2.6 Center of mass and gravity

For every system and at every instant in time, there is a unique location in space that is the average position of the system's mass. This place is called the *center of mass*, commonly designated by cm, c.o.m., COM, G, c.g., or ⊕.

One of the routine but important tasks of many real engineers is to find the center of mass of a complex machine①. Just knowing the location of the center of mass of a car, for example, is enough to estimate whether it can be tipped over by maneuvers on level ground. The center of mass of a boat must be low enough for the boat to be stable. Any propulsive force on a space craft must be directed towards the center of mass in order to not induce rotations. Tracking the trajectory of the center of mass of an exploding plane can determine whether or not it was hit by a massive object. Any rotating piece of machinery must have its center of mass on the axis of rotation if it is not to cause much vibration.

① Nowadays this routine work is often done with CAD (computer aided design) software. But an engineer still needs to know the basic calculation skills, to make sanity checks on computer calculations if nothing else.

Also, many calculations in mechanics are greatly simplified by making use of a system's center of mass. In particular, the whole complicated distribution of near-earth gravity forces on a body is equivalent to a single force at the body's center of mass. Many of the important quantities in dynamics are similarly simplified using the center of mass.

The center of mass of a system is the point at the position $\vec{\boldsymbol{r}}_{\text{cm}}$ defined by

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\sum \vec{\boldsymbol{r}}_i m_i}{m_{\text{tot}}} \quad \text{for discrete systems} \qquad (2.30)$$
$$= \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} \quad \text{for continuous systems}$$

where $m_{\text{tot}} = \sum m_i$ for discrete systems and $m_{\text{tot}} = \int dm$ for continuous systems. See boxes 2.5 and **??** for a discussion of the $\sum$ and $\int$ sum notations.

Often it is convenient to remember the rearranged definition of center of mass as

$$m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} = \sum m_i \vec{\boldsymbol{r}}_i \quad \text{or} \quad m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}} = \int \vec{\boldsymbol{r}}\, dm.$$

For theoretical purposes we rarely need to evaluate these sums and integrals, and for simple problems there are sometimes shortcuts that reduce the calculation to a matter of observation. For complex machines one or both of the formulas **??** must be evaluated in detail.

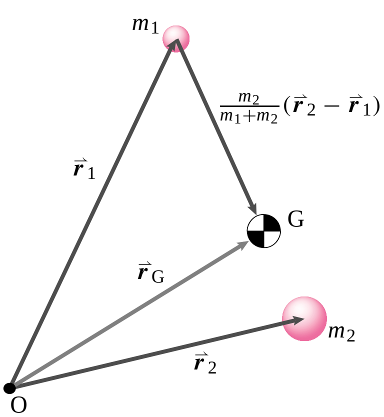


Figure 2.68: Center of mass of a system consisting of two points.

(Filename:tfigure3.com.twomass)

Example: **System of two point masses**

Intuitively, the center of mass of the two masses shown in figure **??** is between the two masses and closer to the larger one. Referring to equation **??**,

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\sum \vec{\boldsymbol{r}}_i m_i}{m_{tot}} = \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2} = \frac{\vec{\boldsymbol{r}}_1(m_1 + m_2) - \vec{\boldsymbol{r}}_1 m_2 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2}$$

$$= \quad \vec{\boldsymbol{r}}_1 + \underbrace{\left(\frac{m_2}{m_1+m_2}\right)}_{\text{the fraction of the distance that the cm is from } \vec{\boldsymbol{r}}_1 \text{ to } \vec{\boldsymbol{r}}_2} \underbrace{(\vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_1)}_{\text{the vector from } \vec{\boldsymbol{r}}_1 \text{ to } \vec{\boldsymbol{r}}_2.}.$$

so that the math agrees with common sense — the center of mass is on the line connecting the masses. If $m_2 \gg m_1$, then the center of mass is near m_2. If $m_1 \gg m_2$, then the center of mass is near m_1. If $m_1 = m_2$ the center of mass is right in the middle at $(\vec{\boldsymbol{r}}_1 + \vec{\boldsymbol{r}}_2)/2$. □

Continuous systems

How do we evaluate integrals like $\int$ (something) dm? In center of mass calculations, (something) is position, but we will evaluate similar integrals where (something) is some other scalar or vector function of position. Most often we label the material by its spatial position, and evaluate dm in terms of increments of position. For 3D solids $dm = \rho dV$ where ρ is density (mass per unit volume). So $\int$(something)dm

2.9 $\int$ means add

As discussed in box 2.5 on page 70 we often add things up in mechanics. For example, the total mass of some particles is

$$m_{\text{tot}} = m_1 + m_2 + m_3 + \ldots = \sum m_i$$

or more specifically the mass of 137 particles is, say, $m_{\text{tot}} = \sum_{i=1}^{137} m_i$.

And the total mass of a bicycle is:

$$m_{\text{bike}} = \sum_{i=1}^{100{,}000{,}000{,}000{,}000{,}000{,}000{,}000} m_i$$

where m_i are the masses of each of the 10^{23} (or so) atoms of metal, rubber, plastic, cotton, and paint. But atoms are so small and there are so many of them. Instead we often think of a bike as built of macroscopic parts. The total mass of the bike is then the sum of the masses of the tires, the tubes, the wheel rims, the spokes and nipples, the ball bearings, the chain pins, and so on. And we would write:

$$m_{\text{bike}} = \sum_{i=1}^{2{,}000} m_i$$

where now the m_i are the masses of the 2,000 or so bike parts. This sum is more manageable but still too detailed in concept for some purposes.

An approach that avoids attending to atoms or ball bearings, is to think of sending the bike to a big shredding machine that cuts it up into very small bits. Now we write

$$m_{\text{bike}} = \sum m_i$$

where the m_i are the masses of the very small bits. We don't fuss over whether one bit is a piece of ball bearing or fragment of cotton from the tire walls. We just chop the bike into bits and add up the contribution of each bit. If you take the letter S, as in SUM, and distort it (S $\sum$ ⟆ $\int$ $\int$) and you get a big old fashioned German 'S' as in $\int \mathcal{UM}$ (sum). So we write

$$m_{\text{bike}} = \int dm$$

to mean the $\int$um of all the teeny bits of mass. More formally we mean the value of that sum in the limit that all the bits are infinitesimal (not minding the technical fine point that its hard to chop atoms into infinitesimal pieces).

The mass is one of many things we would like to add up, though many of the others also involve mass. In center of mass calculations, for example, we add up the positions 'weighted' by mass.

$$\int \vec{\boldsymbol{r}}\, dm \quad \text{which means} \quad \sum_{\lim m_i \to 0} \vec{\boldsymbol{r}}_{\text{i}} m_i.$$

That is, you take your object of interest and chop it into a billion pieces and then re-assemble it. For each piece you make the vector which is the position vector of the piece multiplied by ('weighted by') its mass and then add up the billion vectors. Well really you chop the thing into a trillion trillion . . . pieces, but a billion gives the idea.

