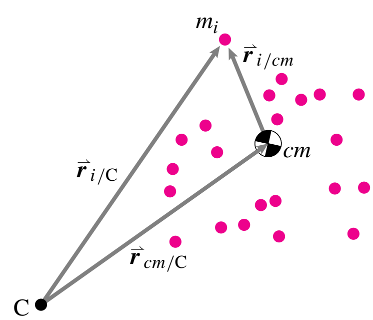


Figure C.2: A system of particles showing its center-of-mass and the i_{th} particle of mass m_i. The i_{th} particle has position $\vec{r}_{i/cm}$ with respect to the center-of-mass. The center-of-mass has position $\vec{r}_{cm/C}$ with respect to the point C

Despite these confusions, the concept of angular momentum allows the solution of many practical problems and eventually becomes somewhat intuitive.

Actually, it is $\dot{\vec{H}}_{/C}$ which is the more fundamental quantity. $\dot{\vec{H}}_{/C}$ is what you use in the equation of motion. You can find $\dot{\vec{H}}_{/C}$ from $\vec{H}_{/C}$ as shown in the box on page 1001. But, in general,

$$\dot{\vec{H}}_{/C} \equiv \sum \vec{r}_{i/C} \times (\mathrm{m}_i \vec{a}_i).$$

A useful theorem about rate of change of angular momentum is the following (see box 13.11 on page 707), applicable to all systems:

$$\dot{\vec{H}}_{/C} = \overbrace{\vec{r}_{cm/C} \times \vec{a}_{cm} m_{\text{tot}}}^{\text{rate of change of angular momentum due to center of mass motion}} + \overbrace{\sum \underbrace{\vec{r}_{i/cm}}_{\vec{r}_{i/cm} \equiv \vec{r}_i - \vec{r}_{cm}} \times \underbrace{\vec{a}_{i/cm}}_{\vec{a}_{i/cm} \equiv \vec{a}_i - \vec{a}_{cm}} m_i}^{\text{rate of change of angular momentum relative to the center of mass}}.$$

This expression is completely analogous to equation C.2 on page 999 and is derived in a manner nearly identical to that shown in box 13.11 on page 707. The rate of change of angular momentum of any system is the same as that of a particle at its center-of-mass *plus* the rate of change of angular momentum associated with motion relative to the center-of-mass. A special point for any system is, as we have mentioned, the center-of-mass. In the above equations for angular momentum we could take C to be a fixed point in space that happens to coincide with the center-of-mass. In this case we would most naturally define $\vec{H}_{\text{cm}} = \int \vec{r}_{/cm} \times \vec{v}\, dm$ with $\vec{v}$ being the absolute velocity. But we have the following theorem:

$$\vec{H}_{\text{cm}} = \int \vec{r}_{/cm} \times \vec{v}\, dm = \int \vec{r}_{/cm} \times \vec{v}_{/cm}\, dm$$

where $\vec{r}_{/cm} = \vec{r} - \vec{r}_{cm}$ and $\vec{v}_{/cm} = \vec{v} - \vec{v}_{cm}$. Similarly,

$$\dot{\vec{H}}_{\text{cm}} = \int \vec{r}_{/cm} \times \vec{a}\, dm = \int \vec{r}_{/cm} \times \vec{a}_{/cm}\, dm.$$

with $\vec{\boldsymbol{a}}_{/cm} = \vec{\boldsymbol{a}} - \vec{\boldsymbol{a}}_{cm}$. That is,

> *the angular momentum and rate of change of angular momentum relative to the center-of-mass, defined in terms of the velocity and acceleration* **relative** *to the center-of-mass, are the same as the angular momentum and the rate of change of angular momentum defined in terms of a fixed point in space that coincides with the center-of-mass.*

The angular momentum relative to the center-of-mass $\vec{\boldsymbol{H}}_{\text{cm}}$ can be calculated with all positions and velocities calculated relative to the center-of-mass. Similarly, the rate of change of angular momentum relative to the center of mass $\dot{\vec{\boldsymbol{H}}}_{\text{cm}}$ can be calculated with all positions and *accelerations* calculated relative to the center-of-mass.

Combining the results above we get the often used result:

$$\sum \vec{\boldsymbol{M}}_{i/cm} = \dot{\vec{\boldsymbol{H}}}_{\text{cm}} \tag{C.3}$$

This formula is the version of angular momentum balance that many people think of as being basic. In this equation, $\dot{\vec{\boldsymbol{H}}}_{\text{cm}}$ can be found using either the absolute acceleration $\vec{\boldsymbol{a}}$ or the acceleration relative to the center-of-mass, $\vec{\boldsymbol{a}}_{/cm}$. The same $\dot{\vec{\boldsymbol{H}}}_{\text{cm}}$ is found both ways. In this book, we do not give equation C.3 quite such central status as equations III where the reference point can be any point C not just the center-of-mass.

C.1 Relation between $\frac{d}{dt}\vec{H}_{/C}$ and $\vec{H}_{/C}$

The expression for $\dot{\vec{\boldsymbol{H}}}_{/C}$ follows from that for $\vec{\boldsymbol{H}}_{/C}$ but requires a few steps of algebra to show. Like the rate of change of linear momentum, $\dot{\vec{\boldsymbol{L}}}$, the derivative of $\vec{\boldsymbol{L}}$, the derivative of angular momentum must be taken with respect to a Newtonian frame in order to be useful in momentum balance equations. Note that since we assumed that C is a point fixed in a Newtonian frame that $\frac{d}{dt}\vec{\boldsymbol{r}}_{i/C} = \vec{\boldsymbol{v}}_{i/C} = \vec{\boldsymbol{v}}_i$. Starting with the definition of $\dot{\vec{\boldsymbol{H}}}_{/C}$, we can calculate as follows:

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/C} &= \frac{d}{dt}\vec{\boldsymbol{H}}_{/C} \\
&= \frac{d}{dt}\sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{v}}_i) \\
&= \sum \frac{d}{dt}\vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{v}}_i) + \vec{\boldsymbol{r}}_{i/C} \times (m_i \frac{d}{dt}\vec{\boldsymbol{v}}_i) \\
&= \sum \overbrace{\underbrace{\vec{\boldsymbol{v}}_i}_{\frac{d}{dt}\vec{\boldsymbol{r}}_{i/C}} \times (m_i \vec{\boldsymbol{v}}_i)}^{\vec{\boldsymbol{0}}} + \vec{\boldsymbol{r}}_{i/C} \times (m_i \frac{d}{dt}\vec{\boldsymbol{v}}_i)
\end{aligned}$$

$$\dot{\vec{\boldsymbol{H}}}_{/C} = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{a}}_i),$$

We have used the fact that the product rule of differentiation works for cross products between vector-valued functions of time. This final formula, $\dot{\vec{\boldsymbol{H}}}_{/C} = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{a}}_i)$, or its integral form, $\dot{\vec{\boldsymbol{H}}}_{/C} = \int \vec{\boldsymbol{r}}_{i/C} \times \vec{\boldsymbol{a}}_i \, dm$ are always applicable. They can be simplified in many special cases which we will discuss in this chapter and those that follow.

Kinetic energy E_K

The equation of mechanical energy balance (III) is:

$$P = \dot{E}_K + \dot{E}_P + \dot{E}_{int}.$$

For discrete systems, the kinetic energy is calculated as

$$\frac{1}{2}\sum m_i\, v_i^2$$

and its rate of change as

$$\frac{d}{dt}\left[\frac{1}{2}\sum m_i\, v_i^2\right].$$

There is also a general result about the kinetic energy that takes advantage of the center-of-mass. The kinetic energy for any system in any motion can be decomposed into the sum of two terms. One is associated with the motion of the center-of-mass of the system and the other is associated with motion relative to the center-of-mass. Namely,

$$\begin{aligned} E_K &= \underbrace{\frac{1}{2}m_{\text{tot}}v_{cm}^2}_{\boxed{\text{kinetic energy due to center-of-mass motion}}} + \underbrace{\frac{1}{2}\sum m_i v_{i/cm}^2}_{\boxed{\text{kinetic energy relative to the center-of-mass}}}, \\ &= \frac{1}{2}m_{tot}v_{cm}^2 + E_{K/cm} \end{aligned}$$

where

$$\begin{aligned} E_{K/cm} &= \frac{1}{2}\sum m_i v_{i/cm}^2 && \text{for discrete systems, and} \\ &= \frac{1}{2}\int (v_{/cm})^2\, dm && \text{for continuous systems.} \end{aligned}$$

C.2 Using $\vec{H}_{/O}$ and $\dot{\vec{H}}_{/O}$ to find $\vec{H}_{/C}$ and $\dot{\vec{H}}_{/C}$

You can find the angular momentum $\vec{H}_{/C}$ relative to a fixed point C if you know the angular momentum $\vec{H}_{/O}$ relative to some other fixed point O and also know the linear momentum of the system $\vec{L}$ (which does not depend on the reference point). The result is:

$$\vec{H}_{/C} = \vec{H}_{/O} + \vec{r}_{O/C} \times \vec{L}.$$

The formula is similar to the formula for the effective moment of a system of forces that you learned in statics: $\vec{M}_C = \vec{M}_O + \vec{r}_{O/C} \times \vec{F}_{tot}$. Similarly, for the rate of change of angular momentum we have:

$$\dot{\vec{H}}_{/C} = \dot{\vec{H}}_{/O} + \vec{r}_{O/C} \times \dot{\vec{L}}$$

So once you have found $\dot{\vec{L}}$ and also $\dot{\vec{H}}_{/O}$ with respect to some point O you can easily calculate the right hand sides of the momentum balance equations using any point C that you like.

The results above can be verified by direct expansion of the basic definitions of E_K and the center-of-mass. To repeat,

> *the kinetic energy of a system is the same as the kinetic energy of a particle with the system's mass at the center-of-mass plus kinetic energy due to motion relative to the center-of-mass.*

C.3 Deriving system momentum balance from the particle equations

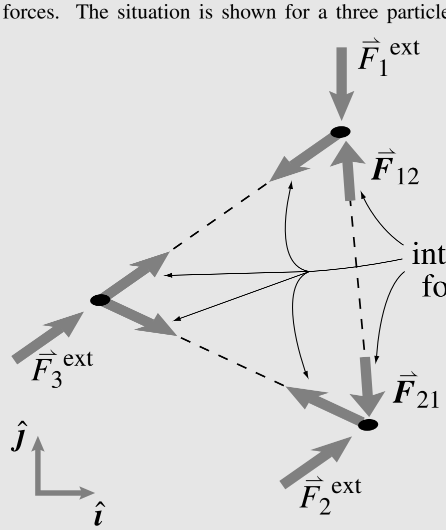
Inside the front cover we asume that the linear and angular momentum balance equations apply to arbitrary systems. Another approach to mechanics is to use the equation

$$\vec{F} = m\vec{a}$$

for every particle in the system and then *derive* the system linear and angular momentum balance equations. This derivation depends on the following assumptions

1. All bodies and systems are composed of point masses.
2. These point masses interact in a pair-wise manner. For every pair of point masses A and B the interaction force is equal and opposite and along the line connecting the point masses.

We then look at any system, which we now assume is a system of point masses, and apply $\vec{F} = m\vec{a}$ to every point mass and add the equations for all point masses in the system. For each point mass we can break the total force into two parts: 1) the interaction forces between the point mass and other point masses in the system, these forces are 'internal' forces ($\vec{F}^{int}$), and 2) the forces acting on the system from the outside, the 'external' forces. The situation is shown for a three particle system below.



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System linear momentum balance

Now lets take the equation $\sum \vec{F} = m\vec{a}$ for each particle and add over all the particles.

$$\underbrace{\sum_{\text{all particles}} \left[\sum_{\text{each particle}} \vec{F} \right]}_{\text{The sum of all forces on the system, internal and external}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

Because all the internal forces come in cancelling pairs we can rewrite this equation as:

$$\underbrace{\sum_{\text{all external forces}} \vec{F}^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} m_i \vec{a}_i$$

That is, we have derived equation I in the front cover from $\vec{F} = m\vec{a}$ for a point mass by assuming the system is composed of point masses with pair-wise equal and opposite forces.

System angular momentum balance

For any particle we can take the equation

$$\sum_{\text{forces on particle i}} \vec{F} = m_i \vec{a}_i$$

and take the cross product of both sides with the position of the particle relative to some point C:

$$\vec{r}_{i/C} \times \left[\sum_{\text{forces on particle i}} \vec{F} \right] = \vec{r}_{i/C} \times [m_i \vec{a}_i].$$

Now we can add this equation up over all the particles to get

$$\underbrace{\sum_{particles} \left\{ \vec{r}_{i/C} \times \left[\sum_{on\ particle\ i} \vec{F} \right] \right\}}_{r/C \times \vec{F} \text{ added up for all forces on the system, internal and external}} = \sum \{ \vec{r}_{i/C} \times [m_i \vec{a}_i] \}.$$

But, by our pair-wise assumption, for every internal force there is an equal and opposite force with the same line of action. So all the internal forces drop out of this sum and we have:

$$\underbrace{\sum_{\text{all external forces}} \vec{r}_{i/C} \times \vec{F}_i^{ext}}_{\text{Only the external forces, the ones acting on the system from the outside.}} = \sum_{\text{all particles}} \vec{r}_{i/C} \times m_i \vec{a}_i.$$

This equation is equation II, the system angular momentum balance equation (assuming we do not allow the application of any pure moments).