① Note: wri
nonsense beca
eter which labe
is no point at r

turns into a standard volume integral $\int_V (\text{something})\rho\, dV$①. For thin flat things like metal sheets we often take ρ to mean mass per unit area A so then $dm = \rho dA$ and $\int (\text{something}) dm = \int_A (\text{something})\rho\, dA$. For mass distributed along a line or curve we take ρ to be the mass per unit length or arc length s and so $dm = \rho ds$ and $\int (\text{something}) dm = \int_{curve} (\text{something})\rho\, ds$.

Example. The center of mass of a uniform rod is naturally in the middle, as the calculations here show (see fig. **??**a). Assume the rod has length $L = 3\,\text{m}$ and mass $m = 7\,\text{kg}$.

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} = \frac{\int_0^L x\hat{\boldsymbol{\imath}}\, \overbrace{\rho dx}^{dm}}{\int_0^L \rho\, dx} = \frac{\rho(x^2/2)|_0^L}{\rho(1)|_1^L}\hat{\boldsymbol{\imath}} = \frac{\rho(L^2/2)}{\rho L}\hat{\boldsymbol{\imath}} = (L/2)\hat{\boldsymbol{\imath}}$$

So $\vec{\boldsymbol{r}}_{\text{cm}} = (L/2)\hat{\boldsymbol{\imath}}$, or by dotting with $\hat{\boldsymbol{\imath}}$ (taking the x component) we get that the center of mass is on the rod a distance $d = L/2 = 1.5\,\text{m}$ from the end. □

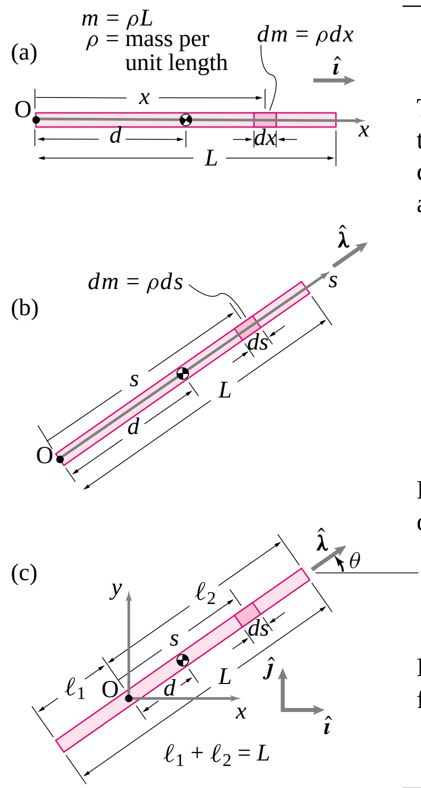


Figure 2.69: Where is the center of mass of a uniform rod? In the middle, as you can find calculating a few ways or by symmetry.
(Filename:tfigure1.rodcm)

The center of mass calculation is *objective*. It describes something about the object that does not depend on the coordinate system. In different coordinate systems the center of mass for the rod above will have different coordinates, but it will always be at the middle of the rod.

Example. Find the center of mass using the coordinate system with s & $\hat{\boldsymbol{\lambda}}$ in fig. **??**b:

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} = \frac{\int_0^L s\hat{\boldsymbol{\lambda}}\, \rho ds}{\int_0^L \rho\, ds}\hat{\boldsymbol{\lambda}} = \frac{\rho(s^2/2)|_0^L}{\rho(1)|_0^L}\hat{\boldsymbol{\lambda}} = \frac{\rho(L^2/2)}{\rho L}\hat{\boldsymbol{\lambda}} = (L/2)\hat{\boldsymbol{\lambda}},$$

again showing that the center of mass is in the middle. □

Note, one can treat the center of mass vector calculations as separate scalar equations, one for each component. For example:

$$\hat{\boldsymbol{\imath}} \cdot \left\{ \vec{\boldsymbol{r}}_{\text{cm}} = \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{\text{tot}}} \right\} \quad \Rightarrow \quad r_{x\text{cm}} = x_{\text{cm}} = \frac{\int x\, dm}{m_{\text{tot}}}.$$

Finally, there is no law that says you have to use the best coordinate system. One is free to make trouble for oneself and use an inconvenient coordinate system.

Example. Use the xy coordinates of fig. **??**c to find the center of mass of the rod.

$$x_{\text{cm}} = \frac{\int x dm}{m_{\text{tot}}} = \frac{\int_{-\ell_1}^{\ell_2} \overbrace{s\cos\theta}^{x}\, \rho ds}{\int_0^L \rho\, ds} = \frac{\rho\cos\theta \frac{s^2}{2}|_{-\ell_1}^{\ell_2}}{\rho(1)|_{-\ell_1}^{\ell_2}} = \frac{\rho\cos\theta\frac{(\ell_2^2-\ell_1^2)}{2}}{\rho(\ell_1+\ell_2)} = \frac{\cos\theta(\ell_2-\ell_1)}{2}$$

Similarly $y_{\text{cm}} = \sin\theta(\ell_2 - \ell_1)/2$ so

$$\vec{\boldsymbol{r}}_{\text{cm}} = \frac{\ell_2 - \ell_1}{2}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})$$

which still describes the point at the middle of the rod. □

The most commonly needed center of mass that can be found analytically but not directly from symmetry is that of a triangle (see box **??** on page **??**). You can find more examples using integration to find the center of mass (or centroid) in your calculus text.

Center of mass and centroid

For objects with uniform material density we have

$$\vec{r}_{\rm cm} = \frac{\int \vec{r}\, dm}{m_{\rm tot}} = \frac{\int_V \vec{r}\, \rho dV}{\int_V \rho dV} = \frac{\rho \int_V \vec{r}\, dV}{\rho \int_V dV} = \frac{\int_V \vec{r}\, dV}{V}$$

where the last expression is just the formula for geometric centroid. Analogous calculations hold for 2D and 1D geometric objects. Thus for objects with density that does not vary from point to point, the geometric centroid and the center of mass coincide.

Center of mass and symmetry

The center of mass respects any symmetry in the mass distribution of a system. If the word 'middle' has unambiguous meaning in English then that is the location of the center of mass, as for the rod of fig. **??** and the other examples in fig. **??**.

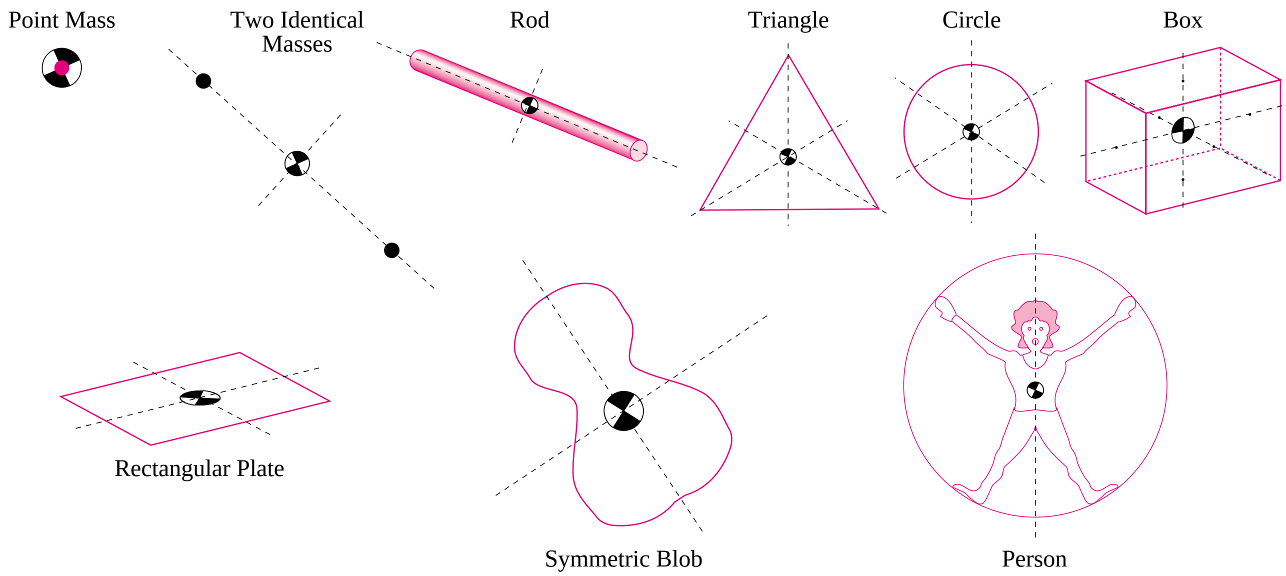


Figure 2.70: The center of mass and the geometric centroid share the symmetries of the object. (Filename:tfigure3.com.symm)

Systems of systems and composite objects

Another way of interpreting the formula

$$\vec{r}_{\rm cm} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2 + \cdots}{m_1 + m_2 + \cdots}$$

is that the m's are the masses of subsystems, not just points, and that the $\vec{r}_{\rm i}$ are the positions of the centers of mass of these systems. This subdivision is justified in box **??** on page **??**. The center of mass of a single complex shaped object can be found by treating it as an assembly of simpler objects.

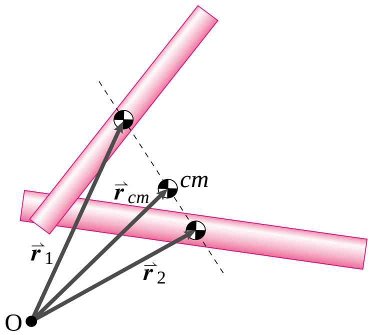


Figure 2.71: Center of mass of two rods

(Filename:tfigure3.com.tworods)

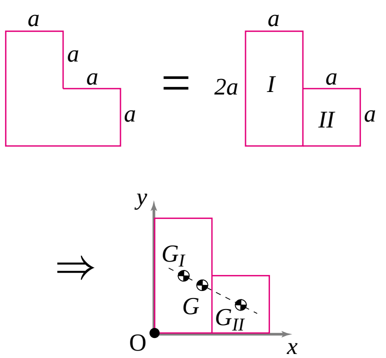


Figure 2.72: The center of mass of the 'L' shaped object can be found by thinking of it as a rectangle plus a square.

(Filename:tfigure3.1.Lshaped)

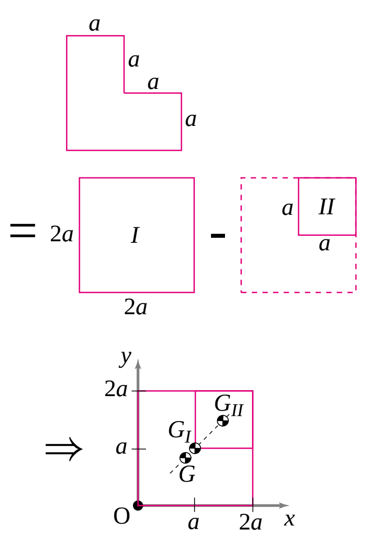


Figure 2.73: Another way of looking at the 'L' shaped object is as a square minus a smaller square in its upper right-hand corner.

(Filename:tfigure3.1.Lshaped.a)

Example: **Two rods**

The center of mass of two rods shown in figure **??** can be found as

$$\vec{r}_{\text{cm}} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2}$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the centers of mass of each rod and m_1 and m_2 are the masses. □

Example: **'L' shaped plate**

Consider the plate with uniform mass per unit area ρ.

$$\begin{aligned}
\vec{r}_{\text{G}} &= \frac{\vec{r}_{\text{I}} m_I + \vec{r}_{\text{II}} m_{II}}{m_I + m_{II}} \\
&= \frac{(\frac{a}{2}\hat{\imath} + a\hat{\jmath})(2\rho a^2) + (\frac{3}{2}a\hat{\imath} + \frac{a}{2}\hat{\jmath})(\rho a^2)}{(2\rho a^2) + (\rho a^2)} \\
&= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).
\end{aligned}$$

□

Composite objects using subtraction

It is sometimes useful to think of an object as composed of pieces, some of which have negative mass.

Example: **'L' shaped plate, again**

Reconsider the plate from the previous example.

$$\begin{aligned}
\vec{r}_{\text{G}} &= \frac{\vec{r}_{\text{I}} m_I + \vec{r}_{\text{II}} m_{II}}{m_I + m_{II}} \\
&= \frac{(a\hat{\imath} + a\hat{\jmath})(\rho(2a)^2) + (\frac{3}{2}a\hat{\imath} + \frac{3}{2}a\hat{\jmath})\overbrace{(-\rho a^2)}^{m_{II}}}{(\rho(2a)^2) + \underbrace{(-\rho a^2)}_{m_{II}}} \\
&= \frac{5}{6}a(\hat{\imath} + \hat{\jmath}).
\end{aligned}$$

□

Center of gravity

The force of gravity on each little bit of an object is $g m_i$ where g is the local gravitational 'constant' and m_i is the mass of the bit. For objects that are small compared to the radius of the earth (a reasonable assumption for all but a few special engineering calculations) the gravity constant is indeed constant from one point on the object to another (see box A.1 on page A.1 for a discussion of the meaning and history of g.)

Not only that, all the gravity forces point in the same direction, down. (For engineering purposes, the two intersecting lines that go from your two hands to the center of the earth are parallel.). Lets call this the $-\hat{\boldsymbol{k}}$ direction. So the net force of gravity on an object is:

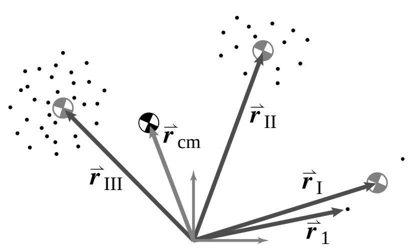
$$\begin{aligned}\vec{\boldsymbol{F}}_{\text{net}} &= \sum \vec{\boldsymbol{F}}_{\text{i}} = \sum m_i g(-\hat{\boldsymbol{k}}) = -mg\hat{\boldsymbol{k}} && \text{for discrete systems, and}\\ &= \int d\vec{\boldsymbol{F}} = \int \underbrace{-g\hat{\boldsymbol{k}}\,dm}_{d\vec{\boldsymbol{F}}} = -mg\hat{\boldsymbol{k}} && \text{for continuous systems.}\end{aligned}$$

That's easy, the billions of gravity forces on an objects microscopic constituents add up to mg pointed down. What about the net moment of the gravity forces? The answer turns out to be simple. The top line of the calculation below poses the question, the last line gives the lucky answer.①