The derivations above are classic and are found in essentially all mechanics books. However, it is more reasonable to take the system linear momentum balance and angular momentum balance as postulates. That way the subject of mechanics does not depend on the unrealistic view of matter being composed of point masses with pairwise equal-and-opposite forces. The real micro-scopic physics is more subtle than that.

C.4 Rigid-object simplifications

We have formulas for the motion quantities $\vec{L}$, $\dot{\vec{L}}$, $\vec{H}_{/C}$, and $\dot{\vec{H}}_{/C}$ and E_K in terms of the positions, velocities, and accelerations of all of the mass bits in a system. Most often in this book we deal with the mechanics of *rigid bodies*, objects with negligible deformation. This assumed simplification means that the relative motions of the 10^{23} or so atoms in a body are highly restricted. In fact, if one knows these five vectors:

- $\vec{r}_{cm}$, the position of the center-of-mass,
- $\vec{v}_{cm}$, the velocity of the center-of-mass,
- $\vec{a}_{cm}$, the acceleration of the center-of-mass
- $\vec{\omega}$, the *angular velocity* of the body, and
- $\vec{\alpha}$, the *angular acceleration* of the body,

then one can find the position, velocity, and acceleration of every point on the body in terms of its position relative to the center-of-mass, $\vec{r}_{/cm} = \vec{r} - \vec{r}_{cm}$.

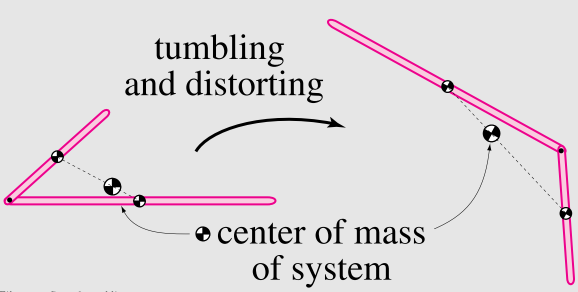
We also use $[I^{cm}]$, the *moment of inertia matrix*. For 2-D problems, $[I^{cm}]$ is just a number. For 3-D problems, $[I^{cm}]$ is a matrix; hence, the square brackets [ ], our notation for a matrix.

These rigid-object concepts lead to a *vast* simplification over the alternative — summing over 10^{23} particles or so.

Note that the formulas for linear momentum $\vec{L}$ and rate of change of linear momentum $\dot{\vec{L}}$ do not really look any simpler for a rigid object than for the general case.

$$\begin{aligned}\vec{L} &= m_{tot}\vec{v}_{cm}\\ \dot{\vec{L}} &= m_{tot}\vec{a}_{cm}\end{aligned}$$

But, these expressions are actually simpler in the following sense. For a general system, when we write $\vec{v}_{cm}$, we are talking about an abstract point that moves in a different way than any point on the system. For example, consider the linked arms below, tumbling in space.



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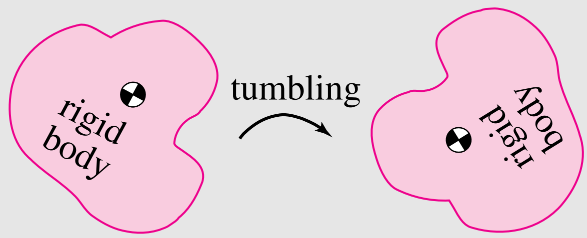
The center-of-mass is not even on any point in the system and, although it represents the average position in the system, it does not move with any point on the system.

On the other hand, for a *rigid body*, the center-of-mass is fixed relative to the body as the body moves,



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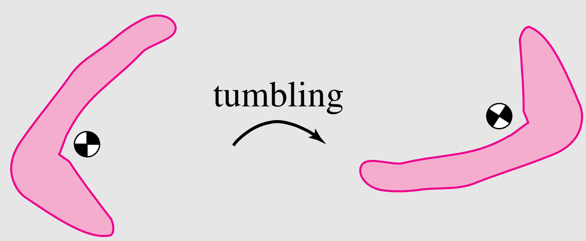
even if the center-of-mass is not on the body, such as for this 'L-shaped' object.



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In this case, the center-of-mass is not literally *on* the body but it is fixed with respect to the body. If you were rigidly attached to the body and fixed your gaze on the location of the center of mass, it would not waver in your view as the body, with you attached, tumbled wildly. In this sense the center-of-mass is fixed "on" a rigid body even if not on the body at all.

Answers to *'d problems

2.1.39) One solution, given in sample 2.7, is $\vec{A} = -16\hat{\imath} + 8\hat{\jmath}$ and $\vec{B} = 15\hat{\imath}$

2.2.17) $r_x = \vec{r} \cdot \hat{\imath} = (3\cos\theta + 1.5\sin\theta)$ ft, $\quad r_y = \vec{r} \cdot \hat{\jmath} = (3\sin\theta - 1.5\cos\theta)$ ft.

2.3.3) No partial credit.

2.3.7) $\vec{N}\,\frac{1000\,\text{N}}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$.

2.3.10) $d = \sqrt{\frac{3}{2}}$.

2.3.14a) $\hat{n} = \frac{1}{3}(2\hat{\imath} + 2\hat{\jmath} + \hat{k})$.

b) $d = 1$.

c) $\frac{1}{3}(-2, 19, 11)$.

2.3.16) $\ell/\sqrt{2}$

2.3.17) To get chicken road sin theta.

2.4.13a) $\hat{\lambda}_{OB} = \frac{1}{\sqrt{50}}(4\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$.

b) $\hat{\lambda}_{OA} = \frac{1}{\sqrt{34}}(3\hat{\jmath} + 5\hat{k})$.

c) $\vec{F}_1 = \frac{5\,\text{N}}{\sqrt{34}}(3\hat{\jmath} + 5\hat{k}), \quad \vec{F}_2 = \frac{7\,\text{N}}{\sqrt{50}}(4\hat{\imath} + 3\hat{\jmath} + 5\hat{k})$.

d) $\angle AOB = 34.45$ deg .

e) $F_{1_x} = 0$

f) $\vec{r}_{DO} \times \vec{F}_1 = \left(\frac{100}{\sqrt{34}}\hat{\jmath} - \frac{60}{\sqrt{34}}\hat{k}\right)$ N·m.

g) $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.

h) $M_\lambda = \frac{140}{\sqrt{50}}$ N·m.(same as (7))

2.5.16) Yes.

2.6.4) $T_{AB} = 75$ N

2.6.7) $\vec{F}_D = (15\sqrt{3}\hat{\imath} - 15\hat{\jmath})$ N, $\vec{M}_D = 30(1 + \sqrt{3})\hat{k}$ N·m

2.6.10) $M = 30\left(\sqrt{3} - 3\right)$ N·m$\hat{k}$

2.6.12) $\vec{F}_C = (-9\hat{\imath} - 12\hat{\jmath})$ N

2.6.12) Through B and also through a point 3 m above A.

2.6.15a) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \hat{k}\, M_1/|\vec{F}_1|^2$, $\vec{F}_2 = \vec{F}_1$.

b) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \hat{k}\, M_1/|\vec{F}_1|^2 + c\vec{F}_1$ where c is any real number, $\vec{F}_2 = \vec{F}_1$.

c) $\vec{F}_2 = \vec{0}$ and $\vec{M}_2 = \vec{M}_1$ applied at any point in the plane.

2.6.16a) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \vec{M}_1/|\vec{F}_1|^2$, $\vec{F}_2 = \vec{F}_1$, $\vec{M}_2 = \vec{M}_1 \cdot \vec{F}_1\vec{F}_1/|\vec{F}_1|^2$. If $\vec{F}_1 = \vec{0}$ then $\vec{F}_2 = \vec{0}$, $\vec{M}_2 = \vec{M}_1$, and $\vec{r}_2$ is any point at all in space.

b) $\vec{r}_2 = \vec{r}_1 + \vec{F}_1 \times \vec{M}_1/|\vec{F}_1|^2 + c\vec{F}_1$ where c is any real number, $\vec{F}_2 = \vec{F}_1$, $\vec{M}_2 = \vec{M}_1 \cdot \vec{F}_1\vec{F}_1/|\vec{F}_1|^2$. See above for the special case of $\vec{F}_1 = \vec{0}$.

2.7.1) $(0.5\,\text{m}, \; -0.4\,\text{m})$

3.1.1a) The forces and moments that show on a free body diagram, the *external* forces and moments.

b) The forces and moments that show on a free body diagram, the *external* forces and moments. No "inertial" or "acceleration" forces show.

3.1.2) You don't.

3.1.19) Note, no couples show on any of the free body diagrams asked for.

4.1.5) $T_1 = Nmg$, $T_2 = (N-1)mg$, $T_N = (1)mg$, and in general $T_n = (N + 1 - n)mg$

4.1.13) (a) is nonsense; others are fine

4.1.20) The cables happen to be co-planar and the force is not in that plane. So there is no solution.

4.1.23) (a)$T_{AB} = 30$ N, (b) $T_{AB} = \frac{300}{17}$ N, (c)$T_{AB} = \frac{5\sqrt{26}}{2}$ N

4.3.12) $\theta \geq \tan^{-1}\left((1-\mu^2)/2\mu\right)$

4.3.15) For this device to hold, $\mu \geq 1$. (Demanding $\mu \geq 1$ is large for a practical device because typical rock friction has $\mu \approx 0.5$. The too-large number follows from the simplified geometry and numbers chosen for a homework problem.)

4.3.19) $T_{AB} = \sqrt{10}\mu mg/(3 + \mu)$

4.3.19) Minimum tension if rope slope is μ (instead of $1/3$)

4.3.21a) $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta + r} = \frac{2\sin\theta}{1+2\cos\theta}$.

b) $T = mg = 2Mg\frac{\sin\theta}{1+2\cos\theta}$.

c) $\vec{F}_C = Mg\left[-\frac{2\sin\theta}{2\cos\theta+1}\hat{\imath}' + \hat{\jmath}'\right]$ (where $\hat{\imath}'$ and $\hat{\jmath}'$ are aligned with the horizontal and vertical directions)

c) $\tan\phi = \frac{\sin\theta}{2+\cos\theta}$. Needs somewhat involved trigonometry, geometry, and algebra.

d) $\tan\psi = \frac{m}{M} = \frac{2\sin\theta}{1+2\cos\theta}$.

4.3.22a) $\frac{m}{M} = \frac{R\sin\theta}{R\cos\theta r} = \frac{2\sin\theta}{2\cos\theta - 1}$.

b) $T = mg = 2Mg\frac{\sin\theta}{2\cos\theta - 1}$.

c) $\vec{F}_C = \frac{Mg}{1-2\cos\theta}[\sin\theta\hat{\imath} + (\cos\theta - 2)\hat{\jmath}]$.

4.3.23a) $\frac{F_1}{F_2} = \frac{R_o + R_i\sin\phi}{R_o - R_i\sin\phi}$

b) For $R_o = 3R_i$ and $\mu = 0.2$, $\frac{F_1}{F_2} \approx 1.14$.

4.4.1) None are true. The tension is 100 N.

4.5.6) Maximum overhang when $n \to \infty is\ell$.

4.5.9) Assuming no side-loads from floor the support from leg AB is 250 N, $T_{AB} = -250$ N.

4.5.10) $T_{IE} = mg/2, T_{CH} = \sqrt{2}mg/2, T_{BH} = -mg/2, A_x = mg/2, A_y = mg/2, A_z = mg$

4.5.13g) $T_{EH} = 0$ as you can find a number of ways.

4.5.14a) Use axis EC.

b) Use axis AH.

c) Use $\hat{\boldsymbol{j}}$ axis through B.

d) Use axis DE.

e) Use axis EH.

f) Can't do in one shot.

4.5.15) $T_{AC} = -\sqrt{2}mg = -1000\sqrt{2}$ N ≈ -1410 N (the bar is in compression)

4.5.15) $T_{IP} = 0$

4.5.15) $T_{KL} = \sqrt{2}mg/6 = \left(1000\sqrt{2}/6\right) N \approx 408$ N (the bar is in tension)

4.5.17) Hint: With reference to a free body diagram of the robot, use moment balance about axis BC.

5.1.9) $T_{AC} = -1000$ N, (AC is in compression)

5.1.10) $T_{AB} = 173$ N

5.1.13) 12 of the 15 bars are zero-force members; all but BD, DG, and GJ. The others carry no load but are needed for stability.

5.2.14) $T_{EB} = -11F/2$

5.2.14) $T_{HI} = -11bF/2a$

5.2.14) $T_{JK} = -35bF/2a$, (more than 3 times the compression of HI)

6.1.1) 1000 N

6.1.2) 0.08 cm

6.1.3) 1160 N

6.1.4) 5 cm

6.1.5) $k_e = 66.7$ N/cm, $\delta = 0.75$ cm

6.1.7) $k = 20$ N/cm

6.1.8) Middle spring: $\delta = 1$ cm; side-springs $\delta = 0.5$ cm

6.1.12) Surprise! This pendulum is in equilibrium for all values of θ.

6.2.19) 200 N

6.3.2) $N = (h(w + d)/d\ell)\, F_h$

6.3.9) Either by looking at part KAP or at part BAQ, if we think of moment balance about A we see that the cutting force has to fight about twice the torque in the gear mechanism as in the ungeared mechanism. For example KAP is aided in its cutting by the torque from the force at G.

6.3.10) The mechanism multiplies the force at B and C by a factor of 2 compared to having the handle hinged at A. The force at G also gets (a shade less than) this force but with half the lever arm. Together they give a force multiplication of (a shade less than) 2+1=3.