① We do the calculation here using the $\int$ notation for sums. But it could be done just as well using $\sum$.

$$\begin{aligned}\vec{\boldsymbol{M}}_{\text{C}} &= \int \vec{\boldsymbol{r}} \times d\vec{\boldsymbol{F}} && \text{The net moment with respect to C.}\\ &= \int \vec{\boldsymbol{r}}_{/\text{C}} \times \left(-g\hat{\boldsymbol{k}}\,dm\right) && \text{A force bit is gravity acting on a mass bit.}\\ &= \left(\int \vec{\boldsymbol{r}}_{/\text{C}} dm\right) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Cross product distributive law } (g, \hat{\boldsymbol{k}} \text{ are constants).}\\ &= (\vec{\boldsymbol{r}}_{\text{cm/C}} m) \times \left(-g\hat{\boldsymbol{k}}\right) && \text{Definition of center of mass.}\\ &= \vec{\boldsymbol{r}}_{\text{cm/C}} \times \left(-mg\hat{\boldsymbol{k}}\right) && \text{Re-arranging terms.}\end{aligned}$$

2.10 *THEORY*
Why can subsystems be treated like particles when finding the center of mass?



Lets look at the collection of 47 particles above and then think of it as a set of three subsystems: I, II, and III with 2, 14, and 31 particles respectively. We treat masses 1 and 2 as subsystem I with center of mass $\vec{\boldsymbol{r}}_{\text{I}}$ and total mass m_I. Similarly, we call subsystem II masses m_3 to m_{16}, and subsystem III, masses m_{17} to m_{47}. We can calculate the center of mass of the system by treating it as 47 particles, or we can re-arrange the sum as follows:

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2 + \cdots + \vec{\boldsymbol{r}}_{46} m_{46} + \vec{\boldsymbol{r}}_{47} m_{47}}{m_1 + m_2 + \cdots + m_{47}}\\ &= \frac{\frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2}(m_1 + m_2)}{m_1 + m_2 + \cdots + m_{47}}\\ &\quad + \frac{\frac{\vec{\boldsymbol{r}}_3 m_3 + \cdots + \vec{\boldsymbol{r}}_{16} m_{16}}{m_3 + \cdots + m_{16}}(m_3 + \cdots + m_{16})}{m_1 + m_2 + \cdots + m_{47}}\\ &\quad + \frac{\frac{\vec{\boldsymbol{r}}_{17} m_{17} + \cdots + \vec{\boldsymbol{r}}_{47} m_{47}}{m_{17} + \cdots + m_{47}}(m_{17} + \cdots + m_{47})}{m_1 + m_2 + \cdots + m_{47}}\\ &= \frac{\vec{\boldsymbol{r}}_{\text{I}} m_I + \vec{\boldsymbol{r}}_{\text{II}} m_{II} + \vec{\boldsymbol{r}}_{\text{III}} m_{III}}{m_I + m_{II} + m_{III}}, \text{ where}\end{aligned}$$

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{I}} &= \frac{\vec{\boldsymbol{r}}_1 m_1 + \vec{\boldsymbol{r}}_2 m_2}{m_1 + m_2},\\ m_I &= m_1 + m_2\\ \vec{\boldsymbol{r}}_{\text{II}} & \quad etc.\end{aligned}$$

The formula for the center of mass of the whole system reduces to one that looks like a sum over three (aggregate) particles.

This idea is easily generalized to the integral formulae as well like this.

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{\int \vec{\boldsymbol{r}}\,dm}{\int dm}\\ &= \frac{\int_{\text{region 1}} \vec{\boldsymbol{r}}\,dm + \int_{\text{region 2}} \vec{\boldsymbol{r}}\,dm + \int_{\text{region 3}} \vec{\boldsymbol{r}}\,dm + \cdots}{\int_{\text{region 1}} dm + \int_{\text{region 2}} dm + \int_{\text{region 1}} dm + \cdots}\\ &= \frac{\vec{\boldsymbol{r}}_{\text{I}} m_I + \vec{\boldsymbol{r}}_{\text{II}} m_{II} + \vec{\boldsymbol{r}}_{\text{III}} m_{III} + \cdots}{m_I + m_{II} + m_{III} + \cdots}.\end{aligned}$$

The general idea of the calculations above is that center of mass calculations are basically big sums (addition), and addition is 'associative.'

$$= \vec{r}_{cm/C} \times \vec{F}_{net} \qquad \text{Express in terms of net gravity force.}$$

Thus the net moment is the same as for the total gravity force acting at the center of mass.

> The near-earth gravity forces acting on a system are *equivalent* to a single force, mg, acting at the system's center of mass.

For the purposes of calculating the net force and moment from near-earth (constant g) gravity forces, a system can be replaced by a point mass at the center of gravity. The words 'center of mass' and 'center of gravity' both describe the same point in space.

Although the result we have just found seems plain enough, here are two things to ponder about gravity when viewed as an inverse square law (and thus not constant like we have assumed) that may make the result above seem less obvious.

- The net gravity force on a sphere is indeed equivalent to the force of a point mass at the center of the sphere. It took the genius Isaac Newton 3 years to deduce this result and the reasoning involved is too advanced for this book.
- The net gravity force on systems that are not spheres is generally *not* equivalent to a force acting at the center of mass (this is important for the understanding of tides as well as the orientational stability of satellites).

A recipe for finding the center of mass of a complex system

You find the center of mass of a complex system by knowing the masses and mass centers of its components. You find each of these centers of mass by

- Treating it as a point mass, or
- Treating it as a symmetric body and locating the center of mass in the middle, or
- Using integration, or
- Using the result of an experiment (which we will discuss in statics), or
- Treating the component as a complex system itself and applying this very recipe.

The recipe is just an application of the basic definition of center of mass (eqn. **??**) but with our accumulated wisdom that the locations and masses in that sum can be the centers of mass and total masses of complex subsystems.

One way to arrange one's data is in a table or spreadsheet, like below. The first four columns are the basic data. They are the x, y, and z coordinates of the subsystem center of mass locations (relative to some clear reference point), and the masses of the subsystems, one row for each of the N subsystems.

Subsys#	1	2	3	4	5	6	7
Subsys 1	x_1	y_1	z_1	m_1	m_1x_1	m_1y_1	m_1z_1
Subsys 2	x_2	y_2	z_2	m_2	m_2x_2	m_2y_2	m_2z_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Subsys N	x_N	y_N	z_N	m_N	m_Nx_N	m_Ny_N	m_Nz_N
Row N+1 sums				$m_{tot} = \sum m_i$	$\sum m_ix_i$	$\sum m_iy_i$	$\sum m_iz_i$

					x_{cm}	y_{cm}	z_{cm}
Result					$\frac{\sum m_ix_i}{m_{tot}}$	$\frac{\sum m_iy_i}{m_{tot}}$	$\frac{\sum m_iz_i}{m_{tot}}$

One next calculates three new columns (5,6, and 7) which come from each coordinate multiplied by its mass. For example the entry in the 6th row and 7th column is the z component of the 6th subsystem's center of mass multiplied by the mass of the 6th subsystem. Then one sums columns 4 through 7. The sum of column 4 is the total mass, the sums of columns 5 through 7 are the total mass-weighted positions. Finally the result, the system center of mass coordinates, are found by dividing columns 5-7 of row N+1 by column 4 of row N+1.