6.3.11) $F_P = 125$ N

6.3.11) $F_P = 125$ N

6.3.11) For the load at I, $F_P = 75$ N. For the load at J, $F_P = 250N$.

6.3.11) With the welded handle there is just a simple lever and the mechanical advantage comes from the horizontal distance between the load and hinge A. For the 4 bar mechanism the force at C is the applied vertical load, no matter where it is applied. So the lever arm is the horizontal distance from A to C.

6.3.12) $F_A = 500$ lbf

6.3.13d) reduce the dimension marked "2 inches". The smaller the less the friction needed.

e) As the "2 inch" dimension is reduced to zero, the needed coefficient of friction goes to zero and the forces squeezing the pipe go to infinity. This is bad because it can damage the pipe. It is also bad because a small pipe deformation will cause the hinge on the wrench to snap through, like a so called "toggle mechanism" and thus not grab at all.

6.3.14) $\vec{\boldsymbol{R}}_A = \vec{\boldsymbol{0}}$

6.3.14) $T = 200$ lbf

6.3.16) $F_N \left(b(a^2 + b^2)/a^2)\right) F = 130F = 1300$ lbf

6.3.16) The mechanism uses three tricks to multiply the force: a lever, a wedge, and a toggle. Each of these multiplies by about 5. Thus the nut-force F_N is on the order of $5^3 = 125$ times as big as F.

7.1.14) $F_{Ay} = -500$ N, $M_A = -500/3$ N·m

7.1.15) $V(\ell/2) = -w\ell/8, M(\ell/2) = w\ell^2/16, M_{max} = M(3\ell/8) = 9wl^2/128$

7.1.17b) [Hint: at every height y the cross sectional area must be big enough to hold the weight plus the wire below that point. From this you can set up and a differential equation for the cross sectional area A as a function of y. Find appropriate initial conditions and solve the equation. Once solved, the volume of wire can be calculated as $V = \int_0^1 0\,\mathrm{mi} A(y)dy$ and the mass as ρV.]

8.1.3) $(117\gamma/2)\,\mathrm{m}^3 = 5.85 * 10^5$ N

8.1.4) Water starts to spill at $h = 3r_{AB} = 3$ m.

8.1.4) Assuming no friction at B, $\vec{\boldsymbol{F}}_A = 2.25 * 10^5 \hat{\boldsymbol{\imath}}$ N

8.1.9a) $\rho g \pi r^2 \ell$

b) $-\rho g \pi r^2 (h - \ell)$, note the minus sign, it now takes force to lift the can.

9.1.1) Some possibilities: a) a car with given thrust and drag, b) a person falling vertically during bungy jumping, c) a speaker cone oscillating due to magnetic forces on its coils and resistance from air pressure.

9.1.2) All points have equal velocity so all have the same velocity as the center of mass. Any point on the car can be used to measure the car's position.

9.1.3) No. You need also to know $v(0)$. Then $v(T) = v(0) + \int_0^T a(t)\,dt$. Knowing $a(t)$ over a given time interval determines the change of v but not the value $v(T)$.

9.1.4) b, changes linearly in time

9.1.5) For part b you need to assume that the linear acceleration starts from zero.

9.1.6) Only c, the change in linear momentum. You could find the displacement only if the initial velocity is also given.

9.1.7) (b)

9.1.8) $t = d/v = (10\,\text{km})/(15\,\text{mi/hr})\underbrace{(1\,\text{mi}/1.61\,\text{km})}_{1}\underbrace{(60\,\text{min/hr})}_{1} = 24.8\,\text{min}$ (What's this, 7th grade again?)

9.1.11) $x(3\,\text{s}) = 20\,\text{m}$

9.1.15) (a) $v(3\,\text{s}) = 2\,\text{m/s}$ in each case. (b) $x(3\,\text{s}) = 3\,\text{m}$ for case (a), $x(3\,\text{s}) = 4\,\text{m}$ for case (b).

9.1.16) b) $F_s = \frac{\pi}{4}F_T$. This is lower than for (a) because for a given peak force the sinusoidal force is bigger at every instant in time. So, to have the same effect (same impulse) the peak must be lower.

9.2.10) 11 ft

9.3.5) Time span $= 3\pi\sqrt{m/k}/2$

9.3.8) (a) $m\ddot{x} + kx = F(t)$, (b) $m\ddot{x} + kx = F(t)$, and (c) $m\ddot{y} + 2ky - 2k\ell_0\frac{y}{\sqrt{\ell_0^2+y^2}} = F(t)$

9.3.10b) $mg - k(x - \ell_0) = m\ddot{x}$

c) $\ddot{x} + \frac{k}{m}x = g + \frac{k\ell_0}{m}$

e) This solution is the static equilibrium position; i.e., when the mass is hanging at rest, its weight is exactly balanced by the upwards force of the spring at this constant position x.

f) $\ddot{\hat{x}} + \frac{k}{m}\hat{x} = 0$

g) $x(t) = \left[D - \left(\ell_0 + \frac{mg}{k}\right)\right]\cos\sqrt{\frac{k}{m}}\,t + \left(\ell_0 + \frac{mg}{k}\right)$

h) period$=2\pi\sqrt{\frac{m}{k}}$.

i) If the initial position D is more than $\ell_0 + 2mg/k$, then the spring is in compression for part of the motion. A floppy spring would buckle when in compression.

9.3.12a) period$= \frac{2\pi}{\sqrt{\frac{k}{m}}} = 0.96\,\text{s}$

b) maximum amplitude=0.75 ft

c) period$= 2\sqrt{\frac{2h}{g}} + \sqrt{\frac{m}{k}}\left[\pi + 2\tan^{-1}\sqrt{\frac{mg}{2kh}}\right] \approx 1.64\,\text{s}$.

9.3.13) LHS of Linear Momentum Balance: $\sum\vec{F} = -(kx + b\dot{x})\hat{\imath} + (N - mg)\hat{\jmath}$.

9.4.10) $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - k_2(x_B - x_A) + c_1(\dot{x}_D - \dot{x}_B) + k_3(x_D - x_B)]\hat{\imath}$.

9.4.11) $\vec{a}_B = \ddot{x}_B\hat{\imath} = \frac{1}{m_B}[-k_4x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2 + k_3)(x_D - x_B)]$.

9.4.12b) If we start off by assuming that each mass undergoes simple harmonic motion at the same frequency but different amplitudes, we will find that this two-degree-of-freedom system has two natural frequencies. Associated with each natural frequency is a fixed ratio between the amplitudes of each mass. Each mass will undergo simple harmonic motion at one of the two natural frequencies only if the initial displacements of the masses are in the fixed ratio associated with that frequency.

9.4.13a) Two normal modes.

b) $x_2 = const * x_1 = const * (A\sin(ct) + B\cos(ct))$, where $const = \pm 1$.

c) $\omega_1 = \sqrt{\frac{3k}{m}}$, $\omega_2 = \sqrt{\frac{k}{m}}$.

9.4.15) $v_A = \sqrt{\frac{m_Bk\delta^2}{m_A^2 + m_Bm_A}}$.

9.4.20a) One normal mode: $[1, 0, 0]$.

b) The other two normal modes: $[0, 1, \frac{1\pm\sqrt{17}}{4}]$.

9.4.23a) $\omega = \sqrt{\frac{2k}{m}}$.

9.5.3) Note: the mass ratio is so high that there is no real loss of accuracy if you take the post-collision mass $m_1 + m_2$ as 50 kg (the 0.004% error is surely *much* smaller than the inaccuracy in any of the other numbers in the problem).

9.5.4) $h_{max} = e^2h$.

9.5.9a) $v_0 = \frac{1}{m}(mv_B + m_Bv_B + m_Av_A)$.

b) $v_1 = \frac{(m+m_B)}{m}v_B$.

c) (1) $E_{loss} = \frac{1}{2}m\left[v_0^2 - \left(\frac{m+m_B}{m}\right)v_B^2\right] - \frac{1}{2}m_Av_A^2$.
(2) $E_{loss} = \frac{1}{2}\left[\frac{(m+m_B)^2}{m}v_B^2 - (m + m_B)v_B^2\right]$.

10.1.3a) $\vec{v}(5\,\text{s}) = (30\hat{\imath} + 300\hat{\jmath})\,\text{m/s}$.

b) $\vec{a}(5\,\text{s}) = (6\hat{\imath} + 120\hat{\jmath})\,\text{m/s}^2$.

10.1.5) $\vec{r}(t) = \left(x_0 + \frac{u_0}{\Omega} - \frac{u_0}{\Omega}\cos(\Omega t)\right)\hat{\imath} + (y_0 + v_0t)\,\hat{\jmath}$.

10.1.11) $\vec{v} = 2t\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{s}}\,\text{m/s}\hat{\jmath}, \vec{a} = 2\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{s}}\,\text{m/s}^2\hat{\jmath}$.

10.1.21) $T_3 = 13\,\text{N}$

10.1.29) Equation of motion: $-mg\hat{\jmath} - b(\dot{x}^2 + \dot{y}^2)\left(\frac{\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$.

10.1.30a) System of equations:

$$\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= -\frac{b}{m} v_x \sqrt{v_x^2 + v_y^2} \\
\dot{v}_y &= -g - \frac{b}{m} v_y \sqrt{v_x^2 + v_y^2}
\end{aligned}$$

11.1.4) No. You need to know the angular momenta of the particles relative to the center of mass to complete the calculation, information which is not given.

11.1.9) The mass would stay on the z axis if the solution was exact.

11.1.9) The solution would be exactly periodic if the ratio of the masses was infinite rather than just 1000. There are special initial conditions for which the motion is periodic for any mass ratio, the oscillations of the light mass need to be synchronous with the in-and-out oscillations of the heavier nearly-circular-motion masses.

11.1.10) These three trajectories are all parts of the same figure 8.

11.1.10) The trajectories trace and retrace the same figure 8. If your integration is not accurate, the curves will not exactly retrace.

11.1.10) The trajectories make a beautiful swirl resembling a figure 8.

11.1.10) The trajectories get wild, possibly ejecting one or more masses off to infinity.

11.2.10) One test problem is this: $\vec{v}_1^- = \hat{\imath}, \vec{v}_2^- = \vec{0}, m_1 = m_2 = 1, e = 0, \theta = 0$. This should have the solution $\vec{v}_1^+ = \vec{v}_2^+ = 0.5\hat{\imath}, \vec{P} = .5\hat{\imath}$.

12.1.2a) $a_B = \left(\frac{m_B - m_A}{m_A + m_B}\right) g$

b) $T = 2\frac{m_A m_B}{m_A + m_B} g$.

12.1.4) $\vec{a}_{\text{A}} = -2\,\text{m/s}^2\,\hat{\jmath}$

12.1.6) (a) $\vec{a}_A = \vec{a}_B = \frac{F}{m}\hat{\imath}$, where $\hat{\imath}$ is parallel to the ground and pointing to the right., (b) $\vec{a}_A = \frac{2F}{m}\hat{\imath}, \vec{a}_B = \frac{4F}{m}\hat{\imath}$, (c) $\vec{a}_A = \frac{F}{2m}\hat{\imath}, \vec{a}_B = \frac{F}{4m}\hat{\imath}$, (d) $\vec{a}_A = \frac{F}{m}\hat{\imath}, \vec{a}_B = -\frac{F}{m}\hat{\imath}$.

12.1.8) $\frac{a_A}{a_B} = 81$.

12.1.9) $a_A = 2F/(5m), a_B = F/(5m)$ to the right.

12.1.11) $P = 2F\dot{x}$

12.1.12) $T_n = \frac{P_t}{v_t}\frac{n}{N}$.

12.1.13) displacement of cart $= \frac{F - \mu m g}{F - \mu(m+M)g}\ell$

12.1.14a) $\vec{a}_A = \frac{5F}{m}\hat{\imath}, \vec{a}_B = \frac{25F}{m}\hat{\imath}$, where $\hat{\imath}$ is parallel to the ground and points to the right.

b) $\vec{a}_A = \frac{g}{(4m_1+m_2)}(2m_2 - \sqrt{3}m_2)\hat{\boldsymbol{\lambda}}_1$, $\vec{a}_B = -\frac{g}{2(4m_1+m_2)}(2m_1 - \sqrt{3}m_2)\hat{\boldsymbol{\lambda}}_2$, where $\hat{\boldsymbol{\lambda}}_1$ is parallel to the slope that mass m_1 travels along, pointing down and to the left, and $\hat{\boldsymbol{\lambda}}_2$ is parallel to the slope that mass m_2 travels along, pointing down and to the right.

12.1.14) $a_A = \frac{5F}{4m}$ and $a_B = \frac{25F}{16m}$ in the direction of F.

12.1.16) $\vec{a}_A = -\frac{a_0}{2}\hat{\imath}$

12.1.18) angular frequency of vibration $\equiv \lambda = \sqrt{\frac{64k}{65m}}$.

12.1.19) The assembly at C has, with the idealizations given, no mass and no net force acting on it. The equation $\vec{F} = m\vec{a}$ says $0 = 0$ and, within the assumptions made, the acceleration is indeterminate. If you built such a machine, the acceleration of C would be determined by the effects which are neglected here, such as the mass of the central assembly and the friction in the pulley bearings. If you built such an assembly you would see that only small additional forces are needed to move point C most any which way.

12.1.25a) $m\ddot{x} + 4kx = A\sin\omega t + mg$, where x is the distance measured from the mass position when the spring is unstretched.

b) The string will go slack if $\omega > \sqrt{\frac{4k}{m}\left(1 - \frac{A}{mg}\right)}$.

12.1.26a) $\vec{a}_A = -\frac{9kd}{m_A}\hat{\imath}$.

b) $v = 3d\sqrt{\frac{k}{m_A}}$

12.2.3) $T_{AB} = \frac{5\sqrt{13}}{28}m(a_y + g)$

12.2.8) $a_1 = a_2 = g\sin\theta$

12.2.8) $T = 0$

12.2.8) $v = \sqrt{2dg\sin\theta}$

12.2.9) $a_{com} = a_1 = a_2 = g(sin(\theta) - \frac{3}{4}\mu cos(\theta))$

12.2.9) $T = \frac{\mu}{4}mgcos(\theta)$

12.2.9) $v = \sqrt{2dg(sin(\theta) - \frac{3}{4}\mu cos(\theta))}$

12.2.9) The upper block would push the lower one down the ramp so the rod tension would be rod compression. But the acceleration would be unchanged.

12.2.11) $a_x > \frac{3}{2}g$

12.2.14) Can't solve for T_{AB}.

12.2.15) a) $\vec{a}_G = \frac{P}{m}\hat{\imath}$ with $\hat{\imath}$ to the right. b) $R_A = R_B = \frac{mg}{2}\hat{\jmath}$, with $\hat{\jmath}$ upwards

12.2.16) The acceleration is still $\vec{a}_G = \frac{P}{m}\hat{\imath}$. But the reactions are changed to $R_A = \frac{mg}{2} - \frac{d}{b}\frac{P}{2}$ and $R_B = \frac{mg}{2} + \frac{d}{b}\frac{P}{2}$

12.2.25d) Normal reaction at rear wheel: $N_r = \frac{mgw}{2(h\mu+w)}$, normal reaction at front wheel: $N_f = mg - \frac{mgw}{2(h\mu+w)}$, deceleration of car: $a_{car} = -\frac{\mu g w}{2(h\mu+w)}$.

e) Normal reaction at rear wheel: $N_r = mg - \frac{mgw}{2(w-\mu h)}$, normal reaction at front wheel: $N_f = \frac{mgw}{2(w-\mu h)}$, deceleration of car: $a_{car} = -\frac{\mu g w}{2(w-\mu h)}$. Car stops more quickly for front wheel skidding. Car stops at same rate for front or rear wheel skidding if $h = 0$.

f) Normal reaction at rear wheel: $N_r = \frac{mg(w/2-\mu h)}{w}$, normal reaction at front wheel: $N_f = \frac{mg(w/2+\mu h)}{w}$, deceleration of car: $a_{car} = -\mu g$.

g) No. Simple superposition just doesn't work.

h) No reaction at rear wheel.

i) Reaction at rear wheel is negative. Not allowing for rotation of the car in the xy-plane gives rise to this impossibility. In actuality, the rear of the car would flip over the front.

12.2.26a) Hint: the answer reduces to $a = \ell_r g/h$ in the limit $\mu \to \infty$.]

12.2.27a) $\vec{a} = g(\sin\phi - \mu\cos\phi)\hat{\imath}$, where $\hat{\imath}$ is parallel to the slope and pointing downwards

b) $\vec{a} = g\sin\phi$

c) $\vec{v} = g(\sin\phi - \mu\cos\phi)t\hat{\imath}, \vec{r} = g(\sin\phi - \mu\cos\phi)\frac{t^2}{2}$

d) $\vec{v} = g\sin\phi t\hat{\imath}, \vec{r} = g\sin\phi\frac{t^2}{2}\hat{\imath}$

12.2.29a) $\vec{R}_A = \frac{(1-\mu)mg\cos\theta}{2}(\hat{\jmath}' - \mu\hat{\imath}')$.

c) No tipping if $N_A = \frac{(1-\mu)mg\cos\theta}{2} > 0$; i.e., no tipping if $\mu < 1$ since $\cos\theta > 0$ for $0 < \theta < \frac{\pi}{2}$.(Here $\mu = 0.9$)

12.2.31) braking acceleration= $g(\frac{1}{2}\cos\theta - \sin\theta)$.

12.2.35a) $v = d\sqrt{\frac{k}{m}}$.

b) The cart undergoes simple harmonic motion for any size oscillation.

12.2.38a) $\vec{a}_{bike} = \frac{F_p L_c}{MR_f}$.

b) $\max(\vec{a}_{bike}) = \frac{ga}{a+b+2R_f}$.

12.2.39) $T_{EF} = 640\sqrt{2}\,\text{lbf}$.

12.2.40a) $T_{BD} = 92.6\,\text{lbm}\cdot\text{ft/s}^2$.

b) $T_{GH} = 5\sqrt{61}\,\text{lbm}\cdot\text{ft/s}^2$.

12.2.41b) $T_{EH} = 0$

c) $(R_{C_x} - T_{AB})\hat{\imath} + (R_{C_y} - \frac{T_{GD}}{\sqrt{2}})\hat{\jmath} + (T_{HE} + R_{C_z} + \frac{T_{GD}}{\sqrt{2}})\hat{k} = m\vec{a} = 10\,\text{N}\hat{k}$.

d) $\sum\vec{M}_{\text{cm}} = (\frac{T_{GD}}{\sqrt{2}} - T_{HE} - R_{C_z})\hat{\imath} + (R_{C_z} - \frac{T_{GD}}{\sqrt{2}} - T_{HE})\hat{\jmath} + (T_{AB} + R_{C_x} - R_{C_y} - \frac{T_{GD}}{\sqrt{2}})\hat{k} = \vec{0}$

e)

$$\begin{aligned}
R_{C_x} - T_{AB} &= 0\\
R_{C_y} - \frac{T_{GD}}{\sqrt{2}} &= 0\\
R_{C_z} + \frac{T_{GD}}{\sqrt{2}} + T_{EH} &= 5\,\text{N}\\
-T_{EH} + \frac{T_{GD}}{\sqrt{2}} - R_{C_z} &= 0\\
-T_{EH} - \frac{T_{GD}}{\sqrt{2}} + R_{C_z} &= 0\\
T_{AB} - \frac{T_{GD}}{\sqrt{2}} + R_{C_x} - R_{C_y} &= 0
\end{aligned}$$

f) $R_{C_x} = 5\,\text{N}$, $R_{C_y} = 5\,\text{N}$, $R_{C_z} = 5\,\text{N}$, $T_{GD} = \frac{10}{\sqrt{2}}\,\text{N}$, $T_{EH} = 0\,\text{N}$, $T_{AB} = 5\,\text{N}$.

g) Find moment about CD axis; e.g., $(\sum\vec{M}_{\text{C}} = \vec{r}_{cm/C} \times \text{m}\vec{a}_{cm}) \cdot \hat{\boldsymbol{\lambda}}_{CD}$, where $\hat{\boldsymbol{\lambda}}_{CD}$ is a unit vector in the direction of axis CD.

12.2.46a) $F_L = \frac{1}{2}m_{\text{tot}}g$.

b) $\vec{a}_P = \frac{1}{m_{\text{tot}}}[2(T - F_D) - D]\hat{\imath}$.

c) $\vec{F} = \left[\frac{m_w}{m_{\text{tot}}}(2T - D - 2F_D) - T + F_D\right]\hat{\imath} + (m_w g - F_L)\hat{\jmath}$ and $\vec{M} = (bF_L - am_w g)\hat{\imath} + \left[(bF_D - cT) + a\frac{m_w}{m_{\text{tot}}}(2T - D - 2F_D)\right]\hat{\jmath}$.

12.2.47) sideways force = $F_B\hat{\imath} = \frac{wma}{2\ell}\hat{\imath}$.

13.1.9) One solution is $\theta = 0$ for all t. Another set of solutions is $\theta = \left(\frac{c(t-t_0)}{2}\right)^2$ where t_0 is an arbitrary constant. To make sense of this second solution set one needs to have $\theta = 0$ until $t = t_0$.

13.2.1) $F = 0.52\,\text{lbf} = 2.3\,\text{N}$

13.2.7b) For $\theta = 0°$,

$$\begin{aligned}
\hat{e}_r &= \hat{\imath}\\
\hat{e}_t &= \hat{\jmath}\\
\vec{v} &= \frac{2\pi r}{\tau}\hat{\jmath}\\
\vec{a} &= -\frac{4\pi^2 r}{\tau^2}\hat{\imath},
\end{aligned}$$

for $\theta = 90°$,

$$\begin{aligned}
\hat{e}_r &= \hat{\jmath}\\
\hat{e}_t &= -\hat{\imath}\\
\vec{v} &= -\frac{2\pi r}{\tau}\hat{\imath}\\
\vec{a} &= -\frac{4\pi^2 r}{\tau^2}\hat{\jmath},
\end{aligned}$$

and for $\theta = 210°$,

$$\begin{aligned}\hat{e}_r &= -\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}\\ \hat{e}_t &= \frac{1}{2}\hat{\imath} - \frac{\sqrt{3}}{2}\hat{\jmath}\\ \vec{v} &= -\frac{\sqrt{3}\pi r}{\tau}\hat{\jmath} + \frac{\pi r}{\tau}\hat{\imath}\\ \vec{a} &= \frac{2\sqrt{3}\pi^2 r}{\tau^2}\hat{\imath} + \frac{2\pi^2 r}{\tau^2}\hat{\jmath}.\end{aligned}$$

c) $T = \frac{4m\pi^2 r}{\tau^2}$.

d) Tension is enough.

13.2.10b) $(\dot{\vec{H}}_{/O})_I = \vec{0}$, $(\dot{\vec{H}}_{/O})_{II} = 0.0080\,\text{N·m}\hat{k}$.

c) Position-A: $(\vec{H}_{/O})_I = 0.012\,\text{N·m}\cdot\text{s}\hat{k}$,$(\vec{H}_{/O})_{II} = 0.012\,\text{N·m}\cdot\text{s}\hat{k}$, Position-B: $(\vec{H}_{/O})_I 0.012\,\text{N·m}\cdot\text{s}\hat{k}$, $(\vec{H}_{/O})_{II} = 0.014\,\text{N·m}\cdot\text{s}\hat{k}$.]

13.2.12) $r = \frac{kr_o}{k - m\omega_o^2}$.

13.2.14) $\ell_0 = 0.2\,\text{m}$

13.2.16b) $T = 0.16\pi^4 N$.

c) $\vec{H}_{/O} = 0.04\pi^2\,\text{kg·m/s}\hat{k}$

d) $\vec{r} = [\frac{\sqrt{2}}{2} - v\cos(\frac{\pi t}{4})]\hat{\imath} + [\frac{\sqrt{2}}{2} + v\sin(\frac{\pi t}{4})]\hat{\jmath}$.

13.2.18a) $2mg$.

b) $\omega = \sqrt{99g/r}$

c) $r \approx 1\,\text{m}$ ($r > 0.98\,\text{m}$)

13.2.21a) $\ddot{\theta} = -(g/L)\sin\theta$

d) $\dot{\omega} = -(g/L)\sin\theta, \quad \dot{\theta} = \omega$

f) $T_{max} = 30N$

13.2.24b) The maximum tension is 3 times the person's weight.

13.2.30a) $\dot{v} = -\mu\frac{v^2}{R}$.

b) $v = v_0 e^{-\mu\theta}$.

13.2.33a) The velocity of departure is $\vec{v}_{dep} = \sqrt{\frac{k(\Delta\ell)^2}{m} - 2GR}\hat{\jmath}$, where $\hat{\jmath}$ is perpendicular to the curved end of the tube.

b) Just before leaving the tube the net force on the pellet is due to the wall and gravity, $\vec{F}_{net} = -mg\hat{\jmath} - m\frac{|\vec{v}_{dep}|^2}{R}\hat{\imath}$; Just after leaving the tube, the net force on the pellet is only due to gravity, $\vec{F}_{net} = -mg\hat{\jmath}$.

13.3.4) $[\vec{r}_B]_{x'y'} = \begin{bmatrix} 1\,\text{m} \\ 1\,\text{m}\end{bmatrix}$, $[\vec{r}_B]_{xy} = \begin{bmatrix} -0.37\,\text{m} \\ 1.37\,\text{m}\end{bmatrix}$, $R = \begin{bmatrix} 0.5 & -0.87 \\ 0.87 & 0.5\end{bmatrix}$

13.4.11a) $\vec{v}_{B/A} = 4\,\text{m/s}(\hat{\imath} + \sqrt{3}\hat{\jmath})$

b) $\vec{a}_{B/A} = -40\,\text{m/s}^2(\sqrt{3}\hat{\imath} - \hat{\jmath})$

13.4.20) $\vec{a}_{B/A} = (-0.353\hat{\imath} + 2.474\hat{\jmath})\,\text{m/s}^2$

13.4.23) $\omega_{min} = 10\,\text{rpm}$ and $\omega_{max} = 240\,\text{rpm}$

13.5.3) $I_{zz} = 0.125\,\text{kg}\cdot\text{m}^2$.

13.5.4a) $0.2\,\text{kg}\cdot\text{m}^2$.

b) $0.29\,\text{m}$.