Of course, there are multiple ways of systematically representing the data. The spreadsheet-like calculation above is just one way to organize the calculation.

Summary of center of mass

All discussions in mechanics make frequent reference to the concept of center of mass because

> *For systems with distributed mass, the expressions for gravitational moment, linear momentum, angular momentum, and energy are all simplified by using the center of mass.*

Simple center of mass calculations also can serve as a check of a more complicated analysis. For example, after a computer simulation of a system with many moving parts is complete, one way of checking the calculation is to see if the whole system's center of mass moves as would be expected by applying the net external force to the system. These formulas tell the whole story if you know how to use them:

$$\vec{\boldsymbol{r}}_{\mathrm{cm}} = \frac{\sum \vec{\boldsymbol{r}}_{\mathrm{i}} m_i}{m_{tot}} \quad \text{for discrete systems or systems of systems}$$

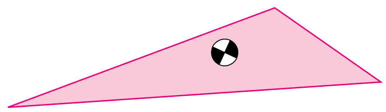
$$= \frac{\int \vec{\boldsymbol{r}}\, dm}{m_{tot}} \quad \text{for continuous systems}$$

$$m_{\mathrm{tot}} = \sum \mathrm{m}_i \quad \text{for discrete systems or systems of systems}$$

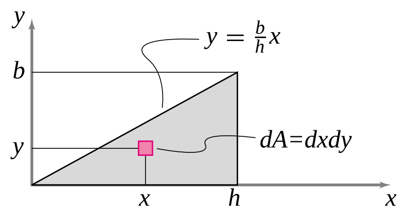
$$= \int dm \quad \text{for continuous systems.}$$

2.11 The center of mass of a uniform triangle is a third of the way up from the base

The center of mass of a 2D uniform triangular region is the centroid of the area.



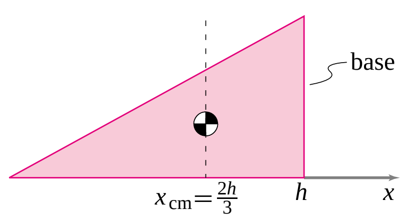
First we consider a right triangle with perpendicular sides b and h



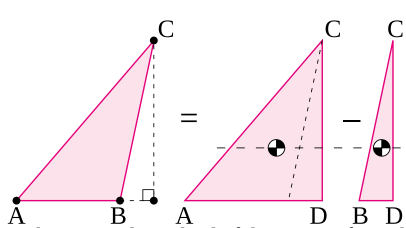
and find the x coordinate of the centroid as

$$
\begin{aligned}
x_{\mathrm{cm}} A &= \int x\, dA \\
&= \int_0^h \left[\int_0^{\frac{b}{h}x} x\, dy \right] dx = \int_0^h [xy]\Big|_{y=0}^{y=\frac{b}{h}x} dx \\
x_{\mathrm{cm}} \left(\frac{bh}{2} \right) &= \int_0^h x \left(\frac{b}{h} x \right) dx = \frac{b}{h} \frac{x^3}{3} \Big|_0^h = \frac{bh^2}{3}
\end{aligned}
$$

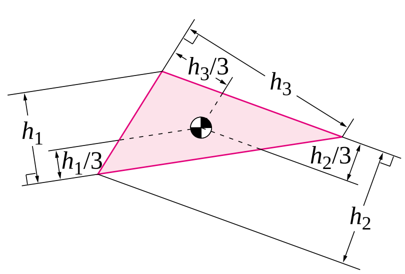
$\Rightarrow \quad x_{\mathrm{cm}} = \frac{2h}{3}$, a third of the way to the left of the vertical base on the right. By similar reasoning, but in the y direction, the centroid is a third of the way up from the base.



The center of mass of an arbitrary triangle can be found by treating it as the sum of two right triangles

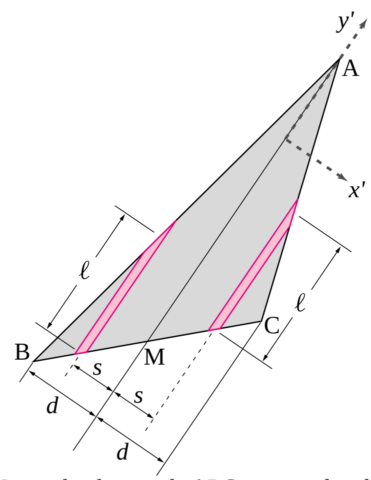


so the centroid is a third of the way up from the base of any triangle. Finally, the result holds for all three bases. Summarizing, the centroid of a triangle is at the point one third up from each of the bases.



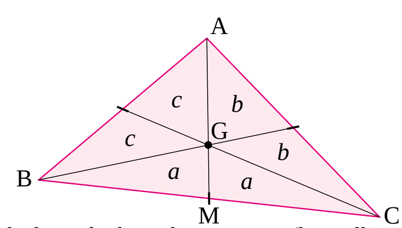
Non-calculus approach

Consider the line segment from A to the midpoint M of side BC.



We can divide triangle ABC into equal width strips that are parallel to AM. We can group these strips into pairs, each a distance s from AM. Because M is the midpoint of BC, by proportions each of these strips has the same length ℓ. Now in trying to find the distance of the center of mass from the line AM we notice that all contributions to the sum come in canceling pairs because the strips are of equal area and equal distance from AM but on opposite sides. Thus the centroid is on AM. Likewise for all three sides. Thus the centroid is at the point of intersection of the three side bisectors.

That the three side bisectors intersect a third of the way up from the three bases can be reasoned by looking at the 6 triangles formed by the side bisectors.



The two triangles marked a and a have the same area (lets call it a) because they have the same height and bases of equal length (BM and CM). Similar reasoning with the other side bisectors shows that the pairs marked b have equal area and so have the pairs marked c. But the triangle ABM has the same base and height and thus the same area as the triangle ACM. So $a + b + b = a + c + c$. Thus $b = c$ and by similar reasoning $a = b$ and all six little triangles have the same area. Thus the area of big triangle ABC is 3 times the area of GBC. Because ABC and GBC share the base BC, ABC must have 3 times the height as GBC, and point G is thus a third of the way up from the base.

Where is the middle of a triangle?

We have shown that the centroid of a triangle is at the point that is at the intersection of: the three side bisectors; the three area bisectors (which are the side bisectors); and the three lines one third of the way up from the three bases.

If the triangle only had three equal point masses on its vertices the center of mass lands on the same place. Thus the ‘middle’ of a triangle seems pretty well defined. *But*, there is some ambiguity. If the triangle were made of bars along each edge, each with equal cross sections, the center of mass would be in a different location for all but equilateral triangles. Also, the three angle bisectors of a triangle do not intersect at the centroid. Unless we define middle to mean centroid, the “middle” of a triangle is not well defined.

SAMPLE 2.39 *Center of mass in 1-D:* Three particles (point masses) of mass 2 kg, 3 kg, and 3 kg, are welded to a straight massless rod as shown in the figure. Find the location of the center of mass of the assembly.