13.5.5) At 0.72ℓ from either end

13.5.6a) $(I_{zz})_{min} = m\ell^2/2$, about the midpoint.

b) $(I_{zz})_{max} = m\ell^2$, about either end

13.5.7a) C

b) A

c) $I_{zz}^A/I_{zz}^B = 2$

d) smaller, $r_{gyr} = \sqrt{I_{zz}^C/(3m)} = \sqrt{2}\ell$

13.5.8a) Biggest: I_{zz}^O; smallest: $I_{yy}^O = I_{xx}^O$.

b) $I_{xx}^O = \frac{3}{2}m\ell^2 = I_{yy}^O$. $I_{zz}^O = 3m\ell^2$.

d) $r_{gyr} = \ell$.

13.5.12a) $I_{zz}^{cm} = 2m\ell^2$

b) $P \equiv A,\ B,\ C$, or D, often, but not always, diagonally opposite the corner with the largest mass

c) $r_{gyr} = \ell/\sqrt{2}$

13.5.13) $I_{xx}^O = I_{yy}^O = 0.3\,\text{kg}\cdot\text{m}^2$.

13.5.15a) $I_{zz}^O = \frac{2m}{bh}\int_0^b\int_0^{hx/b}(x^2 + y^2)\,dy\,dx.$

13.5.19d) Exactly 1% .

13.6.9a) (a) $\vec{F} = 0.33\,\text{N}\hat{\imath} - 0.54\,\text{N}\hat{\jmath}$.

13.6.10a) Net force: $\vec{F}_{net} = -(\frac{3m\omega^2 L}{2})\hat{\imath} - (\frac{m\omega^2 L}{2})\hat{\jmath}$, Net moment: $\vec{M}_{net} = \vec{0}$.

b) Net force: $\vec{F}_{net} = -(\frac{3m\omega^2 L}{2})\hat{\imath} + (2mg - \frac{m\omega^2 L}{2})\hat{\jmath}$, Net moment: $\vec{M}_{net} = \frac{3mgL}{2}\hat{k}$.

13.6.12a) $T(r) = \frac{m\omega^2}{2L}(L^2 - r^2)$

b) at $r = 0$; i.e., at the center of rotation

c) $r = L/\sqrt{2}$

13.6.16a) Point at $2L/3$ from A

b) $mg/4$ directed upwards.

13.6.18) $T_{rev} = \sqrt{\frac{2M_O\ell^2\pi}{3FR_p}}$.

13.6.22a) $P_{in} = 7.33$kilo-watts

b) 500 rpm

c) $M_{out} = 140\,\text{N·m}$

13.6.24a) $F_B = 100\,\text{lbf}$.

c) $v_{right} = v$.

13.6.26) $\vec{a}_m = 0.188\,\text{m/s}^2$

13.6.33b)

(b) $T = 2.29\,\text{s}$

(e) $T = 1.99\,\text{s}$

(b) has a longer period than (e) does since in (b) the moment of inertia about the center of mass (located at the same position as the mass in (e)) is non-zero.

13.6.34c) $T = \frac{2\pi}{\sqrt{g/\ell}}\sqrt{\frac{1}{12(d/\ell)} + \frac{d}{\ell}}$

g) $d = 0.29\,\ell$

13.6.37a) $\ddot{\theta} = 0\ \text{rad/s}^2$.

b) $\ddot{\theta} = \frac{\sin\theta}{m\ell}(Dk - mg)$.

13.6.43) period = $\pi\sqrt{\frac{2m}{k}}$.

13.6.44a) $\omega_n = \sqrt{\frac{gL(M+\frac{m}{2})+K}{(M+\frac{m}{3})L^2+M\frac{R^2}{2}}}$.

b) $\omega_n = \sqrt{\frac{gL(M+\frac{m}{2})+K}{(M+\frac{m}{3})L^2}}$. Frequency higher than in (a)

13.6.45b) $-F(t)\ell\cos\phi - mg\ell\sin\phi + T_m = -m\ell^2\ddot{\phi}$.

14.1.8b) $\vec{\omega} = -\frac{v_0\sin\phi}{\ell\cos(\phi-\theta)}\hat{k}$

14.2.3a) $\vec{a}_{cm} = \frac{F}{m}\hat{\imath}$.

b) $\vec{\alpha} = -\frac{6F}{m\ell}\hat{k}$.

c) $\vec{a}_A = \frac{4F}{m}\hat{\imath}$.

d) $\vec{F}_B = \frac{F}{2}\hat{\imath}$

14.2.6a) $\vec{a} = \frac{F}{m}\hat{\jmath}$.

b) $\vec{\alpha} = \frac{Fd}{m\frac{R^2}{2}}\hat{k}$.

14.2.16a) $F_{out} = \frac{3}{7}$ lb.

b) F_{out} is always less than the F_{in}.

14.4.4a) $\vec{\alpha}_{Disk} = -\frac{4}{3}\ \text{rad/s}^2\hat{k}$.

b) $\vec{a}_A = \frac{8}{3}\ \text{m/s}^2\hat{\imath}$.

14.4.6a) $a_C = (F/m)(1 - R_i/R_O)\hat{\imath}$.

b) $(-R_i/R_O)F\hat{\imath}$.

14.4.11a) speed $v = \frac{R\dot{\theta}}{3}$.

b) The energy lost to friction is $E_{fric} = \frac{mR^2\dot{\theta}_0^2}{6}$. The energy lost to friction is independent of μ for $\mu > 0$. Thus, the energy lost to friction is constant for given m, R, and $\dot{\theta}_0$. As $\mu \to 0$, the transition time to rolling $\to \infty$. It is not true, however, that the energy lost to friction $\to 0$ as $\mu \to 0$. Since the energy lost is constant for any $\mu > 0$, the disk will slip for longer and longer times so that the distance of slip goes to infinity. The dissipation rate $\to 0$ since the constant energy is divided by increasing transition time. The energy lost is zero only for $\mu = 0$.

14.4.12) $V = 2\,\text{m/s}$.

14.4.13) $v_o = 2\frac{\sqrt{2gR}}{3}$.

14.4.19) Accelerations of the center of mass, where $\hat{\imath}$ is parallel to the slope and pointing down: (a) $\vec{a}_{cm} = g\sin\theta\hat{\imath}$, (b) $\vec{a}_{cm} = \frac{2}{3}g\sin\theta\hat{\imath}$, (c) $\vec{a}_{cm} = \frac{1}{2}g\sin\theta\hat{\imath}$, (d) $\vec{a}_{cm} = \frac{2}{3}g\sin\theta\hat{\imath}$. So, the block is fastest, all uniform disks are second, and the hollow pipe is third.

14.5.8) $h = 2L/3$

15.1.1) plot(2): $\theta(t) = br(t)$ and $\vec{v} = \frac{\dot{\theta}}{b}\hat{e}_r + \frac{\theta\dot{\theta}}{b}\hat{e}_t$.

15.1.1) plot(2): $x = r\cos(br)$ and $y = r\sin br$ or $x = \frac{\theta}{b}\cos(\theta)$ and $y = \frac{\theta}{b}\sin\theta$

15.1.5) One situation: $\vec{v} = R\dot{\theta}\hat{e}_\theta$;the case of a particle constrained to move in a circle.

15.1.6) One situation: $\vec{a} = \ddot{R}\hat{e}_R$; the case of no rotation ($\dot{\theta} = \ddot{\theta} = 0$).

15.1.9a)

$$\hat{e}_t = \frac{\frac{2t}{\text{s}}\hat{\imath} + e^{\frac{t}{\text{s}}}\hat{\jmath}}{\sqrt{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}}$$

$$a_t = \frac{4t\frac{\text{m}}{\text{s}^3} + e^{\frac{2t}{\text{s}}}\ \text{m/s}^2}{\sqrt{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}}$$

$$\vec{a}_n = \frac{(2 - 2\frac{t}{\text{s}})e^{\frac{2t}{\text{s}}}\ \text{m/s}^2\hat{\imath} + (4\frac{t^2}{\text{s}^2} - 4\frac{t}{\text{s}})e^{\frac{t}{\text{s}}}\ \text{m/s}^2\hat{\jmath}}{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}$$

$$\hat{e}_n = \frac{e^{\frac{t}{\text{s}}}\hat{\imath} - 2\frac{t}{\text{s}}\hat{\jmath}}{\sqrt{4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}}}}$$

$$\rho = \frac{(4\frac{t^2}{\text{s}^2} + e^{\frac{2t}{\text{s}}})}{(2 - 2\frac{t}{\text{s}})e^{\frac{t}{\text{s}}}}\ \text{m}$$

15.3.4) $\vec{v}_P = \omega_1 L(\cos\theta\hat{\jmath} - \sin\theta\hat{\imath}) + (\omega_1 + \omega_2)r\hat{\imath}$, $\vec{a}_P = -\omega_1^2 L(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + (\omega_1 + \omega_2)^2\hat{\jmath}$.

15.3.5) $\vec{\omega}_D = (\omega_1 + \omega_2)\hat{k}$.

15.3.7) $\vec{a}_b = -(s^2 A + a\omega_D^2 + 2\omega_D v_b)\hat{\imath} - a\dot{\omega}_D\hat{k}$.

15.3.12b) One case: $\vec{a}_P = \vec{a}_{O'}$, where P refers to the bug; the bug stands still on the turntable, the truck accelerates, and the turntable does not rotate.

c) One case: $\vec{a}_P = \vec{a}_{O'} + \vec{a}_{P/\mathcal{D}}$, where $\mathcal{D}$ is the turntable; the bug accelerates along a straight line on the turntable, the truck accelerates, and the turntable does not rotate.

15.4.1a) $\vec{\omega}_{OA} = \dot{\theta}\hat{k}$.

b) $\vec{\alpha}_{OA} = \ddot{\theta}\hat{k}$.

c) $\vec{v}_A = -R\dot{\theta}\sin\theta\hat{\imath} + R\dot{\theta}\cos\theta\hat{\jmath}$.

d) $\vec{a}_A = -(R\dot{\theta}^2\cos\theta + R\ddot{\theta}\sin\theta)\hat{\imath} + (R\ddot{\theta}\cos\theta - R\dot{\theta}^2\sin\theta)\hat{\jmath}$.

e) $\vec{\omega}_{AB} = \frac{-R\dot{\theta}\cos\theta}{\sqrt{\ell^2 - R^2\sin^2\theta}}$.

f) $\theta = (2n+1)\frac{\pi}{2}, n = 0, \pm1, \pm2, \ldots$.

15.4.4) $\vec{\omega}_{DE} = -\frac{\omega}{2}\hat{k} = -1.5\,\text{rad/s}\hat{k}$

15.4.5) $\vec{\alpha}_{DE} = 3.5\,\text{rad/s}^2\hat{k}$

16.1.9a) $\ddot{R}(t) - \omega_o^2 R(t) = 0$.

b) $\frac{d^2R(\theta)}{d\theta^2} - R(\theta) = 0$.

c) $R(\theta = 2\pi) = 267.7\,\text{ft}$, $R(\theta = 4\pi) = 1.43 \times 10^5$ ft.

d) $v = 1682.3\,\text{ft/s}$

e) $E_K = (2.83 \times 10^6) \times (mass)(\text{ft/s})^2$.

16.1.10a) $\ddot{R} = R_0\omega^2 - 2\mu\dot{R}_0\omega$.

b) $\phi = \sin^{-1}\left[\frac{1}{\sqrt{(\sqrt{1+\mu^2}-\mu)^2+1}}\right]$.

16.1.18a) $\omega^- = 4\,\text{rad/s}$, where the minus sign '-' means 'just before leaving'.

b) $\omega^+ = 4\,\text{rad/s}$, where the plus sign '+' means 'just after leaving'.

c) $v_r = 3.841\,\text{m/s}$, $v_t = 2.4\,\text{m/s}$.

d) torque=$0.072\,\text{N}\cdot\text{m}$

16.1.19) $\vec{F} = -0.6\,\text{N}\hat{\boldsymbol{j}}$.

16.1.20a) $T_{BC} = m\frac{\sqrt{34}}{13}(2a_x + g)$.

b) $\vec{a} = \frac{1}{34}[(25a_x + 15g)\hat{\boldsymbol{\imath}} + (15a_x - 25g)\hat{\boldsymbol{j}}]$

c) $T_{BC} = m\frac{1}{\sqrt{34}}(5a_x + 3g)$.

16.2.3a) $\vec{H}_{/\text{O}} = \left[mL^2\omega_1 + \frac{1}{2}mr^2(\omega_1 + \omega_2)\right]\hat{\boldsymbol{k}}$.

b) $\dot{\vec{H}}_{/\text{O}} = \vec{\boldsymbol{0}}$.

c) $\vec{H}_{/\text{C}} = \frac{1}{2}mr^2(\omega_1 + \omega_2)\hat{\boldsymbol{k}}$.

d) $\dot{\vec{H}}_{/\text{C}} = \vec{\boldsymbol{0}}$.

16.2.4) $\vec{F} = m\vec{a} = -109.3\,\text{N}\hat{\boldsymbol{\imath}} - 19.54\,\text{N}\hat{\boldsymbol{j}}$.