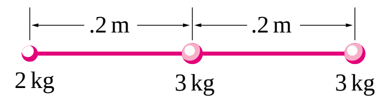


Figure 2.74: (Filename:sfig2.cm.1D)

Solution Let us select the first mass, $m_1 = 2\,\mathrm{kg}$, to be at the origin of our co-ordinate system with the x-axis along the rod. Since all the three masses lie on the x-axis, the center of mass will also lie on this axis. Let the center of mass be located at x_{cm} on the x-axis. Then,

$$
\begin{aligned}
m_{\mathrm{tot}} x_{cm} &= \sum_{i=1}^{3} m_i x_i = m_1 x_1 + m_2 x_2 + m_3 x_3 \\
&= m_1(0) + m_2(\ell) + m_3(2\ell) \\
\Rightarrow \quad x_{cm} &= \frac{m_2 \ell + m_3 2\ell}{m_1 + m_2 + m_3} \\
&= \frac{3\,\cancel{\mathrm{k}}\mathrm{g} \cdot 0.2\,\mathrm{m} + 3\,\cancel{\mathrm{k}}\mathrm{g} \cdot 0.4\,\mathrm{m}}{(2+3+3)\,\cancel{\mathrm{k}}\mathrm{g}} \\
&= \frac{1.8\,\mathrm{m}}{8} = 0.225\,\mathrm{m}.
\end{aligned}
$$

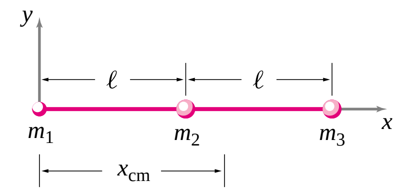


Figure 2.75: (Filename:sfig2.cm.1Da)

$$\boxed{x_{cm} = 0.225\,\mathrm{m}}$$

Alternatively, we could find the center of mass by first replacing the two 3 kg masses with a single 6 kg mass located in the middle of the two masses (the center of mass of the two equal masses) and then calculate the value of x_{cm} for a two particle system consisting of the 2 kg mass and the 6 kg mass (see Fig. **??**):

$$x_{cm} = \frac{6\,\mathrm{kg} \cdot 0.3\,\mathrm{m}}{8\,\mathrm{kg}} = \frac{1.8\,\mathrm{m}}{8} = 0.225\,\mathrm{m}.$$

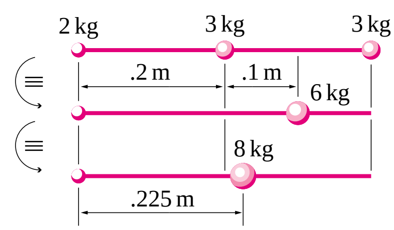


Figure 2.76: (Filename:sfig2.cm.1Db)

SAMPLE 2.40 *Center of mass in 2-D:* Two particles of mass $m_1 = 1\,\mathrm{kg}$ and $m_2 = 2\,\mathrm{kg}$ are located at coordinates (1m, 2m) and (-2m, 5m), respectively, in the xy-plane. Find the location of their center of mass.

Solution Let $\vec{\boldsymbol{r}}_{\mathrm{cm}}$ be the position vector of the center of mass. Then,

$$
\begin{aligned}
m_{\mathrm{tot}} \vec{\boldsymbol{r}}_{\mathrm{cm}} &= m_1 \vec{\boldsymbol{r}}_1 + m_2 \vec{\boldsymbol{r}}_2 \\
\Rightarrow \quad \vec{\boldsymbol{r}}_{\mathrm{cm}} &= \frac{m_1 \vec{\boldsymbol{r}}_1 + m_2 \vec{\boldsymbol{r}}_2}{m_{\mathrm{tot}}} = \frac{m_1 \vec{\boldsymbol{r}}_1 + m_2 \vec{\boldsymbol{r}}_2}{m_1 + m_2} \\
&= \frac{1\,\cancel{\mathrm{k}}\mathrm{g}(1\,\mathrm{m}\hat{\boldsymbol{\imath}} + 2\,\mathrm{m}\hat{\boldsymbol{\jmath}}) + 2\,\cancel{\mathrm{k}}\mathrm{g}(-2\,\mathrm{m}\hat{\boldsymbol{\imath}} + 5\,\mathrm{m}\hat{\boldsymbol{\jmath}})}{3\,\cancel{\mathrm{k}}\mathrm{g}} \\
&= \frac{(1\,\mathrm{m} - 4\,\mathrm{m})\hat{\boldsymbol{\imath}} + (2\,\mathrm{m} + 10\,\mathrm{m})\hat{\boldsymbol{\jmath}}}{3} = -1\,\mathrm{m}\hat{\boldsymbol{\imath}} + 4\,\mathrm{m}\hat{\boldsymbol{\jmath}}.
\end{aligned}
$$

Thus the center of mass is located at the coordinates(-1m, 4m).

$$\boxed{(x_{cm}, y_{cm}) = (-1\,\mathrm{m}, 4\,\mathrm{m})}$$

Geometrically, this is just a 1-D problem like the previous sample. The center of mass has to be located on the straight line joining the two masses. Since the center of mass is a point about which the distribution of mass is *balanced*, it is easy to see (see Fig. **??**) that the center of mass must lie one-third way from m_2 on the line joining the two masses so that $2\,\mathrm{kg} \cdot (d/3) = 1\,\mathrm{kg} \cdot (2d/3)$.

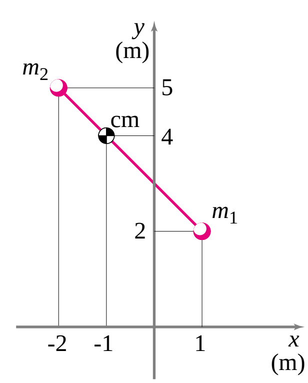


Figure 2.77: (Filename:sfig2.cm.2Da)

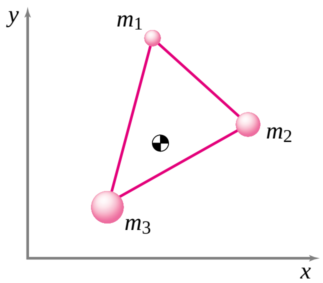


Figure 2.78: (Filename:sfig2.4.2)

SAMPLE 2.41 *Location of the center of mass.* A structure is made up of three point masses, $m_1 = 1\,\text{kg}$, $m_2 = 2\,\text{kg}$ and $m_3 = 3\,\text{kg}$, connected rigidly by massless rods. At the moment of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\text{m/s}\hat{\boldsymbol{\imath}}$, $2\,\text{m/s}(\hat{\boldsymbol{\imath}} - 1.5\hat{\boldsymbol{\jmath}})$ and $1\,\text{m/s}\hat{\boldsymbol{\jmath}}$, respectively. Find the coordinates of the center of mass of the structure.

Solution Just for fun, let us do this problem two ways — first using scalar equations for the coordinates of the center of mass, and second, using vector equations for the position of the center of mass.

(a) **Scalar calculations:** Let (x_{cm}, y_{cm}) be the coordinates of the mass-center. Then from the definition of mass-center,

$$\begin{aligned} x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 1.25\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.75\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{7.5\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.25\,\text{m}. \end{aligned}$$

Similarly,

$$\begin{aligned} y_{cm} &= \frac{\sum m_i y_i}{\sum m_i} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} \\ &= \frac{1\,\text{kg}\cdot 3\,\text{m} + 2\,\text{kg}\cdot 2\,\text{m} + 3\,\text{kg}\cdot 0.5\,\text{m}}{1\,\text{kg} + 2\,\text{kg} + 3\,\text{kg}} \\ &= \frac{8.5\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} = 1.42\,\text{m}. \end{aligned}$$

Thus the center of mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\text{m},\ 1.42\,\text{m})}$$

(b) **Vector calculations:** Let $\vec{r}_{\text{cm}}$ be the position vector of the mass-center. Then,

$$\begin{aligned} m_{\text{tot}}\vec{r}_{\text{cm}} &= \sum_{i=1}^{3} m_i\vec{r}_{\text{i}} = m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 \\ \Rightarrow\quad \vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3} \end{aligned}$$

Substituting the values of m_1, m_2, and m_3, and $\vec{r}_1 = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}$, $\vec{r}_2 = 2\,\text{m}\hat{\boldsymbol{\imath}} + 2\,\text{m}\hat{\boldsymbol{\jmath}}$, and $\vec{r}_3 = 0.75\,\text{m}\hat{\boldsymbol{\imath}} + 0.5\,\text{m}\hat{\boldsymbol{\jmath}}$, we get,

$$\begin{aligned} \vec{r}_{\text{cm}} &= \frac{1\,\text{kg}\cdot(1.25\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})\,\text{m} + 2\,\text{kg}\cdot(2\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{m} + 3\,\text{kg}\cdot(0.75\hat{\boldsymbol{\imath}} + 0.5\hat{\boldsymbol{\jmath}})\,\text{m}}{(1+2+3)\,\text{kg}} \\ &= \frac{(7.5\hat{\boldsymbol{\imath}} + 8.5\hat{\boldsymbol{\jmath}})\,\cancel{\text{kg}}\cdot\text{m}}{6\,\cancel{\text{kg}}} \\ &= 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}} \end{aligned}$$

which, of course, gives the same location of the mass-center as above.

$$\boxed{\vec{r}_{\text{cm}} = 1.25\,\text{m}\hat{\boldsymbol{\imath}} + 1.42\,\text{m}\hat{\boldsymbol{\jmath}}}$$

SAMPLE 2.42 *Center of mass of a bent bar:* A uniform bar of mass 4 kg is bent in the shape of an asymmetric 'Z' as shown in the figure. Locate the center of mass of the bar.

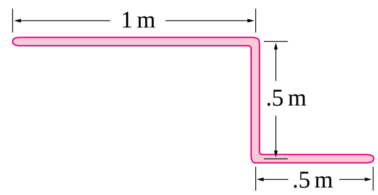


Figure 2.79: (Filename:sfig2.cm.wire)

Solution Since the bar is uniform along its length, we can divide it into three straight segments and use their individual mass-centers (located at the geometric centers of each segment) to locate the center of mass of the entire bar. The mass of each segment is proportional to its length. Therefore, if we let $m_2 = m_3 = m$, then $m_1 = 2m$; and $m_1 + m_2 + m_3 = 4m = 4\,\text{kg}$ which gives $m = 1\,\text{kg}$. Now, from Fig. **??**,

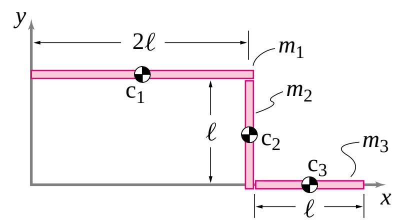


Figure 2.80: (Filename:sfig2.cm.wire.a)

$$\begin{aligned}
\vec{r}_1 &= \ell\hat{\imath} + \ell\hat{\jmath} \\
\vec{r}_2 &= 2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath} \\
\vec{r}_3 &= (2\ell + \frac{\ell}{2})\hat{\imath} = \frac{5\ell}{2}\hat{\imath}
\end{aligned}$$

So,

$$\begin{aligned}
\vec{r}_{\text{cm}} &= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_{\text{tot}}} \\
&= \frac{2m(\ell\hat{\imath} + \ell\hat{\jmath}) + m(2\ell\hat{\imath} + \frac{\ell}{2}\hat{\jmath}) + m(\frac{5\ell}{2}\hat{\imath})}{4m} \\
&= \frac{\not{m}\ell(2\hat{\imath} + 2\hat{\jmath} + 2\hat{\imath} + \frac{1}{2}\hat{\jmath} + \frac{5}{2}\hat{\imath})}{4\not{m}} \\
&= \frac{\ell}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= \frac{0.5\,\text{m}}{8}(13\hat{\imath} + 5\hat{\jmath}) \\
&= 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}.
\end{aligned}$$

$$\boxed{\vec{r}_{\text{cm}} = 0.812\,\text{m}\hat{\imath} + 0.312\,\text{m}\hat{\jmath}}$$

Geometrically, we could find the center of mass by considering two masses at a time, connecting them by a line and locating their mass-center on that line, and then repeating the process as shown in Fig. **??**. The center of mass of m_2 and m_3 (each of

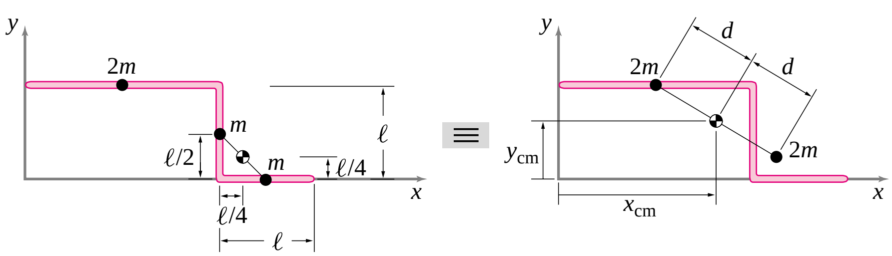


Figure 2.81: (Filename:sfig2.cm.wire.b)

mass m) is at the mid-point of the line connecting the two masses. Now, we replace these two masses with a single mass $2m$ at their mass-center. Next, we connect this mass-center and m_1 with a line and find their combined mass-center at the mid-point of this line. The mass-center just found is the center of mass of the entire bar.

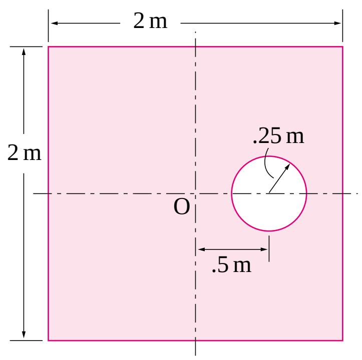


Figure 2.82: (Filename:sfig2.cm.plate)

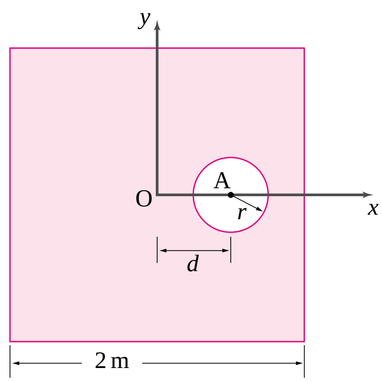


Figure 2.83: (Filename:sfig2.cm.plate.a)

SAMPLE 2.43 *Shift of mass-center due to cut-outs:* A 2 m × 2 m uniform square plate has mass $m = 4\,\text{kg}$. A circular section of radius 250 mm is cut out from the plate as shown in the figure. Find the center of mass of the plate.