16.2.6b) $\ddot{\theta} = -\frac{3g}{2L}\cos\theta$.

c) From the diagram, we see $\dot{\boldsymbol{\omega}}_{AB} = -\ddot{\theta}(t=0)\hat{\boldsymbol{k}} = \frac{3g}{2L}\cos\theta_0$, $\vec{a}_G = \frac{3}{4}g\cos\theta_0[\sin\theta_0\hat{\boldsymbol{\imath}} - \cos\theta_0\hat{\boldsymbol{j}}]$.

d) $\vec{R}_A = \frac{3}{4}mg\sin\theta_0\cos\theta_0\hat{\boldsymbol{\imath}}$, $\vec{R}_B = mg[1 - \frac{3}{4}g\cos^2\theta_0]\hat{\boldsymbol{j}}$.

e) $\vec{a}_B = \frac{3}{2}g\sin\theta_0\cos\theta_0\hat{\boldsymbol{\imath}}$.

f) At $\theta = \frac{\theta_0}{2}$, $\vec{\boldsymbol{\omega}}_{AB} = \sqrt{\frac{3g}{L}(\sin\theta_0 - \sin\frac{\theta_0}{2})}\hat{\boldsymbol{k}}$, $\vec{a}_A = -3g\left[\frac{1}{2}\cos^2\frac{\theta_0}{2} + \sin\frac{\theta_0}{2}(\sin\theta_0 - \sin\frac{\theta_0}{2})\right]\hat{\boldsymbol{j}}$.

16.2.12) There are many solution methods. $-8\pi^2$ poundals to the left. (Note: 1 poundal $= 1\,\text{ft}\cdot\text{lbm}\cdot\text{s}^{-2}$.)

16.2.14a) $\Delta = \sqrt{\frac{(M_c+6M_w)v_0^2}{k}}$.

b) Tangential forces point toward the wall!

c) $\Delta = \sqrt{\frac{(M_c+4M_w)v_0^2}{k}}$.

16.3.3a) $-k(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{j}}) = m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{j}})$.

b) $-kr\hat{\boldsymbol{e}}_r = m(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{e}}_r + m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{e}}_t$.

c) $\frac{d}{dt}(x\dot{y} - y\dot{x}) = 0$.

d) $\frac{d}{dt}(r^2\dot{\theta}) = 0$.

e) dot equation in (a) with $(x\hat{\boldsymbol{j}} - y\hat{\boldsymbol{\imath}})$ to get $(x\ddot{y} - y\ddot{x}) = 0$ which can be rewritten as (c); dot equation in (b) with $\hat{\boldsymbol{e}}_t$ to get $(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$ which can be rewritten as (d).

f) $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2) = const.$

g) $\frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + \frac{1}{2}kr^2$=const.

h) dot equation in (a) with $\vec{v} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{j}}$ to get $m(\ddot{x}\dot{x} + \ddot{y}\dot{y}) + k(\dot{x}x + \dot{y}y) = 0$ which can be rewritten as (f).

i) $x = A_1\sin(\omega t + B_1)$, $y = A_2\sin(\omega t + B_2)$, where $\omega = \sqrt{\frac{k}{m}}$. The general motion is an ellipse.

j) Yes. Consider $B_1 = B_2 = 0$, $A_1 = 1$, and $A_2 = 2$.

16.3.4a) $0.33\,\text{m}\hat{\boldsymbol{\imath}}$

b) $\vec{v}_C = 1.82\,\text{m/s}\hat{\boldsymbol{\imath}}$.

c) $T \approx 240\,\text{N}$.

16.3.10b)

$$\begin{aligned} \ddot{R} - R\dot{\theta}^2 &= 0 \\ (I_{zz} + mR^2)\ddot{\theta} + 2mR\dot{R}\dot{\theta} &= 0 \end{aligned}$$

c) The second equation in part (b) can be rewritten in the form $\frac{d}{dt}\left[I_{zz} + mR^2)\dot{\theta}\right] = 0$. The quantity inside the derivative is angular momentum; thus, it is conserved and equal to a constant, say, $(H_{/o})_0$, which can be found in terms of the initial conditions.

d) $\ddot{R} - R\left[\frac{(H_o)_0^2}{(I_{zz}^O + mR^2)^2}\right] = 0$.

e) $E_0 = \frac{1}{2}m\dot{R}^2 + \frac{1}{2}\omega(H_{/o})_0$.

f) The bead's distance goes to infinity and its speed approaches a constant. The turntable's angular velocity goes to zero and its net angle of twist goes to a constant.

16.3.12a)

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{2mRv\omega}{I_{zz} + mR^2} \\ \dot{R} &= v \\ \dot{v} &= \omega^2 R \end{aligned}$$

16.3.13) $v_{2_x} = \sqrt{\frac{2g(a-b)\tan\phi}{(1+\frac{1}{n}+\left(\frac{n+1}{n}\right)^2\tan^2\phi)}}$,

$v_{2_y} = v_{2_x}\frac{n+1}{n}\tan\phi$, $v_{1_x} = -\frac{v_{2_x}}{n}$, $v_{1_y} = 0$, where 2 refers to the top wedge and 1 refers to the lower wedge.

16.3.14a) $\vec{F} = m(g + a_0)\left[\frac{\sqrt{3}}{4}\hat{\boldsymbol{\imath}} + \frac{3}{4}\hat{\boldsymbol{j}}\right]$.

b) For $a_0 = -g$, the acceleration of the mass is exactly vertical; d_0, v_0, and t could be anything.

16.3.15a) angular speed = $8.8\,\text{rad/s}$.

b) displacement = $0.5\,\text{ft}$.

16.3.26a) For point mass: $\ddot{\phi} = (g/L)\sin\phi - (a_{hand}/L)\cos\phi$.

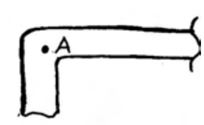
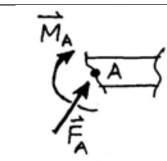
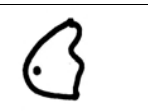
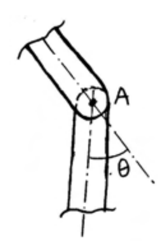
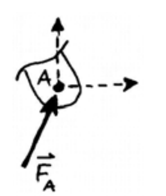
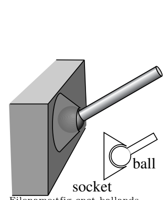
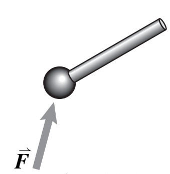
b) There are many correct solutions. Test your solution with a computer simulation. Show your result with appropriate plots.

16.3.27e) $\ddot{\phi} - \left[\frac{\ddot{y}+g}{L}\right]\sin\phi = 0$, where $y(t)$ is the vertical displacement of your hand and $\phi(t)$ is the angle of the broom from the vertical.

Table of common connections

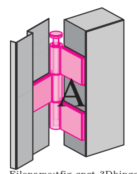
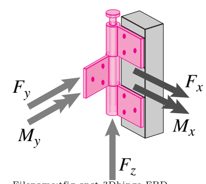
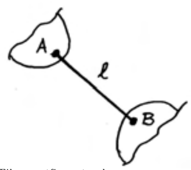
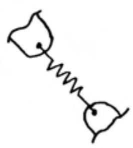
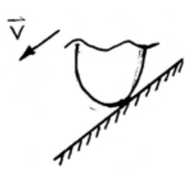
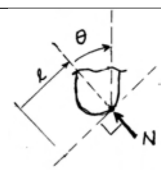
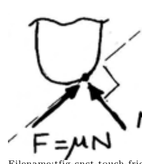
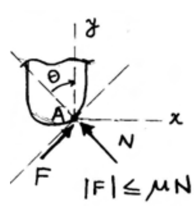
Here are some common models of connections. Fancier connections, such as universal joints and parallel mechanisms, are built out of mixtures of objects connected using these connections. Every connection has jobs to do:

- Transmit some forces and moments.
- Allow some motions. These are the degrees of freedom (DOF).
- Prevent some motions. These are the restrictions. There is always a force or moment associated with a restriction.

Name of connection	Sketch	Forces on FBD	# DOFs	# Restrictions
			2D: # DOF + # Restr. = 3 **3D**: # DOF + # Restr. = 6	
Weld, glue joint, lap-joint, cut of solid piece		 **2D:** 2 force components and 1 moment component **3D:** 3 force components and 3 moment components	0 DOF (no rel. motion)	**2D**: 3 restr: x_A, y_A, θ **3D**: 6 restr: $x_A, y_A, z_A, \theta, \phi, \xi$
No interaction. (no contact & no other mutual forces)		 0 force and 0 moment components	**2D**: 3 DOF: x_A, y_A, θ **3D**: 6 DOF: $x_A, y_A, z_A, \theta, \phi, \xi$	No restrictions (none, zero, nada)
2D pin, hinge, rotary joint			1 DOF: θ	2 restr.: x_A, y_A
3D ball& socket, rod-end		 3 force components, no moment	3 DOF: θ, ϕ, ξ	3 restr.: x_A, y_A, z_A

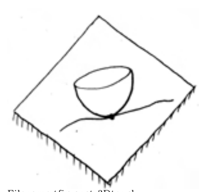
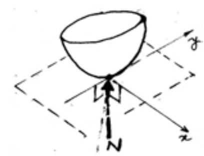
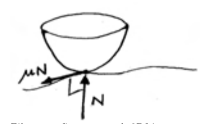
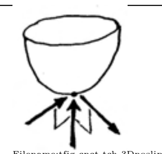
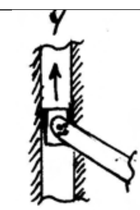
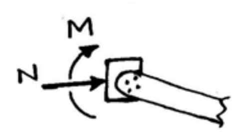
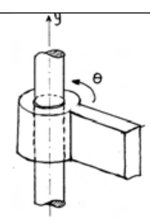
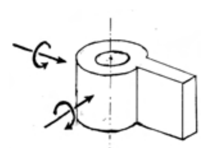
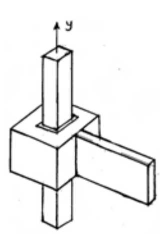
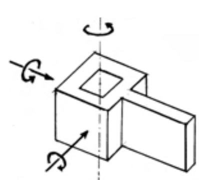
(continued on next page)

(Common connections . . . continued)

Name of connection	Sketch	Forces on FBD	**2D**: # DOF + # Restr. = 3 **3D**: # DOF + # Restr. = 6 # Forces on DOFs	# Restrictions
3D hinge	 Filename:tfig-cnct-3Dhinge	 Filename:tfig-cnct-3Dhinge-FBD 3 force components, 2 moment comps	1 DOF: θ	5 restr.: x_A, y_A, z_A, ϕ, ξ
Rod (massless, hinged); taut string, line, rope or cable	 Filename:tfig-cnct-rod	 Filename:tfig-cnct-rod-FBD	If inextensible: **2D**: 2 DOF **3D**: 5 DOF	If inextensible: **2D**: 1 restr.: $\dot{\ell}$ **3D**: 1 restr.: $\dot{\ell}$
Spring (or dashpot or gravitational attraction)	 Filename:tfig-cnct-spring	 Filename:tfig-cnct-spring-FBD	**2D**: 3 DOF **3D**: 6 DOF	**2D**: 0 restr. **3D**: 0 restr. No constraint on motion
2D point contact, non-conformal touching	 Filename:tfig-connect-touch	 Filename:tfig-connect-touch-nofr frictionless	2 DOF: ℓ, θ	1 restr.: N dir
		 Filename:tfig-cnct-touch-frict frictional slip	2 DOF: ℓ, θ	1 restr.: N dir
		 Filename:tfig-cnct-touch-noslip no slip	1 DOF: θ (like a pin or hinge)	2 restr.: x_A, y_A

(continued on next page)

(Common connections . . . continued)

			2D: # DOF + # Restr. = 3 **3D**: # DOF + # Restr. = 6	
Name of connection	**Sketch**	**Forces on FBD**	# DOFs	# Restrictions
3D point contact, non-conformal touching	 Filename:tfig-cnct-3Dtouch	 Filename:tfig-cnct-3Dtouch-nofr frictionless	5 DOF: x, y, θ, ϕ, ξ	1 restr.: N dir
		 Filename:tfig-cnct-touch-3Dfrict frictional slip	5 DOF: x, y, θ, ϕ, ξ	1 restr.: N dir
		 Filename:tfig-cnct-tch-3Dnoslip no slip, $\|\boldsymbol{F}\| \leq \mu N$	3 DOF: θ, ϕ, ξ (like a ball & socket)	3 restr.: x_A, y_A, z_A
2D keyed slot with weld, glue or lap joint to prevent rotation	 Filename:tfig-cnct-2Dkey	 Filename:tfig-cnct-2Dkey-FBD	1 DOF: y	2 restr.: x_A, θ
3D linear bearing with weld, glue or lap joint, piston in cylinder	 Filename:tfig-cnct-3Drndkey	 Filename:tfig-cnct-3Drndkey-FBD	2 DOF: y, θ	4 restr.: x, z, ϕ, ξ
3D keyed bearing with weld, glue or lap joint, cart on track	 Filename:tfig-cnct-3Dsqrkey	 Filename:tfig-cnct-3Dsqrkey-FBD	1 DOF: y	5 restr.: $x, z, \theta \phi, \xi$

(continued on next page)

(Common connections . . . continued)

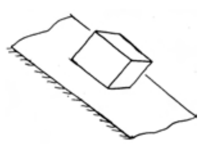
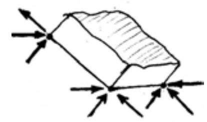
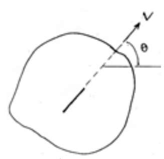
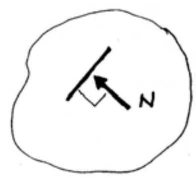
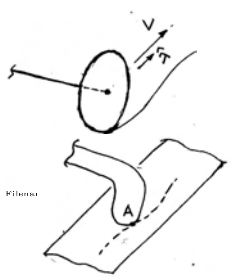
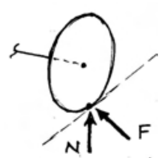
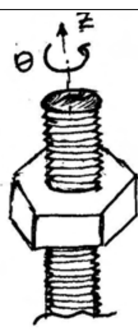
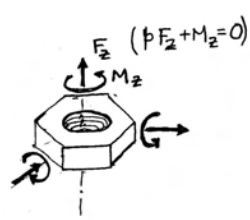
Name of connection	Sketch	Forces on FBD	**2D**: # DOF + # Restr. = 3 **3D**: # DOF + # Restr. = 6 # DOFs	# Restrictions
3D block sliding, Object confined to a plane		 If no friction, then only normal forces	3 DOF: x, y, θ	3 restr.: z, ϕ, ξ Statically indeterminate: there are more reaction components than restrictions.
2D skate. Ideal wheel, looking down. (Non-holonomic system.)			2 velocity DOF: $v, \dot{\theta}$ 3 config DOF: x, y, θ	1 restr.: vel. in N direction.
3D skate. Ideal wheel. (Non-holonomic system.)			4 velocity DOF 5 config DOF: x, y, θ	2 restr.: $v_A \perp$ to path & vertical pos.
3D bolt.			1 DOF: a mixture of z displ & θ rotation	5 restr.: all but the screw motion

TABLE I

Momenta and energy

What system	Linear Momentum: $\vec{L}$ (a)	Linear Momentum: $\dot{\vec{L}} = \frac{d}{dt}\vec{L}$ (b)	Angular Momentum: $\vec{H}_{\rm C}$ (c)	Angular Momentum: $\dot{\vec{H}}_{\rm C} = \frac{d}{dt}\vec{H}_{\rm C}$ (d)	Kinetic Energy: $E_{\rm K}$ (e)	
In General	$L_x\hat{\imath} + L_y\hat{\jmath} + L_z\hat{k}$ $= m_{\rm tot}\vec{v}_{\rm cm}$ $= \frac{d}{dt}.m_{\rm tot}\vec{r}_{\rm cm}/$	$\dot{L}_x\hat{\imath} + \dot{L}_y\hat{\jmath} + \dot{L}_z\hat{k}$ $= m_{\rm tot}\vec{a}_{\rm cm}$ "F = ma"	$H_{Cx}\hat{\imath} + H_{Cy}\hat{\jmath} + H_{Cz}\hat{k}$ $= \vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \vec{H}_{\rm cm}$ $= \frac{d}{dt}$.no such thing/	$\dot{H}_{Cx}\hat{\imath} + \dot{H}_{Cy}\hat{\jmath} + \dot{H}_{Cz}\hat{k}$ $= \vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \dot{\vec{H}}_{\rm cm}$ = (no simple general expression)	$\frac{1}{2} m_{\rm tot}\, v_{\rm cm}^2 + E_{{\rm K}/cm}$	(1)
One Particle P	$m_{\rm P}\vec{v}_{\rm P}$	$m_{\rm P}\vec{a}_{\rm P}$	$\vec{r}_{\rm P/C} \times \vec{v}_P m_{\rm P}$	$\vec{r}_{\rm P/C} \times \vec{a}_{\rm P} m_{\rm P}$	$\frac{1}{2} m_{\rm P} v_{\rm P}^2$	(2)
System of Particles	$\sum_{\text{all particles i}} m_i\vec{v}_i$	$\sum_{\text{all particles i}} m_i\vec{a}_i$	$\sum_{\text{all particles}} \vec{r}_{i/\rm C} \times \vec{v}_i m_i$	$\sum_{\text{all particles}} \vec{r}_{i/\rm C} \times \vec{a}_i m_i$	$\frac{1}{2}\sum_{\text{all particles}} v_i^2 m_i$	(3)
Continuum	$\int_{\text{all mass}} \vec{v}\, dm$	$\int_{\text{all mass}} \vec{a}\, dm$	$\int_{\text{all mass}} \vec{r}_{/\rm C} \times \vec{v}\, dm$	$\int_{\text{all mass}} \vec{r}_{/\rm C} \times \vec{a}\, dm$	$\frac{1}{2}\int_{\text{all mass}} v^2\, dm$	(4)
System of Systems (eg. rigid bodies)	$\sum_{\text{all sub-systems}} m_i\vec{v}_i$	$\sum_{\text{all sub-systems}} m_i\vec{a}_i$	$\sum_{\text{all sub-systems}} \vec{H}_{{\rm C}i}$	$\sum_{\text{all sub-systems}} \dot{\vec{H}}_{{\rm C}i}$	$\sum_{\text{all sub-systems}} E_{{\rm K}i}$	(5)

Rigid Bodies

What system	(a)	(b)	(c)	(d)	(e)	
One rigid body (2D and 3D)	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$\vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \underbrace{[\mathbf{I}^{cm}] \cdot \vec{\omega}}_{\vec{H}_{\rm cm}}$	$\vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot}$ $+ \underbrace{[\mathbf{I}^{cm}] \cdot \dot{\vec{\omega}} + \vec{\omega} \times \vec{H}_{\rm cm}}_{\dot{\vec{H}}_{\rm cm}}$	$\frac{1}{2} m_{\rm tot}\, v_{\rm cm}^2$ $+ \underbrace{\frac{1}{2}\vec{\omega} \cdot [\mathbf{I}^{cm}] \cdot \vec{\omega}}_{E_{{\rm K}/cm}}$	(6)
2D rigid body in xy plane with $\vec{\omega} = \omega\hat{k}$	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$\vec{r}_{\rm cm/C} \times \vec{v}_{\rm cm} m_{\rm tot} + \underbrace{I_{zz}^{cm}\omega\hat{k}}_{\vec{H}_{\rm cm}}$	$\vec{r}_{\rm cm/C} \times \vec{a}_{\rm cm} m_{\rm tot} + \underbrace{I_{zz}^{cm}\dot{\omega}\hat{k}}_{\dot{\vec{H}}_{\rm cm}}$	$\frac{1}{2} m_{\rm tot}\, v_{\rm cm}^2$ $+ \underbrace{\frac{1}{2} I_{zz}^{cm}\omega^2}_{E_{{\rm K}/cm}}$	(7)
One rigid body *if* C is a fixed point (2D and 3D)	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$[\mathbf{I}^C] \cdot \vec{\omega} = \vec{H}_C$	$[\mathbf{I}^C] \cdot \dot{\vec{\omega}} + \vec{\omega} \times \vec{H}_C$	$\frac{1}{2}\vec{\omega} \cdot [\mathbf{I}^C] \cdot \vec{\omega}$	(8)
2D rigid body *if* C is a fixed point with $\vec{\omega} = \omega\hat{k}$	$m_{\rm tot}\vec{v}_{\rm cm}$	$m_{\rm tot}\vec{a}_{\rm cm}$	$I_{zz}^C\omega\hat{k}$	$I_{zz}^C\dot{\omega}\hat{k}$ "$M = I\alpha$"	$\frac{1}{2} I_{zz}^C\omega^2$	(9)

The table has used the following terms:

$m_{\rm tot}$ =total mass of system,

m_i = mass of body or subsystem i,

$\vec{r}_{\rm cm/C}$ = the position of the center of mass relative to point C,

$\vec{v}_i$ = velocity of the center of mass of sub-system or particle i,

$\vec{a}_i$ = acceleration of the center of mass of sub-system i,

$\vec{H}_{{\rm C}i}$ = angular momentum of subsystem i relative to point C.

$\dot{\vec{H}}_{{\rm C}i}$ = rate of change of angular momentum of sub-system i relative to point C.

Filename:table1

$\vec{H}_{\rm cm} = \sum \vec{r}_{i/{\rm cm}} \times .m_i\vec{v}_i/$ angular momentum about the center of mass

$\dot{\vec{H}}_{\rm cm} = \sum \vec{r}_{i/{\rm cm}} \times .m_i\vec{a}_i/$ rate of change of angular momentum about the center of mass

$\vec{\omega}$ is the angular velocity of a rigid body,

$\dot{\vec{\omega}} = \vec{\alpha}$ is the angular acceleration of the rigid body,

$[\mathbf{I}^{cm}]$ is the moment of inertia matrix of the rigid body relative to the center of mass, and

$[\mathbf{I}^{o}]$ is the moment of inertia matrix of the rigid body relative to a fixed point (not moving point) on the body.

Table II. Methods for calculating velocity and acceleration Some facts about path coordinates

The path of a particle is $\vec{\boldsymbol{r}}(t)$.

$$\hat{\boldsymbol{e}}_t \equiv \frac{d\vec{\boldsymbol{r}}(s)}{ds}, \quad \hat{\boldsymbol{e}}_t = \frac{d\vec{\boldsymbol{r}}(t)}{dt}\frac{dt}{ds} = \frac{\vec{\boldsymbol{v}}}{v}, \quad \vec{\boldsymbol{\kappa}} \equiv \frac{d\hat{\boldsymbol{e}}_t}{ds} = \frac{d\hat{\boldsymbol{e}}_t}{dt}\frac{1}{v},$$
$$\hat{\boldsymbol{e}}_n = \frac{\vec{\boldsymbol{\kappa}}}{|\vec{\boldsymbol{\kappa}}|}, \quad \boldsymbol{e}_b \equiv \hat{\boldsymbol{e}}_t \times \hat{\boldsymbol{e}}_n, \quad \rho = \frac{1}{|\vec{\boldsymbol{\kappa}}|}.$$

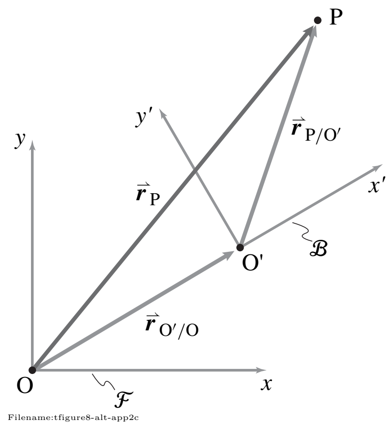


Figure C.3:

Summary of the direct differentiation method

In the direct differentiation method, using moving frame $\mathcal{B}$, we calculate $\vec{\boldsymbol{v}}_P$ by using a combination of the product rule of differentiation and the facts that $\dot{\hat{\boldsymbol{\imath}}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}'$, $\dot{\hat{\boldsymbol{\jmath}}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}'$, and $\dot{\hat{\boldsymbol{k}}} = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{k}}'$, as follows:

$$\begin{aligned}
\vec{\boldsymbol{v}}_P &= \frac{d}{dt}\vec{\boldsymbol{r}}_P = \frac{d}{dt}\left[\vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{P/O'}\right] \\
&= \frac{d}{dt}\left[(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}) + (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' + z'\hat{\boldsymbol{k}}')\right] \\
&= (\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}) + (\dot{x}'\hat{\boldsymbol{\imath}}' + \dot{y}'\hat{\boldsymbol{\jmath}}' + \dot{z}'\hat{\boldsymbol{k}}') + \\
&\left[x'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}') + z'(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{k}}')\right]
\end{aligned}$$

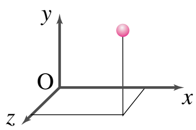
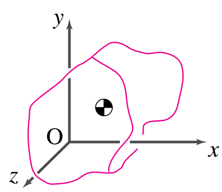
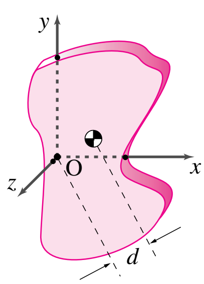
but stop short of identifying these three groups of three terms as

$$\vec{\boldsymbol{v}}_P = \vec{\boldsymbol{v}}_{O'/O} + \dot{\vec{\boldsymbol{r}}}_{rel} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P/O}.$$

We would calculate $\vec{\boldsymbol{a}}_P$ similarly and would get a formula with 15 non-zero terms (3 for each term in the 'five-term' acceleration formula).