Solution Let us use an xy-coordinate system with its origin at the geometric center of the plate and the x-axis passing through the center of the cut-out. Since the plate and the cut-out are symmetric about the x-axis, the new center of mass must lie somewhere on the x-axis. Thus, we only need to find x_{cm} (since $y_{cm} = 0$). Let m_1 be the mass of the plate with the hole, and m_2 be the mass of the circular cut-out. Clearly, $m_1 + m_2 = m = 4\,\text{kg}$. The center of mass of the circular cut-out is at A, the center of the circle. The center of mass of the intact square plate (without the cut-out) must be at O, the middle of the square. Then,

$$\begin{aligned} m_1 x_{cm} + m_2 x_A &= m x_O = 0 \\ \Rightarrow \quad x_{cm} &= -\frac{m_2}{m_1} x_A. \end{aligned}$$

Now, since the plate is uniform, the masses m_1 and m_2 are proportional to the surface areas of the geometric objects they represent, *i.e.*,

$$\frac{m_2}{m_1} = \frac{\pi r^2}{\ell^2 - \pi r^2} = \frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi}.$$

Therefore,

$$\begin{aligned} x_{cm} &= -\frac{m_2}{m_1} d = -\frac{\pi}{\left(\frac{\ell}{r}\right)^2 - \pi} d \qquad (2.31) \\ &= -\frac{\pi}{\left(\frac{2\,\text{m}}{.25\,\text{m}}\right)^2 - \pi} \cdot 0.5\,\text{m} \\ &= -25.81 \times 10^{-3}\,\text{m} = -25.81\,\text{mm} \end{aligned}$$

Thus the center of mass shifts to the left by about 26 mm because of the circular cut-out of the given size.

$$\boxed{x_{cm} = -25.81\,\text{mm}}$$

Comments: The advantage of finding the expression for x_{cm} in terms of r and ℓ as in eqn. (**??**) is that you can easily find the center of mass of any size circular cut-out located at any distance d on the x-axis. This is useful in design where you like to select the size or location of the cut-out to have the center of mass at a particular location.

SAMPLE 2.44 *Center of mass of two objects:* A square block of side 0.1 m and mass 2 kg sits on the side of a triangular wedge of mass 6 kg as shown in the figure. Locate the center of mass of the combined system.

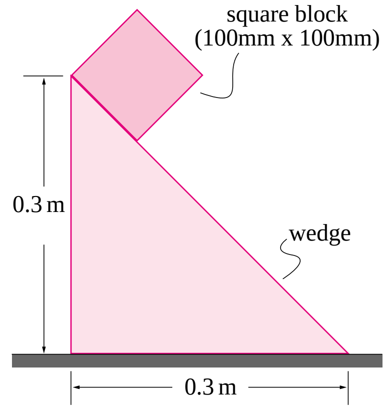


Figure 2.84: (Filename:sfig2.cm.2blocks)

Solution The center of mass of the triangular wedge is located at $h/3$ above the base and $\ell/3$ to the right of the vertical side. Let m_1 be the mass of the wedge and $\vec{\boldsymbol{r}}_1$ be the position vector of its mass-center. Then, referring to Fig. **??**,

$$\vec{\boldsymbol{r}}_1 = \frac{\ell}{3}\hat{\boldsymbol{\imath}} + \frac{h}{3}\hat{\boldsymbol{\jmath}}.$$

The center of mass of the square block is located at its geometric center C_2. From geometry, we can see that the line AE that passes through C_2 is horizontal since $\angle OAB = 45^o$ ($h = \ell = 0.3\,\text{m}$) and $\angle DAE = 45^o$. Therefore, the coordinates of C_2 are ($d/\sqrt{2}, h$). Let m_2 and $\vec{\boldsymbol{r}}_2$ be the mass and the position vector of the mass-center of the block, respectively. Then,

$$\vec{\boldsymbol{r}}_2 = \frac{d}{\sqrt{2}}\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}}.$$

Now, noting that $m_1 = 3m_2$ or $m_1 = 3m$, and $m_2 = m$ where $m = 2\,\text{kg}$, we find the center of mass of the combined system:

$$\begin{aligned}
\vec{\boldsymbol{r}}_{\text{cm}} &= \frac{m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2}{(m_1+m_2)} \\
&= \frac{3m(\frac{\ell}{3}\hat{\boldsymbol{\imath}} + \frac{h}{3}\hat{\boldsymbol{\jmath}}) + m(\frac{d}{\sqrt{2}}\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}})}{3m+m} \\
&= \frac{\not{m}[(\ell + \frac{d}{\sqrt{2}})\hat{\boldsymbol{\imath}} + 2h\hat{\boldsymbol{\jmath}}]}{4\not{m}} \\
&= \frac{1}{4}(\frac{d}{\sqrt{2}} + \ell)\hat{\boldsymbol{\imath}} + \frac{h}{2}\hat{\boldsymbol{\jmath}} \\
&= \frac{1}{4}(\frac{0.1\,\text{m}}{\sqrt{2}} + 0.3\,\text{m})\hat{\boldsymbol{\imath}} + \frac{0.3\,\text{m}}{2}\hat{\boldsymbol{\jmath}} \\
&= 0.093\,\text{m}\hat{\boldsymbol{\imath}} + 0.150\,\text{m}\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{r}}_{\text{cm}} = 0.093\,\text{m}\hat{\boldsymbol{\imath}} + 0.150\,\text{m}\hat{\boldsymbol{\jmath}}}$$

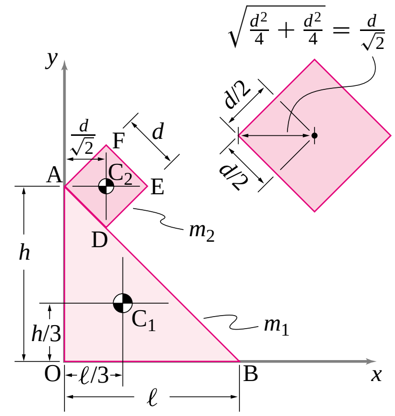


Figure 2.85: (Filename:sfig2.cm.2blocks.a)

Thus, the center of mass of the wedge and the block together is slightly closer to the side OA and higher up from the bottom OB than $C_1(0.1\,\text{m}, 0.1\,\text{m})$. This is what we should expect from the placement of the square block.

Note that we could have, again, used a 1-D calculation by placing a point mass $3m$ at C_1 and m at C_2, connected the two points by a straight line, and located the center of mass C on that line such that $CC_2 = 3CC_1$. You can verify that the distance from $C_1(0.1\,\text{m}, 0.1\,\text{m})$ to $C(.093\,\text{m}, 0.15\,\text{m})$ is one third the distance from C to $C_2(.071\,\text{m}, 0.3\,\text{m})$.