Method	Position	Velocity	Acceleration
In general, as measured relative to the fixed frame $\mathcal{F}$.	$\vec{\boldsymbol{r}}$ or $\vec{\boldsymbol{r}}_P$ or $\vec{\boldsymbol{r}}_{P/O}$	$\vec{\boldsymbol{v}}$ or $\vec{\boldsymbol{v}}_P$ or $\vec{\boldsymbol{v}}_{P/\mathcal{F}}$	$\vec{\boldsymbol{a}}$ or $\vec{\boldsymbol{a}}_P$ or $\vec{\boldsymbol{a}}_{P/\mathcal{F}}$
Cartesian Coordinates	$r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}}$	$v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}} + v_z\hat{\boldsymbol{k}}$ $= \dot{r}_x\hat{\boldsymbol{\imath}} + \dot{r}_y\hat{\boldsymbol{\jmath}} + \dot{r}_z\hat{\boldsymbol{k}}$	$a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}} + a_z\hat{\boldsymbol{k}}$ $= \dot{v}_x\hat{\boldsymbol{\imath}} + \dot{v}_y\hat{\boldsymbol{\jmath}} + \dot{z}_z\hat{\boldsymbol{k}}$ $= \ddot{r}_x\hat{\boldsymbol{\imath}} + \ddot{r}_y\hat{\boldsymbol{\jmath}} + \ddot{r}_z\hat{\boldsymbol{k}}$
Polar Coordinates/ Cylindrical Coordinates	$R\hat{\boldsymbol{e}}_R + z\hat{\boldsymbol{k}}$	$v_R\hat{\boldsymbol{e}}_R + v_\theta\hat{\boldsymbol{e}}_\theta + v_z\hat{\boldsymbol{k}}$ $= \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + \dot{z}\hat{\boldsymbol{k}}$	$a_R\hat{\boldsymbol{e}}_R + a_\theta\hat{\boldsymbol{e}}_\theta + a_z\hat{\boldsymbol{k}}$ $= (\ddot{R} - R\dot{\theta}^2)\hat{\boldsymbol{e}}_R + (R\ddot{\theta} + 2\dot{R}\dot{\theta})\hat{\boldsymbol{e}}_\theta + \ddot{z}\hat{\boldsymbol{k}}$
Path Coordinates	not used	$v\hat{\boldsymbol{e}}_t$	$a_t\hat{\boldsymbol{e}}_t + a_n\hat{\boldsymbol{e}}_n$ $= \dot{v}\hat{\boldsymbol{e}}_t + (v^2/\rho)\hat{\boldsymbol{e}}_n$
Using data from a moving frame $\mathcal{B}$ with origin at O' and angular velocity relative to the fixed frame of $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$. The point P' is glued to $\mathcal{B}$ and instantaneously coincides with P.	$\vec{\boldsymbol{r}}_{O'/O} + \vec{\boldsymbol{r}}_{P/O'}$	$\vec{\boldsymbol{v}}_{P'/\mathcal{F}} + \vec{\boldsymbol{v}}_{P/\mathcal{B}} =$ $\underbrace{\dot{\vec{\boldsymbol{r}}}_{O'/O} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'}}_{\vec{\boldsymbol{v}}_{P'/\mathcal{F}}} + \underbrace{{}^{\mathcal{B}}\dot{\vec{\boldsymbol{r}}}_{P/O'}}_{\vec{\boldsymbol{v}}_{P/\mathcal{B}}}$	$\vec{\boldsymbol{a}}_{P'/\mathcal{F}} + \vec{\boldsymbol{a}}_{P/\mathcal{B}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}} =$ $\overbrace{\ddot{\vec{\boldsymbol{r}}}_{O'/O} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'} + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{P'/O'}}^{\vec{\boldsymbol{a}}_{P'/\mathcal{F}}}$ $+ \underbrace{{}^{\mathcal{B}}\ddot{\vec{\boldsymbol{r}}}_{P/O'}}_{\vec{\boldsymbol{a}}_{P/\mathcal{B}}} + 2\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{v}}_{P/\mathcal{B}}$ 'the 5-term acceleration formula'

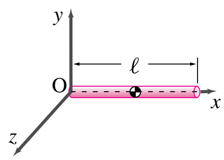
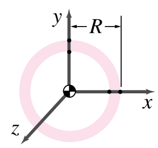
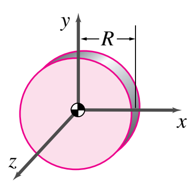
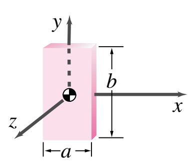
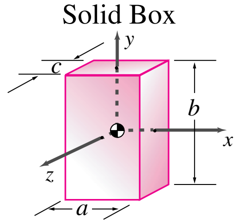
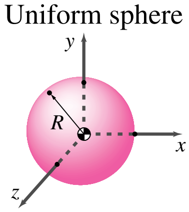
Table III

Object	$[\boldsymbol{I}]$
Point mass 	$[\boldsymbol{I}^{cm}] = m\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $[\boldsymbol{I}^{O}] = m\begin{bmatrix}y^2+z^2 & xy & xz\\ xy & x^2+z^2 & yz\\ xz & yz & x^2+y^2\end{bmatrix}$
General 3D body 	$[\boldsymbol{I}^{cm}] = \int\begin{bmatrix}y_{/cm}^2+z_{/cm}^2 & x_{/cm}y_{/cm} & x_{/cm}z_{/cm}\\ x_{/cm}y_{/cm} & x_{/cm}^2+z_{/cm}^2 & y_{/cm}z_{/cm}\\ x_{/cm}z_{/cm} & y_{/cm}z_{/cm} & x_{/cm}^2+y_{/cm}^2\end{bmatrix}dm$ $\overbrace{[\boldsymbol{I}^{cm}] = \begin{bmatrix}A&0&0\\0&B&0\\0&0&C\end{bmatrix}}^{\text{If the axes are principal axes of the body.}}$ With $A, B, C \geq 0$ and $A+B \geq C$, $B+C \geq A$, and $A+C \geq B$. $[\boldsymbol{I}^{o}] = \int\begin{bmatrix}y_{/o}^2+z_{/o}^2 & x_{/o}y_{/o} & x_{/o}z_{/o}\\ x_{/o}y_{/o} & x_{/o}^2+z_{/o}^2 & y_{/o}z_{/o}\\ x_{/o}z_{/o} & y_{/o}z_{/o} & x_{/o}^2+y_{/o}^2\end{bmatrix}dm$ $\underbrace{[\boldsymbol{I}^{o}] = [\boldsymbol{I}^{cm}] + m\begin{bmatrix}y_{cm/o}^2+z_{cm/o}^2 & x_{cm/o}y_{cm/o} & x_{cm/o}z_{cm/o}\\ x_{cm/o}y_{cm/o} & x_{cm/o}^2+z_{cm/o}^2 & y_{cm/o}z_{cm/o}\\ x_{cm/o}z_{cm/o} & y_{cm/o}z_{cm/o} & x_{cm/o}^2+y_{cm/o}^2\end{bmatrix}}_{\text{The 3D Parallel Axis Theorem}}$
General 2D Body 	$[\boldsymbol{I}^{cm}] = \int\begin{bmatrix}y_{/cm}^2 & x_{/cm}y_{/cm} & 0\\ x_{/cm}y_{/cm} & x_{/cm}^2 & 0\\ 0 & 0 & \underbrace{x_{/cm}^2+y_{/cm}^2}_{I_{zz}^{cm}}\end{bmatrix}dm$ $\overbrace{[\boldsymbol{I}^{cm}] = \begin{bmatrix}A&0&0\\0&B&0\\0&0&C\end{bmatrix}}^{\text{If the axes are principal axes of the body.}}$ With $A+B = C$ (The perpendicular axis theorem). Also, $A \geq 0$, $B \geq 0$. $[\boldsymbol{I}^{o}] = \int\begin{bmatrix}y_{/o}^2 & x_{/o}y_{/o} & 0\\ x_{/o}y_{/o} & x_{/o}^2 & 0\\ 0 & 0 & x_{/o}^2+y_{/o}^2\end{bmatrix}dm$ $\underbrace{[\boldsymbol{I}^{o}] = [\boldsymbol{I}^{cm}] + m\begin{bmatrix}y_{cm/o}^2 & x_{cm/o}y_{cm/o} & 0\\ x_{cm/o}y_{cm/o} & x_{cm/o}^2 & 0\\ 0 & 0 & \underbrace{x_{cm/o}^2+y_{cm/o}^2}_{d^2}\end{bmatrix}}_{\text{The 3D Parallel Axis Theorem. The 2D thm concerns the lower right terms of these 3 matrices.}}$

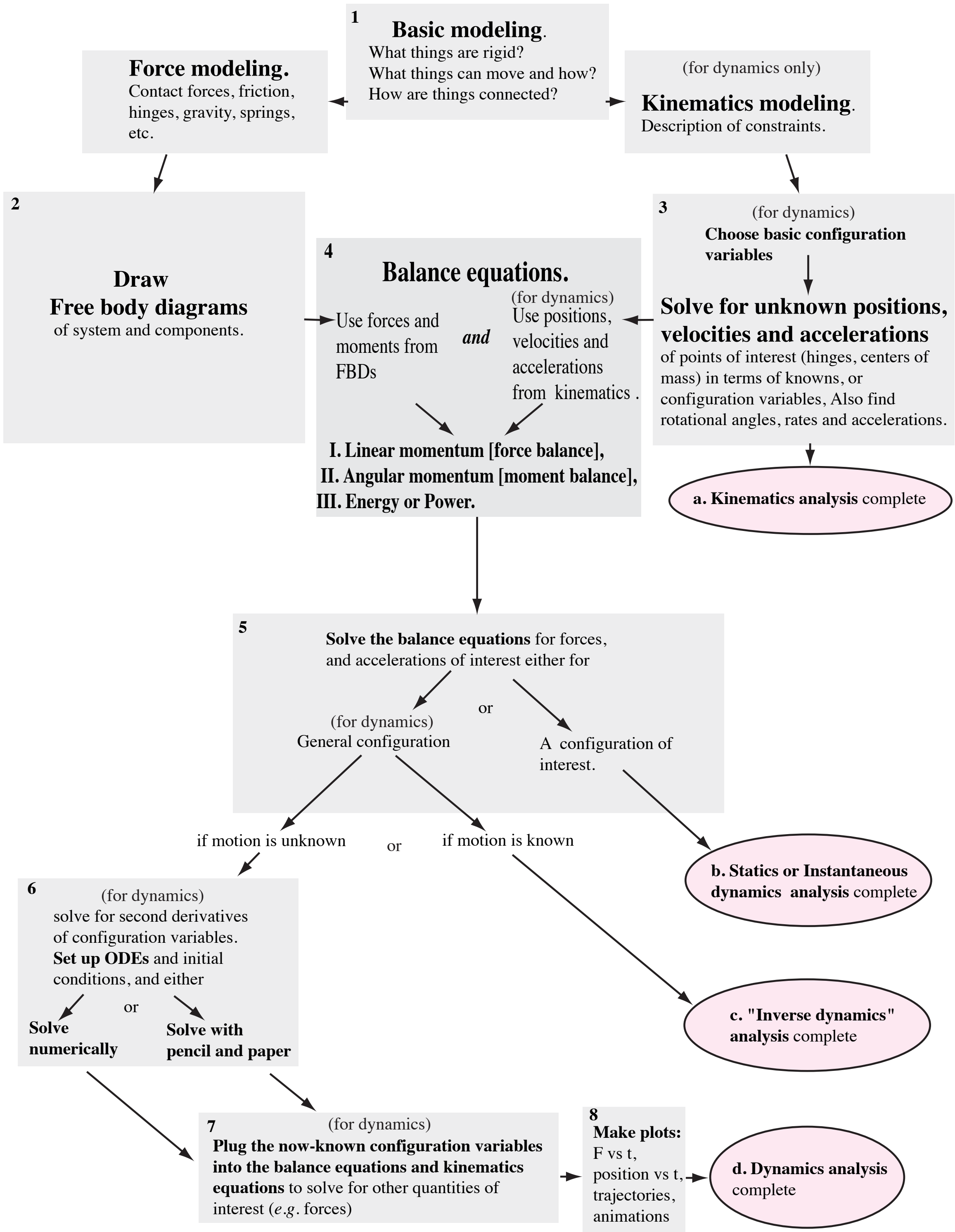
General moments of inertia. The tableshows a point mass, a general 3-D body, and a general 2-D body. The most general cases of the perpeendicual axis theorem and the parallel axis theorem are also shown..

Filename:table3

Table IV
Examples of Moment of Inertia

Object	$[\boldsymbol{I}]$
Uniform rod 	$I^{\mathrm{cm}}_{zz}=\frac{1}{12}m\ell^2,\qquad [\boldsymbol{I}^{cm}]=\frac{1}{12}m\ell^2\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$ $I^{\mathrm{O}}_{zz}=\frac{1}{3}m\ell^2,\qquad [\boldsymbol{I}^{O}]=\frac{1}{3}m\ell^2\begin{bmatrix}0&0&0\\0&1&0\\0&0&1\end{bmatrix}$
Uniform hoop 	$I^{\mathrm{cm}}_{zz}=mR^2,\qquad [\boldsymbol{I}^{cm}]=mR^2\begin{bmatrix}\frac{1}{2}&0&0\\0&\frac{1}{2}&0\\0&0&1\end{bmatrix}$
Uniform disk 	$I^{\mathrm{cm}}_{zz}=\frac{1}{2}mR^2,\qquad [\boldsymbol{I}^{cm}]=mR^2\begin{bmatrix}\frac{1}{4}&0&0\\0&\frac{1}{4}&0\\0&0&\frac{1}{2}\end{bmatrix}$
Rectangular plate 	$I^{\mathrm{cm}}_{zz}=\frac{1}{12}m(a^2+b^2),\qquad [\boldsymbol{I}^{cm}]=\frac{1}{12}m\begin{bmatrix}b^2&0&0\\0&a^2&0\\0&0&a^2+b^2\end{bmatrix}$
Solid Box 	$[\boldsymbol{I}^{cm}]=\frac{1}{12}m\begin{bmatrix}b^2+c^2&0&0\\0&a^2+c^2&0\\0&0&a^2+b^2\end{bmatrix}$
Uniform sphere 	$[\boldsymbol{I}^{cm}]=\frac{2}{5}mR^2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Moments of inertia of some simple objects. For the rod both the $[\boldsymbol{I}^{cm}]$ and $[\boldsymbol{I}^{O}]$ (for the end point at O) are shown. In the other cases only $[\boldsymbol{I}^{cm}]$ is shown. To calculate $[\boldsymbol{I}^{O}]$ relative to other points one has to use the parallel axis theorem. In all the cases shown the coordinate axes are principal axes of the objects.



Filename:tfigure-conceptmap

Basic flow chart for solving dynamics problems.
The methods (boxes 1-8) depend on the goal (ellipses a-d). Statics (b) only uses the solution route $1 \to 2 \to 4 \to 5 \to b$. Dynamics uses other boxes as needed. At first reading this chart shows you the logic of the subject. Later the flow chart in this box should become self-evident.